Lecture 3 - Probability Distributions

Objectives

- Learn about some famous discrete PDFs
- Learn about some famous continous PDFs

Discrete random variables

Coin flipping: Heads = 1, Tails = 0 - The Bernulli Distribution

Let X be the following random variable:

$$X = \begin{cases} 1, & \text{with probability } \theta, \\ 0, & \text{with probability } 1 - \theta, \end{cases}$$

where $\theta \in [0, 1]$ is the probability of getting heads (i.e., 1). The PDF of this random variable has two values:

$$p(X = 1|\theta) = \theta,$$

and

$$p(X = 0|\theta) = 1 - \theta.$$

The expected value is:

$$\mathrm{E}[X|\theta] = \sum_x xp(X = x|\theta) = 1 \times p(X = 1|\theta) + 0 \times p(X = 0|\theta) = \theta.$$

The "coin flipping" random variable is also known as a Bernulli random variable (or Bernulli trial). Typically, we write:

$$X = x|\theta \sim \mathrm{Bern}(X = x|\theta) = \theta^x(1 - \theta)^{1-x}.$$

Let's play with this random variable using scipy.stats (http://docs.scipy.org/doc/scipy/reference/stats.html), and specifically scipy.stats.bernulli (http://docs.scipy.org/doc/scipy/reference/generated/scipy.stats.bernoulli.html#scipy.stats.be

In [21]:
```
# Initialize the random variable
import scipy.stats as st
X = st.bernoulli(0.25)
# Take 10 samples:
X.rvs(20)
```

Out[21]: `array([0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 1, 0, 1, 0, 1, 0, 1])`

In [22]:
```
# You can also compute the mean of a random variable:
X.mean()
```

Out[22]: `0.25`

In [23]:
```
# The variance
X.var()
```

Out[23]: `0.1875`

In [24]:
```
# You can evaluate the PDF anywhere you want
# (called probability mass function):
X.pmf(0.5)
```

Out[24]: `0.0`

In [25]:
```
X.pmf(0.)
```

Out[25]: `0.75`

In [26]:
```
X.pmf(1.)
```

Out[26]: `0.25`

N Coin Flips - The Binomial Distribution

Consider an experiment with two possible outcomes: 1 and 0 (success and faillure). Let θ be the probability of success. Now assume that you perform the experiment N times. Let the random variable that counts the succesfull experiments. We say that X is a Binomial random variable. Using counting arguments, one can show that the PDF of X is:

$$p(X = k|N, \theta) = \mathrm{Bin}(X = k|N, \theta) = \left(\frac{N}{k}\right)\theta^k(1-\theta)^{N-k}.$$

We can also show that the mean is

$$\mathrm{E}[X|N, \theta] = N\theta,$$

and that the variance is

$$\mathrm{V}[X|N, \theta] = N\theta(1-\theta).$$

We can play with this using scipy.stats.binom
(http://docs.scipy.org/doc/scipy/reference/generated/scipy.stats.binom.html#scipy.stats.b

In [31]:
```
X = st.binom(10, 0.25) # N, theta
# Take samples.
X.rvs(10)
```

Out[31]:
```
array([4, 3, 5, 3, 1, 0, 4, 3, 1, 3])
```

In [34]:
```
# Get some statistics
mean, var, skew, kurt = X.stats(moments='mvsk')
print mean
print var
print skew
print kurt
```

```
2.5
1.875
0.36514837167
-0.0666666666667
```

Let's use the binomial we have just constructed to demonstrated yet another attribute of random variables, the **percent point function**. The percent point function, is actually the inverse of the CDF. For example:

In [37]:
```
X.ppf(0.01)
```

Out[37]:
```
0.0
```

which means that with probability 0.01, X, is less than 0, or

In [38]:
```
X.ppf(0.99)
```

Out[38]:
```
6.0
```

which means that with probability 0.99, X, is less than 6, or

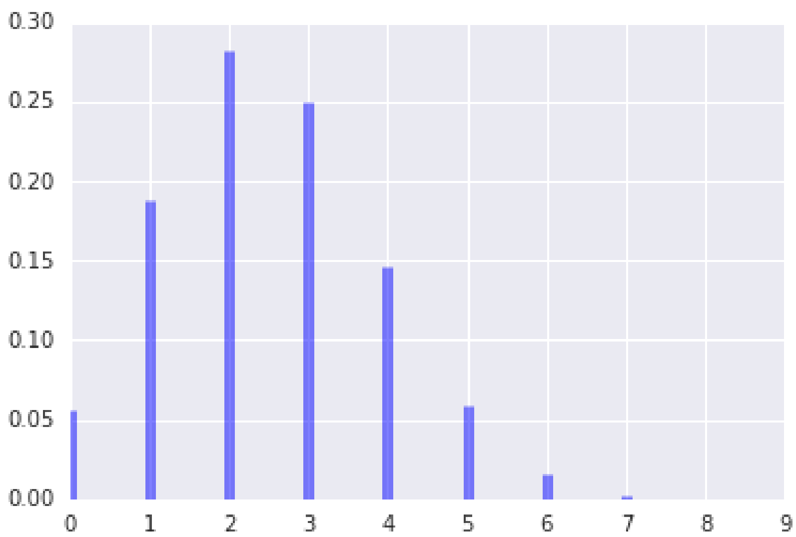
In [39]:
```
X.ppf(0.5)
```

Out[39]:
```
2.0
```

which means that with probability 0.5, X, is less than 2.

Now, let's actually visualize the Binomial distribution:

In [59]:
```
import matplotlib.pyplot as plt
%matplotlib inline
import numpy as np
from ipywidgets import interactive
def visualize_binomial(N=10, theta=0.25):
    X = st.binom(N, theta)
    x = np.arange(0, N)
    plt.vlines(x, 0, X.pmf(x), color='b', alpha=0.5, lw=5)
interactive(visualize_binomial, N=(1, 100, 1), theta=(0, 1., 0.01))
```



Other Important Discrete Random Variables

- Poisson random variable (https://en.wikipedia.org/wiki/Poisson_distribution) (see also scipy.stats.poisson (http://docs.scipy.org/doc/scipy/reference/generated/scipy.stats.poisson.html#scipy.stat which is used to describe the number of events occuring in a fixed time interval.

Continuous Random Variables

Uniform distribution

The uniform distribution over an interval $[a, b]$ can be used to describe our state of knowledge about random variable known to be within this interval, if nothing else is known about them. We say that X is a uniform random variable on the interval $[a, b]$, and write:

$$x|a, b \sim U(x|a, b) = \begin{cases} \frac{1}{b-a}, & \text{if } x \in [a, b], \\ 0, & \text{otherwise.} \end{cases}$$

You can play with it using scipy.stats
(http://docs.scipy.org/doc/scipy/reference/generated/scipy.stats.uniform.html#scipy.stats.uni

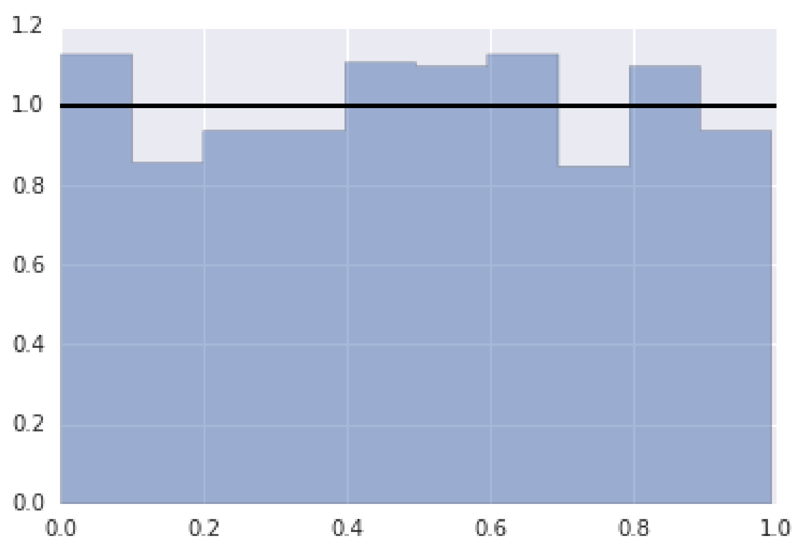
In [60]:
```
X = st.uniform(0, 1)
X.rvs(10)
```

Out[60]:
```
array([ 0.5334961 ,  0.16581931,  0.78658391,  0.85022527,  0.10600555,
        0.76270302,  0.43222264,  0.9725804 ,  0.937761  ,  0.16955123])
```

