Thermoacoustic Tomography in Bounded Domains

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Outline

1. TAT: introduction
2. TAT: models
   - In $\mathbb{R}^n$
   - In Ω: Full Data
   - In Ω: Partial Data

Stand-alone medical imaging modalities

- **High contrast** modalities:
  - Optical Tomography (OT);
  - Electrical Impedance Tomography (EIT);
  - Elastographic Imaging (EI).

$\Longrightarrow$ low resolution

- **High resolution** modalities:
  - Computerized Tomography (CT);
  - Magnetic Resonance Imaging (MRI);
  - Ultrasound Imaging (UI).

$\Longrightarrow$ sometimes low contrast.

Coupled physics medical imaging modalities

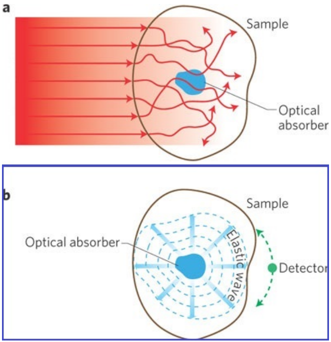
Idea: use physical mechanism that couples two modalities to *improve resolution while keeping the high contrast capabilities*.

High Contrast	High Resolution	Hybrid Inverse Problems
Optical - OT (non-linear)	Ultrasound (non-linear)	Photo-acoustic tomography - PAT
		Thermo-acoustic tomography - TAT
		Ultrasound modulates optical tomography - UMOT or Acoustic optic tomography - AOT
	MRI (linear)	
Electrical - EIT (non-linear)	Ultrasound (non-linear)	Ultrasound modulated electrial impedance tomography - UMEIT or Electro acoustic tomography - EAT
	MRI (linear)	Magnetic resonance electrical impedance tomography - MREIT and Current density impedance imaging - CDII
Elastic	Ultrasound (non-linear)	Transient elastography - TE
	MRI (linear)	Magnetic resonance elastography - MRE

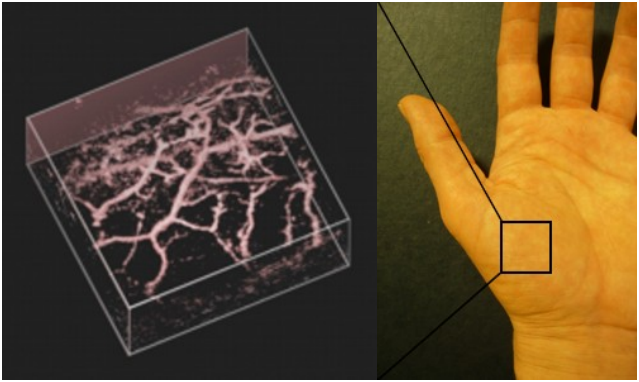
Photo-acoustic effect

Photo-acoustic effect:

Graham Bell: When rapid pulses of light are incident on a sample of matter they can be absorbed and the resulting energy will then be radiated as heat. This heat causes detectable sound waves due to pressure variation in the surrounding medium.



Experimental result



Courtesy UCL (Paul Beard's Lab).

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Model in $\mathbb{R}^n$

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with supp $f \subset \Omega$. Assume the speed $c(x)$ is variable and known in Ω. For $T > 0$, let u solve the problem

$$\begin{cases} (\partial_t^2 - c^2(x)\Delta)u = 0 & \text{in } (0,T)\times\mathbb{R}^n \\ u|_{t=0} = f \\ \partial_t u|_{t=0} = 0. \end{cases}$$

Measurement: $\Lambda f := u|_{[0,T]\times\partial\Omega}$.

Inverse Problem: recover f from Λf.

Literature in $\mathbb{R}^n$

Previous Results: Agranovsky, Ambartsoumian, Anastasio et. al., Burcholzer, Cox et. al., Finch, Grun, Haltmeier, Hofer, Hristova, Jin, Kuchment, Nguyen, Paltauff, Patch, Rakesh, Stefanov, Uhlmann, Wang, Xu, ...

