Analysis of electromagnetic waves in random media

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Problem statement

Analysis of E&M waves in heterogeneous media with microscale is relevant in radar, optical imaging, communications...

Mathematical model: Maxwell's equations

$$\vec{\nabla} \times \vec{\boldsymbol{E}}(\vec{\boldsymbol{x}}) = i\omega\mu_o \vec{\boldsymbol{H}}(\vec{\boldsymbol{x}}),$$
$$\vec{\nabla} \times \vec{\boldsymbol{H}}(\vec{\boldsymbol{x}}) = \vec{\boldsymbol{\mathcal{J}}}(\vec{\boldsymbol{x}}) - i\omega\varepsilon(\vec{\boldsymbol{x}})\vec{\boldsymbol{E}}(\vec{\boldsymbol{x}}),$$

where $\varepsilon(\vec{\boldsymbol{x}})$ has uncertain small scale weak fluctuations.

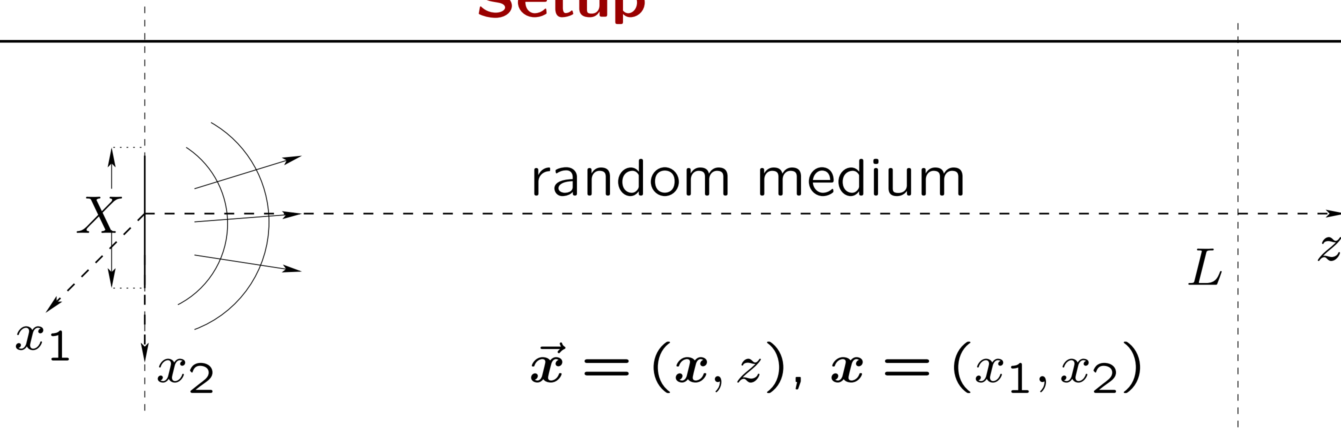
Random model of uncertainty $\varepsilon(\vec{\boldsymbol{x}}) = \varepsilon_o \left[1 + \alpha\nu\left(\frac{\vec{\boldsymbol{x}}}{\ell}\right) 1_{(0,L)}(z)\right]$

where ν is zero mean stationary process, bounded differentiable with bounded derivative a.s. It is mixing with autocorrelation

$$\mathcal{R}(\vec{\boldsymbol{u}}) = \mathbb{E}\left[\nu(\vec{\boldsymbol{u}} + \vec{\boldsymbol{u}}')\nu(\vec{\boldsymbol{u}}')\right]$$

normalized by $\mathcal{R}(0) = 1$ and $\int_{\mathbb{R}^3} d\vec{\boldsymbol{u}}\, \mathcal{R}(\vec{\boldsymbol{u}}) = O(1)$.

Setup



Source $\vec{\mathcal{J}}(\vec{\boldsymbol{x}}) = \vec{\boldsymbol{J}}\left(\frac{\boldsymbol{x}}{X}\right)\delta(z)$ $\rightsquigarrow$ plane waves $e^{ik\vec{\boldsymbol{\kappa}}\cdot\vec{\boldsymbol{x}}-i\omega t}$ with amplitudes proportional to $\int_{\mathbb{R}^2} d\boldsymbol{x}\,\vec{\boldsymbol{J}}\left(\frac{\boldsymbol{x}}{X}\right)e^{-ik\boldsymbol{\kappa}\cdot\boldsymbol{x}} = X^2\widehat{\vec{\boldsymbol{J}}}(Xk\boldsymbol{\kappa})$.

The unit wave vectors $\vec{\boldsymbol{\kappa}} = (\boldsymbol{\kappa}, \sqrt{1-|\boldsymbol{\kappa}|^2})$ are in a cone (beam) of opening angle $\sim (Xk)^{-1}$ where $k = 2\pi/\lambda$ is the wave number.

Evolution of the beam depends on relation between the distance L of propagation, wavelength λ and correlation length ℓ, as well as the amplitude α of the random fluctuations.

Scattering regimes

• **Weak scattering** models like Born or Rytov.

• **Paraxial regime:** waves propagate in a narrow beam. This is well understood in mixing random media (Garnier, Solna). Related study for Schrödinger's equations (Erdös, Yau and Bal, Komorowski, Ryzhik). Paraxial regime captures some randomization effects but not loss of polarization.

• **Radiative transfer regime** where waves propagate in all directions. Energy propagation is modeled by Chandrasekar's transport equations. Not completely mathematically justified. Formal derivations are based on multiple scale asymptotic analysis (Keller, Ryzhik, Papanicolaou) and diagramatic (multiple scattering series) expansions which assume Gaussian fluctuations ν.

• We consider a new **wide-angle regime** which bridges between the paraxial and the radiative transfer regime*.

*Inspired by recent study in waveguides (Alonso and B.)

Wide-angle regime

Scale ordering $\lambda \ll \ell \sim X \ll L$ and small amplitude fluctuations:

- $\epsilon = \frac{\lambda}{L} \ll 1$ so waves propagate over many wavelengths.
- $\gamma = \frac{\lambda}{\ell} \in (0,1)$ is independent of ϵ. When $\gamma \ll 1 \rightsquigarrow$ paraxial.
- $\gamma_J = \frac{\lambda}{X} < 1$ controls opening angle of the emitted beam.
- Standard deviation of the fluctuations is $\alpha = \epsilon^{1/2}$.