Let's take a large sample and draw the histogram:

In [67]:
```
x = np.linspace(0, 1, 100)
plt.plot(x, X.pdf(x), 'k-', lw=2)
plt.hist(X.rvs(1000), normed=True, histtype='stepfilled', alpha=0.5)
```

Out[67]:
```
(array([ 1.12835309,  0.8563394 ,  0.93693605,  0.93693605,  1.10820393,
         1.09812935,  1.12835309,  0.84626482,  1.09812935,  0.9369360
5]),
 array([  5.81261786e-04,   9.98409715e-02,   1.99100681e-01,
          2.98360391e-01,   3.97620101e-01,   4.96879810e-01,
          5.96139520e-01,   6.95399230e-01,   7.94658939e-01,
          8.93918649e-01,   9.93178359e-01]),
 <a list of 1 Patch objects>)
```



The Beta Distribution

The Beta distribution is suitable for parameters that take values in $[0, 1]$, but not necessarily in a uniform manner. The θ describing the probability of success in a Bernulli trial was such a parameter. Our state of knowledge about it, can thus be represented using a Beta distribution. The definition of the Beta distribution is conditional on two parameters $a, b > 0$ that control its shape:

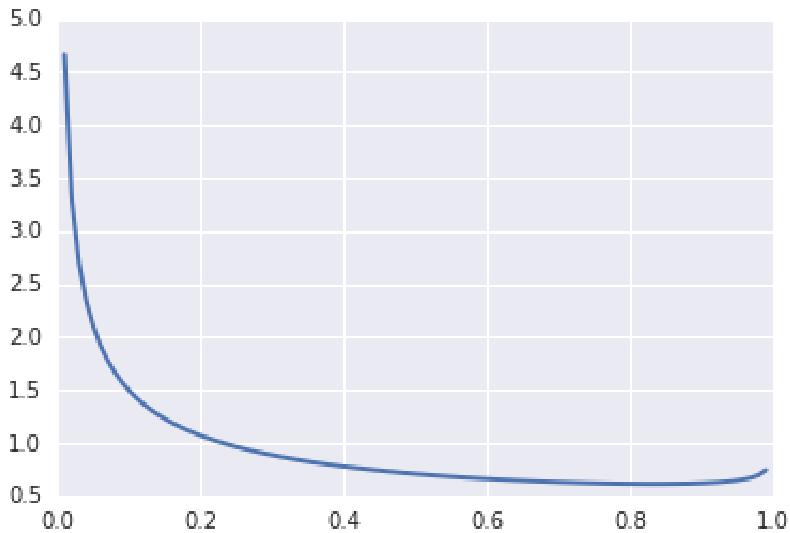
$$\theta | a, b \sim \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \theta^{a-1}(1-\theta)^{b-1},$$

where $\Gamma(\cdot)$ is the Gamma function (https://en.wikipedia.org/wiki/Gamma_function).

In [69]:
```
X = st.beta(0.5, 0.9)
X.rvs(10)
```

Out[69]:
```
array([ 0.00603264,  0.5214056 ,  0.09743458,  0.73979585,  0.06855319,
        0.70332611,  0.01049383,  0.89903704,  0.67018866,  0.01144496])
```

In [73]:
```
def visualize_beta(a=0.5, b=0.9):
    X = st.beta(a, b)
    x = np.linspace(0, 1, 100)
    plt.plot(x, X.pdf(x), lw=2)
interactive(visualize_beta, a=(0.001, 100., 0.1), b=(0.001, 100., 0.1))
```



Questions

Let's say that you wish to analyze a coin flipping experiment and you do not know what the probability of getting heads, θ, is. Therefore, you have to model θ as a random variable and assign a probability density to it. Since θ takes values between 0 and 1, you decide to assign a Beta distribution to it. Using the interactive tool above, try to find parameters a and b that describe the following states of knowledge:

- The coin is fair.
- The coin is slightly biased towards heads.
- The coin is slightly biased towards tails.
- The coin is definitely biased towards heads.
- The coin is definitely biased towards tails.
- The coin is definitely biased, but I don't know how.