For the Riemannian manifold $(\overline{\Omega}, c^{-2}dx^2)$, let

$$T_0 := \max_{\overline{\Omega}} \operatorname{dist}(x, \partial\Omega).$$

$$T_1 := \text{length of longest geodesic in } \overline{\Omega}.$$

Theorem (Stefanov and Uhlmann, 2009)

- $T < T_0 \Rightarrow$ *no uniqueness;*
- $T_0 < T < \frac{T_1}{2} \Rightarrow$ *uniqueness, no stability;*
- $\frac{T_1}{2} < T \Rightarrow$ *stability and explicit reconstruction.*

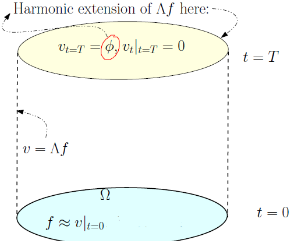
Stefanov and Uhlmann's Time Reversal

Define $A(\Lambda f) := v(0,\cdot)$ (pseudo-inverse of Λ) where v solve the backward problem

$$\begin{cases} (\partial_t^2 - c^2(x)\Delta)v = 0 & \text{in } (0,T)\times\Omega \\ v\big|_{t=T} = \phi \\ \partial_t v\big|_{t=T} = 0 \\ v\big|_{[0,T]\times\partial\Omega} = \Lambda f \end{cases}$$

where ϕ solves

$$\Delta\phi = 0 \quad \text{in } \Omega, \qquad \phi|_{\partial\Omega} = \Lambda f(T,\cdot).$$



Stefanov and Uhlmann's Time Reversal

Denote the error operator by

$$Kf := f - A(\Lambda f)$$

or equivalently

$$(I - K)f = A(\Lambda f).$$

Stefanov and Uhlmann showed that $\|K\| < 1$ when $T > \frac{T_1}{2}$. This leads to the Neumann series reconstruction:

$$f = (I - K)^{-1} A(\Lambda f) = \sum_{m=0}^{\infty} K^m A(\Lambda f).$$

Numerical implementation: Qian-Stefanov-Uhlmann-Zhao (2011)

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- In Ω: Partial Data

Model in Ω

Motivation: placing reflectors around the patient proposed by the UCL photoacoustic group.

Assume the speed $c(x)$ is variable and known in Ω. For $T > 0$, let u solve the problem

$$\begin{cases} (\partial_t^2 - c^2(x)\Delta)u = 0 & \text{in } (0,T)\times\Omega \\ u|_{t=0} = f \\ \partial_t u|_{t=0} = 0 \\ \partial_\nu u|_{(0,T)\times\partial\Omega} = 0. \end{cases}$$

Measurement: $\Lambda f := u|_{[0,T]\times\partial\Omega}$.

Inverse Problem: recover f from Λf.

Literature in Ω

Previous Results: Kunyansky, Holman, Cox, Acosta, Montalto, Nguyen.

For the Riemannian manifold $(\overline{\Omega}, c^{-2}dx^2)$, let

$$T_0 := \max_{\overline{\Omega}} \operatorname{dist}(x, \partial\Omega).$$

$$T_1 := \text{length of longest geodesic in } \overline{\Omega}.$$

Theorem (Stefanov and Y., 2015)

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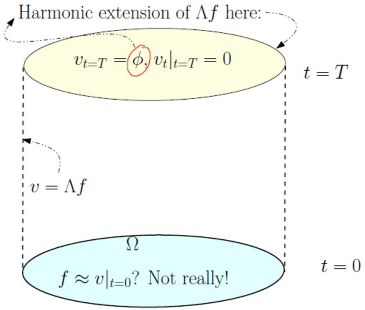
- $T_0 < T < \frac{T_1}{2} \Rightarrow$ *uniqueness, no stability;*
- $\frac{T_1}{2} < T \Rightarrow$ *stability and explicit reconstruction.*

Time reversal in Ω fails!

