4.3 Weak solution of Maxwell's equations

In this section, we consider the existence and uniqueness of the weak solution for the time-harmonic Maxwell equations. Let $\Omega \subset \mathbb{R}^3$ be a Lipschitz continuous bounded domain with boundary $\Gamma = \partial\Omega$. Consider the boundary value problem for the electric field:

$$\nabla \times (\mu^{-1}\nabla \times \mathbf{E}) - \kappa^2\varepsilon\mathbf{E} = \mathbf{F} \quad \text{in } \Omega, \tag{4.7}$$

$$\mathbf{E} \times \mathbf{n} = 0 \quad \text{on } \Gamma, \tag{4.8}$$

where $\mathbf{F} \in (L^2(\Omega))^3$, $\mu \geq 1$, and $\mathrm{Re}\varepsilon \geq 1$. In addition, we assume that

1. $\bar\Omega = \cup_{i=1}^I \bar\Omega_i$, Ω_i is a connected Lipschitz domain and $\Omega_i \cap \Omega_j = \emptyset$ for $i \neq j$.
2. ε, μ are constants in Ω_i and there exists a constant $C_\varepsilon > 0$ such that

$$\mathrm{Im}\varepsilon \geq C_\varepsilon \text{ in } C_\varepsilon \quad \text{or } \mathrm{Im}\varepsilon = 0 \text{ in } \Omega_i,\ 1 \leq i \leq I.$$

The weak formulation of (4.7)–(4.8) is to find $\mathbf{E} \in H_0(\mathrm{curl}, \Omega)$ such that

$$a(\mathbf{E}, \mathbf{v}) = (\mathbf{F}, \mathbf{v}) \quad \text{for all } \mathbf{v} \in H_0(\mathrm{curl}, \Omega), \tag{4.9}$$

where the inner product $(\cdot, \cdot)$ and the bilinear form $a : H_0(\mathrm{curl}, \Omega) \times H_0(\mathrm{curl}, \Omega) \to \mathbb{C}$ are defined:

$$\begin{aligned}(\mathbf{u}, \mathbf{v}) &= \int_\Omega \mathbf{u} \cdot \bar{\mathbf{v}} \quad \text{for all } \mathbf{u}, \mathbf{v} \in (L^2(\Omega))^3,\\ a(\mathbf{u}, \mathbf{v}) &= \int_\Omega \left(\mu^{-1}\nabla \times \mathbf{v} \cdot \nabla \times \bar{\mathbf{v}} - \kappa^2\varepsilon\mathbf{u} \cdot \bar{\mathbf{v}}\right) \quad \text{for all } \mathbf{u}, \mathbf{v} \in H_0(\mathrm{curl}, \Omega).\end{aligned}$$

Lemma 4.3.1. *The space* $H_0(\mathrm{curl}, \Omega)$ *has the following decomposition*

$$H_0(\mathrm{curl}, \Omega) = \nabla H_0^1(\Omega) \oplus H_\perp(\mathrm{curl}, \Omega),$$

where

$$H_\perp(\mathrm{curl}, \Omega) = \{\mathbf{v} \in H_0(\mathrm{curl}, \Omega) : (\varepsilon\mathbf{v}, \nabla p) = 0 \quad \textit{for all } p \in H_0^1(\Omega)\}.$$

Proof. First consider the boundary value problem: Find $p \in H_0^1(\Omega)$ such that

$$(\varepsilon\nabla p, \nabla v) = (\varepsilon\mathbf{v}, \nabla v) \quad \text{for all } v \in H_0^1(\Omega).$$

Since $\mathrm{Re}\varepsilon \geq 1$, it follows from the Lax-Milgram lemma that the above problem has a unique weak solution. It can be verified that $\mathbf{u}_\perp = \mathbf{u} - \nabla p \in H_\perp(\mathrm{curl}, \Omega)$. □

Lemma 4.3.2. *$H_\perp(\mathrm{curl},\Omega)$ is compactly imbedded into $(L^2(\Omega))^3$.*

