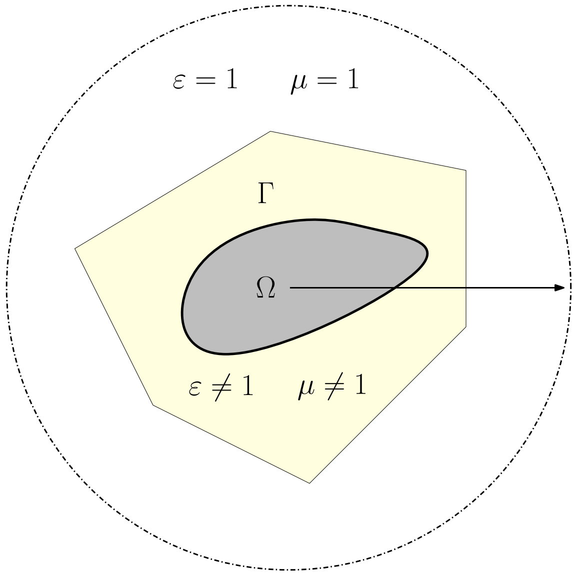

Figure 1.2: Geometry of the scatterer and boundaries for the scattering problem. A bounded scatterer consisting of a perfectly conducting part and a penetrable part where the electromagnetic properties differ from the background is surrounded by air or vacuum.

1.3 Scattering problems and radiation conditions

We shall discuss the scattering from a bounded, inhomogeneous object, e.g., radar scattering from an aircraft. We assume that the object consists of a bounded perfect conductor occupying a bounded domain Ω, perhaps surrounded by an inhomogeneous medium where $\varepsilon \neq 1$ or $\mu \neq 1$. The electromagnetic field occupies the domain $\mathbb{R}^2 \setminus \overline{\Omega}$. We assume that sufficiently far from Ω the object is surrounded in all directions by air or vacuum, so there is a radius R such that $\varepsilon(\mathbf{x}) = \mu(\mathbf{x}) = 1$ when $\rho = |\mathbf{x}| > R$, as seen in Figure 1.2.

On the boundary of Ω, denoted by Γ, we impose the perfect conducting boundary condition. It turns out to be necessary to impose another boundary condition at infinity in order to obtain a well-posed problem. To do this we need to distinguish a given incident field and the resulting scattered field. The incident field is denoted by $\mathbf{E}^{\text{inc}}$ and is assumed to satisfy the Maxwell system in the absence of the scatterer, i.e., in the background medium,

$$\nabla \times (\nabla \times \mathbf{E}^{\text{inc}}) - \kappa^2 \mathbf{E}^{\text{inc}} = 0 \quad \text{in } \mathbb{R}^3. \tag{1.23}$$

A typical example is the plane wave given explicitly by

$$\mathbf{E}^{\text{inc}} = \mathbf{p} e^{i\kappa \mathbf{x}\cdot\mathbf{d}}, \tag{1.24}$$

where $\mathbf{d} \in \mathbb{S}^2$ is a unit vector giving the direction of propagation of the wave, and the vector $\mathbf{p}$ is called the polarization and must be orthogonal to the direction of propagation, i.e., $\mathbf{p} \cdot \mathbf{d} = 0$. The total field $\mathbf{E}$ consists of the incident field $\mathbf{E}^{\mathrm{inc}}$ and the scattered field $\mathbf{E}^{\mathrm{s}}$:

$$\mathbf{E} = \mathbf{E}^{\mathrm{inc}} + \mathbf{E}^{\mathrm{s}}. \tag{1.25}$$

The scattered field is outgoing, i.e., originates at the scatterer and propagates outwards, and this is imposed by requiring the scattered field to satisfy the Silver-Müller radiation condition:

$$\lim_{\rho\to\infty} \rho\left[(\nabla \times \mathbf{E}^{\mathrm{s}}) \times \hat{\mathbf{x}} - \mathrm{i}\kappa \mathbf{E}^{\mathrm{s}}\right] = 0 \tag{1.26}$$

uniformly in all directions $\hat{\mathbf{x}} = \mathbf{x}/|\mathbf{x}|$.

An obvious difficulty with approximating the scattering problem by a finite element method is that the problem is posed on an infinite domain. One simple way to avoid this difficulty is to approximate the scattering by imposing the radiation condition (1.26) on a surface S far from the scatterer. Thus, in this approximation, the domain occupied by the computational electromagnetic field, denoted by Ω, is the region between Γ and S. So we have the perfect conducting boundary condition and impose a boundary condition inspired by the Silver-Müller condition:

$$(\nabla \times \mathbf{E}) \times \mathbf{n} - \mathrm{i}\kappa \mathbf{E}_S = (\nabla \times \mathbf{E}^{\mathrm{inc}}) \times \mathbf{n} - \mathrm{i}\kappa \mathbf{E}_S^{\mathrm{inc}} \quad \text{on } S, \tag{1.27}$$

where $\mathbf{n}$ is the unit outward normal to S and $\mathbf{E}_S = -\mathbf{n} \times (\mathbf{n} \times \mathbf{E})$. This is also known as the impedance boundary condition with a special choice of the impedance. Equation (1.27) is an example of an absorbing boundary condition used to simulate the infinite domain outside Ω.

We shall now summarize the principle boundary value problems for Maxwell's equations in this course. Suppose Ω is a bounded domain with two disjoint connected boundaries Γ and S. We seek to compute the time-harmonic electric field $\mathbf{E}$ corresponding to a given current density $\mathbf{F}$ by solving the time-harmonic electric field equation subject to the perfect conducting boundary condition and the impedance condition as follows:

$$\nabla \times (\mu^{-1} \nabla \times E) - \kappa^2 \varepsilon \mathbf{E} = \mathbf{F} \quad \text{in } \Omega, \tag{1.28}$$

$$\nabla \times \mathbf{E} = 0 \quad \text{on } \Gamma, \tag{1.29}$$

$$\mu^{-1}(\nabla \times \mathbf{E}) \times \mathbf{n} - \mathrm{i}\kappa\lambda E_S = \mathbf{g} \quad \text{on } S, \tag{1.30}$$

where $\mathbf{g}$ is a given tangential vector field on S and λ is the impedance coefficient. We shall allow S to be empty in which case these equations model propagation in a cavity with a perfectly conducting wall.