5.3 Multiple cavity scattering

Now we generalize the model problem and techniques for the two cavity scattering to the case of multiple cavity scattering. The proofs are analogous to those for the two cavity scattering problem. For completeness, we shall briefly discuss the model problem and the solution for the multiple scattering problem.

5.3.1 A model problem

As shown in Figure 5.3, we consider a situation with \( n \) cavities, where the multiple open cavity \( \Omega_1, \ldots, \Omega_n \), enclosed by the aperture \( \Gamma_1, \ldots, \Gamma_n \) and the walls \( S_1, \ldots, S_n \), are placed on a perfectly conducting ground plane \( \Gamma_c \). Above the flat surface \( \{ y = 0 \} = \Gamma_1 \cup \cdots \cup \Gamma_n \cup \Gamma_c \), the medium is assumed to be homogeneous with a positive dielectric permittivity \( \varepsilon_0 \). The medium inside the cavity \( \Omega_j \) is inhomogeneous with a variable dielectric permittivity \( \varepsilon_j(x, y) \). Assume further that \( \varepsilon_j(x, y) \in L^\infty(\Omega) \), Re\( \varepsilon_j > 0 \), Im\( \varepsilon_j \geq 0 \) for \( j = 1, \ldots, n \).

We consider the same model of the two-dimensional Helmholtz equation for the total field:

\[
(5.45) \quad \Delta u + \kappa^2 u = 0, \quad \text{in } \Omega_1 \cup \cdots \cup \Omega_n \cup \mathbb{R}^2_+,
\]

together with the perfectly electric conductor condition

\[
(5.46) \quad u = 0, \quad \text{on } \Gamma_c \cup \Gamma_1 \cup \cdots \cup \Gamma_n.
\]

The total field \( u \) is assumed to consist of the incident field \( u^i \), the reflected field \( u^r \), and the scattered field \( u^s \), where the scattered field is required to satisfy the radiation condition (5.6).

To derive the transparent boundary condition on the aperture \( \Gamma_j, j = 1, \ldots, n \), we reformulate the multiple cavity scattering problem (5.45)–(5.46) into \( n \) single
cavity scattering problems which are coupled through the boundary condition.

\begin{align*}
\Delta u_j + \kappa_j^2 u_j &= 0 \quad \text{in } \Omega_j, \\
u_j &= 0 \quad \text{on } \Gamma_j,
\end{align*}

where \( \kappa_j^2 = \omega^2 \varepsilon_j \mu_0, j = 1, \ldots, n. \)

For \( u_j(x, 0), j = 1, \ldots, n, \) define its zero extension to the whole \( x \)-axis,

\[
\tilde{u}_j(x, 0) = \begin{cases} 
    u_j(x, 0) & \text{for } x \in \Gamma_j, \\
0 & \text{for } x \in \mathbb{R} \setminus \Gamma_j.
\end{cases}
\]

For the total field \( u(x, 0), \) define its extension to the whole \( x \)-axis by

\[
\tilde{u}(x, 0) = \begin{cases} 
    u_j(x, 0) & \text{for } x \in \Gamma_j, \\
0 & \text{for } x \in \Gamma^c.
\end{cases}
\]

It follows from the definition of the extension that we have

\[
\tilde{u} = \sum_{j=1}^n \tilde{u}_j \quad \text{on } \Gamma^c \cup \Gamma_1 \cup \cdots \cup \Gamma_n.
\]

The transparent boundary condition can be written as

\begin{equation}
\partial_n \tilde{u} = T \tilde{u} + g \quad \text{on } \Gamma^c \cup \Gamma_1 \cup \cdots \cup \Gamma_n,
\end{equation}

which leads to the transparent boundary condition for \( u_j: \)

\begin{equation}
\partial_n u_j = T \tilde{u}_j + \sum_{\substack{i=1 \\text{to } n \\text{for } i \neq j}}^n T \tilde{u}_i + g \quad \text{on } \Gamma_j.
\end{equation}

**Lemma 5.3.1.** Let \( u_j \in H^{1/2}(\mathbb{R}), j = 1, \ldots, n. \) It holds that

\[
\text{Re} \sum_{i=1}^n \sum_{j=1}^n \langle Tu_j, u_i \rangle \leq 0
\]

and

\[
\text{Im} \sum_{i=1}^n \sum_{j=1}^n \langle Tu_j, u_i \rangle \geq 0.
\]