Asymptotic analysis for $\epsilon \to 0$.

We use γ, γ_J to control the wide-angle regime i.e., ensure waves propagate in a cone of opening angle less than 180^o.

First result: Wave decomposition

$$\vec{\boldsymbol{E}}(\vec{\boldsymbol{x}}) = \int_{|\boldsymbol{\kappa}|<1} \frac{d(k\boldsymbol{\kappa})}{(2\pi)^2\sqrt{\beta(\boldsymbol{\kappa})}} \Big[a(\boldsymbol{\kappa},z)\,\vec{\boldsymbol{u}}(\boldsymbol{\kappa}) + a^{\perp}(\boldsymbol{\kappa},z)\,\vec{\boldsymbol{u}}^{\perp}(\boldsymbol{\kappa})\Big] e^{ik\vec{\boldsymbol{\kappa}}\cdot\vec{\boldsymbol{x}}}$$

$$\vec{\boldsymbol{H}}(\vec{\boldsymbol{x}}) = \sqrt{\frac{\varepsilon_o}{\mu_o}} \int_{|\boldsymbol{\kappa}|<1} \frac{d(k\boldsymbol{\kappa})}{(2\pi)^2\sqrt{\beta(\boldsymbol{\kappa})}} \Big[a(\boldsymbol{\kappa},z)\,\vec{\boldsymbol{u}}^{\perp}(\boldsymbol{\kappa}) - a^{\perp}(\boldsymbol{\kappa},z)\,\vec{\boldsymbol{u}}(\boldsymbol{\kappa})\Big] e^{ik\vec{\boldsymbol{\kappa}}\cdot\vec{\boldsymbol{x}}}$$

where $\vec{\boldsymbol{\kappa}} = \big(\boldsymbol{\kappa}, \beta(\boldsymbol{\kappa})\big)$, $\boldsymbol{\kappa} \in \mathbb{R}^2$, $\beta(\boldsymbol{\kappa}) = \sqrt{1 - |\boldsymbol{\kappa}|^2}$ and $k = 2\pi/\lambda$.

- Modes distinguished by unit vectors orthogonal to $\vec{\boldsymbol{\kappa}}$

$$\vec{\boldsymbol{u}}(\boldsymbol{\kappa}) = \left(\frac{\beta(\boldsymbol{\kappa})\boldsymbol{\kappa}}{|\boldsymbol{\kappa}|}, -|\boldsymbol{\kappa}|\right), \quad \vec{\boldsymbol{u}}^{\perp}(\boldsymbol{\kappa}) = \left(\frac{\boldsymbol{\kappa}^{\perp}}{|\boldsymbol{\kappa}|}, 0\right), \quad \boldsymbol{\kappa}^{\perp} = (-\kappa_2, \kappa_1).$$

- Random amplitudes $a(\boldsymbol{\kappa}, z)$ and $a^{\perp}(\boldsymbol{\kappa}, z)$ of TM and TE modes model scattering.

Wave decomposition

- The electric and magnetic plane waves

$$\vec{\mathcal{E}}(\boldsymbol{\kappa}, z) = a(\boldsymbol{\kappa}, z)\vec{\boldsymbol{u}}(\boldsymbol{\kappa}) + a^{\perp}(\boldsymbol{\kappa}, z)\vec{\boldsymbol{u}}^{\perp}(\boldsymbol{\kappa})$$
$$\vec{\mathcal{H}}(\boldsymbol{\kappa}, z) = \zeta_o^{-1}\left[a(\boldsymbol{\kappa}, z)\vec{\boldsymbol{u}}^{\perp}(\boldsymbol{\kappa}) - a^{\perp}(\boldsymbol{\kappa}, z)\vec{\boldsymbol{u}}(\boldsymbol{\kappa})\right]$$

are orthogonal to $\vec{\boldsymbol{\kappa}}$ and to each other.

- Their statistical expectation defines the coherent wave.

- The coherence matrix $\mathcal{P}(\boldsymbol{\kappa}, z) = \mathbb{E}\left[\begin{pmatrix} a(\boldsymbol{\kappa}, z) \\ a^{\perp}(\boldsymbol{\kappa}, z)\end{pmatrix}\begin{pmatrix} a(\boldsymbol{\kappa}, z) \\ a^{\perp}(\boldsymbol{\kappa}, z)\end{pmatrix}^{\dagger}\right]$ defines state of polarization.

Diagonal entries of $\mathcal{P}$, the mode powers $\mathbb{E}[|a|^2]$ and $\mathbb{E}[|a^{\perp}|^2]$ give first two components of Stokes' vector. The off-diagonal parts the other two components.

Mathematical justification of decomposition

- Eliminate longitudinal fields from Maxwell's equations

$$H_z(\vec{\boldsymbol{x}}) = -\frac{i}{\omega\mu_o}\nabla^\perp \cdot \boldsymbol{E}(\vec{\boldsymbol{x}}),$$

$$E_z(\vec{\boldsymbol{x}}) = \frac{i}{\omega\varepsilon_o[1+\alpha\nu(\vec{\boldsymbol{x}}/\ell)]}\left[\nabla^\perp \cdot \boldsymbol{H}(\vec{\boldsymbol{x}}) - J_z(\boldsymbol{x}/X)\delta(z)\right],$$

where $\vec{\boldsymbol{x}} = (\boldsymbol{x}, z)$, ∇ is gradient in $\boldsymbol{x}$ and $\nabla^\perp$ its rotation by 90^o.

