1.3 Sobolev spaces

Let Ω be an open set. Denote

$$\begin{aligned} C^m(\Omega) &= \{f : f \text{ has up to } m\text{th order continuous derivatives}\}, \\ \mathrm{supp} f &= \{x \in \Omega : f(x) \neq 0\}, \\ C_0^m(\Omega) &= \{f \in C^m(\Omega) : \mathrm{supp} f \text{ is bounded}\}, \\ C_0^\infty(\mathbb{R}^n) &= \{f \in C^\infty(\mathbb{R}^n) : \mathrm{supp} f \text{ is bounded}\}, \\ L^2(\Omega) &= \{f : f \text{ is square integrable}\}. \end{aligned}$$

Lemma 1.3.1. *Suppose* $f \in L^2(\Omega)$ *and*

$$\int_\Omega f\phi dx = 0 \quad \text{for all } \phi \in C_0^\infty(\Omega),$$

then $f = 0$ *almost everywhere.*

Lemma 1.3.2. $C_0^\infty(\Omega)$ *is dense in* $L^2(\Omega)$.

Denote $\alpha = (\alpha_1, \ldots, \alpha_n)$, where $\alpha_i \geq 0$ are integers, $|\alpha| = \alpha_1 + \cdots + \alpha_n$. Also let $D^\alpha = \partial^\alpha / \partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \ldots \partial x_n^{\alpha_n}$. Then a $L^2(\Omega)$ function is said to have $|\alpha|$th order generalized derivative if there is a $g \in L^2(\Omega)$ such that for any $\phi \in C_0^\infty(\Omega)$

$$\int_\Omega g\phi dx = (-1)^{|\alpha|} \int_\Omega f D^\alpha \phi dx.$$

The function g is called the $|\alpha|$th order generalized derivative of f denoted by $D^\alpha f = g$.

It follows from Lemma 1.3.1 that the generalized derivative must be unique if it exists. Whenever the classic derivatives exist in $L^2(\Omega)$, they are identical to the generalized derivatives of the same order.

It holds the following properties of generalized derivatives:

1. $D^\alpha(af + bg) = aD^\alpha f + bD^\alpha g$ a, b are constants,
2. $D^{\alpha+\beta} f = D^\alpha(D^\beta f)$,
3. $D^\alpha f = 0$ for all $|\alpha| = m$ if and only if f is a polynomial of $m - 1$ order almost everywhere.

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. It is of Lipschitz type if (1) Ω is on one side of $\partial\Omega$; (2) for all $x_0 \in \partial\Omega$, there is an open ball $B_\delta(x_0) = \{x \in \mathbb{R}^n : \| x - x_0 \| < \delta\}$ and a local coordinate transformation $\xi_i = \psi_i(x), i = 1, 2, \ldots, n$ such that $\partial\Omega \cap B_\delta(x_0)$ can be expressed as

$$\xi_i = f(\xi_1, \ldots, \xi_{i-1}, \xi_{i+1}, \ldots, \xi_n)$$

and f is Lipschitz continuous.

Introduce an inner product in $C^\infty(\overline{\Omega})$:

$$(f, g)_m = \sum_{|\alpha| \leq m} (D^\alpha f, D^\alpha g)$$

and a norm

$$\| f \|_m = \left(\sum_{|\alpha| \leq m} \| D^\alpha f \|^2 \right)^{1/2} .$$

Then $C^\infty(\overline{\Omega})$ becomes an inner product space. However, it is not complete hence not a Hilbert space. The completion of $C^\infty(\overline{\Omega})$ in $\| \cdot \|_m$ is called Sobolev spaces and denoted by $H^m(\Omega)$. Equivalently

$$H^m(\Omega) = \{f : D^\alpha f \in L^2(\Omega), |\alpha| \leq m\}.$$

We next present some basic properties of Sobolev spaces. Two functions f and g are equivalent if $f = g$ almost everywhere in Ω.

Theorem 1.3.3 (The Sobolev imbedding theorem). *Suppose $m > n/2$, then any function $f \in H^m(\Omega)$ is equivalent to a function in $C(\Omega)$, also denoted by f, and*

$$\| f \|_{C(\Omega)} \leq c \| f \|_m,$$

where c is a constant independent of f. Moreover, the identity operator $I : H^m(\Omega) \to C(\Omega)$ is compact, i.e., a bounded sequence of functions in the norm of $\| \cdot \|_m$ is compact in the norm of $\| \cdot \|_{C(\Omega)}$ or has a uniformly convergent subsequence.

Theorem 1.3.4 (The compact imbedding theorem). *If $m_1 > m_2$, then the identity operator*

$$I : H^{m_1}(\Omega) \to H^{m_2}(\Omega)$$

is compact.