Furthermore, if \( \tilde{u}_j, j = 1, \ldots, n \) are analytical functions with respect to \( \xi, \) either

\[
\text{Re} \sum_{i=1}^n \sum_{j=1}^n \langle Tu_j, u_i \rangle = 0
\]
or
\[
\text{Im} \sum_{i=1}^{n} \sum_{j=1}^{n} \langle Tu_j, u_i \rangle = 0
\]
implies
\[
\sum_{j=1}^{n} u_j = 0.
\]

Proof. By definitions (5.7) and (5.11), we have
\[
\sum_{i=1}^{n} \sum_{j=1}^{n} \langle Tu_j, u_i \rangle = i \int \beta(\xi) |\sum_{j=1}^{n} \hat{u}_j|^2 d\xi.
\]
Taking the real part gives
\[
\text{Re} \sum_{i=1}^{n} \sum_{j=1}^{n} \langle Tu_j, u_i \rangle = -\int_{|\xi|>\kappa_0} (\xi^2 - \kappa_0^2)^{1/2} |\sum_{j=1}^{n} \hat{u}_j|^2 d\xi \leq 0,
\]
and taking the imaginary part yields
\[
\text{Im} \sum_{i=1}^{n} \sum_{j=1}^{n} \langle Tu_j, u_i \rangle = \int_{|\xi|<\kappa_0} (\kappa_0^2 - \xi^2)^{1/2} |\sum_{j=1}^{n} \hat{u}_j|^2 d\xi \geq 0.
\]
Furthermore,
\[
(5.51) \quad \text{Re} \sum_{i=1}^{n} \sum_{j=1}^{n} \langle Tu_j, u_i \rangle = 0
\]
implies
\[
\sum_{j=1}^{n} \hat{u}_j = 0 \quad \text{for } |\xi| > \kappa_0,
\]
and
\[
(5.52) \quad \text{Im} \sum_{i=1}^{n} \sum_{j=1}^{n} \langle Tu_j, u_i \rangle = 0
\]
implies
\[
\sum_{j=1}^{n} \hat{u}_j = 0 \quad \text{for } |\xi| < \kappa_0.
\]
If \( \hat{u}_j, j = 1, \ldots, n \), are assumed to be analytical functions with respect to \( \xi \), then either (5.51) or (5.52) implies that

\[
(5.53) \quad \sum_{j=1}^{n} \hat{u}_j = 0 \quad \text{for all } \xi \in \mathbb{R}.
\]

The proof is completed by taking the inverse Fourier transform of (5.53).

5.3.2 Well-posedness

We now present a variational formulation for the multiple cavity scattering problem and sketch the proof for the well-posedness of the boundary value problem.

Denote \( \Omega = \Omega_1 \cup \cdots \cup \Omega_n \), \( \Gamma = \Gamma_1 \cup \cdots \cup \Gamma_n \), and \( S = S_1 \cup \cdots \cup S_n \). For simplicity, we shall use the same notation as those adopted in Section 3 for the two cavity scattering problem. The following notation for the general multiple cavity scattering problem are actually consistent with those for the two cavity scattering problem when letting \( n = 2 \).

Define a trace functional space

\[
\tilde{H}^{1/2}(\Gamma) = \tilde{H}^{1/2}(\Gamma_1) \times \cdots \times \tilde{H}^{1/2}(\Gamma_n).
\]

Its norm is characterized by

\[
\| u \|_{\tilde{H}^{1/2}(\Gamma)}^2 = \sum_{j=1}^{n} \| u_j \|_{\tilde{H}^{1/2}(\Gamma_j)}^2.
\]

Denote \( H^{-1/2}(\Gamma) = H^{-1/2}(\Gamma_1) \times \cdots \times H^{-1/2}(\Gamma_n) \), which is the dual space of \( \tilde{H}^{1/2}(\Gamma) \). The norm on the space \( H^{-1/2}(\Gamma) \) is characterized by

\[
\| u \|_{H^{-1/2}(\Gamma)}^2 = \sum_{j=1}^{n} \| u_j \|_{H^{-1/2}(\Gamma_j)}^2.
\]