- Transverse fields $\boldsymbol{E}$ and $\boldsymbol{U} = -\sqrt{\frac{\mu_o}{\varepsilon_o}}\boldsymbol{H}^\perp$ satisfy

$$\partial_z \boldsymbol{E}(\vec{\boldsymbol{x}}) = ik\boldsymbol{U}(\vec{\boldsymbol{x}}) + \frac{i}{k}\nabla\left[\frac{\nabla\cdot\boldsymbol{U}(\vec{\boldsymbol{x}})}{1+\alpha\nu(\vec{\boldsymbol{x}}/\ell)}\right] - \frac{i}{k}\nabla_{\boldsymbol{x}}\left[\frac{J_z(\boldsymbol{x}/X)}{1+\alpha\nu(\vec{\boldsymbol{x}}/\ell)}\right]\delta(z),$$

$$\partial_z \boldsymbol{U}(\vec{\boldsymbol{x}}) = ik[1+\alpha\nu(\vec{\boldsymbol{x}}/\ell)]\boldsymbol{E}(\vec{\boldsymbol{x}}) + \frac{i}{k}\nabla^\perp\left[\nabla^\perp\cdot\boldsymbol{E}(\vec{\boldsymbol{x}})\right] - \boldsymbol{J}(\boldsymbol{x}/X)\delta(z).$$

- Use scaling and Fourier transform in $\boldsymbol{x}$ $\rightsquigarrow$ plane waves.

- Equations to study in limit $\epsilon = \lambda/L \to 0$, for $z > 0$,

$$\partial_z \begin{pmatrix} \widehat{\boldsymbol{E}}^\epsilon(k\boldsymbol{\kappa}, z) \\ \widehat{\boldsymbol{U}}^\epsilon(k\boldsymbol{\kappa}, z) \end{pmatrix} = \frac{ik}{\epsilon}\mathrm{M}(\boldsymbol{\kappa}) \begin{pmatrix} \widehat{\boldsymbol{E}}^\epsilon(k\boldsymbol{\kappa}, z) \\ \widehat{\boldsymbol{U}}^\epsilon(k\boldsymbol{\kappa}, z) \end{pmatrix} + 1_{(0,L)}(z) \left[\mathcal{M}^\epsilon \begin{pmatrix} \widehat{\boldsymbol{E}}^\epsilon \\ \widehat{\boldsymbol{U}}^\epsilon \end{pmatrix}\right](k\boldsymbol{\kappa}, z)$$

for $\boldsymbol{\kappa}$ = transverse wave vector and $k = 2\pi$ (scaled wavenumber).

- Leading matrix $\mathrm{M}(\boldsymbol{\kappa}) = \begin{pmatrix} \mathbf{0} & \mathbf{I} - \boldsymbol{\kappa} \otimes \boldsymbol{\kappa} \\ \mathbf{I} - \boldsymbol{\kappa}^\perp \otimes \boldsymbol{\kappa}^\perp & \mathbf{0} \end{pmatrix}$

- Perturbation by random medium

$$\left[\mathcal{M}^\epsilon \begin{pmatrix} \widehat{\boldsymbol{E}}^\epsilon \\ \widehat{\boldsymbol{U}}^\epsilon \end{pmatrix}\right](k\boldsymbol{\kappa}, z) = \frac{ik}{\sqrt{\epsilon}\,\gamma^2} \int \frac{d(k\boldsymbol{\kappa}')}{(2\pi)^2} \Bigg[\widehat{\nu}\left(k(\boldsymbol{\kappa} - \boldsymbol{\kappa}')/\gamma, \gamma z/\epsilon\right) \begin{pmatrix} \mathbf{0} & \boldsymbol{\kappa} \otimes \boldsymbol{\kappa}' \\ \mathbf{I} & \mathbf{0} \end{pmatrix}$$
$$- \sqrt{\epsilon}\,\widehat{\nu^2}\left(k(\boldsymbol{\kappa} - \boldsymbol{\kappa}')/\gamma, \gamma z/\epsilon\right) \begin{pmatrix} \mathbf{0} & \boldsymbol{\kappa} \otimes \boldsymbol{\kappa}' \\ \mathbf{0} & \mathbf{0} \end{pmatrix}\Bigg] \begin{pmatrix} \widehat{\boldsymbol{E}}^\epsilon(k\boldsymbol{\kappa}', z) \\ \widehat{\boldsymbol{U}}^\epsilon(k\boldsymbol{\kappa}', z) \end{pmatrix}$$

• Wave decomposition uses diagonalization of matrix $\mathbf{M}(\boldsymbol{\kappa})$.

It has two double eigenvalues $\pm\beta(\boldsymbol{\kappa})$, where $\beta(\boldsymbol{\kappa}) = \sqrt{1-|\boldsymbol{\kappa}|^2}$, and the eigenvectors

$$\boldsymbol{\psi}_{\pm}(\boldsymbol{\kappa}) = \begin{pmatrix} \pm\sqrt{\beta(\boldsymbol{\kappa})}\frac{\boldsymbol{\kappa}}{|\boldsymbol{\kappa}|} \\ \\ \frac{1}{\sqrt{\beta(\boldsymbol{\kappa})}}\frac{\boldsymbol{\kappa}}{|\boldsymbol{\kappa}|} \end{pmatrix} \quad \text{and} \quad \boldsymbol{\psi}_{\pm}^{\perp}(\boldsymbol{\kappa}) = \begin{pmatrix} \frac{1}{\sqrt{\beta(\boldsymbol{\kappa})}}\frac{\boldsymbol{\kappa}^{\perp}}{|\boldsymbol{\kappa}|} \\ \\ \pm\sqrt{\beta(\boldsymbol{\kappa})}\frac{\boldsymbol{\kappa}^{\perp}}{|\boldsymbol{\kappa}|} \end{pmatrix}.$$

• The eigenvectors are linearly independent for $|\boldsymbol{\kappa}| \neq 1$, so they form a basis in which we can expand solution

$$\begin{pmatrix} \widehat{\boldsymbol{E}}^{\epsilon}(k\boldsymbol{\kappa}, z) \\ \widehat{\boldsymbol{U}}^{\epsilon}(k\boldsymbol{\kappa}, z) \end{pmatrix} = \left[a(\boldsymbol{\kappa}, z)\boldsymbol{\psi}_{+}(\boldsymbol{\kappa}) + a^{\perp}(\boldsymbol{\kappa}, z)\boldsymbol{\psi}_{+}^{\perp}(\boldsymbol{\kappa})\right]e^{\frac{ik}{\epsilon}\beta(\boldsymbol{\kappa})z} + \left[b(\boldsymbol{\kappa}, z)\boldsymbol{\psi}_{-}(\boldsymbol{\kappa}) + b^{\perp}(\boldsymbol{\kappa}, z)\boldsymbol{\psi}_{-}^{\perp}(\boldsymbol{\kappa})\right]e^{-\frac{ik}{\epsilon}\beta(\boldsymbol{\kappa})z}.$$

