Analytical approximation and numerical studies of one-dimensional elliptic equation with random coefficients

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In this work, we study a one-dimensional elliptic equation with a random coefficient and derive an explicit analytical approximation. We model the random coefficient with a spatially varying random field, $K(x, \omega)$ with known covariance function. We derive the relation between the standard deviation of the solution $T(x, \omega)$ and the correlation length, $\eta$ of $K(x, \omega)$. We observe that, the standard deviation, $\sigma_T$ of the solution, $T(x, \omega)$, initially increases with the correlation length $\eta$ up to a maximum value, $\sigma_{T, \text{max}} \approx \sqrt{\pi(1-x)/3}$ and decreases beyond $\eta_{\text{max}}$. We observe a scaling law between $\sigma_T$ and $\eta$, that is, $\sigma_T \propto \eta^{1/2}$ for $\eta \to 0$ and $\sigma_T \propto \eta^{-1/2}$ for $\eta \to \infty$. We show that, for a small value of coefficient of variation $\varepsilon_x = \sigma_x / \mu_x$ of the random coefficient, the solution $T(x, \omega)$ can be approximated with a Gaussian random field regardless of the underlying probability distribution of $K(x, \omega)$.

This approximation is valid for large value of $\varepsilon_x$, if the correlation length, $\eta$ of input random field $K(x, \omega)$ is small. We compare the analytical results with numerical ones obtained from Monte-Carlo method and polynomial chaos based stochastic collocation method. Under aforementioned conditions, we observe a good agreement between the numerical simulations and the analytical results. For a given random coefficient $K(x, \omega)$ with known mean and variance we can quickly estimate the variance of the solution at any location for a given correlation length. If the correlation length is not available which is the case in most practical situations, we can still use this analytical solution to estimate the maximum variance of the solution at any location.

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1. Introduction

Elliptic equation is fundamental to many physical phenomena such as heat conduction, diffusion, mechanics, electrostatics, Darcy flow and many more. In this paper, we present analytical methods to study the solution of a one-dimensional elliptic equation with a random coefficient. We derive an explicit analytical solution for the stochastic elliptic equation and discuss the importance of the analytical solution and conditions under which this solution is valid. Often, in practice, a quick estimate of the variance of a solution is required to make preliminary decisions prior to a detailed computational analysis. An analytical study helps us to design and test algorithms for computational numerical studies.

In order to compare analytical solution with numerical solution, we solve the stochastic elliptic equation numerically using Monte-Carlo method, polynomial chaos based stochastic collocation method. There are many numerical methods such as Monte-Carlo, Finite Element, Finite Difference, etc., when dealing with stochastic elliptic equations.
as Monte-Carlo methods [1], spectral stochastic Galerkin methods [2–8], stochastic collocation methods [9–14], perturbation methods to solve stochastic partial differential equations. Several numerical methods based on stochastic collocation and Galerkin methods [15–17] to solve elliptic equations with random coefficient. Finite element methods for structures with large variations is studied and applied to problems in solid mechanics in [18]. Mathematical models for random coefficients in a partial differential equations that satisfy physical and mathematical constraints are discussed in [19]. In [20] ANOVA based statistical methods are used to solve stochastic incompressible and compressible flows. A stochastic basis adaptation method is introduced in [21] to address the issue of curse of dimensionality by representing the quantities of interest (QoI) in a low dimensional stochastic space so that the computational cost is reduced significantly. In [22] analytical solution of fractal diffusion equations is studied. In [23], analytical solution of a one-dimensional advection-diffusion equation with several point sources was studied. Random Airy type differential equations and their numerical study based on power series solution is discussed in [24]. In [25], a Bayesian approach was used to recover the stochastic solution in both random and spatial domains.

Our contribution to this work is, to derive an analytical solution and study of the influence of correlation length, η on the standard deviation of the solution. We compare results from analytical study with those from numerical simulations. We show from both numerical and analytical results that the standard deviation of the solution increases as a function of the correlation length, η to a maximum at ηmax and decreases beyond that point. σT follows square-root power law (σT ∝ η1/2) for η → 0 and inverse square-root power law (σT ∝ η−1/2) for η → ∞. Magnitude of σT,max varies for different spatial locations, x. As the magnitude of σT,max increases, the value ηmax shifts towards right. We show that the solution can be approximated to a Gaussian random field regardless of the distribution of the random coefficient, K(x, ω) for small η. As we increase the value of εK and correlation length η, solution becomes non Gaussian and we observe larger error in analytical solution.

The paper is organized as follows. In Section 2, we introduce a steady state elliptic equation with a random coefficient and derive an analytical solution of stochastic elliptic equation and discuss assumptions under which this analytical solution holds. We discuss numerical methods namely, Monte-Carlo method, polynomial chaos based stochastic Galerkin and stochastic collocation methods in Section 3. Standard deviation and probability density function of the solution, obtained from numerical methods and analytical solution is compared in Section 4. Finally we conclude our paper with a discussion on the validity and limitations of the analytical solution and possible extension to the non-homogeneous steady state equation.

2. Analytical solution

Let us consider a one dimensional spatial domain, D = [0, 1], and a complete probability space, (Ω , Σ, P) with sample space Ω , σ-algebra Σ and probability measure P. A stochastic elliptic partial differential equation can be written as: find a random field T(x, ω) : D × Ω → ℝ such that the following equation holds P-almost surely (P-a.s.):

\[
\frac{\partial}{\partial x} \left[ K(x, \omega) \frac{\partial T(x, \omega)}{\partial x} \right] = 0, \quad x \in [0, 1],
\]

\[ T(0, \omega) = T_0, \quad T(1, \omega) = T_1, \]

where x is the position and T(x, ω), is the solution field and we assume that the random coefficient, K(x, ω) is bounded stationary random field and strictly positive, that is,

\[ 0 < K_{\text{min}} \leq K(x, \omega) \leq K_{\text{max}} < \infty \quad \text{in} \quad D \times \Omega, \]

where, K(x, ω) is a random field with known properties such as mean, covariance function and probability distribution. The steady-state solution T(x, ω), is also a stochastic field because of the randomness in the coefficient K(x, ω). Our objective is to find the statistical properties of T(x, ω), such as mean and standard deviation for a given random field K(x, ω).