Let v solve the problem

$$\begin{cases} (\partial_t^2 - c^2(x)\Delta)v &= 0 \quad \text{in } (0,T)\times\Omega \\ v|_{t=T} &= \phi \\ \partial_t u|_{t=T} &= 0 \\ u|_{(0,T)\times\partial\Omega} &= \Lambda f. \end{cases}$$

Define the error operator $Kf := f - v(0)$, but $\|K\| = 1$!



Time reversal in Ω fails!

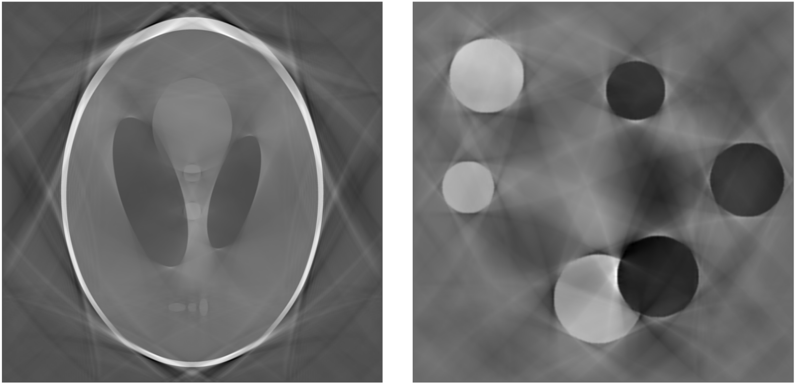


Figure: Failure of the time reversal to resolve all singularities. $T = 0.9 \times$ diagonal, $c = 1$. Increasing T does not help!

Propagation of singularities

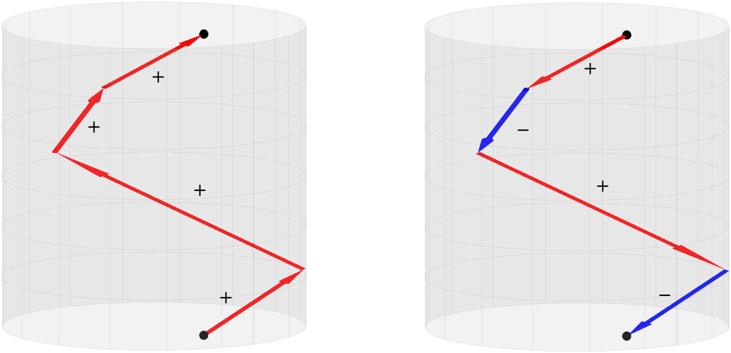


Figure: Propagation of singularities in $[0, T] \times \Omega$ for the positive speed only with Neumann boundary conditions (left) and time reversal with Dirichlet ones (right). In the latter case, the sign changes at each reflection.

Main idea: averaged time reversal

This leads to the following idea: 

Average with respect to T!

Then the error will average as well and some of the positive and negative contributions will cancel out. This will make the error operator a microlocal contraction.

Let $A(\tau)$ be the time reversal over $[0,\tau]$. Define the averaged time reversal operator as

$$\mathcal{A}_0 := \frac{1}{T}\int_0^T A(\tau)\,d\tau.$$

Averaging works!

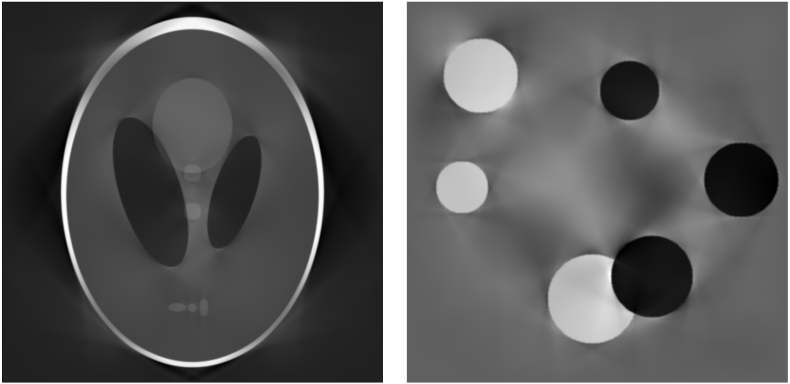


Figure: Averaged time reversal. $T = 0.9 \times$ diagonal, $c = 1$. This is not our inversion yet!