Mathematical justification of decomposition

• For Markovian limit $\epsilon \to 0$ use the propagator $\mathbf{P}^\epsilon$

$$\begin{pmatrix} a(\boldsymbol{\kappa},z) \\ a^\perp(\boldsymbol{\kappa},z) \\ b(\boldsymbol{\kappa},z) \\ b^\perp(\boldsymbol{\kappa},z) \end{pmatrix} = \int d\boldsymbol{\kappa}_o \, \mathbf{P}^\epsilon(\boldsymbol{\kappa},z;\boldsymbol{\kappa}_o) \begin{pmatrix} a(\boldsymbol{\kappa},0) \\ a^\perp(\boldsymbol{\kappa},0) \\ b(\boldsymbol{\kappa},0) \\ b^\perp(\boldsymbol{\kappa},0) \end{pmatrix}$$

satisfying

$$\frac{\partial \mathbf{P}^\epsilon(\boldsymbol{\kappa},z;\boldsymbol{\kappa}_o)}{\partial z} = \frac{ik}{2\gamma^2\sqrt{\epsilon}} \int \frac{d(k\boldsymbol{\kappa}')}{(2\pi)^2} \left[\widehat{\nu}\left(k(\boldsymbol{\kappa}-\boldsymbol{\kappa}')/\gamma, \gamma\frac{z}{\epsilon}\right) \mathbf{F}\left(\boldsymbol{\kappa},\boldsymbol{\kappa}',\frac{z}{\epsilon}\right) + \sqrt{\epsilon}\,\widehat{\nu^2}\left(k(\boldsymbol{\kappa}-\boldsymbol{\kappa}')/\gamma, \gamma\frac{z}{\epsilon}\right) \mathbf{G}\left(\boldsymbol{\kappa},\boldsymbol{\kappa}',\frac{z}{\epsilon}\right) \right] \mathbf{P}^\epsilon(\boldsymbol{\kappa}',z;\boldsymbol{\kappa}_o),$$

with initial condition $\mathbf{P}^\epsilon(\boldsymbol{\kappa},0;\boldsymbol{\kappa}_o) = \mathbf{I}\,\delta(\boldsymbol{\kappa}-\boldsymbol{\kappa}_o)$.

• Source excitation gives $a(\boldsymbol{\kappa},0)$ and $a^\perp(\boldsymbol{\kappa},0)$ and by causality (outgoing condition) $b(\boldsymbol{\kappa},L) = b^\perp(\boldsymbol{\kappa},L) = 0$.

Markov limit (Papanicolaou, Weinryb)

• Let $\mathcal{O}$ be an open set in $\mathbb{R}^d$ and $\mathcal{D}(\mathcal{O}, \mathbb{R}^p)$ the space of infinitely differentiable functions with compact support*.

• Let $\mathbf{Y}^\epsilon(z)$ be the process in $\mathcal{C}([0, L], \mathcal{D}')$, the solution of

$$\frac{d\mathbf{Y}^\epsilon}{dz} = \frac{1}{\sqrt{\epsilon}} \mathcal{F}\left(\frac{z}{\epsilon}, \frac{z}{\epsilon}\right) \mathbf{Y}^\epsilon + \mathcal{G}\left(\frac{z}{\epsilon}, \frac{z}{\epsilon}\right) \mathbf{Y}^\epsilon,$$

with $\mathcal{F}(\zeta, \zeta')$, $\mathcal{G}(\zeta, \zeta')$ random linear operators from $\mathcal{D}'$ to $\mathcal{D}'$:

$$\zeta \to \mathcal{F}(\zeta, \zeta'),\ \mathcal{G}(\zeta, \zeta') = \text{stationary, mixing and } \mathbb{E}[\mathcal{F}(\zeta, \zeta')] = 0,$$

$$\zeta' \to \mathcal{F}(\zeta, \zeta'),\ \mathcal{G}(\zeta, \zeta') = \text{periodic.}$$

• As $\epsilon \to 0$, $\mathbf{Y}^\epsilon(z)$ converges weakly in $\mathcal{C}([0, L], \mathcal{D}')$ to $\mathbf{Y}(z)$, the solution of a martingale problem with generator $\mathcal{L}$.

*In our case $\mathcal{O} = \{\boldsymbol{\kappa} \in \mathbb{R}^2,\ |\boldsymbol{\kappa}| < 1\}$ and $\mathbf{Y}^\epsilon$ is given by concatenation of real and imaginary parts of propagator $\mathbf{P}^\epsilon(\boldsymbol{\kappa}, z; \boldsymbol{\kappa}_o)$.

- For all $\phi \in \mathcal{D}(\mathcal{O}, \mathbb{R}^p)$ and smooth $f : \mathbb{R} \to \mathbb{R}$,

$$\mathbb{E}\left[f(\langle \boldsymbol{Y}(z), \phi \rangle) - \int_0^z dz' \, \mathcal{L} f\Big(\Big\langle \boldsymbol{Y}(z'), \phi \Big\rangle \Big) \Big| \boldsymbol{Y}(0) = \boldsymbol{Y}_o \right] = f(\langle \boldsymbol{Y}_o, \phi \rangle).$$

- The generator is

$$\begin{aligned}
\mathcal{L} f(\langle \boldsymbol{Y}, \phi \rangle) = & \int_0^\infty d\zeta \lim_{Z \to \infty} \frac{1}{Z} \int_0^Z dh \, \mathbb{E}\big[\langle \boldsymbol{Y}, \mathcal{F}^*(0,h)\phi \rangle \langle \boldsymbol{Y}, \mathcal{F}^*(\zeta, \zeta + h)\phi \rangle \big] f''(\langle \boldsymbol{Y}, \phi \rangle) \\
& + \int_0^\infty d\zeta \lim_{Z \to \infty} \frac{1}{Z} \int_0^Z dh \, \mathbb{E}\big[\langle \boldsymbol{Y}, \mathcal{F}^*(0,h)\mathcal{F}^*(\zeta, \zeta + h)\phi \rangle \big] f'(\langle \boldsymbol{Y}, \phi \rangle) \\
& + \lim_{Z \to \infty} \frac{1}{Z} \int_0^Z dh \, \mathbb{E}\big[\langle \boldsymbol{Y}, \mathcal{G}^*(0,h)\phi \rangle \big] f'(\langle \boldsymbol{Y}, \phi \rangle),
\end{aligned}$$

where star denotes adjoint operators.