The random coefficient K(x, ω), in the elliptic equation, Eq. (1) can be written as a series expansion that separates spatial variable, x and stochastic variable, ω. For this, second order statistics such as covariance function is required. A second order stationary process with known covariance function is used to model the random field coefficient. We use following exponential covariance function for the input random field,

\[ \text{cov}_K(x_1, x_2) = \sigma_K^2 \exp \left( -\frac{|x_1 - x_2|}{\eta} \right), \]

where η is the correlation length of the input random field and the coefficient of variation, εK = σK/μK, where μK and σK are the mean and standard deviation of K(x, ω). Random field K(x, ω) can be approximated using Karhunen–Loève expansion [26] as follows:

\[ K(x, \omega) = K_0 + \sum_{i=1}^{\infty} \sqrt{\lambda_i} \xi_i(\omega) K_i(x), \]
where \( \xi \) is a set of uncorrelated random variables with zero mean and unit variance, and \( \langle \lambda_i, K_i(x) \rangle \) is a set of eigen pairs obtained by solving following integral eigenvalue problem,

\[
\int_D \text{cov}(x_1, x_2) K_i(x_2) \, dx_2 = \lambda_i K_i(x_1).
\]

(5)

The eigenvalues \( \lambda_i \) are positive and non-increasing, and the eigenvectors, \( K_i(x) \) are orthonormal, that is,

\[
\int_D K_i(x) K_j(x) \, dx = \delta_{ij},
\]

(6)

where, \( \delta_{ij} \) is the Kronecker delta.

Here, we derive an analytical expression for solution, \( T(x, \omega) \) and describe the assumptions under which such an expression is valid and show the limitations of the analytical solution. We will show with the numerical experiments that the analytical approximation for \( T(x, \omega) \) is valid when either the coefficient of variation \( (\varepsilon \mu = \sigma_i / \mu_i) \) and/or the correlation length \( \eta \) of input random field \( K(x, \omega) \) is small. We start our derivation by first integrating Eq. (1), to obtain,

\[
K(x, \omega) \frac{\partial T(x, \omega)}{\partial x} = A,
\]

(7)

where, \( A \) is a constant that can be determined from the boundary conditions. Substituting the boundary conditions, we get, \( T(0) = T_0 \) and \( T(1) = T_1 \),

\[
T_1 = T_0 + \int_0^1 \frac{A}{K(x, \omega)} \, dx.
\]

(8)

The constant \( A \) from above expression is as follows:

\[
A = \frac{(T_1 - T_0)}{\int_0^1 \frac{1}{K(x, \omega)} dx}.
\]

(9)

Now, we write the solution field \( T(x, \omega) \) explicitly as,

\[
T(x, \omega) = T_0 + (T_1 - T_0) \int_0^x \frac{1}{K(x', \omega)} dx' + \int_0^1 \frac{1}{K(x, \omega)} dx'.
\]

(10)

Here, Eq. (10) gives an exact solution of the random field \( T(x, \omega) \) for a given random coefficient \( K(x, \omega) \). Similar form was also obtained in our previous study for steady and transient heat conduction [27] and upscaling of solute transport in heterogeneous media [31]. From here, we derive an approximation to the exact solution and the standard deviation of the solution, and provide assumptions and limitations. Writing the integral \( \int_0^1 \frac{1}{K(x, \omega)} dx \) as follows:

\[
\int_0^1 \frac{1}{K(x', \omega)} dx' = \int_0^x \frac{1}{K(x', \omega)} dx' + \int_x^1 \frac{1}{K(x', \omega)} dx'.
\]

(11)

and substituting into Eq. (10), we get,

\[
T(x, \omega) = T_0 + \frac{T_1 - T_0}{1 + \frac{1}{x} \frac{S_2}{S_1}} \left[ \frac{1}{\frac{1}{x} + z} - x \right] + \int_0^1 \frac{1}{K(x, \omega)} dx'.
\]

(12)

and

\[
S_1(x, \omega) = \frac{1}{x} \int_0^x \frac{1}{K(x', \omega)} dx',
\]

(13)

\[
S_2(x, \omega) = \frac{1 - x}{1 - x} \int_x^1 \frac{1}{K(x', \omega)} dx',
\]

\[
z(x, \omega) = \frac{S_2(x, \omega)}{S_1(x, \omega)}.
\]

(14)

From Eq. (10), the solution field can be written as:

\[
T(x, \omega) = T_0 + (T_1 - T_0) x + (T_1 - T_0) \left( W(x) - x \right),
\]

(15)

where, we introduce another random variable \( W(x) \) which is a function of random variable \( z \),

\[
W(x) = \frac{x S_1}{x S_1 + (1 - x) S_2} = \frac{1}{1 + \frac{x}{z} z}, \quad x \neq 0.
\]
Next, we can find the probability density function, $f_h$ for the random field $h(x, \omega) = \frac{1}{k(x, \omega)}$,

$$f_h(a) = \begin{cases} \frac{f_k(\frac{1}{a})}{a^2}, & \frac{1}{k_{\text{max}}} \leq a \leq \frac{1}{k_{\text{min}}}, \\ 0, & \text{otherwise}. \end{cases}$$  \hspace{1cm} (16)

Clearly, $S_1$ and $S_2$ in Eq. (13) are the averages of $h(x, \omega)$ in the range of $[0, x]$ and $[x, 1]$, respectively. The mean $\mu_h$ and variance $\sigma_h^2$ of random variable $h(x, \omega)$ can be found for any bounded $K(x, \omega)$ with an arbitrary distribution $f_k$,

$$\mu_h = E(a) = \int_0^\infty \frac{f_k(\frac{1}{a})}{a} \, da,$$  \hspace{1cm} (17)

and

$$\sigma_h^2 = E(h^2) - (E(h))^2 = \int_0^\infty \frac{f_k(\frac{1}{a})}{a} \, da - \left( \int_0^\infty \frac{f_k(\frac{1}{a})}{a} \, da \right)^2.$$  \hspace{1cm} (18)

To find the covariance of $h(x, \omega)$, let us write Karhunen–Loève expansion for $K(x, \omega)$ and substitute in $h(x, \omega) = \frac{1}{k(x, \omega)}$ of Eq. (4),

$$K(x, \omega) = K_0 + \sum_{i=1}^\infty \sqrt{\lambda_i} \xi_i(\omega)K_i(x),$$  \hspace{1cm} (19)

and hence,

$$h(x, \omega) = \frac{1}{K_0} \left( 1 + \sum_{i=1}^\infty \sqrt{\lambda_i} \xi_i(\omega) \frac{K_i(x)}{K_0} \right)^{-1}.$$  \hspace{1cm} (20)

