3.3 Variational form of boundary value problems

Let X be a separable Hilbert space with an inner product $(,)$ and norm $\| \cdot \|$. We identify X with its dual X'. Let V be a linear subspace of X which is dense in X. Usually, V is not complete under $\| \cdot \|$. Assume that a new inner product $\langle , \rangle$ and norm $| \cdot |$ can be introduced so that V is a Hilbert space in terms of the new inner product $\langle , \rangle$.

Assume also that the identity operator $I : V \to X$ is continuous, i.e., there is a constant $\alpha > 0$ such that

$$\| u \| \leq \alpha |u| \quad \text{for all } u \in V. \tag{3.2}$$

In other words, the norm $| \cdot |$ in V is stronger than $\| \cdot \|$ in X.

It is easy to see that

$$V \subset X = X' \subset V'.$$

In fact, for any $f \in X'$, we have

$$\| x_n - x \| \to 0 \quad \text{implies} \quad f(x_n) \to f(x) \quad \text{as } n \to \infty, \tag{3.3}$$

where $x_n, x \in X$. If $x_n, x \in V$, $|x_n - x| \to 0$ then because of (3.2), we have $\| x_n - x \| \to 0$, hence $f(x_n) \to f(x)$ by (3.3). Thus f is a linear functional with respect to the norm in V or $f \in V'$.

Example 1. Let $X = L^2(\Omega), V = H_0^1(\Omega)$, Ω is a bounded domain, V is a linear dense set of X, and the embedding $I : V \to X$ is continuous. For all $f \in X'$, we can verify directly $f \in V'$. In fact, for any $u \in V$,

$$f(u) = \int_\Omega f u dx,$$

where $f \in X, u \in V \subset X$, thus

$$|f(u)| \leq \| f \|_0 \| u \|_0 \leq \| f \|_0 \| u \|_1 .$$

From the Riesz representation theorem, for any $f \in V'$,

$$f(u) = \int_\Omega (f u + f_{x_1} u_{x_1} + f_{x_2} u_{x_2} dx, \tag{3.4}$$

where $f, u \in H_0^1(\Omega)$.

Since $C_0^\infty(\Omega)$ is dense in $H_0^1(\Omega)$, it suffices to for the discussion purpose to assume that $f \in C_0^\infty(\Omega)$, hence

$$f(u) = \int_\Omega (f - f_{x_1 x_1} - f_{x_2 x_2}) u dx. \tag{3.5}$$

Therefore, there are different representations of a linear functional f in terms of different inner products. Using the $H^1(\Omega)$, the representation has the form (3.5), and corresponding to an element $(I - \partial_{x_1}^2 - \partial_{x_2}^2)f$. The $L^2(\Omega)$ inner product leads the corresponding element f. In the study of PDE, the representation (3.5) turns out to be more useful.

Using this correspondence, we denote that V' the dual space of $H_0^1(\Omega) = H^{-1}(\Omega)$.

In general, one may denote $H^{-m}(\Omega)$ as the dual space of $H_0^m(\Omega)$. It is easy to verify hat $C_0^\infty(\Omega)$ is a linear subspace that is dense in $H^{-m}(\Omega)$.

For $f \in C_0^\infty(\Omega) \subset H^{-m}(\Omega)$, define

$$\| f \|_{-m} = \sup_{v \in H_0^m(\Omega)} \frac{(f, v)}{\| v \|_m}, \tag{3.6}$$

thus $H^{-m}(\Omega)$ is the completion of $C_0^\infty(\Omega)$ with respect to the norm (3.6).

Let A be a linear operator. It domain $D(A)$ is a dense linear subset of V, $R(A) \subset V'$. For $f \in V'$, consider the operator equation

$$Au = f. \tag{3.7}$$

In general, Equation (3.7) needs not have a solution in $D(A)$ for any given right hand side. But possible generalized solutions may exist.

Construct the bilinear form:

$$a(u, v) = \langle Au, v \rangle, \quad u, v \in D(A). \tag{3.8}$$

If $a(u, v)$ is bounded in $D(A) \times D(A)$, then there is a constant M such that

$$|a(u, v)| \le M|u|\,|v|, \quad u, v \in D(A). \tag{3.9}$$

Since $D(A)$ is dense in V, $a(u, v)$ can be extended continuously to $V \times V$ so that (3.8) holds for all $u, v \in V$.

Definition 3.3.1. *$u \in V$ is a generalized solution of* (3.7)*, if for any $v \in V$, u satisfies the following variational equation:*

$$a(u, v) = \langle f, v \rangle. \tag{3.10}$$

Since V is dense in X, a generalized solution u becomes a solution of (3.7) if $u \in D(A)$. Such a function is called the classic solution.