- To calculate first moment (mean field) let $f(y) = y$. For second moment $f(y) = y^2$, etc.

Markov limit of the propagator

- Propagator has block structure $\mathbf{P}^{\epsilon} = \begin{pmatrix} \mathbf{P}^{aa,\epsilon} & \mathbf{P}^{ab,\epsilon} \\ \mathbf{P}^{ba,\epsilon} & \mathbf{P}^{bb,\epsilon} \end{pmatrix}$

- Kernel of operator $\mathcal{F}(\zeta, \zeta')$ is $\widehat{\nu}\left(\frac{k(\boldsymbol{\kappa}-\boldsymbol{\kappa}')}{\gamma}, \gamma\zeta\right) \mathbf{F}(\boldsymbol{\kappa}, \boldsymbol{\kappa}', \zeta')$,

$$\mathbf{F}(\boldsymbol{\kappa}, \boldsymbol{\kappa}', \zeta') = \begin{pmatrix} \boldsymbol{\Gamma}^{aa}(\boldsymbol{\kappa}, \boldsymbol{\kappa}') e^{ik[\beta(\boldsymbol{\kappa}')-\beta(\boldsymbol{\kappa})]\zeta'} & \boldsymbol{\Gamma}^{ab}(\boldsymbol{\kappa}, \boldsymbol{\kappa}') e^{ik[\beta(\boldsymbol{\kappa}')+\beta(\boldsymbol{\kappa})]\zeta'} \\ \boldsymbol{\Gamma}^{ba}(\boldsymbol{\kappa}, \boldsymbol{\kappa}') e^{-ik[\beta(\boldsymbol{\kappa}')+\beta(\boldsymbol{\kappa})]\zeta'} & \boldsymbol{\Gamma}^{bb}(\boldsymbol{\kappa}, \boldsymbol{\kappa}') e^{ik[-\beta(\boldsymbol{\kappa}')+\beta(\boldsymbol{\kappa})]\zeta'} \end{pmatrix}$$

where

$$\boldsymbol{\Gamma}^{aa}(\boldsymbol{\kappa}, \boldsymbol{\kappa}') = \begin{pmatrix} \frac{|\boldsymbol{\kappa}||\boldsymbol{\kappa}'|}{\sqrt{\beta(\boldsymbol{\kappa})\beta(\boldsymbol{\kappa}')}} + \frac{\boldsymbol{\kappa}}{|\boldsymbol{\kappa}|} \cdot \frac{\boldsymbol{\kappa}'}{|\boldsymbol{\kappa}'|} \sqrt{\beta(\boldsymbol{\kappa})\beta(\boldsymbol{\kappa}')} & \frac{\boldsymbol{\kappa}}{|\boldsymbol{\kappa}|} \cdot \frac{\boldsymbol{\kappa}'^{\perp}}{|\boldsymbol{\kappa}'|} \sqrt{\frac{\beta(\boldsymbol{\kappa})}{\beta(\boldsymbol{\kappa}')}} \\ \frac{\boldsymbol{\kappa}^{\perp}}{|\boldsymbol{\kappa}|} \cdot \frac{\boldsymbol{\kappa}'}{|\boldsymbol{\kappa}'|} \sqrt{\frac{\beta(\boldsymbol{\kappa}')}{\beta(\boldsymbol{\kappa})}} & \frac{\boldsymbol{\kappa}}{|\boldsymbol{\kappa}|} \cdot \frac{\boldsymbol{\kappa}'}{|\boldsymbol{\kappa}'|} \frac{1}{\sqrt{\beta(\boldsymbol{\kappa}')\beta(\boldsymbol{\kappa})}} \end{pmatrix},$$

and similar for other $\boldsymbol{\Gamma}$ matrices and operator $\mathcal{G}$, quadratic in $\widehat{\nu}$.

- The phases in $\mathbf{F}(\boldsymbol{\kappa}, \boldsymbol{\kappa}', \zeta')$ are important in calculation of generator $\mathcal{L}$, and determine interaction of forward/backward waves.

- As $\epsilon \to 0$, coupling of $\mathbf{P}^{ab,\epsilon}$, $\mathbf{P}^{ba,\epsilon}$ to $\mathbf{P}^{aa,\epsilon}$, $\mathbf{P}^{bb,\epsilon}$ is proportional to

$$\int_{\mathbb{R}^3} d\vec{\boldsymbol{r}}\,\mathcal{R}(\gamma\vec{\boldsymbol{r}})e^{-ik\left[(\boldsymbol{\kappa}-\boldsymbol{\kappa}')\cdot\boldsymbol{r}+(\beta(\boldsymbol{\kappa})+\beta(\boldsymbol{\kappa}'))r_z\right]} = \frac{1}{\gamma^3}\widetilde{\mathcal{R}}\left(\frac{k(\vec{\boldsymbol{\kappa}}-\vec{\boldsymbol{\kappa}'}^{-})}{\gamma}\right)$$

where $\vec{\boldsymbol{\kappa}} = (\boldsymbol{\kappa}, \beta(\boldsymbol{\kappa}))$ and $\vec{\boldsymbol{\kappa}'}^{-} = (\boldsymbol{\kappa}', -\beta(\boldsymbol{\kappa}'))$.

- Coupling of entries within $\mathbf{P}^{aa,\epsilon}$ is proportional to $\frac{1}{\gamma^3}\widetilde{\mathcal{R}}\left(\frac{k(\vec{\boldsymbol{\kappa}}-\vec{\boldsymbol{\kappa}'})}{\gamma}\right)$.

Conclusion: Forward/backward wave coupling is controlled by the support of power spectral density $\widetilde{\mathcal{R}}$ (smoothness assumption on ν) and domain of $|\vec{\boldsymbol{\kappa}}-\vec{\boldsymbol{\kappa}'}^{-}|$.