Here, if we assume that the mean $K_0$ is dominant over the rest of the terms in the Karhunen–Loève expansion in Eq. (20), and hence the coefficient of variation, $\varepsilon_K$ is not very high, we can expand $h(x, \omega)$ using Taylor series about zero and retain only the first order terms and ignore the higher order terms. This approximation can be justified for smaller coefficient of variation, $\varepsilon_K$ which is the case for many practical applications. First order Taylor expansion of $h(x, \omega)$ can be written as:

$$h(x, \omega) \approx \frac{1}{K_0} \left( 1 - \sum_{i=1}^\infty \sqrt{\lambda_i} \xi_i(\omega) \frac{K_i(x)}{K_0} \right)$$

$$= \frac{1}{K_0} + \left( \sum_{i=1}^\infty \sqrt{\lambda_i} \xi_i(\omega) (-K_i(x)) \right)$$

$$= \frac{1}{K_0} + \left( \sum_{i=1}^\infty \sqrt{\lambda_i} \xi_i(\omega) K_i^*(x) \right).$$  \hspace{1cm} (21)

where, $\lambda_i^* = \frac{\lambda_i}{K_0^2}$ and $K_i^*(x) = -K_i(x)$ are the eigenvalues and eigenvectors of the random field $h(x, \omega)$. Using this new set of eigen pairs $(\lambda_i^*, K_i^*(x))$ the covariance function $\text{cov}_h(x_1, x_2)$ of $h(x, \omega)$ can be written as:

$$\text{cov}_h(x_1, x_2) = \sum_{i=1}^\infty \lambda_i^* K_i^*(x_1) K_i^*(x_2),$$

$$= \frac{1}{K_0^4} \sum_{i=1}^\infty \lambda_i K_i(x_1) K_i(x_2),$$

$$= \frac{1}{K_0^4} \text{cov}_K(x_1, x_2),$$  \hspace{1cm} (22)

where, $\text{cov}_K(x_1, x_2) = \sum_{i=1}^\infty \lambda_i K_i(x_1) K_i(x_2)$. By using Eq. (3) for the covariance function $\text{cov}_K(x_1, x_2)$ of $K(x, \omega)$, we get,

$$\text{cov}_h(x_1, x_2) = \sigma_h^2 \exp \left( -\frac{|x_1 - x_2|}{\eta} \right).$$  \hspace{1cm} (23)

where, $\sigma_h^2 = \frac{\sigma_k^2}{K_0^4}$. Clearly $S_1$ and $S_2$ from Eq. (13) are the summation of many correlated random variables with a covariance function shown in Eq. (23). We use the covariance function $\text{cov}_h(x_1, x_2)$ to compute the covariance of $S_1$ and $S_2$ as follows,

$$\text{cov}[S_1(x_1), S_2(x_2)] = \frac{1}{x_1 x_2} \int_{-y}^{x_1} \int_{-y}^{x_2-y} \text{cov}_h(x', y) \, dx' \, dy.$$  \hspace{1cm} (24)
\[ \text{cov}[S_2(x_1), S_2(x_2)] = \frac{1}{(1 - x_1)(1 - x_2)} \int_{x_1}^{1} \int_{x_2 - y}^{x_1} \text{cov}_h(x', y)dx'dy, \]  

\[ \text{cov}[S_1(x_1), S_2(x_2)] = \frac{1}{x_1(1 - x_2)} \int_{x_1}^{1} \int_{x_2 - y}^{x_1} \text{cov}_h(x', y)dx'dy. \]  

Substituting the expression for \( \text{cov}_h(x_1, x_2) \) from Eq. (23) into Eq. (24) and after simplifying, we get the following expression for the variance of \( S_1(x) \) by setting \( x_1 = x_2 \).

\[ \text{var}[S_1(x)] = \frac{2\eta^2 \sigma^2}{x^2} \left( e^{-x/\eta} + \frac{x}{\eta} - 1 \right). \]  

Similarly, the variance of \( S_2(x) \), can be obtained from Eq. (25) as follows by setting \( x_1 = x_2 \).

\[ \text{var}[S_2(x)] = \frac{2\eta^2 \sigma^2}{(1 - x)^2} \left( e^{-(1-x)/\eta} + \frac{(1 - x)}{\eta} - 1 \right). \]  

Since, \( S_1(x) \) and \( S_2(x) \) are the averages of \( h(x, \omega) \) in the ranges of \([0, x]\) and \([x, 1]\), their means can be written as:

\[ \mu_{S_1} = \mu_{S_2} = \mu_h. \]  

From Eqs. (27)–(29), we can observe the limiting cases, \( \text{var}(S_1) \), \( \text{var}(S_2) \rightarrow 0 \) for \( \eta \rightarrow 0 \) and \( \text{var}(S_1) \), \( \text{var}(S_2) \rightarrow \sigma^2_h \) for \( \eta \rightarrow \infty \). Now, using Eqs. (24)–(26), we obtain the correlation coefficient \( \rho(S_1(x), S_2(x)) \) as follows,

\[ \rho(S_1(x), S_2(x)) = \frac{1}{2} \frac{1}{\sqrt{(e^{-x/\eta} + \frac{x}{\eta} - 1)(e^{-(1-x)/\eta} + \frac{1-x}{\eta} - 1)}}. \]  

Again, the limiting cases for \( \rho \) are, \( \rho \rightarrow 0 \) for \( \eta \rightarrow 0 \) and \( \rho \rightarrow 1 \) for \( \eta \rightarrow \infty \). For small values of correlation length, \( \eta \), \( S_1 \) and \( S_2 \) are approximately Gaussian according to Central Limit Theorem (CLT) [28] and hence, the random variable \( z \) from Eq. (13) follows the Gaussian ratio distribution [29] and corresponding probability density function is:

\[ f_z(z) = \frac{b(z)c(z)}{a^2(z)} \frac{1}{\sqrt{2\pi} \sigma_{S_1} \sigma_{S_2}} \left[ 2\Phi \left( \frac{b(z)}{\sqrt{1 - \rho^2 a^2(z)}} \right) - 1 \right] + \sqrt{\frac{1 - \rho^2}{2(1 - \rho^2)}} \exp \left( -\frac{\theta}{2(1 - \rho^2)} \right), \]  

where, \( \mu_{S_1} = \mu_{S_2} = \mu_h \), \( \sigma_{S_1} = \sqrt{\text{var}(S_1)} \) and \( \sigma_{S_2} = \sqrt{\text{var}(S_2)} \).

