Again, assume that no charge or current is present in the field, then the time-harmonic Maxwell equations take the following form:

$$\nabla \times E = i\omega\mu H, \tag{1.17}$$
$$\nabla \times H = -i\omega\varepsilon E, \tag{1.18}$$
$$\nabla \cdot (\varepsilon E) = 0, \tag{1.19}$$
$$\nabla \cdot (\mu H) = 0. \tag{1.20}$$

It follows from (1.8)-(1.11) that the following jump conditions hold: the tangential components of E and H must be continuous crossing an interface, and the normal components of εE and μH must be continuous crossing an interface.

In a homogeneous and isotropic medium, taking the curl of (1.17) and eliminating the magnetic field yield

$$\nabla \times (\nabla \times E) - \omega^2 \varepsilon\mu E = 0, \tag{1.21}$$

which can be further reduce to the Helmholtz equation by using the vector identity of the curl

$$\Delta E + \kappa^2 E = 0, \tag{1.22}$$

where $\kappa = \omega\sqrt{\varepsilon\mu}$ is the wavenumber. Similarly, we may obtain the Helmholtz equation for the magnetic field

$$\Delta H + \kappa^2 H = 0.$$

Note that in a dielectric medium κ is real and positive. It is easy to verify the relation between the wavenumber κ and the wavelength λ: $\kappa = 2\pi/\lambda$.

1.2 Vector spaces

We begin with a review of some basic concepts in functional analysis.

Definition 1.2.1. *Let X be a nonempty set. Support that addition and scalar multiplication are defined on X, i.e., for any $x, y \in X, x + y \in X$ and for $\alpha \in \mathbb{R}$ or $\mathbb{C}$ a scalar, $\alpha x \in X$. Assume also that for any $x, y, z \in X$ and $\alpha, \beta \in K$ (the scalar field):*

1. $x + y = y + x$;
2. $x + (y + z) = (x + y) + z$;

3. *there is a unique* $0 \in X$ *such that* $x + 0 = x$ *for all* $x \in X$*;*
4. *for each* $x \in X$ *there is a unique element* $-x \in X$ *such that* $x + (-x) = 0$*;*
5. $\alpha(x + y) = \alpha x + \alpha y$*;*
6. $(\alpha + \beta)x = \alpha x + \beta y$*;*
7. $(\alpha\beta)x = \alpha(\beta x)$*;*
8. $1\,x = x$.

Then X *is called a vector space with the scalar field* K.

Definition 1.2.2. *A vector space* X *is a normed vector space if there is a real valued function, the norm, on* X *whose value at* $x \in X$ *is denoted by* $\| x \|$ *and satifies*

1. $\| x \| \geq 0$*;*
2. $\| x \| = 0$ *if and only if* $x = 0$*;*
3. $\| \alpha x \| = |\alpha| \, \| x \|$, α *is a scalar;*
4. $\| x + y \| \leq \| x \| + \| y \|$.

Definition 1.2.3. *A complete normed linear vector space is called a Banach space.*

Definition 1.2.4. *A sequence* x_n *in a metric space* $x = (x, d)$ *is said to be Cauchy if for all* $\varepsilon > 0$, *there exists* $N = N(\varepsilon)$ *such that*

$$d(x_m, x_n) < \varepsilon \quad \text{for all } m, n > N,$$

where d *is the metric.*

The space X is said to be complete if every Cauchy sequence in X converges.

Definition 1.2.5. *Let* X *be a complete linear space. It is called an inner product space if for all* $x, y \in X$, *there is a number* (x, y) *such that*

1. $(x, x) \geq 0$ *and* $(x, x) = 0$ *if and only if* $x = 0$*;*
2. $(x + y, z) = (x, z) + (y, z)$*;*
3. $(\alpha x, y) = \alpha(x, y)$, α *is a complex number;*
4. $(x, y) = \overline{(y, x)}$.

Remark 1.2.6. *An inner product space is a normed space.*

Definition 1.2.7. *A complete linear inner product space is called a Hilbert space.*

Remark 1.2.8. *Hilbert spaces are particular examples of Banach spaces.*

Let $f : X \to K$ be a functional or mapping. It is said to be a linear functional if for all $x, y \in X$ and $\alpha, \beta \in K$,

$$f(\alpha x + \beta y) = \alpha f(x) + \beta f(y).$$

In addition if f is continuous, then f is called a continuous linear functional on X.

Proposition 1.2.9. *If a linear functional is continuous at one point, then it is continuous everywhere.*

Definition 1.2.10. *A linear functional $f : X \to K$ is bounded if and only if there is a real number C such that*

$$|f(x)| \leq C \parallel x \parallel \quad \text{for all } x \in X.$$

Proposition 1.2.11. *A linear functional is continuous if and only if it is bounded.*

Definition 1.2.12. *Let X be a normed linear space. The dual space of X, denoted by X', is the normed space of all bounded linear functionals on X. If f, f_1, and f_2 are bounded functionals on X, define $f_1 + f_2$ and αf_1 by*

$$(f_1 + f_2)(x) = f_1(x) + f_2(x),$$
$$(\alpha f)(x) = \alpha f(x) \quad \text{for all } x \in X.$$

Theorem 1.2.13 (The Riesz representation theorm)**.** *If f is a bounded linear functional on a Hilbert space X, the there is a unique element $y \in X$ such that*

$$f(x) = (x, y) \quad \text{for all } x \in X$$

and $\parallel f \parallel = \parallel y \parallel$.

A sequence $\{x_n\} \subset X$ converges weakly to x, denoted by $x_n \to x$, if $\lim_{n\to\infty}(x_n, y) = (x, y)$ for all $y \in X$. The usual convergence, $\parallel x_n - x \parallel \to 0$, is often referred to as strong convergence. It is easily seen that strong convergence implies weak convergence, but not vice versa. Also, the limiting point of a weakly convergent sequence is unique.

A set $S \subset X$ is weakly sequentially compact if every sequence $\{x_n\} \subset S$ contains a convergent sequence. Using the Riesz representation theorem, it can be proved:

Proposition 1.2.14. *Any bounded set of a Hilbert space is weakly sequentially compact.*