Introduce the following space

\[
\mathbb{H}^1_S(\Omega) = H^1_{S_1}(\Omega_1) \times \cdots \times H^1_{S_n}(\Omega_n),
\]

which is a Hilbert space with norm characterized by

\[
\| u \|_{\mathbb{H}^1(\Omega)}^2 = \sum_{j=1}^{n} \| u_j \|_{H^1(\Omega_j)}^2.
\]
5.3. **MULTIPLE CAVITY SCATTERING**

Multiplying the complex conjugate of test function $v_j \in H^1_{S_j}(\Omega_j)$ on both sides of (5.47), integrating over $\Omega_j$, and using the integration by parts and boundary conditions (5.48) and (5.50), we obtain

$$\int_{\Omega_j} (\nabla u_j \cdot \nabla \bar{v}_j - \kappa_j^2 u_j \bar{v}_j) - \sum_{i=1}^{n} \langle T \tilde{u}_i, \bar{v}_j \rangle = \langle g, v_j \rangle_{\Gamma_j}. \quad (5.54)$$

Taking summation of (5.54) for $j = 1, \ldots, n$, we deduce the variational formulation for the multiple cavity scattering problem: find $u \in H^1_\Sigma(\Omega)$ with $u_j = u|_{\Omega_j}$ such that

$$a_3(u, v) = \sum_{j=1}^{n} \langle g, v_j \rangle_{\Gamma_j} \quad \text{for all } v \in H^1_\Sigma(\Omega), \quad (5.55)$$

where the sesquilinear form

$$a_3(u, v) = \sum_{j=1}^{n} \int_{\Omega_j} (\nabla u_j \cdot \nabla \bar{v}_j - \kappa_j^2 u_j \bar{v}_j) - \sum_{j=1}^{n} \sum_{i=1}^{n} \langle T \tilde{u}_i, \bar{v}_j \rangle. \quad (5.56)$$

**Theorem 5.3.2.** The variational problem (5.55) has at most one solution.

**Proof.** It suffices to show that $u_j = 0$ in $\Omega_j$ for $j = 1, \ldots, n$ if $g = 0$. If $u_j$ satisfy the homogeneous variational problem in $\Omega_j$, then we have

$$\sum_{j=1}^{n} \int_{\Omega_j} (|\nabla u_j|^2 - \kappa_j^2 |u_j|^2) - \sum_{j=1}^{n} \sum_{i=1}^{n} \langle T \tilde{u}_i, \bar{u}_j \rangle = 0.$$

Noting $\text{Im} \varepsilon_j \geq 0$ and taking the imaginary part yields

$$\text{Im} \sum_{i=j}^{n} \sum_{i=1}^{n} \langle T \tilde{u}_i, \bar{u}_j \rangle = 0.$$

Since $\tilde{u}_j$ has a compact support on the $x$-axis, $\hat{\tilde{u}}_j$ is analytical with respect to $\xi$. Hence we have from Lemma 5.3.1 that

$$\tilde{u} = \sum_{j=1}^{n} \tilde{u}_j = 0.$$

By the definition of the extensions $\tilde{u}_j$ we obtain

$$\tilde{u}_j = 0 \quad \text{on } \Gamma^c \cup \Gamma_1 \cup \cdots \cup \Gamma_n.$$

The transparent boundary condition (5.49) yields that $\partial_n \tilde{u} = 0$ on $\Gamma^c \cup \Gamma_1 \cup \cdots \cup \Gamma_n$. An application of Holmgren uniqueness theorem yields $u = 0$ in $\mathbb{R}^2_+$. A unique continuation result in [?] concludes that $u_j = 0$ in $\Omega_j$ for $j = 1, \ldots, n$. \qed
We have the following well-posedness result for the general multiple cavity scattering problem. The proof is similar in nature as that of the two cavity model problem and is omitted here for brevity.

**Theorem 5.3.3.** The variational problem (5.55) has a unique weak solution $u$ in $H^1_S(\Omega)$ and the solution satisfies the estimate

$$\| u \|_{H^1(\Omega)} \leq C \| g \|_{H^{-1/2}(\Gamma)},$$

where $C$ is a positive constant.