\[ a(z) = \sqrt{\frac{z^2}{\sigma_{S_1}^2} - \frac{2\rho z}{\sigma_{S_1} \sigma_{S_2}} + \frac{1}{\sigma_{S_2}^2}}, \]

\[ b(z) = \frac{\mu_{S_1} z - \rho(\mu_{S_1} + \mu_{S_2} z)}{\sigma_{S_1} \sigma_{S_2}} + \frac{\mu_{S_2}}{\sigma_{S_2}^2}, \]

\[ \theta = \frac{\mu_{S_1}^2}{\sigma_{S_1}^2} - \frac{2\rho \mu_{S_1} \mu_{S_2}}{\sigma_{S_1} \sigma_{S_2}} + \frac{\mu_{S_2}^2}{\sigma_{S_2}^2}, \]

\[ c(z) = \exp \left[ \frac{b^2(z) - \theta a^2(z)}{2(1 - \rho^2) a^2(z)} \right], \]

and

\[ \Phi(x) = \frac{1}{2} \left[ 1 + \text{erf} \left( \frac{x}{\sqrt{2}} \right) \right]. \]

Rewriting the Eq. (31) as follows,

\[ f_z(z) = \frac{b(z)c(z)}{a^2(z)} \frac{1}{\sqrt{2\pi} \sigma_{S_1} \sigma_{S_2}} [2\Phi(r(z)) - 1] + \frac{\sqrt{1 - \rho^2}}{a^2(z) \pi \sigma_{S_1} \sigma_{S_2}} \exp (-r_0). \]
where, \( r(z) = \frac{h_z}{\sqrt{1 - \rho^2 \mu(z)}} \) and \( r_0 = \frac{\theta}{2(1 - \rho^2)} \). Substituting the values of \( \theta \) in \( r_0 \), we get,

\[
 r_0 = \frac{\mu_h^2}{2(1 - \rho^2)} \frac{\sigma_{S_z}^2 - 2 \rho \sigma_{S_z} \sigma_{S_z} (1 + z) + z \sigma_{S_z}^2}{\sigma_{S_z}^2 \sigma_{S_z}^2} .
\]  

(38)

Fig. 1 shows the plot of \( r_0 \) as a function of \( \eta \) for \( x = 0.5 \) and \( \varepsilon_h = \sigma_h/\mu_h = 0.1, 0.5 \) and 1.0. We can observe from the plot, that the value of \( r_0 \) is very large (\( \sim 10^8 \)) for small \( \varepsilon_h \) and \( \eta \) and hence, the term \( \exp(-r_0) \) in “term 2” is very small compared to the exponential term in “term 1” in Eq. (37) and can be neglected. Rewriting \( r(z) \) as follows:

\[
r(z) = \frac{\mu_h}{\sqrt{1 - \rho^2 \sigma_{S_z} \sigma_{S_z}}} \frac{\sigma_{S_z}^2 - \rho \sigma_{S_z} \sigma_{S_z} (1 + z) + z \sigma_{S_z}^2}{\sqrt{\sigma_{S_z}^2 - 2 \rho \sigma_{S_z} \sigma_{S_z} + z^2 \sigma_{S_z}^2}} .
\]

(39)

and substituting, \( z = 1 + z' \), we get,

\[
r(z') = \frac{\mu_h}{\sqrt{1 - \rho^2 \sigma_{S_z} \sigma_{S_z}}} \frac{\sigma_{S_z}^2 - \rho \sigma_{S_z} \sigma_{S_z} (2 + z') + (1 + z') \sigma_{S_z}^2}{\sqrt{\sigma_{S_z}^2 - 2 \rho (1 + z') \sigma_{S_z} \sigma_{S_z} + (1 + z')^2 \sigma_{S_z}^2}} .
\]

(40)

For \( z = \frac{S_z}{S_z} \rightarrow 1 \) or \( z' \rightarrow 0 \), we get \( r(z') \rightarrow \sqrt{2r_0} \). Again from Fig. 1 as \( r_0 \) increases, \( r(z') \) increases and hence the error function, \( \Phi(r(z)) \) approaches 1.0. Therefore, neglecting “term 2” in Eq. (37) and substituting, 1.0 for \( \Phi(r(z)) \), we get,

\[
f_{z'}(z') \approx \frac{\mu_h}{\sigma_{S_z} \sigma_{S_z} \sigma_{S_z}^2} \left( \frac{1}{\sigma_{S_z}^2} - \frac{2 \rho \sigma_{S_z} \sigma_{S_z}}{\sigma_{S_z}^2} + \frac{(1 + z')^2}{\sigma_{S_z}^2} \right)^{3/2} \frac{1}{\sqrt{2\pi \sigma_{S_z} \sigma_{S_z}}} \exp \left[ -\frac{1}{2} \frac{(z')^2}{\sigma_{S_z}^2 - 2 \rho (1 + z') \sigma_{S_z} + (1 + z')^2 \sigma_{S_z}^2} \right] .
\]

(41)

Ignoring the higher order terms beyond the second order in the exponential term in Eq. (41) for \( z' \rightarrow 0 \), we get following Gaussian probability density functions for \( z' \) and \( z \),

\[
f_{z'}(z') \approx \frac{1}{\sigma_{z'} \sqrt{2\pi}} e^{-\frac{1}{2} \frac{z'^2}{\sigma_{z'}^2}} .
\]

(42)

\[
f_{z}(z) \approx \frac{1}{\sigma_{z} \sqrt{2\pi}} e^{-\frac{1}{2} \frac{z^2}{\sigma_{z}^2}} .
\]

(43)

respectively, where,

\[
\sigma_{z} = \frac{\sqrt{\sigma_{S_z}^2 - 2 \rho \sigma_{S_z} \sigma_{S_z} + \sigma_{S_z}^2}}{\mu_h} .
\]

(44)
We test the validity of this approximation by comparing the probability density function of \( z \) obtained from Eqs. (31) and (43) and that from numerical simulations. We present this comparison in Section 4. In Eq. (42), \( z' \sim N(0, \sigma_z) \). Substituting \( z = 1 + z' \) in Eq. (15), we get,
\[
W = \frac{1}{1 + (1 + z')(1-x)^{-1}},
\]
\[
= \frac{x}{1 + z'(1-x)},
\]
\[
= x[1 + z'(1-x)]^{-1}.
\]
(45)

Using first order Taylor series expansion, \( W \) in Eq. (45) can be written as,
\[
W \approx x[1 - z'(1-x)].
\]
(46)

Since, \( W \) in Eq. (46) is a linear function of \( z' \), it follows a Gaussian distribution with following probability density function,
\[
f_W(W) = \frac{1}{\sqrt{2\pi} \sigma_W} e^{-\frac{(W - \mu_W)^2}{2 \sigma_W^2}},
\]
(47)

where the mean and standard deviation are:
\[
\mu_W = x,
\]
\[
\sigma_W = \frac{x(1-x)}{\mu_h} \sqrt{\sigma_s^2 - 2\rho\sigma_s \sigma_s + \sigma_s^2}.
\]
(48)