Comparison with non-averaged time reversal

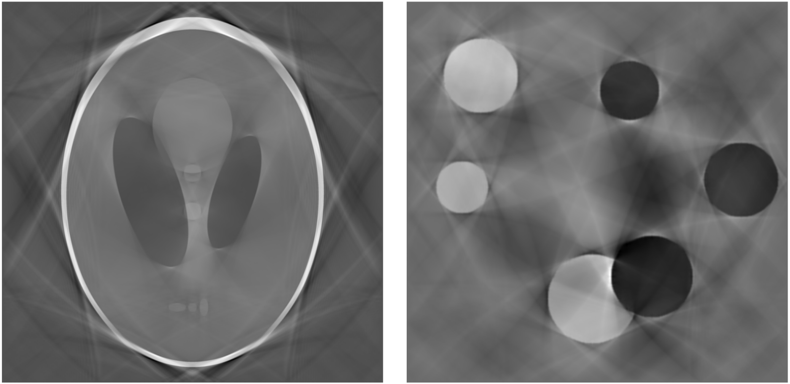


Figure: For comparison: Failure of the time reversal to resolve all singularities. $T = 0.9 \times \text{diagonal}$, $c = 1$.

Explicit Inversion

Let T_1 be the length of the longest geodesic in $(\overline{\Omega}, c^{-2}dx^2)$.

Theorem (Stefanov-Y., 2015)

Let $(\Omega, c^{-2}e)$ be non-trapping, strictly convex, and let $T > T_1$. Let $\Omega_0 \Subset \Omega$. Then $\mathcal{A}_0\Lambda = Id - \mathcal{K}_0$ on $H_D(\Omega_0)$, where $\|\mathcal{K}_0\|_{\mathcal{L}(H_D(\Omega_0))} < 1$. In particular, $Id - \mathcal{K}_0$ is invertible on $H_D(\Omega_0)$, and the inverse problem has an explicit solution of the form

$$f = \sum_{m=0}^{\infty} \mathcal{K}_0^m \mathcal{A}_0 h, \quad h := \Lambda f.$$

Neumann series inversion

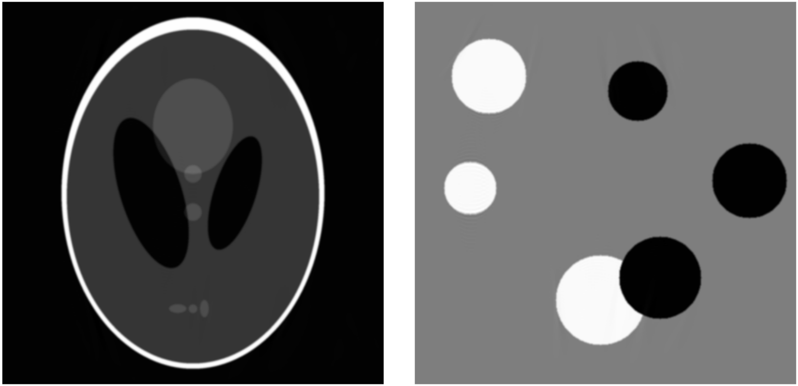


Figure: Full data Neumann series inversion, 10 terms, $T = 5$, on the square $[-1,1]^2$, variable $c = 1 + 0.3\sin(\pi x^1) + 0.2\cos(\pi x^2)$.
The artifacts are mainly due to the presence of corners. The L^2 error on the left is 0.44%; and on the right: 0.34%. The L^∞ error on the left is about 1.2%; and about 3% on the right.

Partial data

Assume the speed $c(x)$ is variable and known in Ω. For $T > 0$, let u solve the problem

$$\begin{cases} (\partial_t^2 - c^2(x)\Delta)u &= 0 \quad \text{in } (0,T)\times\Omega \\ u|_{t=0} &= f \\ \partial_t u|_{t=0} &= 0 \\ \partial_\nu u|_{(0,T)\times\partial\Omega} &= 0. \end{cases}$$

Partial Data Measurement: $\Lambda f := u|_{[0,T]\times\Gamma}$ where Γ is an open subset of $\partial\Omega$.