Forward scattering approximation

• $\tilde{\mathfrak{R}}$ supported in ball of radius 1, and source emits waves with $\vec{\boldsymbol{\kappa}} = (\boldsymbol{\kappa}, \beta(\boldsymbol{\kappa}))$ where* $|\boldsymbol{\kappa}| \le \gamma_J/k < 1$ and $\beta(\boldsymbol{\kappa}) = \sqrt{1 - |\boldsymbol{\kappa}|^2}$.

As long as $|\boldsymbol{\kappa}| \le \kappa_M \in (\gamma_J/k, 1)$, s.t. $\frac{k|\vec{\boldsymbol{\kappa}} - \vec{\boldsymbol{\kappa}}'^{-}|}{\gamma} \ge \frac{2k\beta(\kappa_M)}{\gamma} > 1$,

there is no coupling of forward to backward waves.

• Energy distribution in $\boldsymbol{\kappa}$ obeys transport equations: makes random walk (diffusion) with diffusion coefficient $\sim \gamma = \lambda/\ell$. Energy reaches κ_M at scaled distance $\sim \kappa_M^2/\gamma$.

• Evanescent waves couple with propagating ones with $|\boldsymbol{\kappa}| \approx 1$. In our regime these waves do not get excited.

*Recall that $\gamma_J = \lambda/X$ and that $k = 2\pi$ is the scaled wavenumber.

Result: Characterization of the coherent (mean) field

- The mean amplitudes $\begin{pmatrix} \mathbb{E}[a(\boldsymbol{\kappa}, z)] \\ \mathbb{E}[a^{\perp}(\boldsymbol{\kappa}, z)] \end{pmatrix} = \exp\left[\mathbf{Q}(\boldsymbol{\kappa}) z\right] \begin{pmatrix} a_o(\boldsymbol{\kappa}) \\ a_o^{\perp}(\boldsymbol{\kappa}) \end{pmatrix}$.

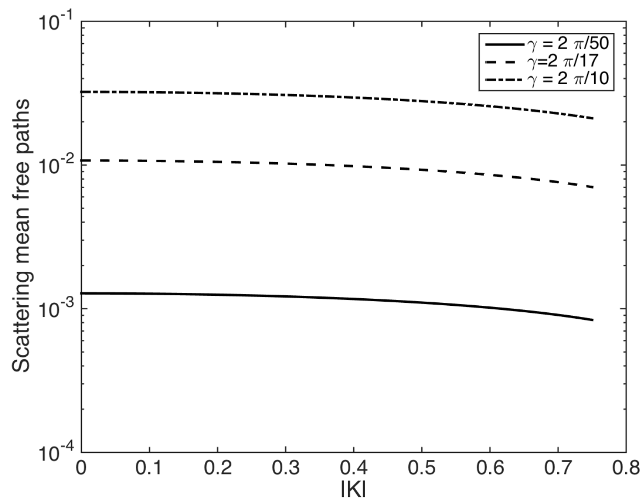
- Effect of random medium is in complex symmetric matrix

$$\mathbf{Q}(\boldsymbol{\kappa}) = -\frac{k^2}{4\gamma^3} \int \frac{d(k\boldsymbol{\kappa}')}{(2\pi)^2} \boldsymbol{\Gamma}(\boldsymbol{\kappa}, \boldsymbol{\kappa}') \boldsymbol{\Gamma}(\boldsymbol{\kappa}, \boldsymbol{\kappa}')^T \int_{\mathbb{R}^2} d\boldsymbol{x} \int_0^\infty dz \, \mathcal{R}(\vec{\boldsymbol{x}}) \, e^{-i\frac{k(\vec{\boldsymbol{\kappa}} - \vec{\boldsymbol{\kappa}}')}{\gamma} \cdot \vec{\boldsymbol{x}}} - \frac{ik}{2} \mathcal{R}(\mathbf{0}) \frac{|\boldsymbol{\kappa}|^2}{\beta(\boldsymbol{\kappa})} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

- $\mathbf{Q}(\boldsymbol{\kappa})$ has negative definite real part $\rightsquigarrow$ decay of mean field. Scales of decay in z = scattering mean free paths.

- In statistically isotropic media Re$[\mathbf{Q}(\boldsymbol{\kappa})]$ is a multiple of identity and depends only on $|\boldsymbol{\kappa}|$.

Scattering mean free paths $\mathcal{S}(\boldsymbol{\kappa})$ in isotropic media



• $\mathcal{S}(\boldsymbol{\kappa})$ decrease monotonically with $|\boldsymbol{\kappa}|$ because waves with larger $|\boldsymbol{\kappa}|$ travel a longer path to reach the same range.

• $\mathcal{S}(\boldsymbol{\kappa})$ are shorter for $\gamma = \lambda/\ell$ small. High frequency waves lose coherence faster:

$$\mathcal{S}(\boldsymbol{\kappa}) = \frac{4\gamma\sqrt{1-|\boldsymbol{\kappa}|^2}}{k^2 \int_0^\infty dr \mathcal{R}(r)} \left[1 + O(\gamma)\right].$$

Results: Wigner transform (energy density)

$$\mathcal{W}(\boldsymbol{\kappa},\vec{\boldsymbol{x}}) = \int \frac{d(k\boldsymbol{q})}{(2\pi)^2} e^{ik\boldsymbol{q}\cdot\left(\boldsymbol{x}+\nabla\beta(\boldsymbol{q})z\right)} \mathbb{E}\left[\begin{pmatrix} a\left(\boldsymbol{\kappa}+\frac{\boldsymbol{q}}{2},z\right) \\ a^{\perp}\left(\boldsymbol{\kappa}+\frac{\boldsymbol{q}}{2},z\right)\end{pmatrix}\begin{pmatrix} a\left(\boldsymbol{\kappa}-\frac{\boldsymbol{q}}{2},z\right) \\ a^{\perp}\left(\boldsymbol{\kappa}-\frac{\boldsymbol{q}}{2},z\right)\end{pmatrix}^{\dagger}\right]$$