Proof. Let $\{\mathbf{v}_n\}_{n=0}^\infty \subset H_\perp(\mathrm{curl},\Omega)$ be a bounded sequence in $\|\cdot\|_{H(\mathrm{curl},\Omega)}$, it suffices to prove that $\{\mathbf{v}_n\}_{n=0}^\infty$ has a convergent subsequence in $(L^2(\Omega))^3$. For any $\mathbf{v}_n$, let $\psi_n \in H_0^1(\Omega)$ be the unique solution of the following problem

$$(\nabla\psi_n, \nabla v) = (\mathbf{v}_n, \nabla v) \quad \text{for all } v \in H_0^1(\Omega).$$

Clearly, $\mathbf{w}_n = \mathbf{v}_n - \nabla\psi_n \in X_N(\Omega)$ and satisfies

$$\nabla\cdot\mathbf{w}_n = 0 \text{ in } \Omega, \quad \| \mathbf{w}_n \|_X = \| \mathbf{w}_n \|_{H(\mathrm{curl},\Omega)} \leq \mathbf{v}_n \|_{H(\mathrm{curl},\Omega)},$$

where

$$\begin{aligned} X_N &= H_0(\mathrm{curl},\Omega) \cap H(\mathrm{div},\Omega), \\ \| \mathbf{u} \|_X^2 &= \| \mathbf{u} \|_{(L^2(\Omega))^3}^2 + \| \nabla\times\mathbf{u} \|_{(L^2(\Omega))^3}^2 + \| \nabla\cdot\mathbf{u} \|_{(L^2(\Omega))^3}^2 . \end{aligned}$$

Hence $\{\mathbf{w}_n\}_{n=0}^\infty$ is a bounded sequence in $\|\cdot\|_X$. Since $X_N(\Omega)$ is compactly imbedded into $(L^2(\Omega))^3$, there exists $\mathbf{w} \in X_N(\Omega)$ such that $\mathbf{w}_n \to \mathbf{w}$ in $(L^2(\Omega))^3$. It follows from the Helmholtz decomposition that

$$\mathbf{w} = \mathbf{v} + \nabla\phi, \quad \mathbf{v} \in H_\perp(\mathrm{curl},\Omega),\ \phi \in H_0^1(\Omega).$$

Therefore

$$\begin{aligned} (\varepsilon(\mathbf{v}_n - \mathbf{v}), \mathbf{v}_n - \mathbf{v}) &= (\varepsilon(\mathbf{v}_n - \mathbf{v}), \mathbf{w}_n + \nabla\psi_n - \mathbf{w} - \nabla\phi) \\ &= (\varepsilon(\mathbf{v}_n - \mathbf{v}), \mathbf{w}_n - \mathbf{w}) \to 0 \quad \text{as } n \times \infty. \end{aligned}$$

Since $\mathrm{Re}\varepsilon \geq 1$, we have $\| \mathbf{v}_n - \mathbf{v} \|_{(L^2(\Omega))^3} \to 0$ as $n \to \infty$.

□

Lemma 4.3.3. *Let $D \subset \mathbb{R}^3$, $u_1, \ldots, u_J \in H^2(D)$ be real functions and there exists a constant $C > 0$ such that*

$$|\Delta u_i| \leq C \sum_{j=1}^{J} (|u_j| + |\nabla u_j|) \quad in`D,\ j = 1, \ldots, J.$$

If $u_1, \ldots, u_J$ is zero in a neighborhood of $x_0 \in D$, then

$$u_j = 0 \quad in\ D,\ 1 \leq j \leq J.$$

Theorem 4.3.4. *If $\mathrm{Im}\varepsilon > 0$ in some Ω_i, then the variational problem (4.9) has at most one solution.*

Proof. It suffices to prove that $\mathbf{E} = 0$ if $\mathbf{F} = 0$. Without the loss of generality, we assume that

- $\mathrm{Im}\varepsilon > 0$ in Ω_1 and $\mathrm{Im}\varepsilon = 0$ in Ω_2;
- $\bar{\Omega}_1 \cap \bar{\Omega}_2$ is a Lipschitz surface with positive measurement.