Inverse Problem: recover f from Λf.

Partial data: uniqueness

Uniqueness: follows from unique continuation. Let

$$T_0 := \max_{\overline{\Omega}} \operatorname{dist}(x, \Gamma).$$

Theorem (Uniqueness)

$\Lambda f = 0$ for some $f \in H_D(\Omega)$ implies $f(x) = 0$ for $\operatorname{dist}(x, \Gamma) < T$. In particular, if $T \geq T_0$, then $f = 0$.

Partial data: stability

Stability: follows from boundary control by Bardos-Lebeau-Rauch.

Theorem (Stability)

If each broken geodesic $\gamma(t)$ hits Γ for $|t| \leq T \implies$ stability.
If some does not hit $\bar{\Gamma} \implies$ no stability.

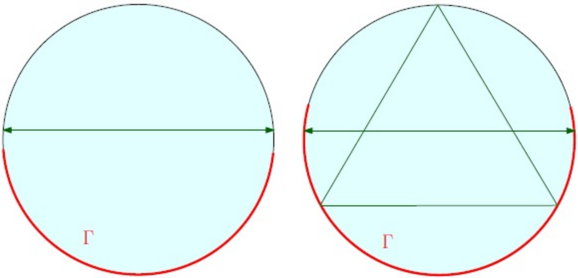


Figure: Bardos-Lebeau-Rauch condition: Left: unstable. Right: stable

Partial data: smooth wave speed reconstruction

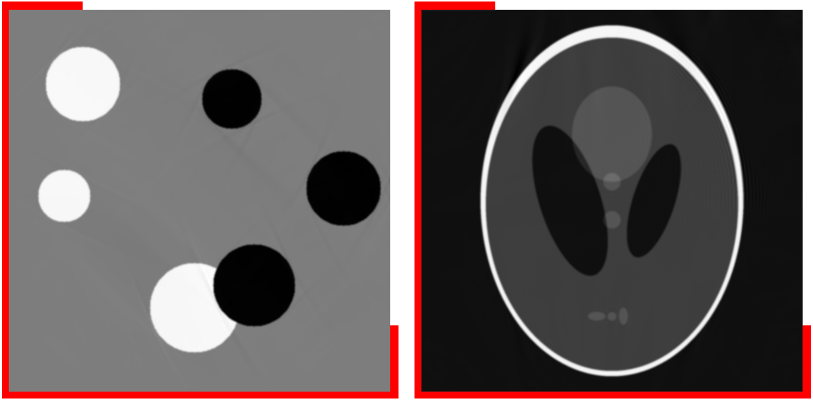
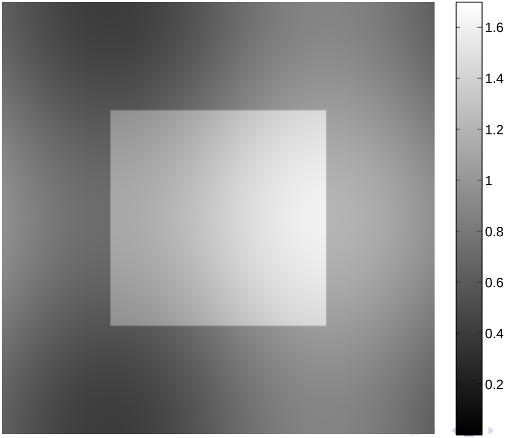