satisfies transport equation

$$\partial_z \mathcal{W}(\boldsymbol{\kappa},\vec{\boldsymbol{x}}) - \nabla\beta(\boldsymbol{\kappa})\cdot\nabla_{\boldsymbol{x}}\mathcal{W}(\boldsymbol{\kappa},\vec{\boldsymbol{x}}) = \mathbf{Q}(\boldsymbol{\kappa})\mathcal{W}(\boldsymbol{\kappa},\vec{\boldsymbol{x}}) + \mathcal{W}(\boldsymbol{\kappa},\vec{\boldsymbol{x}})\mathbf{Q}(\boldsymbol{\kappa})^{\dagger} + \frac{k^2}{4\gamma^3}\int_{|\boldsymbol{\kappa}'|<1}\frac{d(k\boldsymbol{\kappa}')}{(2\pi)^2}\boldsymbol{\Gamma}(\boldsymbol{\kappa},\boldsymbol{\kappa}')\mathcal{W}(\boldsymbol{\kappa}',\vec{\boldsymbol{x}})\boldsymbol{\Gamma}(\boldsymbol{\kappa}',\boldsymbol{\kappa})\,\widetilde{\mathcal{R}}\left(\frac{k(\vec{\boldsymbol{\kappa}}-\vec{\boldsymbol{\kappa}}')}{\gamma}\right)$$

• Because of polarization we do not have scalar valued differential and total scattering cross-sections, but linear operators.

• Connects to Chandrasekhar's radiative transport equations in isotropic media $\rightsquigarrow$ rigorous derivation in our regime and also to paraxial regime in the limit $\gamma = \lambda/\ell \to 0$.

Results: State of polarization

• The Hermitian, positive definite coherence matrix, which quantifies the state of polarization, is

$$\mathcal{P}(\boldsymbol{\kappa}, z) = \begin{pmatrix} \mathbb{E}[|a(\boldsymbol{\kappa}, z)|^2] & \mathbb{E}[a(\boldsymbol{\kappa}, z)\overline{a^{\perp}(\boldsymbol{\kappa}, z)}] \\ \mathbb{E}[\overline{a(\boldsymbol{\kappa}, z)}a^{\perp}(\boldsymbol{\kappa}, z)] & \mathbb{E}[|a_{\perp}(\boldsymbol{\kappa}, z)|^2] \end{pmatrix} = \int_{\mathbb{R}^2} d\boldsymbol{x}\, \mathcal{W}(\boldsymbol{\kappa}, \vec{\boldsymbol{x}}).$$

Two important effects displayed by the evolution of $\mathcal{P}(\boldsymbol{\kappa}, z)$:

1. Exchange of power between the TM and TE modes.

2. Diffusion of power in $\boldsymbol{\kappa}$.

• Illustration for initial condition* $a(\boldsymbol{\kappa}, 0) = a_o(|\boldsymbol{\kappa}|)$, $a^{\perp}(\boldsymbol{\kappa}, 0) = 0$, due to current source $(\boldsymbol{J}, J_z)$ with $\boldsymbol{J} = \nabla\phi$.

In isotropic media the coherence matrix remains diagonal.

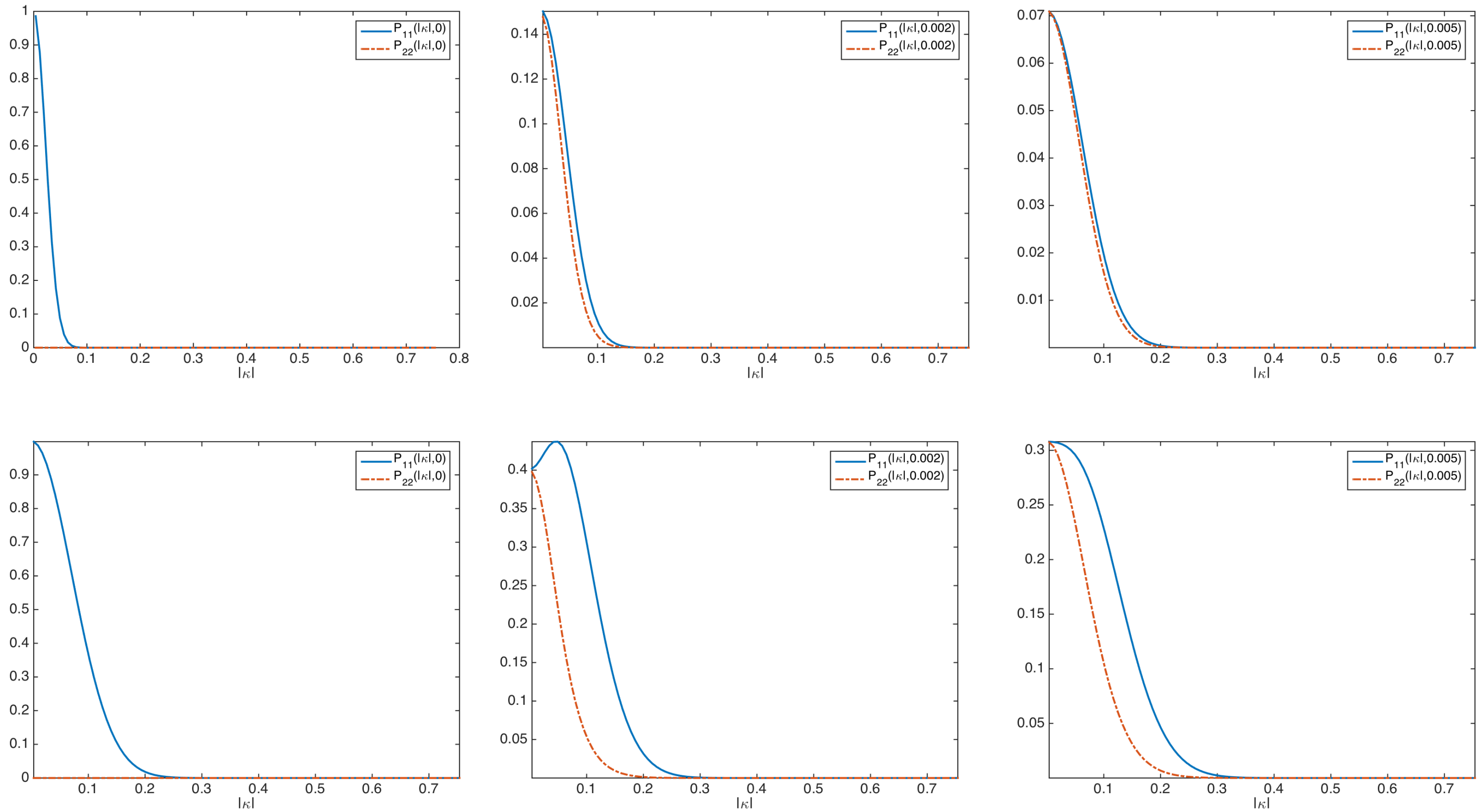
*This gives linear initial polarization with Stokes vector $\mathfrak{S}(\boldsymbol{\kappa}, 0) \sim (1, 1, 0, 0)$.