Taking $\mathbf{v} = \mathbf{E}$ in (4.9) yields

$$\| \mu^{-1}\nabla \times \mathbf{E} \|^2_{(L^2(\Omega))^3} - \kappa^2(\varepsilon\mathbf{E}, \mathbf{E}) = 0.$$

Noticing $\mathrm{Im}\varepsilon > 0$ in Ω_1. Taking the imaginary part of above equation gives $\mathbf{E} = 0$ in Ω_1.

Since $\varepsilon|_{\Omega_2}$ is a constant, we define

$$\tilde{\varepsilon} = \varepsilon_{\Omega_2}, \quad \tilde{\mu} = \mu|_{\Omega_2} \quad \text{in } \bar{\Omega}_1 \cup \bar{\Omega}_2.$$

It follows from $\mathbf{E} = 0$ in Ω_1 that $\mathbf{E}$ satisfies

$$\nabla \times (\tilde{\mu}\nabla \times \mathbf{E}) - \kappa^2\tilde{\mathbf{v}}\mathbf{E} = 0 \quad \text{in } \bar{\Omega}_1 \cup \bar{\Omega}_2.$$

Taking the divergence on both sides gives $\nabla \cdot \mathbf{E} = 0$. Therefore

$$-\Delta\mathbf{E} = \nabla \times (\nabla \times \mathbf{E}) - \nabla(\nabla \cdot \mathbf{E}) = \nabla \times (\nabla \times \mathbf{E}) = \kappa^2\tilde{\mu}\tilde{\varepsilon}\mathbf{E} \quad \text{in } \bar{\Omega}_1 \cup \bar{\Omega}_2.$$

Using Lemma 4.3.3 and $\mathbf{E} = 0$ in Ω_1, we have $\mathbf{E} = 0$ in $\Omega_1 \cup \Omega_2$. The proof is completed by repeating the process for Ω_i.

□

By Lemma 4.3.1, we have

$$\mathbf{E} = \mathbf{E}_\perp + \nabla\psi, \quad \mathbf{E}_\perp \in H_\perp(\mathrm{curl}, \Omega), \ \psi \in H_0^1(\Omega).$$

Here ψ is the unique solution of the following problem:

$$(\varepsilon\nabla\psi, \nabla v) = (\varepsilon\mathbf{E}, \nabla v) = -\kappa^{-2}(\mathbf{F}, \nabla v) \quad \text{for all } \mathbf{v} \in H_0^1(\Omega).$$

Since $\mathrm{Re}\varepsilon \geq 1$, taking $v = \psi$ gives

$$|\psi|_{1,\Omega} \leq \kappa^{-2} \| \mathbf{F} \|_{0,\Omega} .$$

Thus we have an equivalent formulation to (4.9): Find $\mathbf{E}_\perp \in H_\perp(\mathrm{curl}, \Omega)$ such that

$$a(\mathbf{E}_\perp, \mathbf{v}) = (\mathbf{F}, \mathbf{v}) + \kappa^2(\varepsilon\nabla\psi, \mathbf{v}) \quad \text{for all } \mathbf{v} \in H_\perp(\mathrm{curl}, \Omega). \tag{4.10}$$

Define a bilinear form: $a_\perp : H_\perp(\mathrm{curl}, \Omega) \times H_\perp(\mathrm{curl}, \Omega) \to \mathcal{C}$ as follows

$$a_\perp(\mathbf{u}, \mathbf{v}) = (\mu^{-1}\nabla \times \mathbf{u}, \nabla \times \mathbf{v}) + \kappa^2(\varepsilon\mathbf{u}, \mathbf{v}) \quad \text{for all } \mathbf{u}, \mathbf{v} \in H_\perp(\mathrm{curl}, \Omega). \tag{4.11}$$

Let $\mathbf{w}$ be the unique weak solution of the problem

$$a_\perp(\mathbf{w}, \mathbf{v}) = (\mathbf{F}, \mathbf{v}) + \kappa^2(\varepsilon\nabla\psi, \mathbf{v}) \quad \text{for all } \mathbf{v} \in H_\perp(\mathrm{curl}, \Omega). \tag{4.12}$$