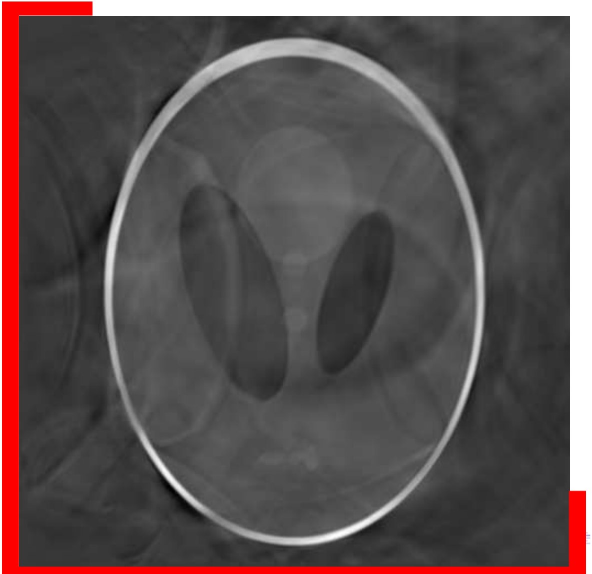
Figure: Partial data inversion with data on the indicated part of $\partial\Omega$. Neumann series inversion with 10 terms, $T = 5$, $\Omega = [-1,1]^2$. Left: constant speed $c = 1$, L^2 error $= 0.7\%$. Right: variable speed $c = 1 + 0.3\sin(\pi x^1) + 0.2\cos(\pi x^2)$, L^2 error $= 2\%$. Again, the most visible artifacts can be explained by the presence of corners.

Partial data: discontinous speed

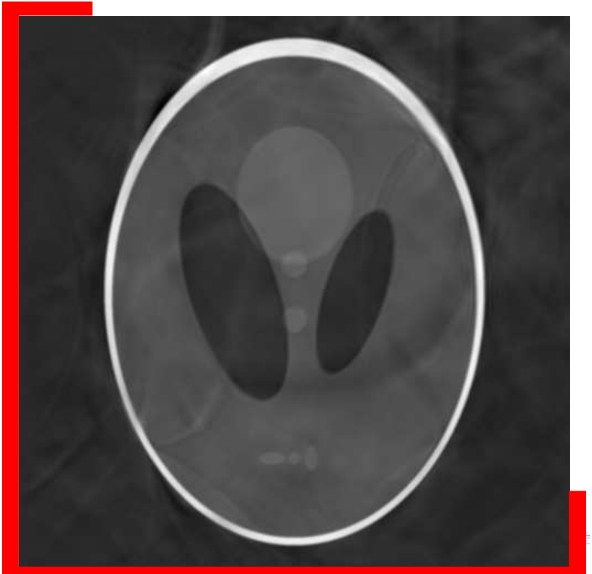
It works well with the following discontinuous speed AND partial data.



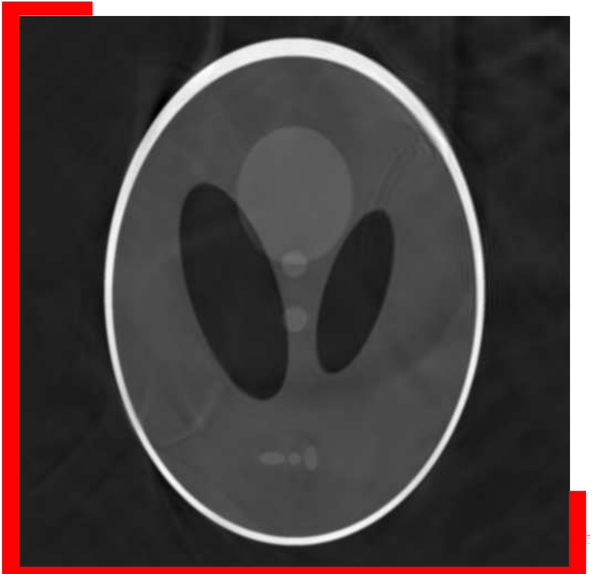
Partial data: discontinuous speed, Iteration = 0