Results: State of polarization

$z = 0$ | $z \approx 2\mathcal{S}(0)$ | $z \approx 5\mathcal{S}(0)$

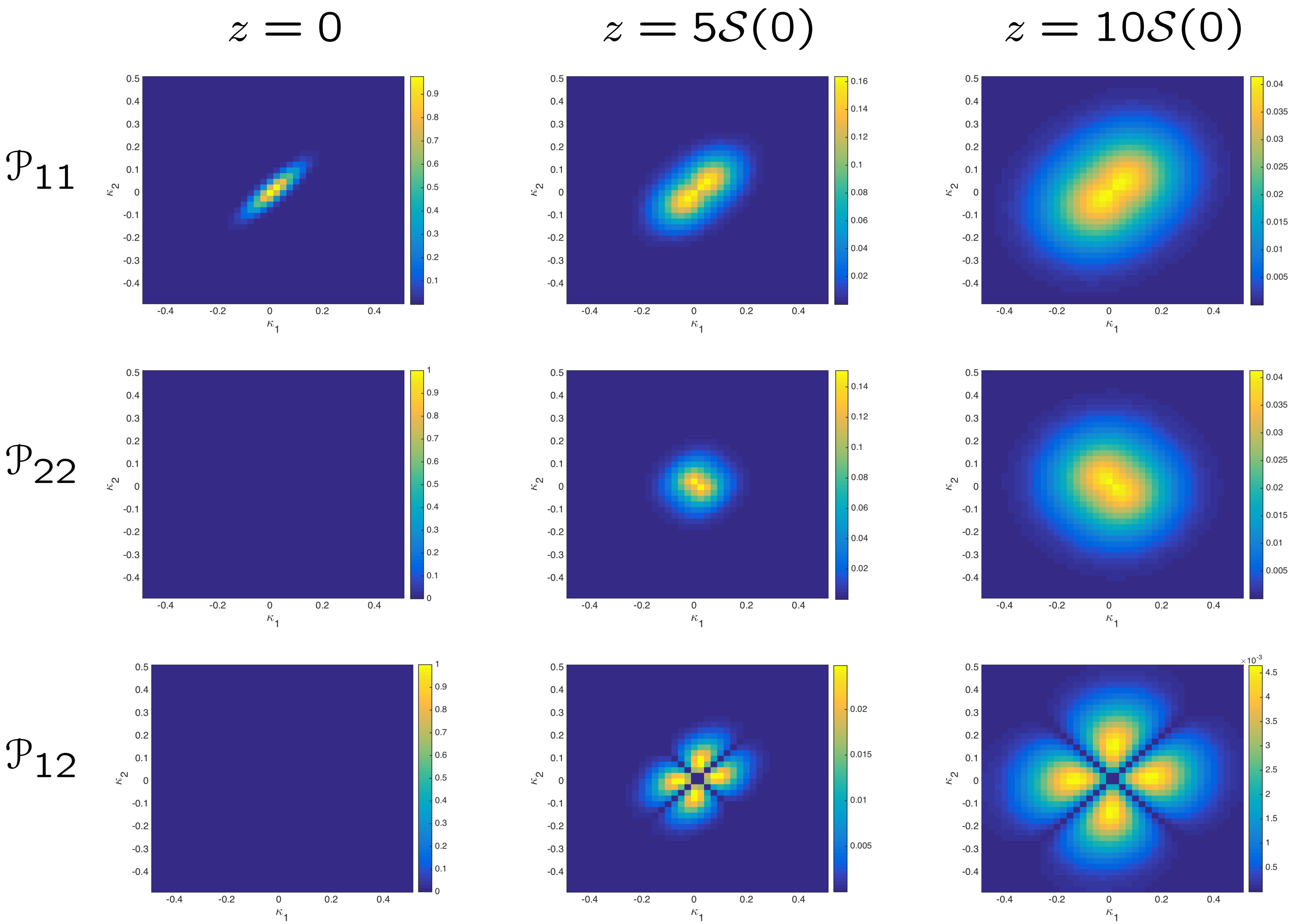
$X = \ell$

$X = \frac{\ell}{4}$



- Coherence matrix is diagonal in this case.
- TM and TE mode powers couple strongly at smaller $|\boldsymbol{\kappa}|$.
- Diffusion spreads power to larger $|\boldsymbol{\kappa}|$.

Results: Anisotropic initial condition.



At high frequency $\gamma = \lambda/\ell \ll 1$ the transport equations simplify:

- We can change basis from $\{\vec{\boldsymbol{u}}(\boldsymbol{\kappa}), \vec{\boldsymbol{u}}^{\perp}(\boldsymbol{\kappa}), \vec{\boldsymbol{\kappa}}\}$ (that gave the TM and TE modes) to $\{\vec{\mathrm{e}}_1, \vec{\mathrm{e}}_2, \vec{\mathrm{e}}_z\}$ for small $|\boldsymbol{\kappa}|$, as propagation is basically along $\vec{\mathrm{e}}_z$.

- This diagonalizes the coupling term in the equations $\rightsquigarrow$ there is no polarization exchange between fields along $\vec{\mathrm{e}}_1$ and $\vec{\mathrm{e}}_2$ in the cross-range plane. Agreement with paraxial results.

We need a large enough opening angle in order to see polarization exchange. This study considers a wide angle regime which bridges between classic radiative transfer and paraxial.

Summary

- We presented a mathematical study of electromagnetic wave propagation in random media in a wide-angle propagation regime.
- The main advantage is that we have a main direction of propagation and we can reduce the problem to the study of the random forward propagating mode amplitudes. This can be carried using the Markovian limit.
- As a result we could justify mathematically the radiative transport equations with polarization. This is in the forward scattering regime, which is relevant in optical imaging.