Clearly \( f_W(W) \) is the Gaussian probability density function of \( W \) in Eq. (47). From Eq. (14), \( T \) is a linear transformation of \( W \) and hence it follows the Gaussian distribution with following mean, \( \mu_T \) and standard deviation, \( \sigma_T \),
\[
\mu_T = T_0 + (T_1 - T_0)x.
\]
(49)

and
\[
\sigma_T = (T_1 - T_0)\sqrt{2}\eta^2 \frac{\sigma_h}{\mu_h} \alpha \beta \sqrt{\frac{\alpha - 1 + e^{-\beta} - 1}{\alpha^2} - \frac{1 + e^{-1/\eta} - e^{-\alpha} - e^{-\beta}}{\alpha \beta}},
\]
(50)

where, \( \alpha = x/\eta \) and \( \beta = (1-x)/\eta \). Fig. 2a shows a plot of standard deviation of the solution as a function of spatial location according to Eq. (50). Though the magnitude of the standard deviation for a correlation length of 0.01 is comparable with that for a correlation length of 10, the shapes of the plots are different. Plot with a smaller correlation length is flatter. We can observe from the Fig. 2b that the standard deviation of the solution at a point is smaller for small correlation length and increases with the correlation length to a maximum at \( \eta_{max} \) and decreases beyond \( \eta_{max} \). We also observe a shift in \( \sigma_{T, max} \) with respect to \( \eta \) towards right with increase in magnitude as a function of location, \( x \) (Fig. 2b).

In Eq. (50), we need to compute the value of coefficient of variation, \( \varepsilon_h = \frac{\sigma_h}{\mu_h} \) for a given \( \varepsilon_K \). For a small value of \( \varepsilon_h \), it can be approximated with an explicit function. Using Eqs. (6.5) and (7.3) of [30], mean and variance of \( h = 1/K \) can be obtained as follows,
\[
\mu_h \approx \frac{1}{\mu_K} \left( 1 + \frac{\sigma_h^2}{\mu_K^2} \right),
\]
\[
\sigma_h^2 \approx \frac{\sigma_K^2}{\mu_K^2}.
\]
(51)

Now, the coefficient of variation, \( \varepsilon_h \), can be approximated by using the values of \( \mu_h \) and \( \sigma_h \) from Eq. (51) as follows,
\[
\varepsilon_h = \frac{1}{\mu_h} \frac{\sigma_h}{\mu_h} + \frac{\sigma_h}{\mu_h}.
\]
(52)

Substituting \( \varepsilon_h \) from Eq. (52) into Eq. (50), we get following approximation for \( \sigma_T \),
\[
\sigma_T \approx (T_1 - T_0)\sqrt{2}\eta^2 \frac{\sigma_h}{\mu_h} \alpha \beta \sqrt{\frac{\alpha - 1 + e^{-\beta} - 1}{\alpha^2} + \frac{e^{-\alpha} + \alpha - 1}{\alpha^2} - \frac{1 + e^{-1/\eta} - e^{-\alpha} - e^{-\beta}}{\alpha \beta}}.
\]
(53)

From Eq. (50), limiting cases of \( \sigma_T \) as \( \eta \to 0 \) and \( \eta \to \infty \) can be obtained by Taylor expansion of parameters \( \alpha \) and \( \beta \) around zero,
\[
\sigma_T = \sqrt{2} \frac{\sigma_h}{\mu_h} \sqrt{x(1-x) \sqrt{\eta}(T_1 - T_0)}, \quad \text{for } \eta \to 0
\]
(54)

\[
\sigma_T = \sqrt{2} \frac{\sigma_h}{\mu_h} \sqrt{x(1-x)} \frac{1}{3\sqrt{\eta}}(T_1 - T_0), \quad \text{for } \eta \to \infty
\]
(55)
where, we found two scaling laws, $\sigma_T \propto \eta^{1/2}$ for $\eta \to 0$ and $\sigma_T \propto \eta^{-1/2}$ for $\eta \to \infty$. We found maximum value of the standard deviation, $\sigma_{T, \text{max}}$, and corresponding correlation length, $\eta_{\text{max}}$ from the intersection point of two scaling laws, as shown in Fig. 3.

$$
\eta_{\text{max}} \approx \sqrt{\frac{x(1-x)}{3}},
$$

and

$$
\sigma_{T, \text{max}} = \sqrt{\frac{2}{\mu_h} \frac{\sigma_h}{\sigma_K} \frac{|x(1-x)|^{3/4}}{3^{1/4}}},
$$

respectively. Similarly, $\sigma_{T, \text{max}}$ can be approximated as follows by using Eq. (52),

$$
\sigma_{T, \text{max}} \approx \sqrt{\frac{2}{\mu_h} \frac{\sigma_h}{\sigma_K} \frac{|x(1-x)|^{3/4}}{3^{1/4}}}
$$

For a given random field $K(x, \omega)$, with known mean, $\mu_K$ and variance, $\sigma_K^2$, we can use Eq. (53) to estimate the variance of the solution, $\sigma_T$ at any location $x$ for a given correlation length $\eta$. However, if there is no information about $\eta$, we still can use Eq. (58) to quickly estimate the maximum variance of the solution, $\sigma_{T, \text{max}}$ at any location $x$ which can be used for practical engineering design.

We can observe in Fig. 3 and from Eq. (50), that $\sigma_T$ follows square-root power law with respect to $\eta$ for $\eta \to 0$ and follows inverse square-root power law for $\eta \to \infty$. Here, we compare $\sigma_T$ computed from Eq. (50) and from limiting cases ($\eta \to 0$ and $\eta \to \infty$) in Eqs. (54) and (55). We can clearly observe that the approximate expressions for $\sigma_T$ from Eqs. (54) and (55) match well with the expression in Eq. (50) near the limiting values of $\eta$. 

---

**Fig. 2.** Standard deviation of the solution, $\sigma_T$ computed from the analytical solution (Eq. (50)) (a) as function of spatial location, $x$ and (b) as a function of correlation length, $\eta$ when, coefficient of variation, $\varepsilon_h = 0.5$. 

---
3. Numerical methods

In this section, we discuss about various numerical methods, namely Monte-Carlo method, polynomial chaos based stochastic Galerkin and collocation methods used to compare against analytical solution.