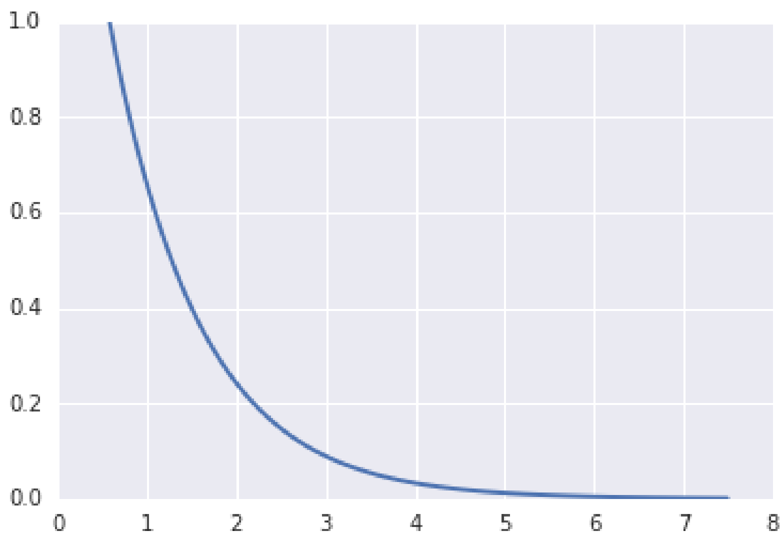
Exponential Random Variable

The Exponential Distribution (https://en.wikipedia.org/wiki/Exponential_distribution) is suitable for random variables that describe the time between events in a process in which events occure continuously and at a constant rate. In general, it can be used for random variables that are positive as a special case of the Gamma Distribution (https://en.wikipedia.org/wiki/Gamma_distribution). We write:

$$x|r \sim \mathrm{E}(x|r) = re^{-rx},$$

where $r > 0$ is known as the *rate parameter*.

In [110]:
```
def visualize_exponential(r=0.5):
    X = st.expon(r)
    x = np.linspace(X.ppf(0.001), X.ppf(0.999), 100)
    plt.plot(x, X.pdf(x), lw=2)
interactive(visualize_exponential, r=(0.001, 10, 0.01))
```



The Gaussian (Normal) Distribution

The Gaussian distribution is suitable for continuous random variables with known mean, μ, and variance, σ^2, with no other restrictions posed on them. We write:

$$x|\mu,\sigma \sim \mathrm{N}(x|\mu,\sigma) = \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}.$$

Of course:

$$\mathrm{E}[X|\mu,\sigma] = \mu,$$

and

$$\mathrm{V}[X|\mu,\sigma] = \sigma^2.$$

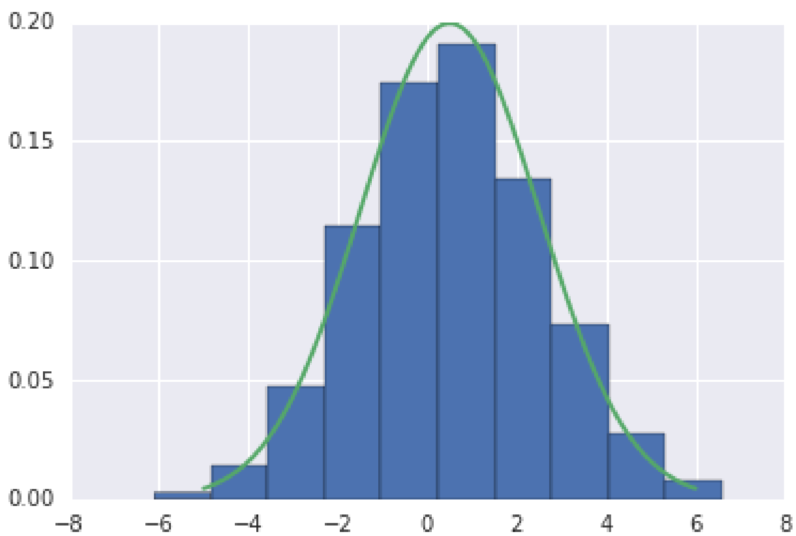
In [74]:
```
X = st.norm(loc=0.5, scale=2.)
```

In [75]:
```
print X.rvs(10)
```

```
[-1.40480312  0.32419139  3.67686951 -1.15332344  3.35935014  1.3678408
 -3.91793104 -0.37546934  2.39550956 -0.15666293]
```