Given a fixed $u \in V$, let

$$g_u(v) = a(u, v), \quad v \in V.$$

From (3.9), $g_u(v)$ is a bounded linear functional on V. By the Riesz representation theorem, there is a $Ju \in V$ such that

$$a(u,v) = \langle Ju, v\rangle, \quad v \in V. \tag{3.11}$$

Clearly, $J : V \to V$ is a linear operator and

$$|\langle Ju, v\rangle| = |a(u,v)| \le M|u|\,|v|. \tag{3.12}$$

Choosing particularly $v = Ju$ in (3.12), we verify that J is bounded, i.e.,

$$|Ju| \le M|u|.$$

For any $f \in V'$, consider the linear functional on V

$$h(v) = \langle f, v\rangle.$$

Again by using the Riesz representation theorem, there is a $Kf \in V$ such that

$$\langle f, v\rangle = \langle Kf, v\rangle. \tag{3.13}$$

where $K : V' \to V$ is a bounded linear operator. From (3.11) and (3.13), the variational equation (3.10) has an equivalent form

$$Ju = Kf. \tag{3.14}$$

The following results are concerned with the well-posedness of (3.14).

Theorem 3.3.2 (Babuška)**.** *Let*

$$\inf_{u\in V,|u|=1} \sup_{v\in V,|v|=1} |a(u,v)| \ge \gamma > 0, \tag{3.15}$$

and

$$\sup_{u\in V} |a(u,v)| > 0, \quad v \ne 0, v \in V. \tag{3.16}$$

Then for any $f \in V'$, *Equation* (3.10) *or* (3.14) *has a unique solution* $u \in V$ *and*

$$|u| \le \frac{1}{\gamma}|Kf|. \tag{3.17}$$

Proof. By (3.11), the condition (3.15) may be rewritten as

$$\inf_{u\in V,|u|=1} \sup_{v\in V,|v|=1} |\langle Ju, v\rangle| = \inf_{u\in V,|u|=1} |Ju| \ge \gamma > 0,$$

i.e., the operator J has a bounded inverse J^{-1} and

$$|J^{-1}| \leq \gamma^{-1}, \tag{3.18}$$

the domain of J is a closed linear subspace of V. Now if $R(J)$ is not V then there is $v_0 \in V, v_0 \neq 0$ such that

$$\langle Ju, v_0 \rangle = a(u, v_0) = 0 \quad \text{for all } u \in V,$$

which contradicts (3.16).

Therefore Equation (3.10) or (3.14) has a unique solution $u = J^{-1}Kf$. The estimate (3.17) follows from (3.18). □

In practice, Condition (3.15) is difficult to check. We next state an important result in which the hypothesis is stronger but relatively easier to check.

Theorem 3.3.3 (Lax–Milgram)**.** *Suppose that $a(u, v)$ satisfies the following coercivity condition, i.e., there is a constant $\gamma > 0$ such that*

$$|a(u, u)| \geq \gamma |u|^2 \quad \text{for all } u \in V. \tag{3.19}$$

Then for any $f \in V'$, Equation (3.10) *or* (3.14) *has a unique solution u and the estimate* (3.17) *holds.*

The proof is a simple consequence of the previous theorem.

Theorem 3.3.4. *Assume that $A = A_1 + A_2$, where $D(A_1) = D(A), D(A_2) \supset D(A), R(A), R(A_1), R(A_2) \subset V'$. Assume also that the bilinear form $a_1(u, v) = \langle A_1 u, v \rangle, u, v \in D(A_1)$ satisfies*

1. $|a_1(u, v)| \leq M_1 |u|\,|v|, \quad u, v \in D(A_1)$
2. $|a_1(u, u)| \geq \gamma |u|^2, \quad u \in V,$

where γ is a constant. Moreover, $T = A_1^{-1} A_2$ is a compact operator from V to V and -1 is not an eigenvalue of T. Then for any $f \in V'$, Equation (3.10) *or* (3.14) *has a unique solution u, and*

$$|u| \leq \gamma^{-1} |(I + T)^{-1}|\,|Kf|.$$

Proof. It follows from the previous proof that there is a bounded operator $J_1 : V \to V$ whose inverse J_1^{-1} is also a bounded operator, such that

$$a_1(u, v) = \langle J_1 u, v \rangle.$$

Hence

$$\begin{gathered} a(u,v) = \langle (A_1 + A_2)u, v\rangle = \langle A_1(I+T)u, v\rangle \\ a_1((I+T)u, v) = \langle J_1(I+T)u, v\rangle. \end{gathered}$$

Thus the variational equation (3.10) may be rewritten as

$$\langle J_1(I+T)u, v\rangle = \langle Kf, v\rangle$$

or

$$J_1(I+T)u = Kf.$$

Therefore a unique solution exists

$$u = (I+T)^{-1} J_1^{-1} K f$$

and

$$|u| \le |(I+T)^{-1}|\,|J_1^{-1}|\,|Kf|.$$

□