3.1. Monte-Carlo methods

We use Monte-Carlo method due to their simplicity and robustness to compare the analytical solution. We use following truncated Karhunen–Loève expansion to sample the random field,

\[ K(x, \omega) = K_0(x) + \sum_{i=1}^{M} \sqrt{\lambda_i} \xi_i(\omega) K_i(x), \]

(59)

where, \( M \) is the number of random variables spanning the random space that contain the random field \( K(x, \omega) \). We assume that the random field \( K(x, \omega) \) has an exponential covariance function shown in Eq. (3). Since our focus of this work is to study the dependence of the solution on the correlation length, \( \eta \), we vary \( \eta \) and accordingly, vary the number of random variables, \( M \) such that the input uncertainty (standard deviation) of \( K(x, \omega) \) for each value of \( \eta \) is approximately equal. We consider three cases to model uncertainty in \( K(x, \omega) \), (i) random coefficient modeled with a Gaussian random field and accordingly \( \{\xi_i\}_{i=1}^{M} \) in Eq. (59) are modeled as, independent Gaussian random variables with zero mean and unit variance, (ii) \( K(x, \omega) \) modeled with log-normal random field and (iii) the random variables, \( \{\xi_i\}_{i=1}^{M} \) in Eq. (59) are modeled with, independent uniformly distributed random variables with zero mean and unit variance. In all three cases, as we vary correlation length, \( \eta \), we also vary number of terms retained in K–L expansion in Eq. (59) so that, the coefficient of variation, \( \varepsilon_k \) of input random field remains constant. Solution is obtained by solving following deterministic elliptic equation for each sample, \( \xi^j \) for \( j = 1, \ldots, n \) and \( \xi = \{\xi_1, \ldots, \xi_M\}^T \).

\[ \frac{\partial}{\partial x} \left[ K(x) \frac{\partial T(x)}{\partial x} \right] = 0, \quad x \in [0, 1]. \]

(60)

\[ T(0) = T_0, \quad T(1) = T_1. \]

where, \( K(x) = K(x, \xi^j) \) and \( T(x) = T(x, \xi^j) \). We compute mean, \( \mu_T \) and variance, \( \sigma_T^2 \) of the solution as:

\[ \mu_T(x) = \frac{1}{n} \sum_{j=1}^{n} T(x), \]

(61)

\[ \sigma_T^2(x) = \frac{1}{n-1} \sum_{j=1}^{n} [T(x) - \mu_T(x)]^2. \]

For small \( \eta \), it is more efficient to use Monte-Carlo methods than polynomial chaos based stochastic Galerkin and collocation methods discussed in the next section.
3.2. Polynomial chaos expansion

The K–L expansion can be used for bounded random variables such as uniform random variables for which the positivity of the random field can be maintained. For unbounded random variables, such as log-normal random variables, a nonlinear polynomial chaos expansion preserves the positivity. The polynomial chaos expansion approximates a random field in terms of multi-variate orthogonal polynomials. In spectral stochastic Galerkin and collocation methods, polynomial chaos expansion is a good choice to model the input random field for unbounded random variables, and for the solution field, \( T(x, \omega) \). The polynomial expansion of the solution field in terms of multi-variate orthogonal polynomials, can be written as,

\[
T(x, \omega) = \hat{T}_0(x) + \sum_{i=1}^{\infty} \hat{T}_i(x) \Gamma_1(\xi_i(\omega)) + \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \hat{T}_{ij}(x) \Gamma_2(\xi_i(\omega), \xi_j(\omega)) + \cdots \tag{62}
\]

where, \( \Gamma_n(\xi_1, \ldots, \xi_n) \) is the multidimensional polynomial chaos of order \( n \) in random variables, \( (\xi_1, \ldots, \xi_n) \). Introducing a one-to-one mapping of polynomials \( \{\Gamma_i\} \) to a set of polynomials with ordered indices, \( \psi(\xi) \), we get following series expansion for \( T(x, \xi) \):

\[
T(x, \xi) = \hat{T}_0(x) + \sum_{i=1}^{\infty} T_i(x) \psi_i(\xi), \tag{63}
\]

where, \( \xi = [\xi_1, \xi_2, \ldots, \xi_M]^T \), when the K–L expansion in Eq. (4) truncated up to \( M \) random variables. For computational purposes, we truncate the series in Eq. (63) to \( N_\xi \) terms as,

\[
T(x, \xi) = \hat{T}(x, \xi) \approx \hat{T}_0(x) + \sum_{i=1}^{N_\xi} T_i(x) \psi_i(\xi). \tag{64}
\]

The polynomials, \( \psi_i(\xi) \) in Eq. (63) are orthogonal with respect to the inner product defined by expectation in the stochastic space, that is,

\[
\langle \psi_i(\xi), \psi_j(\xi) \rangle = \int_\Omega \psi_i(\xi(\omega)) \psi_j(\xi(\omega)) dP(\omega) = \delta_{ij}. \tag{65}
\]

Here, we discuss the numerical methods used to compare the solution obtained from analytical approximation. There are several numerical methods such as perturbation methods, Monte-Carlo methods, spectral stochastic Galerkin methods and spectral stochastic Galerkin methods to solve stochastic partial differential equations. We use stochastic collocation and Monte-Carlo methods to compare the analytical results. Stochastic Galerkin and collocation methods have the limitation that, when the correlation length is very small, the dimension of the stochastic space is very high and the size of the problem grows exponentially. Hence, we avoid collocation method for cases when the correlation length, \( \eta \) is small (\( \eta < 0.2L \)).

1. Stochastic Galerkin method

In stochastic Galerkin method, random coefficient \( K(x, \omega) \) is approximated with Karhunen–Loève expansion (Eqs. (59)) and the solution field is approximated with polynomial chaos expansion (Eq. (64)). When we substitute Eqs. (59) and (64) in to stochastic elliptic equation (Eq. (1)) and project is resulting expansion on to the subspace spanned by first \( N_\xi \) polynomial chaos basis, we get following coupled set of deterministic elliptic equations.

\[
\begin{align*}
\sum_{j=0}^{N_\xi} \sum_{i=0}^{M} c_{ijk} \frac{\partial}{\partial x} \left[ K_i(x) \frac{\partial T_j(x)}{\partial x} \right] &= 0, & x \in [0, 1], \\
T_0(0) &= T_0, T_1(1) = T_1, \\
T_k(0) &= 0, T_k(1) = 0, & k = 1, \ldots, N_\xi,
\end{align*}
\tag{66}
\]

where, \( c_{ijk} = \mathbb{E}[\xi_i \psi_j(\xi) \psi_k(\xi)] \) and \( \mathbb{E}[\cdot] \) is the mathematical expectation. These coupled set of deterministic elliptic equations can be discretized using finite element or finite difference methods that result in a coupled set of algebraic equations. We solve these system of linear equations using Krylov subspace methods. We compute mean and variance of the solution using stochastic Galerkin method as follows:

\[
\begin{align*}
\mu_T(x) &= T_0(x), \\
\sigma_T^2(x) &= \sum_{i=1}^{N_\xi} T_i^2(x) \mathbb{E}[\psi_i^2(\xi)].
\end{align*}
\tag{67}
\]
Fig. 4. Standard deviation of the solution, $\sigma_T$ vs correlation length, $\eta$, when the random coefficient, $K(x, \omega)$ is modeled with uniform random variables, log-normal random field and Gaussian random field. Numerical results are computed using (a) Monte-Carlo method and (b) polynomial chaos based collocation method.