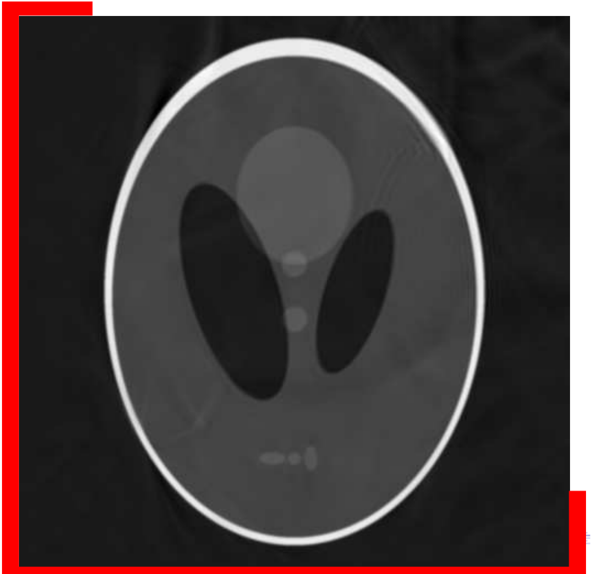
Partial data: discontinuous speed, Iteration = 1



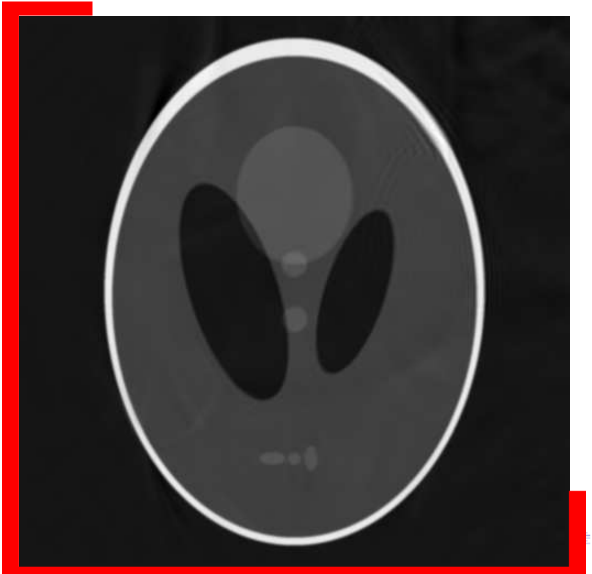
Partial data: discontinuous speed, Iteration = 2



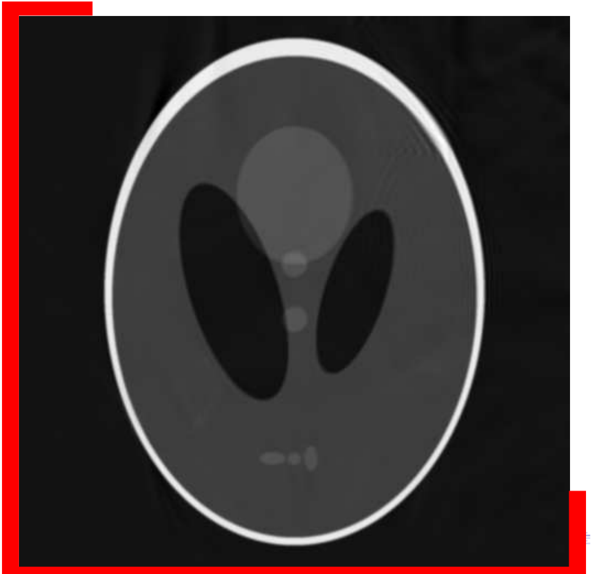
Partial data: discontinuous speed, Iteration = 3



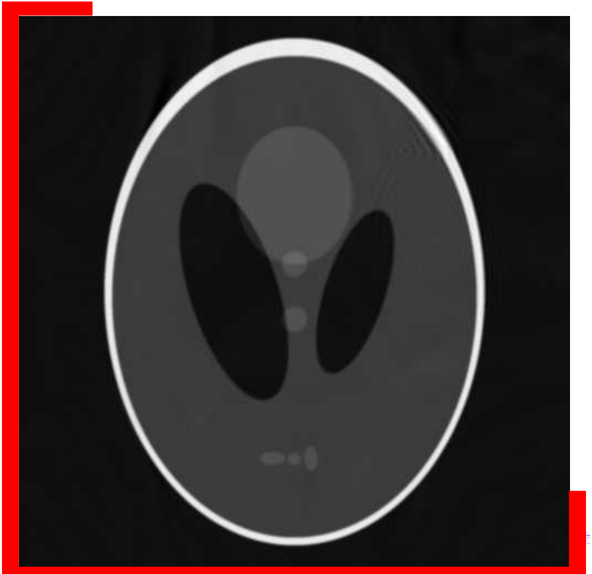
Partial data: discontinuous speed, Iteration = 4



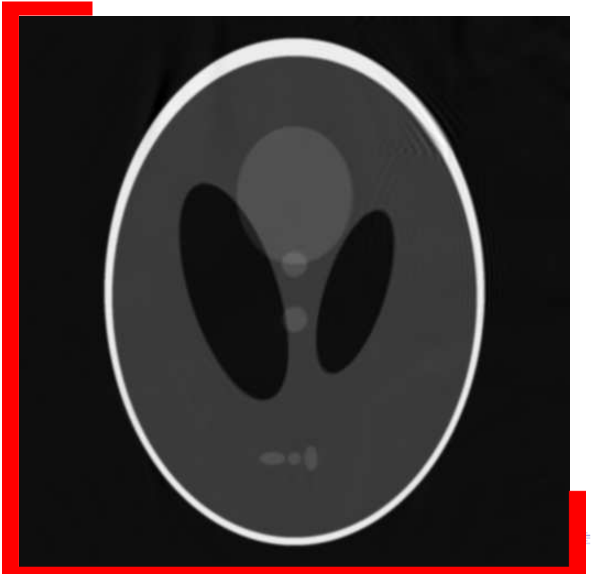
Partial data: discontinuous speed, Iteration = 5



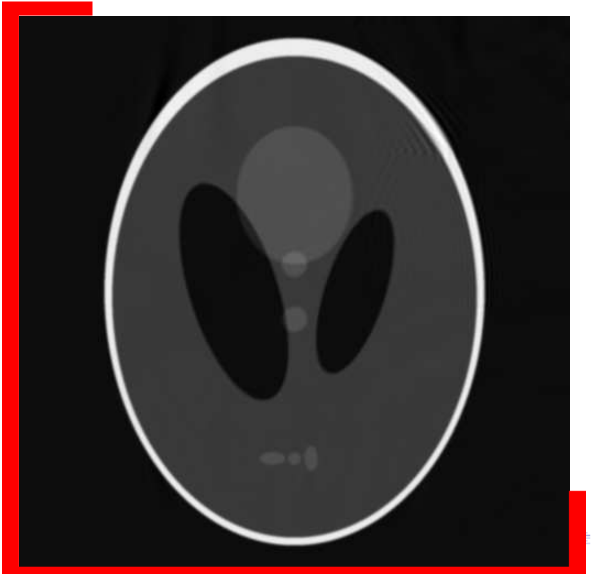
Partial data: discontinuous speed, Iteration = 6



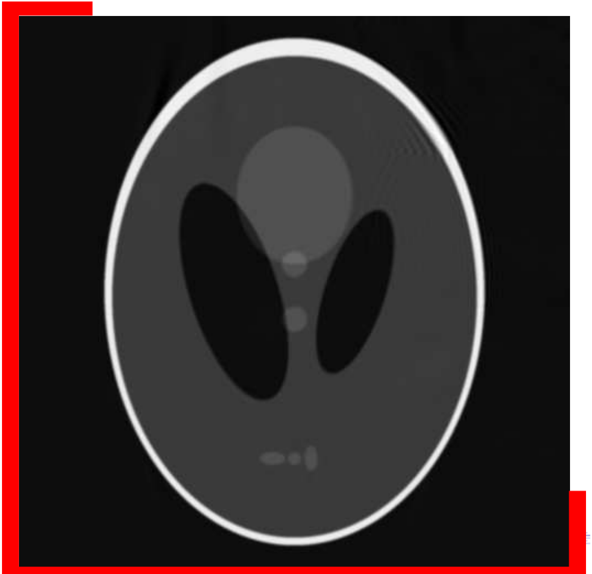
Partial data: discontinuous speed, Iteration = 7



Partial data: discontinuous speed, Iteration = 8



Partial data: discontinuous speed, Iteration = 9




Thank you!