Thus there exist two positive constants \( c_1 \) and \( c_2 \) such that
\[
\text{Rea}(u, u) \geq c_1 \| \nabla u \|_{L^2(\Omega)}^2 - c_2 \| u \|_{L^2(\Omega)}^2.
\]
There the Fredholm alternative holds. The existence then follows from the uniqueness of the solution. \( \Box \)

## 3.4 TM polarization

Similarly, we can derive the variational formulation in TM polarization. However, in this case, the function \( \partial_n u \) rather than the function \( u \) is compactly supported. Consequently, the transparent boundary condition becomes slightly more complicated.

The scattered field \( v \) satisfies the homogeneous Helmholtz equation in \( \{ x_2 > 0 \} \) as well as the boundary condition \( \partial_n v = 0 \) on \( \Gamma_g \). By taking the Fourier transform of the model equation in \( \{ x_2 > 0 \} \) and observing that the function \( \partial_n v \) is compactly supported on the \( x_1 \) axis, we obtain after some similar computation that
\[
v(x_1, 0) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{1}{\beta_0(\xi)} \partial_{x_2} \hat{v}(\xi, 0) e^{i\xi x_1} d\xi.
\]

Define for \( f \in H^{-1/2}(\mathbb{R}) \),
\[
T_{TM}(f) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{1}{\beta_0(\xi)} \hat{f}(\xi) e^{i\xi x_1} d\xi.
\]

We arrive at the transparent boundary condition
\[
v = T_{TM}(\partial_n v).
\]

**Lemma 3.4.1.** The following identity holds
\[
\int_{\gamma} \partial_n v T_{TM}(\widehat{\partial_n v}) = -i \int_{-\infty}^{\infty} \frac{1}{\beta_0(\xi)} |\widehat{\partial_n v}|^2,
\]
where \( \widehat{\partial_n v} \in H^{-1/2}(\mathbb{R}) \) is an extension of \( \partial_n v \) with \( \widehat{\partial_n v} = 0 \) on \( \Gamma_g \).

Introduce a functional space
\[
\tilde{H}_0^1(\Omega) = \{ w \in H^1(\Omega), \partial_n w \in \tilde{H}^{-1/2}(\Gamma), w = T_{TM}(\widehat{\partial_n w}) \text{ on } \Gamma \},
\]
where

\[ \tilde{H}^{-1/2}(\Gamma) = \{ w \in H^{-1/2}(\Gamma), \int_{-\infty}^{\infty} \frac{1}{\beta_0(\xi)} \tilde{w} e^{ix} d\xi \in H^{1/2}(K) \text{ for all } K \subset \mathbb{R} \} \]

The scattering problem can be formulated as

\[ \nabla \cdot (\kappa^{-2} \nabla v) + v = f \quad \text{in } \Omega, \]
\[ \partial_n v = -\partial_n u^{\text{ref}} \quad \text{on } S, \]
\[ v = T_{\text{TM}}(\tilde{\partial}_n v) \quad \text{on } \Gamma, \]

where

\[ f = -\nabla \cdot (\kappa^{-2} \nabla u^{\text{ref}}) + u^{\text{ref}}. \]

An equivalent weak form is to find \( v \in \tilde{H}_0^1(\Omega) \) such that

\[ a_{\text{TM}}(u, v) = \int_S \kappa^{-1} \partial_n u^{\text{ref}} v - \int_\Omega f v \quad \text{for all } v \in \tilde{H}_0^1(\Omega), \]

where the bilinear form is defined by

\[ a_{\text{TM}}(u, v) = \int_\Omega \kappa^{-2} \nabla u \cdot \nabla v - \int_\Omega u v - \int_\Gamma \partial_n u T_{\text{TM}}(\tilde{\partial}_n v). \]

From the bilinear form, it is easy to verify that

\[ \text{Re} a_{\text{TM}}(u, u) = \text{Re} \left[ \int_\Omega \kappa^{-2} |\nabla u|^2 - \int_\Omega |u|^2 - \int_\Gamma \partial_n u T_{\text{TM}}(\tilde{\partial}_n u) \right] \geq c_1 \| \nabla u \|^2_{L^2(\Omega)} - c_2 \| u \|^2_{L^2(\Omega)}, \]

where \( c_1 \) and \( c_2 \) are positive constants. Thus again, the Fredholm alternative holds.

To state the uniqueness results, we need to make some technique assumption about the function \( \varepsilon(x) \in C^{1,\gamma}(\Omega) \). In order to establish the uniqueness of the solution, it suffices to show that \( a_{\text{TM}}(u, u) = 0 \) implies \( u = 0 \). From the bilinear form, we have

\[ \text{Im} b(u, u) = -\text{Im} \int_\Gamma \partial_n u T_{\text{TM}}(\tilde{\partial}_n u) = \text{Re} \int_{-\infty}^{\infty} \frac{1}{\beta_0} |\tilde{\partial}_n u|^2 d\xi. \]

Hence

\[ \tilde{\partial}_n u|_{x_2=0} = 0 \]

and

\[ \partial_n v = 0 \quad \text{on } \Gamma. \]

It follows from the transparent boundary condition that \( u = 0 \) on \( \Gamma \). We can conclude that \( u = 0 \) in \( \Omega \) by unique continuation and the assumption about the regularity of the function \( \varepsilon \).
Theorem 3.4.2. Assume $\varepsilon \in C^{1,\gamma}(\Omega)$, the cavity scattering problem for the TM case attains a unique weak solution in $\widetilde{H}^1_0(\Omega)$.

3.5 Overfilled cavity