2. Stochastic collocation method

In collocation method the PDE is sampled at a pre-selected set of points called collocation points, $(\xi^{(1)}, \ldots, \xi^{(N)})$. Stochastic solution is obtained by interpolating at the collocation points.

$$T(x, \xi) \approx \sum_{i=1}^{N} T_i(x)L_i(\xi),$$

(68)

where, $\{L_i(\xi)\}$ are Lagrange interpolatory polynomials such that $L_i(\xi^{(j)}) = \delta_{ij}$ and $T_i(x)$ is the solution of following deterministic elliptic equation,

$$\frac{\partial}{\partial x} \left[ K(x, \xi) \frac{\partial T_i(x)}{\partial x} \right] = 0, \quad x \in [0, 1].$$

(69)

The collocation points can be chosen as tensor products of 1-d Gaussian quadrature points and the interpolating polynomials as tensor products of 1-d Lagrange interpolating functions. However, number of collocation points grow exponentially.
To alleviate this, an alternate method is to use Smolyak sparse grid quadrature, where the collocation points that do not increase asymptotic accuracy are removed from tensor product grid. This results in fewer collocation points. We compute mean and variance of the solution using stochastic Galerkin method as follows,

$$\mu_T(x) = \sum_{i=1}^{N_x} f_i(x) w_i,$$
Fig. 6. Standard deviation of the solution, $\sigma_T$, vs correlation length, $\eta$, when the random coefficient is modeled with the log-normal random field with coefficient of variation, $\varepsilon_K$ of 0.1, 0.5 and 1.0. Here, legend “Analytical-Eq.55” is used when $\sigma_T$ is approximated with Eq. (53), “Analytical-Eq.52” is used when $\sigma_T$ is obtained from Eq. (50), where, $\varepsilon_h = \sigma_h/\mu_h$ is computed numerically, and “Numerical-logN” is used when the solution is computed using Monte-Carlo method.

$$\sigma_T^2(x) = \sum_{i=1}^{N_q} T_i^2(x)w_i - \mu_T^2,$$

where, $N_q$ is the number of collocation points and $w_i$ are the weights corresponding to each collocation point.

4. Results and discussion

Here, we show the numerical results to support our analytical study described in Section 2. We compare analytical solution with the numerical one and test the validity under different sets of parameters ($\eta, \varepsilon_K$). We compare the probability
density functions of the quantities, \( z \) (Eq. (13)) and \( T \) (Eq. (14)) from analytical solution and numerical simulation. We compare the standard deviation of the solution, \( \sigma_T \) as a function of the correlation length, \( \eta \), of the input random field, for both analytical and numerical solutions. Numerical and analytical results show that the standard deviation of the solution is approximated very well when the coefficient of the variation, \( \varepsilon_K = \sigma_K / \mu_K \) is small over entire range of \( \eta \) and for small \( \eta \) over entire range of \( \varepsilon_K \).

For numerical simulations, we use stochastic collocation and Monte-Carlo methods. For consistency we use truncated K-L expansion in Eq. (59) for sampling input random field for Monte-Carlo and stochastic collocation methods. As we vary correlation length, \( \eta \), we also vary number of terms retained in K-L expansion in Eq. (59) so that, the coefficient of variation, \( \varepsilon_K \) of input random field remains constant. For small value of \( \eta \), we need to retain large number of terms in Eq. (59). With large number of random variables, it becomes prohibitive to use stochastic Galerkin and stochastic collocation methods due to high dimensionality and therefore, we avoid stochastic Galerkin and collocation methods for cases when the correlation length, \( \eta \) is small (\( \eta < 0.2L \)). For \( \eta < 0.2L \), we use Monte-Carlo method as the convergence of the Monte-Carlo solution does not depend on the dimension of the problem.

We performed numerical tests for a mean value, \( \mu_K \) of 5.0 and coefficient of variation, \( \varepsilon_K \) values of 0.1, 0.5 and 1.0, and the correlation length, \( \eta \) is varied from 0.001 to 10. We discretized the spatial domain, \([0.0, 1.0]\) into \( n = 1000 \) elements for numerical simulation. For numerical tests, we considered three models for the random coefficient \( K(x, \omega) \), namely, (i) Gaussian random field, (ii) log-normal random field and (iii) random field with uniformly distributed random variables, \( \xi \) in Eq. (59). For (i) and (ii) the bounded condition in Eq. (2) can be approximately satisfied if the coefficient of variation is sufficiently small. Fig. 4a and b show the plots of standard deviation of solution vs correlation length, for input random field modeled as cases (i), (ii) and (iii) in Eq. (59), computed analytically and from numerical simulations. In Fig. 4a numerical results are obtained from Monte-Carlo method and in Fig. 4b, they are obtained from polynomial chaos based stochastic collocation method. Due to curse of dimensionality associated with polynomial chaos based methods, we did not compute the solution for very small correlation length (\( \eta < 0.2L \)) with these methods, while Monte-Carlo methods can be used for small values of \( \eta \). Here, we use a coefficient of variation \( \varepsilon_K \) of 0.1 for these plots. We can observe in these plots that the standard deviation from analytical solution matches well with that from numerical solutions for all three cases (i), (ii) and (iii) regardless of the probability distribution of the input random field, \( K(x, \omega) \). Fig. 5 shows the surface plots of \( \sigma_T \) at \( x = 0.5 \) as a function of \( \eta \) and \( \varepsilon_K \). In Fig. 5a, \( \sigma_T \) is computed from analytical solution, and in Fig. 5b, it is computed from Monte-Carlo method. Fig. 5c shows the plot of relative error in \( \sigma_T \) computed from analytical and Monte-Carlo simulations. We can observe from the plots that, error is small when either \( \eta \) or \( \varepsilon_K \) are small and the error is large only when both \( \eta \) and \( \varepsilon_K \) are large.