Let $\mathbf{n}$ be the normal vector of $\partial\Omega$. For all $f \in C^m(\overline{\Omega})$, one may define the trace operator as:

$$\gamma_j f = \partial_{\mathbf{n}}^j f|_{\partial\Omega}, \quad 0 \leq j \leq m - 1.$$

Theorem 1.3.5. *There is a constant c independent of f such that*

$$\| \gamma_j f \|_{L^2(\partial\Omega)} \leq c \| f \|_{H^{j+1}}, \quad j = 0, 1, \ldots, n-1$$

for all $f \in H^m(\overline{\Omega})$.

Let $H_0^m(\Omega)$ be the completion of $C_0^\infty(\Omega)$ or $C_0^m(\Omega)$ by $\| \cdot \|_m$. They are Sobolev spaces and subspaces of $H^m(\Omega)$:

$$H_0^m(\Omega) = \{f \in H^m(\Omega) : \gamma_j f = \partial_{\mathbf{n}}^j f|_{\partial\Omega}, \quad 0 \leq j \leq m-1\}.$$

In many applications, real indexes Sobolev spaces, $H^s(\Omega)$ and $H^s(\partial\Omega)$, s is a real number, may also be needed. A direct way to introduce such spaces is by means of Fourier transforms. The following is the trace theorem in a sharp form.

Theorem 1.3.6. *For all $f \in H^{s+1}(\Omega), s \geq 0$, the operator*

$$\gamma_j : H^{s+1}(\Omega) \rightarrow H^{s+\frac{1}{2}}(\partial\Omega), \quad j = 0, 1, \ldots, [s],$$

is well-defined and

$$\| \gamma_j f \|_{H^{s+\frac{1}{2}}(\partial\Omega)} \leq c \| f \|_{H^{s+1}(\Omega)}, \quad j = 0, 1, \ldots, [s].$$

Moreover, γ_j is a surjection but not an injection from $H^{s+1}(\Omega)$ to $H^{s+\frac{1}{2}}(\partial\Omega)$. There is an $f \in H^{s-1}(\Omega)$ depending on γ_j such that

$$\| f \|_{H^{s+1}(\Omega)} \leq \| \gamma_j f \|_{H^{s+\frac{1}{2}}(\partial\Omega)}, \quad j = 0, 1, \ldots, [s].$$

Definition 1.3.7. *Let A be a linear operator $A : X \rightarrow X$. It is called compact if it maps any bounded sequence $\{x_n\}$ into a compact sequence $\{Ax_n\}$, i.e., $\{Ax_n\}$ has a convergent subsequence.*

It holds the following properties of compact operators:

1. A is compact if its range $R(A)$ is a finite dimensional space
2. An operator is bounded if it is compact
3. If S is a bounded operator and A is a compact operator, then SA is compact.

Denote

$$L(X, Y) = \{A : A \text{ is a linear continuous operator from } X \text{ to } Y\}.$$

Lemma 1.3.8. *Suppose that* $A_n, A \in L(X, Y)$ *and* $\{A_n\}$ *is a sequence of compact operators. Then* A *is a compact operator if*

$$\lim_{n \times \infty} \| A_n - A \| = 0.$$

Let $A \in L(X, X)$ be a compact operator. Consider

$$x - \lambda A x = f, \tag{1.23}$$

where λ is a parameter, $f \in X$ is the right hand side.

Theorem 1.3.9 (The Fredholm alternative)**.** *Let* A *be a compact operator, then one of the following two conclusions holds*

1. *for all* $f \in X$*, Equation* (1.29) *has a unique solution and the corresponding homogeneous equation only has trivial solution*
2. *It is not solvable for all* $f \in X$*. The homogeneous equation has nontrivial solutions*

In the first case, $(I - \alpha A)^{-1}$ *exists and is bounded; the second case holds when* $1/\lambda$ *is an eigenvalue of* A*. In this case, the set of eigenvalues of* A *is countable and has no finite limiting point.*

1.4 Variational form of boundary value problems

Let X be a separable Hilbert space with an inner product $(,)$ and norm $\| \cdot \|$. We identify X with its dual X'. Let V be a linear subspace of X which is dense in X. Usually, V is not complete under $\| \cdot \|$. Assume that a new inner product $[,]$ and norm $| \cdot |$ can be introduced so that V is a Hilbert space in terms of the new inner product $[,]$.

Assume also that the identity operator $I : V \to X$ is continuous, i.e., there is a constant $\alpha > 0$ such that

$$\| u \| \leq \alpha |u| \quad \text{for all } u \in V. \tag{1.24}$$

In other words, the norm $| \cdot |$ in V is stronger than $\| \cdot \|$ in X.

It is easy to see that

$$V \subset X = X' \subset V'.$$