Define an operator $\mathcal{K} : (L^2(\Omega))^3 \to H_\perp(\mathrm{curl}, \Omega)$ as follows: For any $\mathbf{f} \in (L^2(\Omega))^3$, $\mathcal{K}\mathbf{f}$ satisfies

$$\mathcal{K}\mathbf{f} \in H_\perp(\mathrm{curl}, \Omega), \quad a_\perp(\mathcal{K}\mathbf{f}, \mathbf{v}) = 2\kappa^2(\varepsilon\mathbf{f}, \mathbf{v}) \quad \text{for all } \mathbf{v} \in H_\perp(\mathrm{curl}, \Omega).$$

Thus (4.10) can be written as

$$a_\perp(\mathbf{E}_\perp, \mathbf{v}) - a_\perp(\mathcal{K}\mathbf{E}_\perp, \mathbf{v}) = a_\perp(\mathbf{w}, \mathbf{v}) \quad \text{for all } \mathbf{v} \in H_\perp(\mathrm{curl}, \Omega),$$

which is equivalent to the operator equation

$$(\mathcal{I} - \mathcal{K})\mathbf{E}_\perp = \mathbf{w} \quad \text{in } H_\perp(\mathrm{curl}, \Omega). \tag{4.13}$$

Theorem 4.3.5. *Suppose* $\mathrm{Im}\varepsilon > 0$ *in some* Ω_0*, then* $(\mathcal{I} - \mathcal{K})^{-1}$ *is a bounded linear operator. Given* $\mathbf{F} \in (L^2(\Omega))^3$*, the variational problem* (4.9) *has a unique weak solution* $\mathbf{E} \in H_0(\mathrm{curl}, \Omega)$ *which satisfies*

$$\| \mathbf{E} \|_{H(\mathrm{curl},\Omega)} \leq C \| \mathbf{F} \|_{0,\Omega} .$$

Proof. The existence and uniqueness for the solution $\psi \in H_0^1(\Omega)$ is obvious from the above discussion. The uniqueness of the weak solution $\mathbf{E}$ is proved. We only need to prove the existence of the weak solution $\mathbf{E}_\perp$.

Clearly $a_\perp$ is coercive in $H_0(\mathrm{curl}, \Omega)$. It follows from the Lax-Milgram lemma that there exist unique $\mathbf{w}, \mathcal{K}\mathbf{f} \in H_\perp(\mathrm{curl}, \Omega)$, and satisfy

$$\| \mathcal{K}\mathbf{f} \|_{H(\mathrm{curl},\Omega)} \leq C \| \mathbf{f} \|_{0,\Omega},$$
$$\| \mathbf{w} \|_{H(\mathrm{curl},\Omega)} \leq C\,(\| \mathbf{F} \|_{0,\Omega} + |\psi|_{0,\Omega}) \leq C \| \mathbf{F} \|_{0,\Omega} .$$

The operator $\mathcal{K} : (L^2(\Omega))^3 \to H_\perp(\mathrm{curl}, \Omega)$ is a bounded operator. By Lemma 4.3.2, $H_\perp(\mathrm{curl}, \Omega)$ is compactly imbedded into $(L^2(\Omega))^3$. So $\mathcal{K} : H_\perp(\mathrm{curl}, \Omega) \to H_\perp(\mathrm{curl}, \Omega)$ is a compact operator. It follows from the Fredholm alternative that $\mathcal{I} - \mathcal{K}$ is invertible and we have

$$\| \mathbf{E}_\perp \|_{H(\mathrm{curl},\Omega)} \leq \| (\mathcal{I} - \mathcal{K})^{-1}\mathbf{w} \|_{H(\mathrm{curl},\Omega)} \leq C \| \mathbf{w} \|_{H(\mathrm{curl},\Omega)} \leq C \| \mathbf{F} \|_{0,\Omega} .$$

Therefore we obtain the existence of the weak solution and the estimate

$$\| \mathbf{E} \|_{H(\mathrm{curl},\Omega)} \leq \| \mathbf{E}_\perp \|_{H(\mathrm{curl},\Omega)} + |\psi|_{1,\Omega} \leq C \| \mathbf{F} \|_{0,\Omega} + |\psi|_{1,\Omega} \leq C \| \mathbf{F} \|_{0,\Omega} .$$

$\square$