Fig. 6a–c show the plots of standard deviation, \( \sigma_T \) of the solution vs correlation length, \( \eta \) for coefficient of variation, \( \varepsilon_K \) of 0.1, 0.5 and 1.0, respectively. Random coefficient, \( K(x, \omega) \) is modeled with log-normal random field for these plots. Here, analytical solution is computed with two variations. In the first case, as shown in legend, “Analytical-Eq.55” is used when \( \sigma_T \) is computed with Eq. (53), and in the second case, “Analytical-Eq.52” is used when \( \sigma_T \) obtained from Eq. (50), where, \( \varepsilon_h = \sigma_h / \mu_h \) is computed numerically, and “Numerical-logN” is used when the solution is computed using Monte-Carlo method. In Fig. 6a, \( \varepsilon_K \) of 0.1 is used. We can observe for small \( \varepsilon_K \) the standard deviation, \( \sigma_T \) from both analytical
solutions match very well with that from numerical simulations over entire range of correlation length $\eta$. In Fig. 6b and c, $\sigma_T$ from analytical solution (Eq. (50)) matches with numerical one when $\eta$ is very small and it has larger discrepancy for larger $\eta$ for both analytical solutions. While Eq. (50) overestimates $\sigma_T$, Eq. (53) underestimates $\sigma_T$. As $\varepsilon_{\kappa}$ increases, the discrepancy increases between numerical and analytical results for larger $\eta$.

Fig. 7 shows the probability density function of the ratio, $z$ computed analytically and from numerical method for correlation length, $\eta$ of 0.1 and coefficient of variation, $\varepsilon_{\kappa}$ of 0.1 of the input random coefficient. Fig. 8 shows the probability density function of the ratio, $z$ for $\eta$ of 1.0 and $\varepsilon_{\kappa}$ of 0.5. We can observe from Figs. 7 and 8 that, the ratio term $z$ follows Gaussian distribution and its probability density function matches well with the numerical solution, for small $\eta$ of 0.1, and small $\varepsilon_{\kappa}$ of 0.1, while for large $\eta$ of 1.0, and large $\varepsilon_{\kappa}$ of 0.5, probability density function of $z$ doesn’t follow Gaussian distribution and hence the analytical solution does not match well with the numerical one. For large $\eta$ of 1.0, and large $\varepsilon_{\kappa}$ of 0.5, even the Gaussian ratio distribution in Eq. (31) does not match with the numerically computed distribution because the numerator ($S_2$) and denominator ($S_1$) in Eq. (13) do not follow Gaussian distribution. In these plots, the legend, “Numeri-
Fig. 10. Pdf of $T$ at $x = 0.5$, obtained for coefficient of variation, $\varepsilon_K$ of 0.1 and correlation length, $\eta$ of 10. Here, legend “Numerical Soln” is used when the solution is computed using Monte-Carlo method, “Analytical-Eq.55” is used when $\sigma_T$ is approximated with Eq. (53) and “Analytical-Eq.52” is used when $\sigma_T$ obtained from Eq. (50), where, $\varepsilon_h = \sigma_h/\mu_h$ is computed numerically.

Fig. 11. Pdf of $T$ at $x = 0.5$ obtained for coefficient of variation, $\varepsilon_K$ of 0.5 and correlation length, $\eta$ of 0.001. Here, legend “Numerical Soln” is used when the solution is computed from Monte-Carlo method, “ratio Gauss-Eq.33” is used when it is computed from Eq. (31) and “Gauss approx-Eq.45” is used when it is computed from Eq. (43).

Figs. 9–12 show the probability density function of the solution $T$ at the center of the spatial domain, for correlation lengths, 0.001, 0.1, 1.0 and 10 and coefficient of variation, $\varepsilon_K$, of 0.1 and 0.5 of the input random coefficient. In these plots, the legend, “Numerical Soln” is used when the solution is computed using Monte-Carlo method, “Analytical-Eq.55” is used when $\sigma_T$ is approximated with Eq. (53) and “Analytical-Eq.52” is used when $\sigma_T$ obtained from Eq. (50) with $\varepsilon_h = \sigma_h/\mu_h$ computed numerically. From Figs. 9–11 we can observe that the probability density function of the solution $T$ computed from analytical solution in Eq. (50) and Monte-Carlo method matches well when either $\eta$ or $\varepsilon_K$ is small while the analytical solution in Eq. (53) matches well only when $\varepsilon_K$ is small because in Eq. (53), $\varepsilon_h$ is approximated with $\varepsilon_h = \frac{1}{\eta^2 \varepsilon_K^2 + \frac{1}{h^2}}$ which is valid only for small $\varepsilon_K$. However from Fig. 12, we can observe a discrepancy in probability density function of $T$ computed from both analytical solutions (Eqs. (50) and (53)) and numerical solution when both $\eta$ and $\varepsilon_K$ are large.
5. Conclusions

In this work, we present analytical solution of one-dimensional elliptic equation with random coefficient. We explore the influence of correlation length, $\eta$ of the random coefficient, $K(x, \omega)$ on the variance of the solution, $\sigma_T^2$. We derive an explicit analytical expression that quantifies uncertainty of the solution of the stochastic elliptic equation which is our main contribution in this work. We compare the probability density function and the standard deviation of the solution with results from numerical simulations from Monte-Carlo and stochastic collocation methods.

We show both numerically and analytically that, the standard deviation, $\sigma_T$ of the solution, initially increases with $\eta$ up to a maximum value, $\sigma_{T, \text{max}}$ at $\eta_{\text{max}} \approx \frac{\sqrt{\pi}}{\sqrt{1-x}}$ and decreases beyond $\eta_{\text{max}}$. We show analytically that, there exists a scaling law, $\sigma_T \propto \eta^{1/2}$ for very small ($\eta \to 0$) and $\sigma_T \propto \eta^{1/2}$ for very large ($\eta \to \infty$) values of correlation length.

We show that, when the coefficient of variation $\xi_K$ is small, we can approximate the solution $T(x, \omega)$ with a Gaussian distribution regardless of the distribution of $K(x, \omega)$. From the plots shown in Section 4, we can observe that, the probability density function and the standard deviation of the solution computed from analytical solution matches well with those computed numerically over entire range of correlation length ($\eta$) under study, when the coefficient of variation, $\xi_K$ is small. However, as we increase, $\xi_K$, $\sigma_T$ from analytical solution matches well for small value of $\eta$, and there is a larger discrepancy in $\sigma_T$ and probability density function of $T(x, \omega)$ when both $\eta$ and $\xi_K$ are large.

For many practical problems, where $\xi_K \ll 1.0$, we can use the analytical solution for a quick estimate of the uncertainty using Eq. (50) or Eq. (53) when the information about the correlation length, $\eta$ is available and Eqs. (57) and (58) when $\eta$ is unknown. Analytical solution gives a quick insight into the physics of the problem. Here, we derive the solution for a homogeneous elliptic equation with a random coefficient. We can extend this work for non-homogeneous elliptic equations by following a similar procedure.

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