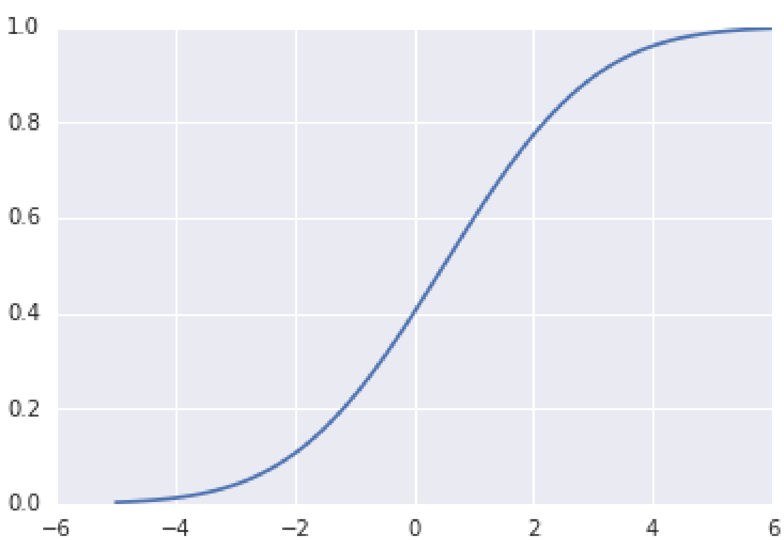
In [79]:
```
plt.hist(X.rvs(2000), normed=True)
x = np.linspace(-5, 6, 100)
plt.plot(x, X.pdf(x), lw=2)
```

Out[79]:
```
[<matplotlib.lines.Line2D at 0x110a94390>]
```



In [80]:
```
# The CDF of the Gaussian
plt.plot(x, X.cdf(x))
```

Out[80]:
```
[<matplotlib.lines.Line2D at 0x11009d610>]
```



The Central Limit Theorem

The sum of N independent and indentically distributed (iid) random variables starts to look like a Gaussian for large N. Mathematically, let $X_1, X_2, \dots$ be iid random variables with mean μ and variance σ^2 and define the running average of the random variables:

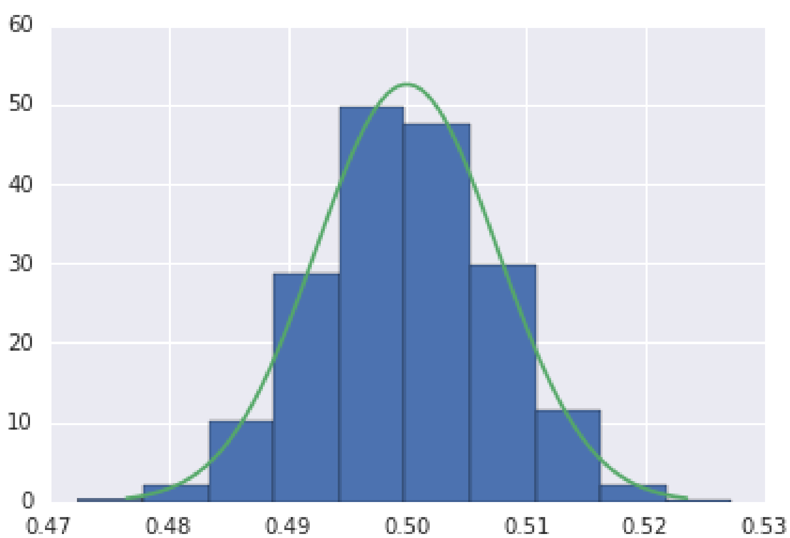
$$S_N = \frac{X_1 + \dots + X_N}{N}.$$

The Central Limit Theorem (CLT), states that:

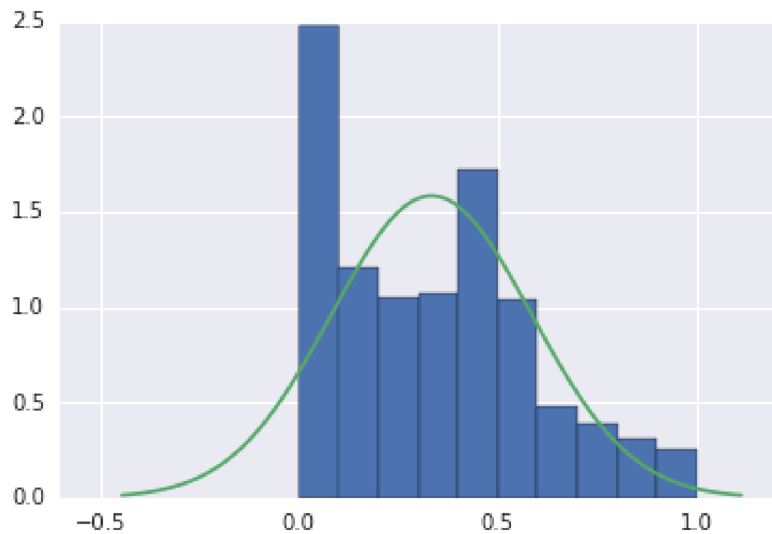
$$S_N \sim \mathrm{N}(S_N|\mu, \frac{\sigma^2}{N}),$$

for large N. We are not going to prove this, but let us test it numerically:

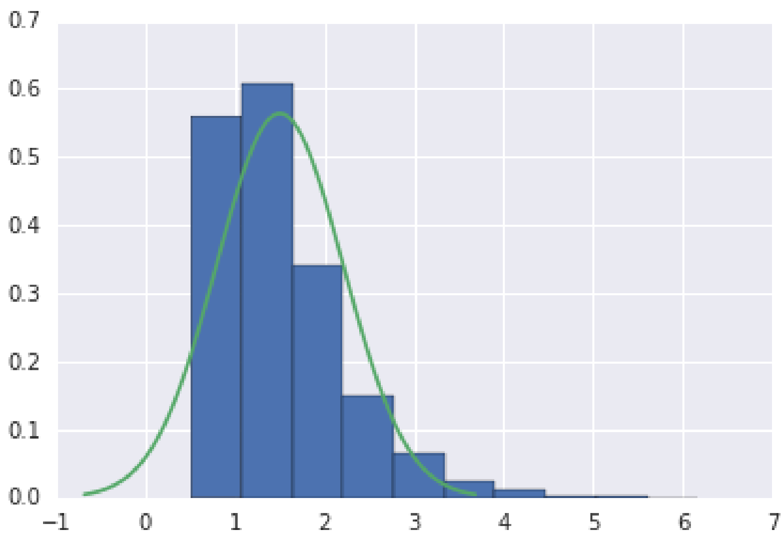
In [100]:
```
# Testing CLT for iid uniform random variables:
def visualize_uniform_clt(N=2):
    X = st.uniform()
    mu = X.mean()
    sigma = X.std()
    x = X.rvs((N, 10000))
    s = x.mean(axis=0)
    S_N_approx = st.norm(loc=mu, scale=sigma / np.sqrt(N))
    plt.hist(s, normed=True)
    xx = np.linspace(S_N_approx.ppf(0.001), S_N_approx.ppf(0.999), 100)
    plt.plot(xx, S_N_approx.pdf(xx))
interactive(visualize_uniform_clt, N=(1, 10000, 1))
```



In [102]:
```
# Testing CLT for iid beta random variables:
def visualize_beta_clt(N=2, a=0.25, b=0.5):
    X = st.beta(a, b)
    mu = X.mean()
    sigma = X.std()
    x = X.rvs((N, 10000))
    s = x.mean(axis=0)
    S_N_approx = st.norm(loc=mu, scale=sigma / np.sqrt(N))
    plt.hist(s, normed=True)
    xx = np.linspace(S_N_approx.ppf(0.001), S_N_approx.ppf(0.999), 100)
    plt.plot(xx, S_N_approx.pdf(xx))
interactive(visualize_beta_clt, N=(1, 10000, 1), a=(0.01, 10, 0.01), b=(0.0
```



In [105]:
```
# Testing CLT for iid exponential random variables:
def visualize_exponential_clt(N=2, r=0.5):
    X = st.expon(r)
    mu = X.mean()
    sigma = X.std()
    x = X.rvs((N, 10000))
    s = x.mean(axis=0)
    S_N_approx = st.norm(loc=mu, scale=sigma / np.sqrt(N))
    plt.hist(s, normed=True)
    xx = np.linspace(S_N_approx.ppf(0.001), S_N_approx.ppf(0.999), 100)
    plt.plot(xx, S_N_approx.pdf(xx))
interactive(visualize_exponential_clt, N=(1, 10000, 1), a=(0.01, 10, 0.01),
```



In [ ]: