Sensitivity analysis and stochastic simulations of non-equilibrium plasma flow

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SUMMARY

We study the parametric uncertainties involved in plasma flows and apply stochastic sensitivity analysis to rank the importance of all inputs to guide large-scale stochastic simulations. Specifically, we employ different gradient-based sensitivity methods, namely Morris, multi-element probabilistic collocation method on sparse grids, Quasi-Monte Carlo and Monte Carlo methods. These approaches go beyond the standard ‘One-At-a-Time’ sensitivity analysis and provide a measure of the non-linear interaction effects for the uncertain inputs. The objective is to perform systematic stochastic simulations of plasma flows treating only as stochastic processes the inputs with the highest sensitivity index, hence reducing substantially the computational cost. Two plasma flow examples are presented to demonstrate the capability and efficiency of the stochastic sensitivity analysis. The first one is a two-fluid model in a shock tube whereas the second one is a one-fluid/two-temperature model in flow past a cylinder. Copyright © 2009 John Wiley & Sons, Ltd.

KEY WORDS: stochastic simulations; generalized polynomial chaos; collocation projection; two-fluid model

1. INTRODUCTION

There exist many sources of uncertainty in magneto-hydro-dynamics (MHD) applications, e.g. due to unknown equation of state, random geometric roughness, noise in boundary/initial conditions, lack of precise values in the transport coefficients, and other external or interaction forcing. The
computational cost is prohibitive to consider all inputs as stochastic, which may be processes, i.e. varying in space and/or time. Sensitivity analysis is a powerful tool in ranking the importance of the random inputs and quantifying their interactions.

The sensitivity approach developed in [1] uses a few sample points in a large parametric domain to rank the random inputs, which is computationally efficient for linear models. However, when the system has strong non-linearities, more sample points covering the entire parametric domain have to be used. The sensitivity methods we investigated here are based on One-At-a-Time (OAT) designs, where the impact of changing the values of each factor is evaluated in turn. Many screening methods have been developed, e.g. Morris’s method [2–4], Cotter’s method [5], Andres’-iterated fractional factorial method [6], Bettonvil’s sequential bifurcation [7] and variance-based sensitivity analysis [4]. Recently the polynomial chaos method has been applied for sensitivity analysis [8, 9]. In the present work, we employ the Morris method and compare it against other approaches, which can efficiently identify the sensitive parameters e.g. multi-element probabilistic collocation method (ME-PCM) on sparse grids, Quasi-Monte Carlo (QMC) and Monte Carlo (MC) methods; see also [10] for applications to electromechanical systems.

The most important random factor is identified through the stochastic sensitivity analysis using the aforementioned four numerical methods. Stochastic simulations are performed based on the most important random factor. A high-order probabilistic collocation method (PCM) [8, 11–28] is employed, which combines the strengths of MC methods and stochastic Galerkin methods. By taking advantage of the existing theory on multivariate polynomial interpolations (see [29, 30]), fast convergence is achieved using PCM, when the solutions possess sufficient smoothness in the random space. Additionally, implementation of PCM is straightforward, as it only requires solutions of the corresponding deterministic problems at pre-selected sampling points. The choice of these sampling or collocation points is based on the sparse grid obtained from the Smolyak algorithm [31]. Sparse grids offer high-order accuracy with convergence rate not as strongly dependent on dimensionality. However, the sparse grids technique depends on the regularity of the solution in parameter space. Thus, systems with discontinuous dependence on random parameters cause difficulties for the convergence of these methods. To this end, to address problems related to the aforementioned discontinuities in random space, the Wiener–Haar method was developed in [32–34] based on the multi-wavelet polynomial chaos. Wan and Karniadakis [35] introduced the multi-element generalized polynomial chaos method (ME-gPC), which has been demonstrated to be able to efficiently handle problems with discontinuous dependence on parameter space and this work was extended further in [36]. In the present work, we follow the latest developments in [36] where ME-PCM is developed as an extension of ME-gPC with collocation projection. ME-PCM combines the strengths of domain decomposition in random space and also the computational ease of sampling-based methods.

The paper is organized as follows: In the next section, the Karhunen–Loève (KL) expansion is introduced to represent the random inputs. In Section 3, the high-order ME-PCM on sparse grids is introduced. In Section 4, we present four gradient-based sensitivity methods, namely the Morris method, the sparse PCM, and the QMC and MC methods. A comparison between different numerical methods for stochastic sensitivity analysis of a non-linear function is presented. In Sections 5 and 6, two plasma flows are studied to demonstrate the capability and efficiency of the stochastic sensitivity analysis, which can be used as a pre-screening technique for reducing the dimensionality and hence the cost of large-scale stochastic simulations. We conclude in Section 7 with a few remarks.
2. STOCHASTIC REPRESENTATION OF RANDOM INPUTS

We assume that the random input $Y(x; \omega)$ is a second-order random process with multivariate uniform distribution. The KL expansion provides the best linear approximation in the mean square sense for a second-order random process and is easy to couple with PCM [37]. Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a probability space, where $\Omega$ is the sample space, $\mathcal{F}$ is the $\sigma$-algebra of subsets of $\Omega$ and $\mathcal{P}$ is a probability measure. We assume $Y(x; \omega)$ is continuous in mean square on a closed domain $D$. Then $Y(x; \omega)$ can be expanded using the KL decomposition [38]:

$$Y(x; \omega) = \bar{Y}(x) + \sum_{i=1}^{\infty} \sqrt{\lambda_i} \psi_i(x) \xi_i(\omega)$$

where $\lambda_i$ and $\psi_i(x)$ are eigenvalues and corresponding eigenfunctions, which can be obtained from the Fredholm equation:

$$\int_{D} C_Y(t,s) \psi(t) \, dt = \lambda \psi(s)$$

where $C_Y(t,s)$ is the covariance kernel. We denote by $A$ the correlation length of the random process. In numerical simulations, we need to truncate the series in Equation (1) up to a finite index number $M$ as

$$Y_M(x; \omega) = \bar{Y}(x) + \sum_{i=1}^{M} \sqrt{\lambda_i} \psi_i(x) \xi_i(\omega)$$

The corresponding mean-square-error is

$$\int_{D} \mathbb{E}[(Y(x; \omega) - Y_M(x; \omega))^2] \, dx = \sum_{i=M+1}^{\infty} \lambda_i$$

where the error is completely determined by the decay rate of the eigenvalues $\lambda_i$ for a fixed $M$. In general, for a multi-dimensional problem with arbitrary geometry, the eigenvalue problem for KL expansion has to be solved numerically. In the present work, we employ the covariance kernels introduced in [39], where an analytical expression of the eigenpairs is available. For one-dimensional two-fluid plasma flow, an one-dimensional space-dependent random process is described as

$$Y(x; \omega) = \bar{Y}(x) + \varepsilon \sum_{i=0}^{M} \sqrt{\Lambda_i} \psi_i(x) \xi_i$$

where $\xi_i$ is a set of uncorrelated random variables with uniform distribution $\xi_i \in [-\sqrt{3}, \sqrt{3}]$ and $\Lambda_i = (b_i^2/2) L$ and $\psi_i(x) = \sqrt{2/L} \sin(2i \pi x/L)$, $b_i^2 = 2A^4/L(1/(1+(i \pi A/L))^2)$, where $A$, $L$ are the correlation length and kernel domain length, respectively.

For two-dimensional single-fluid/two-temperature plasma flow, we assume $Y$ can be described as

$$Y(x, y; \omega) = \sum_{i=0}^{d_x} \sum_{j=0}^{d_y} \sqrt{\Lambda_{ij}} \psi_{ij}(x, y) \xi_{ij}$$
where $d_x$ and $d_y$ are the number of random dimensions in $x$ and $y$ directions, $\Lambda_{ij} = 2/(L_x L_y (I_1 + k^2))$, $k = \sqrt{L_x L_y / A}$, $I_1 = \sqrt{(i \pi / L_x)^2 + (j \pi / L_y)^2}$ and

$$\psi_{ij} = \frac{2}{\sqrt{L_x L_y}} \sin \left( \frac{i \pi (x - 0.5 L_x)}{L_x} \right) \sin \left( \frac{j \pi (y - 0.5 L_y)}{L_y} \right)$$

Here $A$, $L_x$ and $L_y$ are the correlation length and kernel domain lengths in $x$ and $y$ directions, respectively.

3. PROBABILISTIC COLLOCATION METHOD (PCM)

Polynomial Chaos and more recently PCM have been investigated extensively in the literature [8, 11–28]. We consider the stochastic equation $L(x, t, \xi; u) = f(x, t; \xi)$ with a general (non-linear) differential operator $L$, where $x \in \mathbb{R}^{d_x}$, $d_x = 1, 2, 3$, is a physical space and $t$ is time. In contrast to the Galerkin projection, in the collocation formulation we employ Delta functions $\delta(\xi - \xi_k)$ as test functions, $k = 0, \ldots, M - 1$, where $\{\xi_k\}$ is a proper set of grid (quadrature) points on the support of $\xi$ and $M$ is the number of random dimensions. By applying the collocation projection on both sides of the equation, we obtain

$$L(x, t, \xi_k; u) = f(x, t; \xi_k) \quad (7)$$

The general procedure for PCM is

1. Generate $N_c$ prescribed number of collocation points as independent random inputs based on a cubature formula.
2. Solve the deterministic problem at each collocation point.
3. Evaluate the statistics of the random solution using the corresponding quadrature rule, e.g.

$$\mathbb{E}[u(x, t)] = \int u(x, t, \xi) f(\xi) d\xi \approx \sum_{k=1}^{N_c} u(x, t, \xi_k) w_k$$

$$\sigma(u)(x, t) = \sqrt{\sum_{k=1}^{N_c} u(x, t, \xi_k) w_k - \mathbb{E}[u]^2}$$

where $f(\xi)$ is the probability density function (PDF) of random variable $\xi$, $N_c$ is the number of sparse grid points and $\{w_k\}$ are the integration weights, which are the combination of corresponding integration weight in each random dimension. In the second step of PCM procedure, any pre-existing deterministic legacy code can be used. We use cubature point set as the collocation points and employ the cubature formula to evaluate the solution statistics; this is easy to implement on standard grids or on sparse grids. We refer the interested readers to [40] for extensive reviews on construction of cubature formula.

3.1. Choices of collocation points

In the present paper we consider multidimensional random spaces. Without loss of generality, we assume that the random vector variables, $\xi_k, k = 1, N_M$, have a bounded support and can be
represented by a $M$-hypercube, where $M$ is the number of random dimensions (components) of vector $\xi_k = [\xi_{1,k}, \xi_{2,k}, \ldots, \xi_{M,k}]^T$. For the prescribed number of collocation points, $N_c$, the computational cost of PCM is $N_c$ times the cost of the corresponding deterministic problem. Thus, for a given required accuracy, our goal is to choose the collocation point set with the minimal number of collocation points. In the following sections, we describe two different methods for the selection of collocation point set, i.e. tensor products of one-dimensional collocation point sets and sparse grids.

**Tensor products of one-dimensional collocation point sets:** Owing to its simplicity, the tensor product of one-dimensional set is a straightforward choice of a collocation point set. For each random dimension $i = 1, \ldots, M$, a good one-dimensional interpolation formula can be constructed for a smooth function $f$:

$$
\mathcal{U}^i(f) = \sum_{k=1}^{n_i} f(X^i_k) \cdot a^i_k
$$

based on collocation point sets

$$
\Theta^i = (X^i_1, \ldots, X^i_{n_i})
$$

where $\mathcal{U}^i(f)$ is a one-dimensional interpolation formula to approximate smooth function $f$, $X^i_k$ is the collocation point $k$ in random dimension $i$, $a^i_k = a_k(X^i)$ is the weight and $n_i$ is the number of collocation points in dimension $i$. A sequence of Equation 8 can be obtained for each random dimension $i$. In the multivariate case ($M > 1$), the tensor product formulas are

$$
\mathcal{I}(f) = (\mathcal{U}^{i_1} \otimes \cdots \otimes \mathcal{U}^{i_M})(f) = \sum_{k_1=1}^{n_{i_1}} \cdots \sum_{k_M=1}^{n_{i_M}} f(X^{i_1}_{k_1}, \ldots, X^{i_M}_{k_M}) \cdot (a^{i_1}_{k_1} \otimes \cdots \otimes a^{i_M}_{k_M})
$$

Equation (10) needs $N_c$ collocation points, where $N_c = n_{c_1} \cdots n_{c_M}$ and $n_{c_i}$ is the number of collocation points in dimension $i$, $i \in [1, M]$. Since $N_c$ grows exponentially as the random dimensions $M$ increases, tensor product of one-dimensional collocation point sets becomes less efficient for high random dimensions $M$ case.

**High dimensionality and sparse grids:** Owing to its high degree of accuracy, the quadrature rule can be efficient for one-dimensional integration. However, the number of random dimensions increases very fast as the correlation length decreases and if the number of random dimensions is not adequate, erroneous oscillations appear for both the mean and the variance. In multi-dimensions, using grid sets based on tensor products of one-dimensional constructs leads to prohibitively large number of collocation points. In this work, we use the Smolyak algorithm [31], which is a linear combination of tensor product formulas, and the resulting grid set has a significantly smaller number of grids compared with the full-tensor product rule. Recently, Xiu and Hesthaven [13] constructed a PCM extension based on sparse grids using the Smolyak algorithm [31]. Sparse grids do not depend as strongly on the dimensionality of the random space, and hence, they are more suitable for applications with large dimensional random inputs.

Let us consider an interpolation formula $\mathcal{I}(f)$ to approximate function $f(x)$ in the $M$-dimensional domain $\Gamma = \Gamma_1 \times \Gamma_2 \times \cdots \times \Gamma_M$, where the $\Gamma_i$ can be unbounded. Smolyak’s algorithm is given by the linear combination of the tensor product formulas in such a way that an interpolation property for $M = 1$ is preserved for $M > 1$ case. Only products with a relatively small
number of points are used and the resulting collocation point set has significantly less number of points compared with the full-tensor product formulas (10). The Smolyak algorithm is given by

$$I(f) = A(q, M) = \sum_{q-M+1 \leq |i| \leq q} (-1)^{q-|i|} \binom{M-1}{q-|i|} (\mathcal{U}^i_1 \otimes \cdots \otimes \mathcal{U}^i_M)$$

where $\mathcal{U}^i_1$ is the one-dimensional interpolation formula obtained from Equation (8) for each $i$. $I(f)$ is the Smolyak interpolation formula and the sparseness parameter $q \geq M$ determines the order of the algorithm, $i = (i_1, \ldots, i_M)$ and $|i| = i_1 + i_2 + \cdots + i_M$. Here $i_j$ ($j \in [1, M]$) ranges from 1 to the number of collocation points in random dimension $j$. We set $k = q - M$, as the ‘level’ of the Smolyak construction. To compute $A(q, M)$, we only need to evaluate functions on the ‘sparse grid’

$$\Theta_N = \bigcup_{q-N+1 \leq |i| \leq q} (\Theta^i_1 \times \cdots \times \Theta^i_M)$$

where $\Theta^i_j$ represents one-dimensional collocation point set at random dimension $j$ and $\Theta_N$ is denoted as the $M$-dimensional collocation point set. Figure 1 demonstrates the distribution of sparse grid points (two-random-dimension). Figure 1(a), (b) are for sparse grids (levels $k = 5$ and $6$) and corresponding full-tensor products, respectively. Significant less number of points for sparse grids is observed, compared with the number of points in the full-tensor products.

**Nested sparse grids:** The extreme points of the Chebyshev polynomials are employed in the Clenshaw–Curtis formulas. The set of sparse grid points for $\Gamma_j$, $\vartheta^j$ is nested, i.e., $\vartheta^j \in \vartheta^{j+1}$. The number of points in each random dimension $i$ is given by $m_i = 2^{i-1} + 1, i \geq 2$. These nodes are given as

$$s^j_i = -\cos\left(\frac{\pi(j-1)}{m_i - 1}\right), \quad s^1_i = 0, \quad j = 1, \ldots, m_i$$

Figure 1. Two-random-dimensional ($M = 2$) sparse grid points versus full-tensor products: (a) sparse grid level $k = 5$ and 6 (the cross and circle nodes represent $k = 5$ and 6 sparse grid points, respectively) and (b) corresponding full-tensor products.


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and the corresponding weights are

\[
\omega_j^i = \frac{2}{m_i-1} \left( 1 + 2 \sum_{j=1}^{(m_i-1)/2} \frac{1}{1 - 4j^2} \cos \frac{2\pi j (i-1)}{m_i-1} \right), \quad 2 \leq i \leq m_i - 1
\]

where \( \sum' \) represents that the last term of the sum is halved.

**ME-PCM on sparse grids**: Based on the generalized polynomial chaos (gPC), the ME-gPC with Galerkin projection has been developed in [41]. ME-PCM is an extension of ME-gPC with collocation projection, which decomposes the random space into finite elements as in the deterministic finite element method. To illustrate this idea here we consider the one-dimensional case, see Figure 2. We assume that the support \([a, b]\) of the one-dimensional random variable \(\xi\) is decomposed into elements \(e_k := [a_k, b_k]\). We then define a new random variable \(\xi_k, k = 1, 2, \ldots, N_e\), in each random element, \(e_k\), as

\[
\xi = \frac{b_k - a_k + b_k + a_k}{2} \xi_k + \frac{b_k + a_k}{2}
\]

with a re-scaled PDF

\[
f_k(\xi_k) = \frac{f(\xi(\xi_k))}{Pr(\xi \in e_k)} \frac{b_k - a_k}{2}, \quad k = 1, \ldots, N_e
\]

where \(N_e\) is the number of random elements and \(f(\xi)\) is the PDF of \(\xi\), \(Pr(\xi \in e_k) = \int_{e_k} f(\xi) \, d\xi\) is the probability that \(\xi\) is located in random element \(e_k\). Such decomposition of the random space can be easily extended to multi-random dimensions. Practically, \(\xi_k\) is mapped to the space \([-1, 1]^M\). The desired random field \(u(\xi)\) is approximated locally within each random element by ME-PCM, where the degree of perturbation is effectively decreased by the above linear transform from \(O(1)\) to \(O((b_k - a_k) / 2)\). Subsequently, we gather the information from all random elements with appropriate weight to obtain the statistics of \(u(\xi)\).

Compared with PCM, ME-PCM is especially favorable for such problems with discontinuities in the random space or long-time integration [36]. The general procedure is as follows:

1. Obtain sparse nodal set and corresponding weights for each random element.
2. Evaluate the solution \(u(x; \tau_i)\) at each point \(\tau_i\) using a deterministic solver.
(3) Calculate the moments of the local solution $u_k$,

\[ \mathbb{E}[u_k(x)] = \sum_{i \in \vartheta_k} u(x, \tau_i) \omega_c(\tau_i) \]  

(15)

where $\omega_c$ are the weights corresponding to $\vartheta_k$.

(4) Calculate the moments of the global solution,

\[ \mathbb{E}[u(x)] = \sum_{k=1}^{N_c} Pr(\xi \in e_k) \mathbb{E}[u_k(x)] \]  

(16)

For random input described as a random process with uniform random variables, the $M$-dimensional random vector has constant PDF, $(1/2)^M$. Thus, there is no need to construct orthogonal basis for each element and corresponding quadrature points. A nested sparse nodal set has been employed in the present work.

4. STOCHASTIC SENSITIVITY ANALYSIS

In general, it is computationally quite very intensive to perform stochastic simulations with all inputs treated as random, especially when the system has hundreds or thousands of random inputs $x_i$. Therefore, parameter screening is crucial to examine the effect on the output due to different random inputs and to rank the random inputs accordingly. We can make use of the OAT global sensitivity analysis to construct a more effective method. To this end, the elementary effect of $i$ random input on $j$ output ($d_{ij}^*$) is defined as the approximated gradient when only the $i$ random input deviates from its nominal value with magnitude. To avoid averaging out the effects from the $i$ random input in the $d_{ij}^*$ computation, we use the absolute difference of the $j$ output,

\[ d_{ij}^* = \frac{|y_j(x_1, \ldots, x_i \pm \Delta, \ldots, x_m) - y_j|}{\Delta} \]  

(17)

where $x_i$ with $i = 1, \ldots, m$ and $y_j$ with $j = 1, \ldots, n$. Here $m$ is the number of random inputs and $n$ is the number of outputs. Using the local gradient computation, when $\partial y_j/\partial x_i$ is equal to zero, a non-zero constant or a non-constant function of random input parameters, the effects of $x_i$ on $y_j$ are negligible, linear and additive, or non-linear and coupled, respectively. By randomizing all values of $x$ in computing $d_{ij}^*$, the interaction effects can be discovered from the variation of the $d_{ij}^*$ distribution. Next, we present four gradient-based methods, i.e. Morris, MC, QMC and collocation method on sparse grids.

**Explicit solution:** The gradient-based mean sensitivity can be derived as follows:

\[ \mathbb{E}_{\text{ext}}[d_{ij}^*] = \int_0^1 \cdots \int_0^1 \frac{|y_j(x_1, \ldots, x_i \pm \Delta, \ldots, x_m) - y_j|}{\Delta} \, dx_1 \cdots dx_m \]

\[ = \int_0^1 \cdots \int_0^{1/2} \cdots \int_0^1 \frac{|y_j(x_1, \ldots, x_i + \Delta, \ldots, x_m) - y_j|}{\Delta} \, dx_1 \cdots dx_m \]


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\[ + \int_0^1 \ldots \int_{1/2}^1 \ldots \int_0^1 \frac{|y_j(x_1, \ldots, x_i - \Delta, \ldots, x_m) - y_j|}{\Delta} \, dx_1 \ldots dx_m \\
= 2 \int_0^1 \ldots \int_{1/2}^1 \ldots \int_0^1 \frac{|y_j(x_1, \ldots, x_i + \Delta, \ldots, x_m) - y_j|}{\Delta} \, dx_1 \ldots dx_m \]  

(18)

Similarly, the gradient-based standard deviation can be obtained by

\[ \sigma_{\text{ext}}(d_i^j) = \sqrt{\int_0^1 \ldots \int_0^1 \frac{|y_j(x_1, \ldots, x_i \pm \Delta, \ldots, x_m) - y_j|^2}{\Delta^2} \, dx_1 \ldots dx_m - \mathbb{E}[d_i^j]^2} \]

\[ = \sqrt{2 \int_0^1 \ldots \int_{1/2}^1 \ldots \int_0^1 \frac{|y_j(x_1, \ldots, x_i + \Delta, \ldots, x_m) - y_j|^2}{\Delta^2} \, dx_1 \ldots dx_m - \mathbb{E}[d_i^j]^2} \]  

(19)

where \( \Delta \) is chosen to be \( \frac{1}{2} \).

**Morris method:** The Morris method considers the OAT \( d_i^j \) to identify the significant first-order and interaction effects of random input parameters with only a few evaluations of \( d_i^j \), which is proportional to \( m \) random inputs. The basic methodology of this approach is to randomly select an initial condition on the grid points and construct a randomized trajectory along this grid structure in a high-dimensional random input space for \( r \) trials. Thus, the mean and standard deviation of \( d_i^j \) represent the first-order effect of \( i \) random input and interaction of other random inputs with the \( i \) random input. Originally, all random input parameters in the Morris method [42] were assumed to be independent uniformly distributed, but the method is applicable to parameters that follow a normal distribution as well. The Morris method becomes more efficient when the number of random inputs is much larger than the number of outputs. The procedure to construct the OAT randomized trajectory is as follows:

First, set the value of \( \Delta \) as \( p/(2(p-1)) \). Here \( p \) is even and \( p \geq 2 \) such that \( \Delta \) optimally covers the \( m \)-dimensional \( p \)-level random space with an equal probability. It is clear that the randomized trajectory is contained within the range of random input variation.

Second, select an \((m+1) \times m\) lower-triangular sampling matrix \( B \), which is expressed in the following form:

\[ B_{ij} = \begin{cases} 0 & \text{for } i \leq j \\ 1 & \text{for } i > j \end{cases} \]  

(20)

The key property of matrix \( B \) is that each row of the \( B \) matrix is different from its adjacent row by only one element; therefore, the row difference of the \( B \) matrix times \( \Delta \), called \( \Delta B \), forms the deterministic trajectory of the elementary effect, as shown in Figure 3. To construct an OAT randomized matrix, \( B^* \), from the \( B \) matrix, Morris [42] proposed the following formulation:

\[ B^* = \left( J_{m+1,1} x^* + \frac{\Delta}{2} ((2B - J_{m+1,m})D^* + J_{m+1,m}) \right) P^* \]  

(21)

where \( x^* \) is the random chosen initial \( m \) dimensions vector for which each element is randomly assigned a value from \([0, 1/(p-1), 2/(p-1), \ldots, 1] \), \( D^* \) is an \( m \)-dimensional diagonal matrix in
which each diagonal element is either +1 or −1 with equal probability, $P^*$ is an $m \times m$ random permutation of an identity matrix in which each column contains one element equal to 1 and all others equal to 0 and no two columns have 1’s in the same position and each such matrix has an equal probability of selection. The role of $P^*$ matrix is to guarantee an equal probability of the random elementary effect in each random input direction. Here $J$ is a matrix with all elements equal to one.

The last step is to construct $r$ trials of this randomized $B^*$ matrix such that the $r$ random trajectories can be obtained from the row difference of $B^*$, called $\Delta B^*$ matrix. To distinguish between the row difference of $B$, $\Delta B$, and the row difference of $B^*$, $\Delta B^*$, we show one of the $r$ trajectories from $\Delta B$ and $\Delta B^*$ for $p=4$ in Figure 3.

The total computational cost consists of two parts: generating the randomized trajectory in the order of $O(r \times m)$ and evaluating the $d^i_j$ for one output using the randomized trajectories in the order of $O(r \times (m+1))$. Note that as we increase the $p$-level in this $m$-dimensional space, the value of $\Delta$ approaches $\frac{1}{2}$. This minimum $\Delta=\frac{1}{2}$ seems to constrain the randomized trajectory on the boundary more than the interior of the random input domain.

The Morris method was first developed for ranking the sensitivity of a multi-input function. To extend the capability of the Morris method for solving PDEs, we can view the computation of the elementary effect at each time step as a random perturbation in each random input dimension with fixed length $\Delta$. For the spatial domain, if we are interested in the output at some specific location, the physical value at the specific point can be chosen as the output for sensitivity analysis. Otherwise, the $L_2$ or $L_\infty$ norm in the spatial domain can be chosen to be the output. Owing to the requirement of random trajectory generated in this approach, $(m+1) \times r$ deterministic PDEs must be solved for $r$ random initial conditions.

**MC sampling method:** Instead of computing the statistics of the $d^i_j$ from the randomized trajectories on the $p$-level grid as in the Morris method, the MC sampling method can be used to randomly generate the $N$ sampling points in the $m$-dimensional random inputs, and then the elementary effect in each direction can be computed at these $N$ random sampling points. The mean of each $i$ elementary effect or $E[d^i_j]$ can be directly used to rank random input parameters. In addition, if we observe a substantial standard deviation $\sigma[d^i_j]$ relative to the mean $E[d^i_j]$, it indicates potentially

Figure 3. Sketch of Morris method for sensitivity analysis: trajectories from $\Delta B$ and $\Delta B^*$.  

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extensive patterns of random input interaction and non-linearity on the output, which has also been observed in [2]. To obtain smooth convergence in the result, the same random realizations in the small \( N \) case are reused in the large \( N \) case.

To demonstrate the methodology, Figure 4 shows nine realizations of random sampling points and directions in the \( d_i^j \) computation with fixed \( \Delta \) in a three-dimensional random input space. The total evaluation of output for \( m \) elementary effects is on the order of \( O(N \times m) \) for the \( m \) random inputs. Therefore, the accuracy of \( \mathbb{E}[d_i^j] \) and \( \sigma[d_i^j] \) depends on the convergence rate of the MC method, approximately \( 1/\sqrt{N} \). Applying this technique to analyze the sensitivity of PDEs requires solving \( N \) problems. We perturb the system random inputs one at a time with a fixed \( \Delta \). Similar to a maximum limit of the \( \Delta \) magnitude in the Morris method, which equals \( 1/2 \) for an random input range between \([0, 1]\), we assign the \( \Delta \) magnitude to be half of the parameter-variation range.

**QMC sampling method:** The QMC sampling method follows the same procedure as MC sampling method. The only difference lies in choosing the sampling points. The convergence rate for QMC sampling method is \( 1/N \) for small number of parameters.

**ME-PCM on sparse grids:** To further improve the accuracy and efficiency of the parameter screening technique, ME-PCM approach can be employed, in which the sampling points of the \( d_i^j \) computation can be selected at the sparse grid points (see details on ME-PCM on sparse grids in the previous section). The procedure for this approach is similar to the MC sampling method. First, decompose the random domain along the discontinuity or bifurcation in the parametric space. Second, we specify a distance of \( \Delta \) in computing the \( d_i^j \) and the level of sparse grid points. Third, the elementary effect in each \( i \)-input direction is calculated at the sparse grid points with a random direction of \( \Delta \). Finally, the mean and standard deviation of \( d_i^j \) are obtained by combining the results from each random element. The computational cost for collocation method on sparse grids is \( O(N \times N_c \times m) \), where \( N \) is the number of random elements and \( N_c \) is the number of sparse grid points.

To demonstrate the convergence of these sensitivity algorithms, we systematically test the parametric sensitivity and interaction of some functions with three random input parameters.
The test function with three random input parameters is

\[ y = 63e^{4x_1} - 70e^{3x_2} + 15e^{2x_3} \]  

(22)

Here, we define the mean error and \( \sigma \) error as

\[
\text{Mean-error} = \frac{|E_{\text{num}}[d_i^j] - E_{\text{ext}}[d_i^j]|}{|E_{\text{ext}}[d_i^j]|}, \quad \sigma\text{-error} = \frac{|\sigma_{\text{num}}(d_i^j) - \sigma_{\text{ext}}(d_i^j)|}{|\sigma_{\text{ext}}(d_i^j)|}
\]

(23)

where \( E_{\text{ext}}[d_i^j] \) and \( \sigma_{\text{ext}}(d_i^j) \) are defined in Equations (18) and (19), and \( E_{\text{num}}[d_i^j] \) and \( \sigma_{\text{num}}(d_i^j) \) are obtained from gradient-based numerical methods. Figure 5 demonstrates the convergence of mean and standard deviation solution obtained by MEPCM on sparse grids. Fast convergence of both mean and standard deviation solution obtained by MEPCM on sparse grids is observed compared with the Morris and MC method. It is worthwhile to note that there is bifurcation while \( y_j(x_1, \ldots, x_i \pm \Delta, \ldots, x_m) - y_j \) in Equation 17 changes the sign when computing the elementary effect \( d_i^j \) in the absolute value. ME-PCM [43], ME-gPC [35] or wavelets [32–34] in stochastic space have to be employed to achieve fast convergence.

In the following section, we first discuss sensitivity analysis and stochastic simulation of one-dimensional two-fluids plasma flow. Subsequently, sensitivity analysis and stochastic simulation of two-dimensional single-fluid/two-temperatures plasma flow are considered.

5. SENSITIVITY ANALYSIS AND STOCHASTIC SIMULATION OF ONE-DIMENSIONAL TWO-FLUID PLASMA FLOW WITH UNCERTAINTIES

MHD plasma models have certain limitations, e.g. the Hall effect and diamagnetic terms, which model contributions to ion current and the finite Larmor radius of the plasma constituents.
The Larmor radius is the radius of gyration of charged particle in a magnetic field as it undergoes cyclotron motion. The Larmor radius is defined as

\[ r_{Lz} = \frac{v_{Tz}}{\omega_c} = \frac{m_z v_T}{e B_0} \]  

where \( v_{Tz} \) is the thermal velocity and \( \omega_c \) is the cyclotron frequency, \( B_0 \) is the characteristic magnetic field strength and \( e \) is the species charge. \( m_z \) is the species mass. Here the subscript \( z \) is either \( e \) for electrons or \( i \) for ions. These effects are important in space physics and Hall current thrusters. By taking moments of the Boltzmann equation for each species, the two-fluid plasma model can be derived, which captures the separate motion of the electrons and ions without the added complexity of the kinetic model. An approximate Riemann solver for the two-fluid plasma model was developed in [44].

5.1. Two-fluid plasma model

The governing equations for the fluid and electromagnetic models are

Fluid model

- Ion and electron continuity equations

\[ \frac{\partial n_i}{\partial t} + \nabla \cdot (n_i \mathbf{v}_i) = 0 \]

\[ \frac{\partial n_e}{\partial t} + \nabla \cdot (n_e \mathbf{v}_e) = 0 \]  

where \( n_i, n_e \) are the ion, electron number density and \( \mathbf{v}_i, \mathbf{v}_e \) are the ion and electron fluid velocities.

- Ion and electron momentum equations

\[ n_i m_i \left( \frac{\partial \mathbf{v}_i}{\partial t} + (\mathbf{v}_i \cdot \nabla) \mathbf{v}_i \right) = -\nabla p_i + n_i e (\mathbf{E} + \mathbf{v}_i \times \mathbf{B}) + \mathbf{R}_{ei} \]

\[ n_e m_e \left( \frac{\partial \mathbf{v}_e}{\partial t} + (\mathbf{v}_e \cdot \nabla) \mathbf{v}_e \right) = -\nabla p_e + n_e e (\mathbf{E} + \mathbf{v}_e \times \mathbf{B}) - \mathbf{R}_{ei} \]  

where \( \mathbf{E} \) and \( \mathbf{B} \) are the electric and magnetic fields, \( p_i \) and \( p_e \) are the ion and electron partial pressures, and \( \mathbf{R}_{ei} \) is the electron to ion momentum transfer vector. By assuming that the plasma is perfectly conducting, \( \mathbf{R}_{ei} \) is set to zero.

- Ion and electron temperature equations

\[ \frac{1}{\gamma-1} n_i \left( \frac{\partial T_i}{\partial t} + (\mathbf{v}_i \cdot \nabla) T_i \right) = -p_i \nabla \mathbf{v}_i \]

\[ \frac{1}{\gamma-1} n_e \left( \frac{\partial T_e}{\partial t} + (\mathbf{v}_e \cdot \nabla) T_e \right) = -p_e \nabla \mathbf{v}_e \]  

where \( \gamma = \frac{5}{3} \), which is the ratio of the specific heats and \( T_i \) and \( T_e \) are the ion and electron temperatures. An adiabatic equation of state is assumed. The temperatures are related to the partial pressures by \( p_x = n_x T_x \) for \( x = i, e \).
Electromagnetic model

- The Ampere law

\[
\varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} = \nabla \times \mathbf{B}/\mu_0 + \mathbf{j}_i + \mathbf{j}_e
\]

where \( \mathbf{j}_i = n_i e \mathbf{v}_i \) and \( \mathbf{j}_i = n_e e \mathbf{v}_e \).

- The Faraday law

\[
\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E}
\]

- The Poisson equation

\[
\nabla \cdot \mathbf{E} = e(n_i - n_e)/\varepsilon_0
\]

- Magnetic flux

\[
\nabla \cdot \mathbf{B} = 0
\]

Non-dimensionalizing the two-fluid plasma model: The two-fluid plasma model is non-dimensionalized by selecting characteristic variables for the length \( x_0 \), plasma size and velocity \( v_{T_i} \), ion thermal speed, which produces a characteristic time of \( x_0/v_{T_i} \). The partial pressures and total energies are normalized by the ion dynamic pressure \( m_i n_0 v_{T_i}^2 \), where \( n_0 \) is a characteristic number density. The electric field is normalized by the product of the characteristic magnetic field and velocity \( B_0 v_{T_i} \).

The non-dimensionalized fluid equations are written as

\[
\begin{pmatrix}
\frac{\partial n_i}{\partial t} \\
\frac{\partial n_e}{\partial t} \\
\frac{\partial \mathbf{j}_i}{\partial t} \\
\frac{\partial \mathbf{j}_e}{\partial t} \\
\frac{\partial \mathbf{v}_i}{\partial t} \\
\frac{\partial \mathbf{v}_e}{\partial t}
\end{pmatrix}
+ \nabla \cdot \begin{pmatrix}
\mathbf{j}_i \\
-\mathbf{j}_e \\
\frac{\mathbf{j}_i \mathbf{j}_j + p_i \mathbf{I}}{n_i} \\
\frac{\mathbf{j}_i \mathbf{j}_j - m_i \mathbf{p}_i \mathbf{I}}{m_e n_e} \\
\frac{(\mathbf{v}_i + p_i) \mathbf{j}_i}{n_i} \\
-\frac{(\mathbf{v}_e + p_e) \mathbf{j}_e}{n_e}
\end{pmatrix}
= \begin{pmatrix}
0 \\
0 \\
\frac{1}{\ell_L} (n_i \mathbf{E} + \mathbf{j}_i \times \mathbf{B}) \\
\frac{m_i}{m_e \ell_L} (n_e \mathbf{E} - \mathbf{j}_e \times \mathbf{B}) \\
\frac{1}{\ell_L} \mathbf{j}_i \cdot \mathbf{E} \\
\frac{1}{\ell_L} \mathbf{j}_e \cdot \mathbf{E}
\end{pmatrix}
\]

The non-dimensionalized electromagnetic field equations are written as

\[
\frac{\partial}{\partial t} \begin{pmatrix}
\mathbf{B} \\
\mathbf{E}
\end{pmatrix}
+ \nabla \times \begin{pmatrix}
\mathbf{E} \\
-\mathbf{E} \times \mathbf{B}
\end{pmatrix}
= \begin{pmatrix}
0 \\
-\frac{1}{\ell_L^2} (\mathbf{j}_i + \mathbf{j}_e)
\end{pmatrix}
\]
Physically significant similarity parameters appear in the non-dimensional equation: ion to electron mass ratio $m_i/m_e$, normalized speed of light $\hat{c}=c/v_T$, normalized ion Larmor radius $\hat{r}_i=r_{Li}/x_0$ and the normalized Debye length $\hat{\lambda}_D=\lambda_D/r_i$, where $\lambda_D=\sqrt{\varepsilon_0 m_i v_T^2 / ne}$. The electron and ion Debye lengths are equal for equal ion and electron temperatures. The ion Larmor radius is a measure of importance of two-fluid physics. In Equations (32) and (33), there are two independent non-dimensional parameters, i.e. $\hat{\lambda}_D$, $\hat{r}_i$.

A fifth-order weighted essentially non-oscillatory (WENO) method has been employed to solve the homogeneous advection parts of Equations (32) and (33). When the source terms on the right-hand side of Equations (32) and (33) are small, they are treated explicitly, i.e.

$$
\frac{Q_{i}^{n+1}-Q_{i}^{n}}{\Delta t} = - \frac{3}{2} \left( \frac{F_{i+1/2}^{n} - F_{i-1/2}^{n}}{\Delta x} \right) + \frac{1}{2} \left( \frac{F_{i+1/2}^{n-1} - F_{i-1/2}^{n-1}}{\Delta x} \right) + S_{i}^{n}
$$

where $Q_{i}^{n}$ is the complete vector of solution variables (two-fluid and electromagnetic) at time step $n$ in cell $i$, $F_{i+1/2}^{n}$ is the complete numerical flux vector and $S_{i}^{n}$ is the complete vector of the source terms. When the source terms are large, they are treated implicitly to alleviate the stiffness of the problem. A second-order implicit-explicit (IMEX) scheme (Crank–Nicolson, Adams–Bashforth) is employed, i.e.

$$
\frac{Q_{i}^{n+1}-Q_{i}^{n}}{\Delta t} = - \frac{3}{2} \left( \frac{F_{i+1/2}^{n} - F_{i-1/2}^{n}}{\Delta x} \right) + \frac{1}{2} \left( \frac{F_{i+1/2}^{n-1} - F_{i-1/2}^{n-1}}{\Delta x} \right) + \frac{S_{i}^{n+1}+S_{i}^{n}}{2}
$$

By expanding the source terms in a Taylor series to first-order, the Newton iteration is obtained as follows:

$$
\left( \frac{1}{\Delta t} - \frac{1}{2} \left( \frac{\partial S}{\partial Q} \right)_{i} \right) \Delta Q = - \frac{Q_{i}^{k}-Q_{i}^{n}}{\Delta t} - \frac{3}{2} \left( \frac{F_{i+1/2}^{n} - F_{i-1/2}^{n}}{\Delta x} \right) + \frac{1}{2} \left( \frac{F_{i+1/2}^{n-1} - F_{i-1/2}^{n-1}}{\Delta x} \right) + S_{i}^{k}
$$

where $\Delta Q=Q_{i}^{n+1}-Q_{i}^{n}$ and $k$ is the iteration variable. A fifth-order WENO procedure is employed to calculate the numerical fluxes $F^n$ explicitly, while the source terms are solved implicitly by a symmetric Gauss–Seidel method. Since $(\partial S/\partial Q)^k_i$ is a local variable within each cell, the computational work for solving Equation (36) increases linearly with the number of cells.

5.2. Deterministic numerical results for electromagnetic plasma shock

The Brio and Wu shock problem [45] is a well-known benchmark for one-dimensional MHD Riemann solvers; it is also a good test case for the two-fluid algorithm to model asymptotic MHD-like behavior and to demonstrate the departure of the solution when the two-fluid effects are employed. By an appropriate choice of the non-dimensional parameters, i.e. $\hat{\lambda}_D$, $m_e/m_i$, $\hat{r}_L$ and $\hat{c}$, the two-fluid model can either achieve MHD-type structure or approach gas dynamics behavior. The two-fluid model allows the ion and electron fluid to have a more physically correct coupling to the magnetic field as characterized by the ion and electron Larmor radii. The two-fluid variables can be written in terms of species mass densities, species velocities and species pressures.
The MHD variables are related to these two-fluid variables as

\[
\begin{pmatrix}
\rho \\
v_x \\
v_y \\
v_z \\
p \\
B_x \\
B_y \\
B_z
\end{pmatrix} = \lim_{m_e/m_i \to 0} \begin{pmatrix}
\rho_i + \rho_e \\
\rho_i v_{ix} + \rho_e v_{ex} \\
\rho_i v_{iy} + \rho_e v_{ey} \\
\rho_i v_{iz} + \rho_e v_{ez} \\
\rho_i + \rho_e \\
\rho_i p_i + \rho_e p_e \\
B_x \\
B_y \\
B_z
\end{pmatrix} = \begin{pmatrix}
\rho_i \\
v_{ix} \\
v_{iy} \\
v_{iz} \\
p_i + p_e \\
B_x \\
B_y \\
B_z
\end{pmatrix} \tag{37}
\]

It is assumed that \(n_i = n_e\) and \(p_i = p_e\) in the MHD limit. For the two-fluid model, the corresponding initial conditions for the electromagnetic plasma shock are

\[
(\rho_e, v_{ex}, v_{ey}, v_{ez}, p_e, \rho_i, v_{ix}, v_{iy}, v_{iz}, p_i, B_x, B_y, B_z, E_x, E_y, E_z)_{\text{left}} = \left(\frac{m_e}{m_i}, 0, 0, 0, 0.5, 1.0, 0, 0, 0, 0.5, 0.75, 1.0, 0, 0, 0, 0\right),
\]

\[
(\rho_e, v_{ex}, v_{ey}, v_{ez}, p_e, \rho_i, v_{ix}, v_{iy}, v_{iz}, p_i, B_x, B_y, B_z, E_x, E_y, E_z)_{\text{right}} = \left(0.125 \frac{m_e}{m_i}, 0, 0, 0, 0.05, 0.125, 0, 0, 0, 0.05, 0.75, -1.0, 0, 0, 0, 0\right) \tag{38}
\]

where \(\lambda_D = 0.01\), \(m_i/m_e = 1836\), \(\varepsilon = 100\) and \(\hat{r}_L\) is variable.

Figures 6 and 7 present the total mass density two-fluid solutions of the electromagnetic plasma shock problem for \(\hat{r}_L = 10, 1, 0.1\) and 0.0003 at \(t = 0.1\). 800 cells are used in the spatial domain. In Figures 6 and 7, \(\hat{r}_L = \infty\) represents gas dynamics case while \(\hat{r}_L = 0\) is the MHD case. The shock–wave structure for \(\hat{r}_L = 0\) MHD case is following: left rarefaction wave, compound wave, contact discontinuity, slow shock and right rarefaction wave (from left to right). For \(\hat{r}_L = \infty\) gas dynamics case, the compound wave structure cannot be observed, which is caused by the electromagnetic field. The two-fluid solution with finite Larmor radius \((\hat{r}_L = 10, 1, 0.1\) and 0.0003) is different from the gas dynamics case and the MHD case, which is caused by the finite Debye length and Larmor radius effects. It can be observed that the two-fluid solution at \(x = 0\) is not constrained to 1.0 due to the Neuman boundary condition at \(x = 0\). The wiggles in Figure 7 (a) are caused by the finite Debye length and Larmor radius effects; this was also observed in [44].

The total mass density two-fluid solutions demonstrate that by decreasing the Larmor radius, the two-fluid solution approaches the MHD solution, while by increasing the Larmor radius, the two-fluid solution achieves the gas dynamics structure. For large Larmor radius, both ion and electron are decoupled from each other and the plasma species are decoupled from the magnetic field for large ion and electron Larmor radii. Therefore, the ion fluid behaves like a neutral gas.
Figure 6. Total mass density two-fluid solutions of the electromagnetic plasma shock problem for $\hat{r}_L = 10$ (a) and $\hat{r}_L = 1$ (b) are compared with the two asymptotic solutions of the gas dynamic limit $\hat{r}_L = \infty$ and the MHD limit $\hat{r}_L = 0$.

Figure 7. Total mass density two-fluid solutions of the electromagnetic plasma shock problem for $\hat{r}_L = 0.1$ (a) and $\hat{r}_L = 0.003$ (b) are compared with the two asymptotic solutions of the gas dynamic limit $\hat{r}_L = \infty$ and the MHD limit $\hat{r}_L = 0$.

and the total mass density approaches the gas dynamics solution. On the other hand, for small ion and electron Larmor radii, the ion and electron fluid and magnetic field are coupled through the source terms. The two-fluid solution approaches the MHD solution. By comparing with the MHD solution, fast moving plasma waves are observed for two-fluid solution due to the two-fluid effects. The two-fluid plasma model is more general than the standard MHD model, which has some limitations, e.g. the Hall effect and diamagnetic current.
5.3. Sensitivity analysis for two-fluid plasma flow with uncertainties

For the two-fluid plasma model, there are two non-dimensional parameters that may be considered uncertain, i.e. $\hat{r}_L$ and $\hat{\lambda}_D$. Stochastic sensitivity analysis is performed to rank all the random inputs by QMC, MC, Morris and sparse collocation methods. In the spatial domain, 800 cells are used. Fifth-order WENO scheme and second-order IMEX scheme are employed for the spatial and temporal discretization, respectively. For QMC, MC, Morris methods, 200 samples are chosen, while for sparse collocation method, 29 samples are used.

To comprehensively present the parametric sensitivity analysis over the spatial domain $[0,1]$, we define the average sensitivity and interaction indices in the $L_2$ norm as $\text{AS}_i^j$ and $\Pi_i^j$, respectively, i.e.

$$\text{AS}_i^j = \| E[d_i^j] \|_2$$
$$\Pi_i^j = \left\| \sqrt{ E^2[d_i^j] + \sigma^2(d_i^j) } \right\|_2$$

Figures 8–10 present the average sensitivity and interaction indices for two random inputs $\hat{\lambda}_D$ and $\hat{r}_L$ in the two-fluid plasma model. There are ten outputs of the sensitivity analysis in the $x$ coordinate, which are $\rho, u, v, w, \rho, p, E_x, E_y, E_z, B_y$ and $B_z$. The $y$ coordinate corresponds to the two random inputs $\hat{r}_L$ and $\hat{\lambda}_D$. First, Figures 8–10 demonstrate that random input $\hat{\lambda}_D$ has larger average sensitivity and interaction indices than $\hat{r}_L$ at three different times $t=0.05, 0.1$ and $0.2$. Additionally, we observed that the electric field in $y$ direction, $E_y$, has the largest average sensitivity and interaction indices at three different times $t=0.05, 0.1$ and $0.2$, which means that the electric field in $y$ direction, $E_y$, is very sensitive to the two random inputs. At a later time $t=0.2$, the streamwise velocity $u$ and the magnetic field in $z$ direction have larger average sensitivity and interaction indices than the ones at the early time.

![Figure 8](image)

Figure 8. (a) Average sensitivity and (b) interaction indices of the 10 outputs (indicated along $x$ coordinate) at $t=0.05$ for two random inputs $\hat{r}_L$ (top) and $\hat{\lambda}_D$ (bottom, indicated along $y$ coordinate) in the two-fluid plasma model by collocation method on sparse grids. The $x$ coordinate represents the outputs of the sensitivity analysis and the $y$ coordinate is the two inputs of the sensitivity analysis.
5.4. Stochastic simulation of two-fluid plasma flow with uncertainties

Debye length is described as a random variable: From Figures 8–10, we observe that random input $\hat{\lambda}_D$ has larger mean and standard deviation of elementary effects than random input $\hat{r}_L$. Thus, the Debye length $\hat{\lambda}_D$ is the most important random factor, and hence stochastic simulations are performed based on the Debye length $\hat{\lambda}_D$ considered to be a random variable as follows:

$$\hat{\lambda}_D = \varepsilon (1 + 0.2 \xi)$$  \hspace{1cm} (40)
Figure 11. The mean and standard deviation of ion and electron number density as a function of $x$ with the Debye length $\hat{\lambda}_D = 1 \times 10^{-2}(1 + 0.2\xi)$ described as a random variable, obtained from the two-fluid solution of the electromagnetic plasma shock problem.

where $\xi$ is chosen to be $1 \times 10^{-2}$. Figure 11 presents the mean and standard deviation of ion and electron number density as a function of $x$ with the Debye length $\hat{\lambda}_D = 1 \times 10^{-2}(1 + 0.2\xi)$ described as a random variable. Electrons and ions have very similar profile, except around the compound wave and contact discontinuity region, where the ions accelerate due to the electric field created by the electrons.

**Larmor radius is described as a random variable:** Here the Larmor radius is described as a random variable $\hat{r}_L = 0.003 + (2 - 0.003)\xi$, and a stochastic simulation based on the Larmor radius is performed. Here $\xi$ is a random variable with uniform distribution $\xi \in [-\sqrt{3}, \sqrt{3}]$. Figure 12 demonstrates the total mass density of the two-fluid solution as a function of $x$ with three different Larmor radii, $\hat{r}_L$ for the electromagnetic plasma shock problem. From Figure 12, we observe that by decreasing the Larmor radius, the transition from gas dynamics-type shock structure to MHD-like shock structure is a smooth transition as indicated in [44]. Figure 13 plots the mean and standard deviation of ion and electron number density as a function of $x$ with Larmor radius $\hat{r}_L$ described as a random variable.

**Larmor radius is described as a space-dependent random process:** Similar to the one-dimensional second-order stochastic process with boundary constraints introduced in [46], we assume that the Larmor radius can be described as a space-dependent random process as follows:

$$
\hat{r}_L(x; \omega) = 0.01 \left( 1 + 0.4 \sum_{i=0}^{2} \sqrt{\Lambda_i} \psi_i(x) \xi_i \right)
$$

(41)
Figure 12. The total mass density as a function of $x$ with three different Larmor radii, $\hat{r}_L$, obtained from the two-fluid solution of the electromagnetic plasma shock problem.

Figure 13. The mean and standard deviation of ion and electron pressure as a function of $x$ with Larmor radius $\hat{r}_L$ described as a random variable, obtained from the two-fluid solution of the electromagnetic plasma shock problem.

where $\xi_i$ is a set of independent random variables with uniform distribution $\xi_i \in [-\sqrt{3}, \sqrt{3}]$ and $A_i = (b_i^2/2)L$ and $\psi_i(x) = \sqrt{(2/L)} \sin(2i\pi x/L)$, $b_i^2 = (2A^4/L)(1/[1+(i\pi A/L)^2])$ and correlation length $A = 1$, $L = x_0 = 1$. Figure 14 presents the mean and standard deviation of ion and electron pressure.
Figure 14. The mean and standard deviation of ion and electron pressure as a function of $x$ with Larmor radius $\hat{r}_L$ described as a random process, obtained from the two-fluid solution of the electromagnetic plasma shock problem.

number density as a function of $x$, with the Larmor radius $\hat{r}_L$ described as a random process. Similar to the random variable case, the ion number density has similar profile as the electron number density, except around the compound wave and contact discontinuity region.

6. SENSITIVITY ANALYSIS AND STOCHASTIC SIMULATION OF SINGLE-FLUID/TWO-TEMPERATURE PLASMA FLOW WITH UNCERTAINTIES

In the two-dimensional single-fluid/two-temperature plasma flow simulation [47], there are possibly five parametric uncertainty sources, i.e. $Re_i$, $Re_e$, $Pr_i$, $Pr_e$ and $Re_{mag}$. We consider as a benchmark example the supersonic flow $Ma=2$ past a cylinder. To efficiently capture the primary uncertainty source from the above five random inputs, sensitivity analysis is performed. We define $E[Re_i]=100$, $E[Re_e]=200$, $E[Pr_i]=0.72$, $E[Pr_e]=0.72$ and $E[Re_{mag}]=100$. The computational domain is $[-15, 35] \times [-20, 20]$ and the center of the cylinder is located at $(0, 0)$; its radius is 0.5. We perform simulations using an unstructured mesh consisting of 1132 triangular elements; a discontinuous Galerkin method is employed for the spatial discretization [47]. The initial conditions are as follows:

$$u = 1, \quad v = 0, \quad w = 0, \quad p_i = 0.15, \quad p_e = 1.1p_i$$
$$\rho = 1, \quad b_x = 0.1, \quad b_y = 0, \quad b_z = 0$$

The physical variables at two locations are considered as the output. One location is at $(0, 1.5)$, which is on top of the cylinder and the other location is at $(1.5, 0)$, which is in the wake region. Figure 15 presents the sensitivity analysis results for density and streamwise velocity at $(0, 1.5)$.

obtained by the Morris method, sparse collocation method, MC method and QMC method. For QMC, MC, Morris methods, 200 samples are chosen, while for sparse collocation method, 61 samples are used. Here we note that in Figures 15 and 16, 1 represents $Re_i$, 2 represents $Re_e$, 3 denotes $Pr_i$, 4 denotes $Pr_e$ and 5 represents $Re_{mag}$. Figure 15 presents the sensitivity analysis results for density and streamwise velocity at (0, 1.5). Obviously, for output chosen as $\rho$ or $u$ at (0, 1.5), $Re_i$ has the largest mean elementary effect. Figure 16 plots the sensitivity analysis results for ion and electron temperature at (0, 1.5). Different from the sensitivity analysis results obtained by the Morris method, sparse collocation method, Monte Carlo method and Quasi-Monte Carlo method.
for output $\rho$ or $u$, $Pr_i$ has the largest mean elementary effect while $T_i$ or $T_e$ is chosen as the output. Similarly, Figures 17 and 18 show sensitivity analysis results for density, streamwise velocity, and ion and electron temperature at $(1.5, 0)$.

**Ion Reynolds number $Re_i$ is described as a space-dependent random process:** Assuming that we are interested in the variation of the density and considering the sensitivity analysis results, then $Re_i$ has the largest mean elementary effect. Thus, $Re_i$ is represented as a random process and corresponding stochastic simulations are performed. Here we assume that $Re_i$ can be described as

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follows:

\[
R(x, y; \omega) = \sum_{i=0}^{d_x} \sum_{j=0}^{d_y} \sqrt{\Lambda_{ij}} \psi_{ij}(x, y) \zeta_{ij}
\]

\[
Re_i(x, y; \omega) = 100 \left( 1 + \frac{\varepsilon}{R_{\text{max}}} R(x, y; \omega) \right)
\]

where \( \Lambda_{ij} = 2/(L_x L_y (I_1 + k^2)) \), \( k = \sqrt{(L_x L_y)/A} \), \( I_1 = (i \pi/L_x)^2 + (j \pi/L_y)^2 \) and

\[
\psi_{ij} = \frac{2}{\sqrt{L_x L_y}} \sin \left( \frac{i \pi (x - 0.5L_x)}{L_x} \right) \sin \left( \frac{j \pi (y - 0.5L_y)}{L_y} \right)
\]

and \( R_{\text{max}} = \max_{x,y}(\sigma(R)) \). Here the correlation length \( A = 1 \), \( d_x = d_y = 1 \), \( \varepsilon = 0.5 \) and \( L_x = L_y = 4 \) (Figures 19 and 20).

Figure 21 presents the mean of ion and electron temperature contours for \( Re_i \) described as a random process with correlation length \( A = 1 \) and \( \varepsilon = 0.5 \).

**Ion Prandtl number** \( Pr_i \) **is described as a space-dependent random process**: Based on the sensitivity analysis for stochastic single-fluid/two-temperature plasma flow, \( Pr_i \) has the largest mean elementary effect when the temperature is chosen as the output. Thus, \( Pr_i \) is represented as a random process. Here we assume \( Pr_i \) can be described as follows:

\[
S(x, y; \omega) = \sum_{i=0}^{d_x} \sum_{j=0}^{d_y} \sqrt{\Lambda_{ij}} \psi_{ij}(x, y) \zeta_{ij}
\]

\[
Pr_i(x, y; \omega) = 100 \left( 1 + \frac{\varepsilon}{S_{\text{max}}} S(x, y; \omega) \right)
\]
Figure 20. (a) Mean and (b) standard deviation of streamwise velocity contours for \( Re_l \) described as a random process with correlation length \( A = 1 \) and \( \varepsilon = 0.5 \).

Figure 21. Mean of (a) ion and (b) electron temperature contours for \( Re_l \) described as a random process with correlation length \( A = 1 \) and \( \varepsilon = 0.5 \).

where \( \Lambda_{ij} = 2/(L_x L_y (I_1 + k^2)) \), \( k = \sqrt{(L_x L_y)} / A \), \( I_1 = \sqrt{(i \pi/L_x)^2 + (j \pi/L_y)^2} \) and

\[
\psi_{ij} = \frac{2}{\sqrt{L_x L_y}} \sin \left( \frac{i \pi (x - 0.5 L_x)}{L_x} \right) \sin \left( \frac{j \pi (y - 0.5 L_y)}{L_y} \right)
\]

and \( S_{\text{max}} = \max_{x,y} (\sigma(S)) \). Here the correlation length is \( A = 1, \Delta_x = \Delta_y = 1, \varepsilon = 0.5 \) and \( L_x = L_y = 4 \).

Figure 22 presents the mean of ion and electron temperature contours for \( Pr_i \) described as a random process with correlation length \( A = 1 \) and \( \varepsilon = 0.5 \). Compared with the previous case in
which we described the ion Reynolds number $Re_i$ as a random process, the values of ion and electron temperature are of similar magnitude.

7. SUMMARY

In summary, stochastic sensitivity analysis can help to identify the most important uncertain parameters, reduce dimensions in parametric space and guide large-scale stochastic simulations and potentially new experimental work. An ME-PCM has been developed to efficiently perform sensitivity analysis, which can achieve fast convergence compared with the Morris and MC methods for problems with moderate number of random parameters (20–100). For problems with very large number of random parameters (beyond 100), the Morris and MC methods may be more efficient. To effectively model stochastic plasma flows with lower computational cost, we demonstrate in this paper how to couple sensitivity analysis and stochastic simulations. The procedure is as follows: First, we use sensitivity analysis to rank the significance of the random inputs, which are all modeled as random variables. We then include only the random inputs with the largest impact as random processes (represented by many random variables via a KL expansion) and perform stochastic simulations. The two MHD examples presented demonstrate the capability and efficiency of the stochastic sensitivity analysis, which can be used as a pre-screening technique for reducing the dimensionality and hence the cost in stochastic simulations.

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