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ELECTROMAGNETIC SCATTERING BY UNBOUNDED ROUGH SURFACES*

PEIJUN LI†, HAIJUN WU‡, AND WEIYING ZHENG§

Abstract. This paper is concerned with the analysis of electromagnetic wave scattering in inhomogeneous medium with infinite rough surfaces. Consider a time-harmonic electromagnetic field generated by either a magnetic dipole or an electric dipole incident on an infinite rough surface. The dielectric permittivity is assumed to have a positive imaginary part which accounts for the energy absorption. The scattering problem is modeled as a boundary value problem governed by the Maxwell equations, with transparent boundary conditions proposed on plane surfaces with inhomogeneity in between. The existence and uniqueness of the weak solution for the model problem are established by using a variational approach. The perfectly matched layer (PML) method is investigated to truncate the unbounded rough surface electromagnetic scattering problem in the direction away from the rough surfaces. It is shown that the truncated PML problem attains a unique solution. An explicit error estimate is given between the solution of the scattering problem and that of the truncated PML problem. The error estimate implies that the PML solution converges exponentially to the scattering solution by increasing either the PML medium parameter or the PML layer thickness. The convergence result is expected to be useful for determining the PML medium parameter in the computational electromagnetic scattering problem.

Key words. rough surface scattering, variational formulation, Maxwell's equations, transparent boundary condition, perfectly matched layer

AMS subject classifications. 35J20, 78A45, 78M30

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1. Introduction.

This paper is concerned with the mathematical analysis of an infinite rough surface electromagnetic wave scattering problem for Maxwell's equations. An infinite rough surface is referred to as a surface which is a nonlocal perturbation of an infinite plane surface such that the whole surface lies within a finite distance of the original plane.

The importance of rough surface scattering problems is clear, since they are related to technology with significant industrial and military applications, e.g., modeling acoustic and electromagnetic wave propagation over outdoor ground and sea surfaces or, at a very different scale, optical scattering from the surface of materials in near-field optics or nano-optics and detection of underwater mines, especially those buried in soft sediments. These problems are widely studied in the engineering literature, and a considerable amount of information is available concerning their solutions via both rigorous methods of computation and approximate, asymptotic, or statistical methods; see, e.g., the reviews and monographs by Ogilvy [42], Voronovich [49], Saillard

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†Department of Mathematics, Purdue University, West Lafayette, IN 47907 (lipeijun@math.purdue.edu). This author's research was supported in part by the NSF grants EAR-0724656, DMS-0914595, and DMS-1042958.

‡Department of Mathematics, Nanjing University, Jiangsu, 210093 China (hjw@nju.edu.cn). This author's research was supported by the national basic research program under grant 2005CB321701 and by the China NSF grants 11071116 and 10971096.

§Institute of Computational Mathematics, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing, 100190 China (zwy@lsec.cc.ac.cn). This author's research was supported in part by China NSF grant 60873177 and by the National High Technology Research and Development Program of China under grant 2009AA01A134.

and Sentenac [45], Warnick and Chew [50], DeSanto [28], Elfouhaily and Guerin [34], and references cited therein.

In this paper, we study the electromagnetic wave scattering problem in an inhomogeneous medium with infinite rough surfaces. Specifically, we consider a time-harmonic electromagnetic field generated by either a magnetic dipole or an electric dipole incident on an infinite rough surface. The scattering problem is modeled as a boundary value problem for electromagnetic wave propagation governed by the three-dimensional Maxwell equations, with transparent boundary conditions proposed on plane surfaces with the inhomogeneity in between. As a part of the boundary value problem the radiation condition is required. Due to the infinite rough surfaces, the usual Silver–Müler radiation condition is no longer valid. Instead, the following radiation is employed: above (below) the rough surface of the inhomogeneous medium, the solution can be represented in integral form as a superposition of upward (downward) traveling and evanescent plane waves. This radiation condition is equivalent to the upward propagating radiation condition proposed for a two-dimensional rough surface scattering problem by Zhang and Chandler-Wilde [52], and has recently been analyzed carefully by Arens and Hohage [6]. The existence and uniqueness of the weak solution for the model problem are established by using a variational approach. Our method enjoys a great generality in the sense that it allows very general surface structures as well as complex materials. In particular, the analysis requires neither the smoothness nor being the graphs of functions for the rough surfaces, and the material coefficients, i.e., the dielectric permittivity and the magnetic permeability, can be general spatially varying bounded measurable functions. Throughout the paper we restrict our analysis to the case of a lossy medium, where the dielectric permittivity is assumed to have a positive imaginary part accounting for the energy absorption. We refer to Ritterbusch [43] for a related electromagnetic scattering problem in the lossless case where weighted Sobolev spaces are studied for unbounded domains.

This paper is closest to the recent study by Chandler-Wilde and Monk [17, 18], Chandler-Wilde, Monk, and Thomas [19], and Lechleiter and Ritterbusch [37], who consider the variational approach to solving a two- or three-dimensional rough surface scattering problem governed by the Helmholtz equation which models time-harmonic acoustic wave scattering by a layer of homogeneous or inhomogeneous medium above a sound soft surface. The present paper is devoted to the analysis of the scattering problem for the vector form of Maxwell's equations with dielectric surfaces, which models the time-harmonic electromagnetic wave by three layers of inhomogeneous medium with two infinite rough surfaces. It is evident that the present model problem is considerably more difficult than the scalar Helmholtz equation. The two-dimensional scalar model problem has been considered by integral equation methods in two cases. The first case assumes that the medium is homogeneous and the surface is the graph of a sufficiently smooth bounded function, when the boundary integral equation methods are applicable; see, e.g., Chandler-Wilde and coworkers [16, 20], Zhang and Chandler-Wilde [51, 53], and DeSanto and Martin [29, 30, 31]. The second case studied assumes that the surface is a straight line; see, e.g., Chandler-Wilde and Zhang [21] and Li [38].

The PML technique, which was first proposed by Berenger [12, 13], is an important and popular mesh termination technique in computational wave propagation due to its effectiveness, simplicity, and flexibility; see, e.g., Chen and Liu [23], Chen and Wu [24], Collino and Monk [25], Lassas and Somersalo [36], Teixeira and Chew [47], and Turkel and Yefet [48]. The idea is to surround the computational domain by a nonphysical PML medium which has the remarkable property of being reflectionless for incident waves of any frequency or any incident direction, and waves decay exponentially in magnitude into the PML medium. In practice, the PML medium must be truncated, and the truncation boundary generates reflected waves which can pollute the solution in the computational domain. We refer to Bao and Wu [10], Bramble and Pasciak [14, 15], and Chen and Chen [22] for convergence analysis of the PML problems for three-dimensional Maxwell's equations. We shall use a PML to truncate in the direction vertically away from the rough surfaces. A practical calculation also requires truncation on vertical side boundaries, which we do not consider here. Under a proper assumption on the PML medium parameter, we prove that the truncated PML problem attains a unique solution and obtain an error estimate between the solution of the scattering problem and the solution of the truncated PML problem in the computational domain. The error estimate implies particularly that the PML solution converges exponentially to the scattering problem when either the PML medium parameter or the thickness of the layer is increased.

Related work for the scattering of electromagnetic waves in a grating (periodic surface) structure (diffractive optics) and in a cavity (with a local perturbation of a plane surface) have been studied extensively by either integral equation methods or variational approaches by Ammari and Bao [2], Bao [7], Bao and Dobson [9], Bao, Li, and Wu [11], Dobson and Friedman [32], Nedelec and Starling [41], Ammari, Bao, and Wood [5], and references therein. A recent review on diffractive optics technology and its mathematical modeling can be found in Bao, Cowsar, and Masters [8]. The diffraction grating problem or the cavity problem is simpler since the variational formulation is on either a single periodic cell or a bounded domain, a compact set, as a consequence of which the sesquilinear form satisfies a Gårding inequality, so that the associated linear operator is Fredholm of index zero, and well-posedness follows from uniqueness. More recently, existence of the solution to the acoustic and electromagnetic scattering problem in an infinite periodic surface perturbed by a single inhomogeneous object placed inside the periodic structure is established via the integral equation method by Ammari and Bao [3, 4]. We refer to Sun and Zheng [46], Ehrhardt, Sun, and Zheng [33], and Joly, Li, and Fliss [35] for some numerical results for scattering problems with local perturbation of periodic structure. One may consult Colton and Kress [26, 27], Nédélec [40], and Monk [39] for extensive accounts of integral equation methods and finite element methods for acoustic and electromagnetic scattering problems.

The outline of this paper is as follows. In section 2, the model problem is introduced, and some regularity properties of the trace operator are discussed. Section 3 is devoted to the derivation of the transparent boundary condition. Some estimates of the capacity operator are introduced. In section 4, a variational formulation for the infinite rough surface electromagnetic scattering problem is introduced by using the transparent boundary condition. The existence and uniqueness of the weak solution for the variational problem are established. The PML formulation and convergence analysis are presented in section 5. The paper is concluded with some general remarks and directions for future research in section 6.

2. A model problem.

In this section we shall introduce a mathematical model and define some notation for the rough surface scattering problem. Let us first specify the problem geometry. Figure 1 shows the structure of the problem geometry, where S_1 and S_2 are two Lipschitz continuous surfaces imbedded in the strip

$$\Omega = \{\mathbf{x} = (x, y, z) \in \mathbb{R}^3 : z_2 < z < z_1\} = \mathbb{R}^2 \times [z_2, z_1],$$

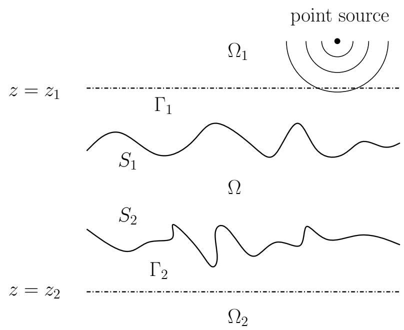

FIG. 1. *Problem geometry.*

where z_1 and z_2 are two constants. The medium in the region Ω between S_1 and S_2 may be inhomogeneous. Above the surface S_1 and below the surface S_2, the media are assumed to be homogeneous. Let $\Omega_1 = \{\mathbf{x} \in \mathbb{R}^3 : z > z_1\}$ and $\Omega_2 = \{\mathbf{x} \in \mathbb{R}^3 : z < z_2\}$. Define the boundaries $\Gamma_1 = \{z = z_1\}$ and $\Gamma_2 = \{z = z_2\}$. The surfaces S_1 and S_2 divide Ω into three connected components.

Suppose that the whole space is filled with material with the dielectric permittivity ε and magnetic permeability μ. The electromagnetic field is governed by the time-harmonic Maxwell equations (time dependence $e^{-\mathrm{i}\omega t}$):

$$\operatorname{curl} \mathbf{E} - \mathrm{i}\omega\mu\mathbf{H} = 0, \quad \operatorname{curl} \mathbf{H} + \mathrm{i}\omega\varepsilon\mathbf{E} = 0 \quad \text{in } \mathbb{R}^3, \tag{2.1}$$

where $\mathbf{E}$ and $\mathbf{H}$ denote the electric field and magnetic field in $\mathbb{R}^3$, and ω is the angular frequency. Throughout the paper we assume that $\varepsilon \in L^\infty(\mathbb{R}^3)$ and $\mu \in L^\infty(\mathbb{R}^3)$ with $\operatorname{Re}\varepsilon > 0$, $\operatorname{Im}\varepsilon > 0$, and $\mu > 0$. We restrict our attention to the case $\operatorname{Im}\varepsilon > 0$, which accounts for materials which absorb energy. The more sophisticated case $\operatorname{Im}\varepsilon = 0$ will be considered in a separate work. Since the medium is homogeneous away from region Ω that contains the inhomogeneity, there exist constants ε_j and μ_j such that

$$\begin{aligned} \varepsilon(\mathbf{x}) &= \varepsilon_1, \quad \mu(\mathbf{x}) = \mu_1 \quad \text{in } \Omega_1, \\ \varepsilon(\mathbf{x}) &= \varepsilon_2, \quad \mu(\mathbf{x}) = \mu_2 \quad \text{in } \Omega_2, \end{aligned}$$

which satisfy $\operatorname{Re}(\varepsilon_j) > 0$, $\operatorname{Im}(\varepsilon_j) > 0$, and $\mu_j > 0$ for $j = 1, 2$.

Let $\mathbf{p}$ be a constant vector, known as the polarization vector. The pair of functions

$$\mathbf{E}_{\mathrm{m}}(\mathbf{x}) := \operatorname{curl}_{\mathbf{x}} [\mathbf{p}G(\mathbf{x}, \mathbf{y})], \quad \mathbf{H}_{\mathrm{m}}(\mathbf{x}) := \frac{1}{\mathrm{i}\omega\mu_1} \operatorname{curl} \mathbf{E}_{\mathrm{m}}(\mathbf{x}) \tag{2.2}$$

represent the electromagnetic field generated by a magnetic dipole located at the point $\mathbf{y}$, and

$$\mathbf{H}_{\mathrm{e}}(\mathbf{x}) := \operatorname{curl}_{\mathbf{x}} [\mathbf{p}G(\mathbf{x}, \mathbf{y})], \quad \mathbf{E}_{\mathrm{e}}(\mathbf{x}) := -\frac{1}{\mathrm{i}\omega\varepsilon_1} \operatorname{curl} \mathbf{H}_{\mathrm{e}}(\mathbf{x}) \tag{2.3}$$

represent the electromagnetic field generated by an electric dipole located at the point $\mathbf{y}$, where G is the fundamental solution for the three-dimensional Helmholtz equation, given explicitly as

$$G(\mathbf{x}, \mathbf{y}) = \frac{1}{4\pi} \frac{e^{\mathrm{i}\kappa|\mathbf{x}-\mathbf{y}|}}{|\mathbf{x} - \mathbf{y}|},$$

where $\kappa^2 = \omega^2\varepsilon_1\mu_1$ with $\mathrm{Im}\kappa > 0$ and κ is called the wavenumber. Evidently both (2.2) and (2.3) satisfy the Maxwell equations (2.1) for $\mathbf{x} \neq \mathbf{y}$, and $\varepsilon = \varepsilon_1,\ \mu = \mu_1$ in $\mathbb{R}^3$. The incident electric field $\mathbf{E}_{\rm inc}$ and the magnetic field $\mathbf{H}_{\rm inc}$ are taken as the electromagnetic fields generated by either a magnetic dipole or an electric dipole at some point $\mathbf{y} \in \Omega_1$; i.e., $\mathbf{E}^{\rm inc} = \mathbf{E}_{\rm m}$ and $\mathbf{H}^{\rm inc} = \mathbf{H}_{\rm m}$ or $\mathbf{E}^{\rm inc} = \mathbf{E}_{\rm e}$ and $\mathbf{H}^{\rm inc} = \mathbf{H}_{\rm e}$. To ensure uniqueness, an appropriate radiation condition is required. Due to the infinite rough surfaces, the usual Silver–Müler radiation condition is no longer valid. Instead, the following radiation condition is employed: $(\mathbf{E}, \mathbf{H})$ is composed of bounded outgoing waves in Ω_1 and Ω_2 plus the incident wave $(\mathbf{E}^{\rm inc}, \mathbf{H}^{\rm inc})$ in Ω_1.

To describe the boundary value problem and derive its variational formulation, we introduce some Sobolev space notation. The reader is referred to, e.g., Adams [1] for more details. For $u \in L^2(\Gamma_j)$, which is identified with $L^2(\mathbb{R}^2)$, we denote by $\hat{u}$ the Fourier transform of u defined by

$$\hat{u}(\xi) = \frac{1}{2\pi}\int_{\mathbb{R}^2} u(\rho)e^{-{\rm i}\rho\cdot\xi}d\rho,$$

where $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$ and $\rho = (x, y) \in \mathbb{R}^2$. We define $C_\rho^\infty(\Omega)$ to be the linear space of infinitely differential functions with compact support with respect to the variable ρ on Ω. Denote by $L^2(\Omega)$ the space of complex square integrable functions on Ω with the norm

$$\| u \|_{0,\,\Omega} = \left[\int_{z_2}^{z_1}\int_{\mathbb{R}^2} |u(\rho, z)|^2 d\rho dz\right]^{1/2} = \left[\int_{z_2}^{z_1}\int_{\mathbb{R}^2} |\hat{u}(\xi, z)|^2 d\xi dz\right]^{1/2}.$$

We define the Sobolev space $H^s(\Omega) = \{D^\alpha u \in L^2(\Omega) \text{ for all } |\alpha| \leq s\}$, which is a Banach space for the norm

$$\| u \|_{s,\,\Omega} = \left[\int_{z_2}^{z_1}\sum_{l+m\leq s}\left(\int_{\mathbb{R}^2}(1+|\xi|^2)^l|D_z^m\hat{u}(\xi, z)|^2 d\xi\right)dz\right]^{1/2},$$

where $l, m \in \mathbb{N}$ and D_z^m is the mth derivative with respect to z. These norms, given in the spatial-frequency domain, are equivalent to the usual Sobolev norms in the entire spatial domain due to the Parseval identity.

The following two lemmas are concerned with the density of $C_\rho^\infty(\Omega)$. This is important, particularly for our case of unbounded slab Ω, since it allows us to prove results for smooth function with compact support and extend them by limiting argument to more general functions.

LEMMA 2.1. *$C_\rho^\infty(\Omega)$ is dense in $H^s(\Omega)$.*

Proof. Noting that $C_0^\infty(\mathbb{R}^3)$ is dense in $H^s(\mathbb{R}^3)$, we have that $C_0^\infty(\mathbb{R}^3)|_\Omega$ is dense in $H^s(\mathbb{R}^3)|_\Omega$. From the Sobolev extension theorem, $H^s(\mathbb{R}^3)|_\Omega = H^s(\Omega)$. Therefore $C_\rho^\infty(\Omega) \supseteq C_0^\infty(\mathbb{R}^3)|_\Omega$ is dense in $H^s(\Omega)$. ∎

Introduce the space

$$H(\mathrm{curl},\,\Omega) = \{\mathbf{u} \in (L^2(\Omega))^3,\ \mathrm{curl}\,\mathbf{u} \in (L^2(\Omega))^3\},$$

which is clearly a Hilbert space for the norm

$$\| \mathbf{u} \|_{H(\mathrm{curl},\,\Omega)} = \left(\| \mathbf{u} \|_{0,\,\Omega}^2 + \| \mathrm{curl}\,\mathbf{u} \|_{0,\,\Omega}^2\right)^{1/2}.$$

Given $\mathbf{u} = (u_1(\rho,z), u_2(\rho,z), u_3(\rho,z))^\top$ in $H(\mathrm{curl}, \Omega)$, it has the following inverse Fourier transform:

$$\mathbf{u}(\rho, z) = \frac{1}{2\pi}\int_{\mathbb{R}^2} (\hat{u}_1(\xi,z), \hat{u}_2(\xi,z), \hat{u}_3(\xi,z))^\top e^{\mathrm{i}\rho\cdot\xi} d\xi.$$

A simple calculation yields an explicit characterization of the norm in $H(\mathrm{curl}, \Omega)$ via Fourier coefficients:

$$\begin{aligned}\| \mathbf{u} \|^2_{H(\mathrm{curl},\,\Omega)} = \int_{b_2}^{b_1}\int_{\mathbb{R}^2} [&|\hat{u}_1(\xi,z)|^2 + |\hat{u}_2(\xi,z)|^2 + |\hat{u}_3(\xi,z)|^2 + |\mathrm{i}\xi_2\hat{u}_3(\xi,z) - \hat{u}_2'(\xi,z)|^2 \\ &+ |\hat{u}_1'(\xi,z) - \mathrm{i}\xi_1\hat{u}_3(\xi,z))|^2 + |\xi_1\hat{u}_2(\xi,z) - \xi_2\hat{u}_1(\xi,z)|^2] d\xi dz,\end{aligned} \tag{2.4}$$

where $\hat{u}_j'(\xi,z) = \frac{\partial}{\partial z}\hat{u}_j(\xi,z)$, $j = 1,2$. Using an argument similar to that of Lemma 2.1, we have the following lemma. The proof is omitted.

LEMMA 2.2. *$(C^\infty_\rho(\Omega))^3$ is dense in $H(\mathrm{curl},\Omega)$.*

Therefore when dealing with function spaces $H^1(\Omega)$ or $H(\mathrm{curl},\Omega)$, we may prove assertions on the dense subset $C^\infty_\rho(\Omega)$ or $(C^\infty_\rho(\Omega))^3$ and then pass the limit through approximations from $C^\infty_\rho(\Omega)$ or $(C^\infty_\rho(\Omega))^3$. The density lemmas are useful for the variational formulation in the domain Ω, where the lateral boundary integrals will vanish as $\rho \to \infty$ when integration by parts is applied.

For any vector field $\mathbf{u} = (u_1, u_2, u_3)^\top$, denote the tangential component on the surface Γ_j by

$$\mathbf{u}_{\Gamma_j} = -\mathbf{n}_j \times (\mathbf{n}_j \times \mathbf{u}) = (u_1(x_1,x_2,b_j), u_2(x_1,x_2,b_j), 0)^\top,$$

where $\mathbf{n}_j$ is the unit outward normal on Γ_j; i.e., $\mathbf{n}_1 = (0,0,1)^\top$ and $\mathbf{n}_2 = (0,0,-1)^\top$. For any smooth vector $\mathbf{u} = (u_1,u_2,u_3)^\top$ defined on Γ_j, denote by $\mathrm{div}_{\Gamma_j}\mathbf{u} = \partial_x u_1 + \partial_y u_2$ and $\mathrm{curl}_{\Gamma_j}\mathbf{u} = \partial_x u_2 - \partial_y u_1$ the surface divergence and the surface scalar curl, respectively, of the field $\mathbf{u}$. For a smooth function u, denote by $\nabla_{\Gamma_j} u = (\partial_x u, \partial_y u, 0)^\top$ the surface gradient.

To describe the capacity operator and transparent boundary condition in the formulation of the boundary value problem, we introduce some trace functional spaces. Denote by $H^{-1/2}(\Gamma_j)$ the standard Sobolev space, the completion of $L^2(\Gamma_j)$ in the norm $\|\cdot\|_{H^{-1/2}(\Gamma_j)}$ characterized by

$$\| u \|^2_{H^{-1/2}(\Gamma_j)} = \int_{\mathbb{R}^2} (1+|\xi|^2)^{-1/2} |\hat{u}|^2 d\xi. \tag{2.5}$$

We then introduce the following two vector trace spaces:

$$\begin{aligned}H^{-1/2}_{\mathrm{div}}(\Gamma_j) &= \{\mathbf{u} \in (H^{-1/2}(\Gamma_j))^3 : u_3 = 0,\, \mathrm{div}_{\Gamma_j}\mathbf{u} \in H^{-1/2}(\Gamma_j)\},\\ H^{-1/2}_{\mathrm{curl}}(\Gamma_j) &= \{\mathbf{u} \in (H^{-1/2}(\Gamma_j))^3 : u_3 = 0,\, \mathrm{curl}_{\Gamma_j}\mathbf{u} \in H^{-1/2}(\Gamma_j)\}.\end{aligned}$$

Using Fourier modes, the norms on the spaces $H^{-1/2}_{\mathrm{div}}(\Gamma_j)$ and $H^{-1/2}_{\mathrm{curl}}(\Gamma_j)$ can be characterized by

$$\| \mathbf{u} \|^2_{H^{-1/2}_{\mathrm{div}}(\Gamma_j)} = \int_{\mathbb{R}^2} (1+|\xi|^2)^{-1/2} \left[|\hat{u}_1|^2 + |\hat{u}_2|^2 + |\xi_1\hat{u}_1 + \xi_2\hat{u}_2|^2\right] d\xi, \tag{2.6}$$

$$\| \mathbf{u} \|^2_{H^{-1/2}_{\mathrm{curl}}(\Gamma_j)} = \int_{\mathbb{R}^2} (1+|\xi|^2)^{-1/2} \left[|\hat{u}_1|^2 + |\hat{u}_2|^2 + |\xi_1\hat{u}_2 - \xi_2\hat{u}_1|^2\right] d\xi. \tag{2.7}$$

To simplify proofs, we shall employ positive constants C and C_i as generalized constants whose precise values are not required and may change line by line but should always be clear from the context.

The following lemma shows that the spaces $H_{\mathrm{div}}^{-1/2}(\Gamma_j)$ and $H_{\mathrm{curl}}^{-1/2}(\Gamma_j)$ are mutually adjoint under the dual paring $\langle\cdot,\cdot\rangle$ defined by

$$\langle \mathbf{u}, \mathbf{v}\rangle = \int_{\mathbb{R}^2} (\hat{u}_1\overline{\hat{v}}_1 + \hat{u}_2\overline{\hat{v}}_2)d\xi. \tag{2.8}$$

Note that, from the Parseval formula, the dual paring $\langle \mathbf{u}, \mathbf{v}\rangle$ is the $(L^2(\mathbb{R}^2))^3$ inner product between $\mathbf{u}$ and $\mathbf{v}$ if $\mathbf{u}, \mathbf{v} \in (L^2(\mathbb{R}^2))^3$ (cf. [44, p. 189]).

LEMMA 2.3. *The spaces $H_{\mathrm{div}}^{-1/2}(\Gamma_j)$ and $H_{\mathrm{curl}}^{-1/2}(\Gamma_j)$ are mutually adjoint with respect to the scalar product in $(L^2(\Gamma_j))^3$.*

Proof. The proof is similar to what is used in [40, Lemma 5.3.1]. It is easy to show that

$$\begin{aligned}\langle \mathbf{u}, \mathbf{v}\rangle = \int_{\mathbb{R}^2} (1+|\xi|^2)^{-1}\Big(&\big((\xi_1+\mathrm{i})\hat{u}_1 + \xi_2\hat{u}_2\big)\cdot\overline{\big((\xi_1+\mathrm{i})\hat{v}_1 + \xi_2\hat{v}_2\big)}\\ &+ \big((\xi_1-\mathrm{i})\hat{u}_2 - \xi_2\hat{u}_1\big)\cdot\overline{\big((\xi_1-\mathrm{i})\hat{v}_2 - \xi_2\hat{v}_1\big)}\Big)d\xi.\end{aligned} \tag{2.9}$$

It is clear that

$$\frac{1}{3}\big|(\xi_1+\mathrm{i})\hat{u}_1 + \xi_2\hat{u}_2\big|^2 - \frac{1}{2}|\hat{u}_1|^2 \le |\xi_1\hat{u}_1 + \xi_2\hat{u}_2|^2 \le \frac{4}{3}\big|(\xi_1+\mathrm{i})\hat{u}_1 + \xi_2\hat{u}_2\big|^2 + 4|\hat{u}_1|^2.$$

Noting that

$$|\hat{u}_1|^2 + |\hat{u}_2|^2 = (1+|\xi|^2)^{-1}\Big(\big|(\xi_1+\mathrm{i})\hat{u}_1 + \xi_2\hat{u}_2\big|^2 + \big|(\xi_1-\mathrm{i})\hat{u}_2 - \xi_2\hat{u}_1\big|^2\Big),$$

we have the following equivalent norm on the spaces $H_{\mathrm{div}}^{-1/2}(\Gamma_j)$:

$$\begin{aligned}\| \mathbf{u} \|^2_{H_{\mathrm{div}}^{-1/2}(\Gamma_j)} \simeq \int_{\mathbb{R}^2} \Big(&(1+|\xi|^2)^{-1/2}\big|(\xi_1+\mathrm{i})\hat{u}_1 + \xi_2\hat{u}_2\big|^2\\ &+ (1+|\xi|^2)^{-3/2}\big|(\xi_1-\mathrm{i})\hat{u}_2 - \xi_2\hat{u}_1\big|^2\Big)d\xi.\end{aligned} \tag{2.10}$$

Similarly,

$$\begin{aligned}\| \mathbf{u} \|^2_{H_{\mathrm{curl}}^{-1/2}(\Gamma_j)} \simeq \int_{\mathbb{R}^2} \Big(&(1+|\xi|^2)^{-3/2}\big|(\xi_1+\mathrm{i})\hat{u}_1 + \xi_2\hat{u}_2\big|^2\\ &+ (1+|\xi|^2)^{-1/2}\big|(\xi_1-\mathrm{i})\hat{u}_2 - \xi_2\hat{u}_1\big|^2\Big)d\xi.\end{aligned} \tag{2.11}$$

From (2.9)–(2.11) and the Cauchy–Schwarz inequality, there exists a constant C such that

$$\langle \mathbf{u}, \mathbf{v}\rangle \le C \| \mathbf{u} \|^2_{H_{\mathrm{div}}^{-1/2}(\Gamma_j)} \| \mathbf{v} \|^2_{H_{\mathrm{curl}}^{-1/2}(\Gamma_j)}\ . \tag{2.12}$$

That is $\langle \mathbf{u}, \mathbf{v}\rangle$ is well defined for any $\mathbf{u} \in H_{\mathrm{div}}^{-1/2}(\Gamma_j), \mathbf{v} \in H_{\mathrm{curl}}^{-1/2}(\Gamma_j)$.

Define two operators R and L by

$$\begin{aligned}\widehat{R\mathbf{u}} = &\begin{pmatrix} -\xi_2 \\ \xi_1+\mathrm{i}\end{pmatrix}(1+|\xi|^2)^{-3/2}\big((\xi_1-\mathrm{i})\hat{u}_2 - \xi_2\hat{u}_1\big)\\ &+ \begin{pmatrix} \xi_1-\mathrm{i} \\ \xi_2\end{pmatrix}(1+|\xi|^2)^{-1/2}\big((\xi_1+\mathrm{i})\hat{u}_1 + \xi_2\hat{u}_2\big),\end{aligned}$$

$$\widehat{L\mathbf{v}} = \begin{pmatrix} -\xi_2 \\ \xi_1 + \mathrm{i} \end{pmatrix} (1+|\xi|^2)^{-1/2}\big((\xi_1 - \mathrm{i})\hat{v}_2 - \xi_2\hat{v}_1\big) + \begin{pmatrix} \xi_1 - \mathrm{i} \\ \xi_2 \end{pmatrix} (1+|\xi|^2)^{-3/2}\big((\xi_1 + \mathrm{i})\hat{v}_1 + \xi_2\hat{v}_2\big).$$

We have

$$\begin{aligned}
(\xi_1 + \mathrm{i})\big(\widehat{R\mathbf{u}}\big)_1 + \xi_2\big(\widehat{R\mathbf{u}}\big)_2 &= (1+|\xi|^2)^{1/2}\big((\xi_1+\mathrm{i})\hat{u}_1 + \xi_2\hat{u}_2\big),\\
(\xi_1 - \mathrm{i})\big(\widehat{R\mathbf{u}}\big)_2 - \xi_2\big(\widehat{R\mathbf{u}}\big)_1 &= (1+|\xi|^2)^{-1/2}\big((\xi_1-\mathrm{i})\hat{u}_2 - \xi_2\hat{u}_1\big),\\
(\xi_1 + \mathrm{i})\big(\widehat{L\mathbf{v}}\big)_1 + \xi_2\big(\widehat{L\mathbf{v}}\big)_2 &= (1+|\xi|^2)^{-1/2}\big((\xi_1+\mathrm{i})\hat{v}_1 + \xi_2\hat{v}_2\big),\\
(\xi_1 - \mathrm{i})\big(\widehat{L\mathbf{v}}\big)_2 - \xi_2\big(\widehat{L\mathbf{v}}\big)_1 &= (1+|\xi|^2)^{1/2}\big((\xi_1-\mathrm{i})\hat{v}_2 - \xi_2\hat{v}_1\big).
\end{aligned}$$

Therefore, from (2.9) – (2.11), we have

$$\langle L\mathbf{v}, R\mathbf{u}\rangle = \overline{\langle \mathbf{u}, \mathbf{v}\rangle}$$

and

$$\| R\mathbf{u} \|^2_{H^{-1/2}_{\mathrm{curl}}(\Gamma_j)} \simeq \| \mathbf{u} \|^2_{H^{-1/2}_{\mathrm{div}}(\Gamma_j)}, \qquad \| L\mathbf{v} \|^2_{H^{-1/2}_{\mathrm{div}}(\Gamma_j)} \simeq \| \mathbf{v} \|^2_{H^{-1/2}_{\mathrm{curl}}(\Gamma_j)} .$$

This completes the proof of the lemma. ∎

The following trace regularity results in $H^{-1/2}_{\mathrm{curl}}(\Gamma_j)$ and $H^{-1/2}(\Gamma_j)$ are useful in our subsequent analysis.

LEMMA 2.4. *Let* $\gamma_1 = \max\big\{\sqrt{1+(z_1-z_2)^{-1}}, \sqrt{2}\big\}$*. Then the following estimate holds:*

$$\| \mathbf{u}_{\Gamma_j} \|_{H^{-1/2}_{\mathrm{curl}}(\Gamma_j)} \le \gamma_1 \| \mathbf{u} \|_{H(\mathrm{curl},\,\Omega)}$$

for all $\mathbf{u} \in H(\mathrm{curl},\,\Omega)$.

Proof. First we have

$$\begin{aligned}
(z_1 - z_2)|\zeta(z_j)|^2 &= \int_{z_2}^{z_1} |\zeta(z)|^2 dz + \int_{z_2}^{z_1}\int_{z}^{z_j} \frac{d}{dt}|\zeta(t)|^2 dt dz\\
&\le \int_{z_2}^{z_1} |\zeta(z)|^2 dz + (z_1 - z_2)\int_{z_2}^{z_1} 2|\zeta(z)||\zeta'(z)|dz,
\end{aligned} \tag{2.13}$$

which implies by the Cauchy–Schwarz inequality that

$$(1+|\xi|^2)^{-1/2}|\zeta(z_j)|^2 \le \gamma_1^2 \int_{z_2}^{z_1} |\zeta(z)|^2 dz + (1+|\xi|^2)^{-1}\int_{z_2}^{z_1} |\zeta'(z)|^2 dz. \tag{2.14}$$

Given $\mathbf{u}$ in $H(\mathrm{curl},\,\Omega)$, it follows from the definition (2.7) that

$$\| \mathbf{u}_{\Gamma_j} \|^2_{H^{-1/2}_{\mathrm{curl}}(\Gamma_j)} = \int_{\mathbb{R}^2} (1+|\xi|^2)^{-1/2}[|\hat{u}_1(\xi,z_j)|^2 + |\hat{u}_2(\xi,z_j)|^2 + |\xi_1\hat{u}_2(\xi,z_j) - \xi_2\hat{u}_1(\xi,z_j)|^2]d\xi.$$

Using (2.14), we obtain

$$(1+|\xi|^2)^{-1/2}[|\hat{u}_1(\xi,z_j)|^2 + |\hat{u}_2(\xi,z_j)|^2 + |\xi_1\hat{u}_2(\xi,z_j) - \xi_2\hat{u}_1(\xi,z_j)|^2] \le I_1 + I_2,$$

where

$$I_1 = \gamma_1^2 \int_{z_2}^{z_1} [|\hat{u}_1(\xi, z)|^2 + |\hat{u}_2(\xi, z)|^2 + |\xi_1 \hat{u}_2(\xi, z) - \xi_2 \hat{u}_1(\xi, z)|^2] dz$$

and

$$\begin{aligned} I_2 = (1 + |\xi|^2)^{-1} \int_{z_2}^{z_1} & [2|\hat{u}_1'(\xi, z) - \mathrm{i}\xi_1 \hat{u}_3(\xi, z)|^2 + 2\xi_1^2 |\hat{u}_3(\xi, z)|^2 \\ & + 2|\mathrm{i}\xi_2 \hat{u}_3(\xi, z) - \hat{u}_2'(\xi, z)|^2 + 2\xi_2^2 |\hat{u}_3(\xi, z)|^2 \\ & + |\xi_1 \hat{u}_2'(\xi, z) - \mathrm{i}\xi_1\xi_2 \hat{u}_3(\xi, z) + \mathrm{i}\xi_1\xi_2 \hat{u}_3(\xi, z) - \xi_2 \hat{u}_1'(\xi, z)|^2] dz. \end{aligned}$$

A simple calculation yields

$$\begin{aligned} I_2 &\leq 2(1 + |\xi|^2)^{-1} \int_{z_2}^{z_1} [(1 + \xi_2^2)|\hat{u}_1'(\xi, z) - \mathrm{i}\xi_1 \hat{u}_3(\xi, z)|^2 \\ &\qquad + (1 + \xi_1^2)|\mathrm{i}\xi_2 \hat{u}_3(\xi, z) - \hat{u}_2'(\xi, z)|^2 + |\xi|^2 |\hat{u}_3(\xi, z)|^2] dz \\ &\leq 2 \int_{z_2}^{z_1} [|\hat{u}_1'(\xi, z) - \mathrm{i}\xi_1 \hat{u}_3(\xi, z)|^2 + |\mathrm{i}\xi_2 \hat{u}_3(\xi, z) - \hat{u}_2'(\xi, z)|^2 + |\hat{u}_3(\xi, z)|^2] dz. \end{aligned} \tag{2.15}$$

The proof is complete by combining the above estimates and noting the definition of the norm in $H(\mathrm{curl}, \Omega)$ (2.4). ∎

Lemma 2.5. *For any $\eta > 0$ there is a constant $C(\eta)$ depending only on η, z_1, and z_2 such that the following estimate holds:*

$$\| \mathbf{u}_{\Gamma_j} \|^2_{H^{-1/2}(\Gamma_j)} \leq \eta \| \mathrm{curl}\, \mathbf{u} \|^2_{0,\Omega} + C(\eta) \| \mathbf{u} \|^2_{0,\Omega}$$

for all $\mathbf{u} \in H(\mathrm{curl}, \Omega)$.

Proof. For any $\eta > 0$, we have from (2.13) that

$$\begin{aligned} (z_1 - z_2)|\zeta(z_j)|^2 \leq & \int_{z_2}^{z_1} |\zeta(z)|^2 dz + (z_1 - z_2) \\ & \times \left(\frac{2(1 + |\xi|^2)^{1/2}}{\eta} \int_{z_2}^{z_1} |\zeta(z)|^2 dz + \frac{\eta}{2(1 + |\xi|^2)^{1/2}} \int_{z_2}^{z_1} |\zeta'(z)|^2 dz \right), \end{aligned}$$

which implies that

$$(1 + |\xi|^2)^{-1/2} |\zeta(z_j)|^2 \leq C(\eta) \int_{z_2}^{z_1} |\zeta(z)|^2 dz + \frac{\eta}{2}(1 + |\xi|^2)^{-1} \int_{z_2}^{z_1} |\zeta'(z)|^2 dz. \tag{2.16}$$

Given $\mathbf{u}$ in $H(\mathrm{curl}, \Omega)$, it follows from the definition (2.5) that

$$\| \mathbf{u}_{\Gamma_j} \|^2_{H^{-1/2}(\Gamma_j)} = \int_{\mathbb{R}^2} (1 + |\xi|^2)^{-1/2} [|\hat{u}_1(\xi, z_j)|^2 + |\hat{u}_2(\xi, z_j)|^2] d\xi.$$

Using (2.16) yields

$$(1 + |\xi|^2)^{-1/2} [|\hat{u}_1(\xi, z_j)|^2 + |\hat{u}_2(\xi, z_j)|^2] \leq J_1 + J_2,$$

where

$$J_1 = C(\eta) \int_{z_2}^{z_1} [|\hat{u}_1(\xi, z)|^2 + |\hat{u}_2(\xi, z)|^2] dz$$

and

$$\begin{aligned} J_2 &= \frac{\eta}{2}(1+|\xi|^2)^{-1}\int_{z_2}^{z_1}[|\hat{u}_1'(\xi,z)|^2+|\hat{u}_2'(\xi,z)|^2]dz \\ &\le \eta(1+|\xi|^2)^{-1}\int_{z_2}^{z_1}[|\hat{u}_1'(\xi,z)-\mathrm{i}\xi_1\hat{u}_3(\xi,z)|^2+|\hat{u}_2'(\xi,z)-\mathrm{i}\xi_2\hat{u}_3(\xi,z)|^2 \\ &\qquad\qquad + (\xi_1^2+\xi_2^2)|\hat{u}_3(\xi,z)|^2]dz \\ &\le \eta\int_{z_2}^{z_1}[|\hat{u}_1'(\xi,z)-\mathrm{i}\xi_1\hat{u}_3(\xi,z)|^2+|\mathrm{i}\xi_2\hat{u}_3(\xi,z)-\hat{u}_2'(\xi,z)|^2+|\hat{u}_3(\xi,z)|^2]dz. \end{aligned}$$

The proof is complete by combining the above inequalities and (2.4). ∎

3. Transparent boundary condition. In this section, we introduce a transparent boundary condition by using a capacity operator which maps the value of the tangential component of the electric field to the value of the tangential trace of the magnetic field.

We wish to reduce the problem to the domain Ω. The radiation condition for the scattering problem insists that $(\mathbf{E},\mathbf{H})$ is composed of bounded outgoing waves in Ω_1 and Ω_2 plus the incident plane wave $(\mathbf{E}^{\mathrm{inc}},\mathbf{H}^{\mathrm{inc}})$ in Ω_1. Since the derivation of the transparent boundary conditions on Γ_1 and Γ_2 is parallel, we will show only how to deduce the transparent boundary condition on Γ_1, and state the corresponding transparent boundary condition on Γ_2 without derivation.

Observe that the medium is homogeneous in Ω_1. The scattered electric field $\mathbf{E}^{\mathrm{sc}}=\mathbf{E}-\mathbf{E}^{\mathrm{inc}}$ and the scattered magnetic field $\mathbf{H}^{\mathrm{sc}}=\mathbf{H}-\mathbf{H}^{\mathrm{inc}}$ satisfy

$$\operatorname{curl}\mathbf{E}^{\mathrm{sc}}-\mathrm{i}\omega\mu_1\mathbf{H}^{\mathrm{sc}}=0,\qquad \operatorname{curl}\mathbf{H}^{\mathrm{sc}}+\mathrm{i}\omega\varepsilon_1\mathbf{E}^{\mathrm{sc}}=0. \tag{3.1}$$

Let $\mathbf{E}^{\mathrm{sc}}=(E_1,E_2,E_3)^\top$. Denote by $\mathbf{E}^{\mathrm{sc}}_{\Gamma_1}=(E_1(\rho,b_1),E_2(\rho,b_1),0)^\top$ the tangential component of the electric field on Γ_1. Denote by $\mathbf{H}^{\mathrm{sc}}\times\mathbf{n}_1=(H_2(\rho,b_1),-H_1(\rho,b_1),0)^\top$ the tangential trace of the magnetic field on Γ_1, where

$$\begin{aligned} H_2(\rho,b_1) &= \frac{1}{\mathrm{i}\omega\mu_1}\left[\partial_z E_1(\rho,b_1)-\partial_x E_3(\rho,b_1)\right], \\ -H_1(\rho,b_1) &= \frac{1}{\mathrm{i}\omega\mu_1}\left[\partial_z E_2(\rho,b_1)-\partial_y E_3(\rho,b_1)\right]. \end{aligned}$$

It is clear that

$$\hat{H}_2(\xi,b_1)=\frac{1}{\mathrm{i}\omega\mu_1}\left[\partial_z\hat{E}_1(\xi,b_1)-\mathrm{i}\xi_1\hat{E}_3(\xi,b_1)\right], \tag{3.2}$$

$$-\hat{H}_1(\xi,b_1)=\frac{1}{\mathrm{i}\omega\mu_1}\left[\partial_z\hat{E}_2(\xi,b_1)-\mathrm{i}\xi_2\hat{E}_3(\xi,b_1)\right]. \tag{3.3}$$

Eliminating the magnetic field from (3.1) and noting $\nabla\cdot\mathbf{E}^{\mathrm{sc}}=0$ in Ω_1, we arrive at the equation for the electric field for $j=1,2,3$,

$$\begin{aligned} \Delta E_j+\omega^2\varepsilon_1\mu_1E_j &= 0 \quad \text{in } \Omega_1, \\ E_j &= E_j(\rho,b_1) \quad \text{on } \Gamma_1. \end{aligned}$$

Taking the Fourier transform with respect to ρ yields

$$\begin{aligned} \partial_z^2\hat{E}_j+(\omega^2\varepsilon_1\mu_1-|\xi|^2)\hat{E}_j &= 0 \quad \text{for } z>b_1, \\ \hat{E}_j &= \hat{E}_j(\xi,b_1) \quad \text{for } z=b_1. \end{aligned}$$

Using the radiation condition to eliminate the incoming wave term, we obtain

$$\hat{E}_j(\xi, z) = \hat{E}_j(\xi, b_1)e^{\mathrm{i}\beta_1(\xi)(z-b_1)}, \tag{3.4}$$

where

$$\beta_1^2(\xi) = \omega^2\varepsilon_1\mu_1 - |\xi|^2 \quad \text{with } \mathrm{Im}\beta_1(\xi) > 0.$$

It is clear that

$$\partial_z\hat{E}_j(\xi, b_1) = \mathrm{i}\beta_1(\xi)\hat{E}_j(\xi, b_1).$$

Noting that $\nabla \cdot \mathbf{E}^{\mathrm{sc}} = \partial_x E_1 + \partial_y E_2 + \partial_z E_3 = 0$ in Ω_1 and $\beta_1(\xi) \neq 0$ for all ξ, we deduce that

$$\hat{E}_3(\xi, b_1) = \frac{1}{\mathrm{i}\beta_1(\xi)}\partial_z\hat{E}_3(\xi, b_1) = \frac{-1}{\beta_1(\xi)}[\xi_1\hat{E}_1(\xi, b_1) + \xi_2\hat{E}_2(\xi, b_1)].$$

Therefore, we have from (3.2)–(3.3)

$$\begin{aligned}
\hat{H}_2(\xi, b_1) &= \frac{1}{\omega\mu_1}\left[\beta_1(\xi)\hat{E}_1(\xi, b_1) + \frac{\xi_1}{\beta_1(\xi)}(\xi_1\hat{E}_1(\xi, b_1) + \xi_2\hat{E}_2(\xi, b_1))\right],\\
-\hat{H}_1(\xi, b_1) &= \frac{1}{\omega\mu_1}\left[\beta_1(\xi)\hat{E}_2(\xi, b_1) + \frac{\xi_2}{\beta_1(\xi)}(\xi_1\hat{E}_1(\xi, b_1) + \xi_2\hat{E}_2(\xi, b_1))\right],
\end{aligned}$$

or equivalently

$$\begin{aligned}
\hat{H}_2(\xi, b_1) &= \frac{1}{\omega\mu_1\beta_1}\left[\omega^2\varepsilon_1\mu_1\hat{E}_1(\xi, b_1) + \xi_2(\xi_1\hat{E}_2(\xi, b_1) - \xi_2\hat{E}_1(\xi, b_1))\right],\\
-\hat{H}_1(\xi, b_1) &= \frac{1}{\omega\mu_1\beta_1}\left[\omega^2\varepsilon_1\mu_1\hat{E}_1(\xi, b_1) - \xi_1(\xi_1\hat{E}_2(\xi, b_1) - \xi_2\hat{E}_1(\xi, b_1))\right].
\end{aligned}$$

For any tangential vector $\mathbf{u} = (u_1, u_2, 0)^\top$ on Γ_1, define the following capacity operator T_1:

$$T_1\mathbf{u} = (v_1, v_2, 0)^\top,$$

where

$$\begin{aligned}
\hat{v}_1 &= \frac{1}{\omega\mu_1}\left[\beta_1\hat{u}_1 + \frac{\xi_1}{\beta_1}(\xi_1\hat{u}_1 + \xi_2\hat{u}_2)\right],\\
\hat{v}_2 &= \frac{1}{\omega\mu_1}\left[\beta_1\hat{u}_2 + \frac{\xi_2}{\beta_1}(\xi_1\hat{u}_1 + \xi_2\hat{u}_2)\right],
\end{aligned} \tag{3.5}$$

or equivalently

$$\begin{aligned}
\hat{v}_1 &= \frac{1}{\omega\mu_1\beta_1}\left[\omega^2\varepsilon_1\mu_1\hat{u}_1 + \xi_2(\xi_1\hat{u}_2 - \xi_2\hat{u}_1)\right],\\
\hat{v}_2 &= \frac{1}{\omega\mu_1\beta_1}\left[\omega^2\varepsilon_1\mu_1\hat{u}_2 - \xi_1(\xi_1\hat{u}_2 - \xi_2\hat{u}_1)\right].
\end{aligned} \tag{3.6}$$

Similarly, for any tangential vector $\mathbf{u} = (u_1, u_2, 0)^\top$ on Γ_2, define the capacity operator T_2:

$$T_2\mathbf{u} = (v_1, v_2, 0)^\top,$$

where

$$\begin{aligned}\hat{v}_1 &= \frac{1}{\omega\mu_2}\left[\beta_2\hat{u}_1 + \frac{\xi_1}{\beta_2}(\xi_1\hat{u}_1 + \xi_2\hat{u}_2)\right],\\ \hat{v}_2 &= \frac{1}{\omega\mu_2}\left[\beta_2\hat{u}_2 + \frac{\xi_2}{\beta_2}(\xi_1\hat{u}_1 + \xi_2\hat{u}_2)\right],\end{aligned}\tag{3.7}$$

or equivalently

$$\begin{aligned}\hat{v}_1 &= \frac{1}{\omega\mu_2\beta_2}\left[\omega^2\varepsilon_2\mu_2\hat{u}_1 + \xi_2(\xi_1\hat{u}_2 - \xi_2\hat{u}_1)\right],\\ \hat{v}_2 &= \frac{1}{\omega\mu_2\beta_2}\left[\omega^2\varepsilon_2\mu_2\hat{u}_2 - \xi_1(\xi_1\hat{u}_2 - \xi_2\hat{u}_1)\right].\end{aligned}\tag{3.8}$$

Here

$$\beta_2^2(\xi) = \omega^2\varepsilon_2\mu_2 - |\xi|^2 \quad \text{with } \mathrm{Im}\beta_2(\xi) > 0.$$

For any vector field $\mathbf{E} \in H(\mathrm{curl}, \Omega)$, it follows from Lemma 2.4 that its tangential components $\mathbf{E}_{\Gamma_j} \in H_{\mathrm{curl}}^{-1/2}(\Gamma_j)$. Using the capacity operator, we may propose the following transparent boundary conditions:

$$T_1(\mathbf{E}_{\Gamma_1} - \mathbf{E}_{\Gamma_1}^{\mathrm{inc}}) = (\mathbf{H} - \mathbf{H}^{\mathrm{inc}}) \times \mathbf{n}_1 \quad \text{on } \Gamma_1,\tag{3.9}$$

$$T_2(\mathbf{E}_{\Gamma_2}) = \mathbf{H} \times \mathbf{n}_2 \quad \text{on } \Gamma_2,\tag{3.10}$$

which map the tangential components of the electric fields to the tangential traces of the magnetic fields.

To present some estimates of the capacity operators, it is useful to introduce the following notation. Define

$$\omega^2\varepsilon_j\mu_j = \varphi_j + \mathrm{i}\psi_j,$$

where

$$\varphi_j = \omega^2\,\mathrm{Re}(\varepsilon_j\mu_j) \quad \text{and} \quad \psi_j = \omega^2\,\mathrm{Im}(\varepsilon_j\mu_j).$$

Denote

$$\beta_j^2 = \omega^2\varepsilon_j\mu_j - |\xi|^2 = \phi_j + \mathrm{i}\psi_j,\tag{3.11}$$

where

$$\phi_j = \omega^2\,\mathrm{Re}(\varepsilon_j\mu_j) - |\xi|^2 = \varphi_j - |\xi|^2.\tag{3.12}$$

A simple calculation gives

$$\beta_j = a_j + \mathrm{i}b_j,$$

where

$$\begin{aligned}a_j &= \mathrm{Re}\beta_j = \left(\frac{\sqrt{\phi_j^2 + \psi_j^2} + \phi_j}{2}\right)^{1/2},\\ b_j &= \mathrm{Im}\beta_j = \left(\frac{\sqrt{\phi_j^2 + \psi_j^2} - \phi_j}{2}\right)^{1/2}.\end{aligned}\tag{3.13}$$

LEMMA 3.1. *The operator* $T_j : H^{-1/2}_{\rm curl}(\Gamma_j) \to H^{-1/2}_{\rm div}(\Gamma_j)$ *is continuous.*

Proof. For any $\mathbf{u} = (u_1, u_2, 0)^\top$, $\mathbf{w} = (w_1, w_2, 0)^\top \in H^{-1/2}_{\rm curl}(\Gamma_j)$, let $T_j\mathbf{u} = (v_1, v_2, 0)^\top$. It follows from the definitions (2.8), (3.6), and (3.8) that

$$\begin{aligned}\langle T_j\mathbf{u}, \mathbf{w}\rangle &= \int_{\mathbb{R}^2} \left(\hat{v}_1\overline{\hat{w}}_1 + \hat{v}_2\overline{\hat{w}}_2\right) d\xi \\ &= \int_{\mathbb{R}^2} \frac{1}{\omega\mu_j\beta_j}\left[\omega^2\varepsilon_j\mu_j(\hat{u}_1\overline{\hat{w}}_1 + \hat{u}_2\overline{\hat{w}}_2) - (\xi_1\hat{u}_2 - \xi_2\hat{u}_1)(\xi_1\overline{\hat{w}}_2 - \xi_2\overline{\hat{w}}_1)\right] d\xi.\end{aligned}$$

To prove the lemma, it is required to estimate

$$\frac{(1+|\xi|^2)^{1/2}}{|\beta_j|} = \left[\frac{(1+\varphi_j-\phi_j)^2}{\psi_j^2+\phi_j^2}\right]^{1/4}.$$

Let

$$F_j(t) = \frac{(1+\varphi_j - t)^2}{\psi_j^2 + t^2}.$$

It can be verified that $F_j(t)$ increases for $t < K_j = -\psi_j^2/(1+\varphi_j)$ and decreases for $K_j < t \le \varphi_j$. Therefore,

$$\frac{(1+\varphi_j-\phi_j)^2}{\psi_j^2+\phi_j^2} = F_j(\phi_j) \le F_j(K_j) = 1 + \frac{(1+\varphi_j)^2}{\psi_j^2}.$$

Combining the above estimates yields

$$|\langle T_j\mathbf{u}, \mathbf{w}\rangle| \le C_j \parallel \mathbf{u} \parallel_{H^{-1/2}_{\rm curl}(\Gamma_j)} \parallel \mathbf{w} \parallel_{H^{-1/2}_{\rm curl}(\Gamma_j)},$$

where

$$C_j = \frac{(F_j(K_j))^{1/4}}{\omega\mu_j}\max\left\{\sqrt{\varphi_j^2+\psi_j^2}, 1\right\}.$$

Thus, from Lemma 2.3, we have

$$\parallel T_j\mathbf{u} \parallel_{H^{-1/2}_{\rm div}(\Gamma_j)} \le C \sup_{\mathbf{w}\in H^{-1/2}_{\rm curl}(\Gamma_j)} \frac{|\langle T_j\mathbf{u}, \mathbf{w}\rangle|}{||\mathbf{w}||_{H^{-1/2}_{\rm curl}(\Gamma_j)}} \le CC_j||\mathbf{u}||_{H^{-1/2}_{\rm curl}(\Gamma_j)}. \qquad \square$$

LEMMA 3.2. *Let* $\mathbf{u}$ *be in* $H^{-1/2}_{\rm curl}(\Gamma_j)$. *It holds that* $\mathrm{Re}\,\langle T_j\mathbf{u}, \mathbf{u}\rangle \ge 0$. *If* $\mathrm{Re}\,\langle T_j\mathbf{u}, \mathbf{u}\rangle = 0$, *then* $\mathbf{u} = 0$.

Proof. By definitions (2.8), (3.5), and (3.7), we obtain

$$\langle T_j\mathbf{u}, \mathbf{u}\rangle = \frac{1}{\omega\mu_j}\int_{\mathbb{R}^2}\left[\beta_j(|\hat{u}_1|^2 + |\hat{u}_2|^2) + \frac{1}{\beta_j}|\xi_1\hat{u}_1 + \xi_2\hat{u}_2|^2\right] d\xi.$$

Taking the real part gives

$$\mathrm{Re}\,\langle T_j\mathbf{u}, \mathbf{u}\rangle = \frac{1}{\omega\mu_j}\int_{\mathbb{R}^2}\left[a_j(|\hat{u}_1|^2 + |\hat{u}_2|^2) + \frac{a_j}{a_j^2+b_j^2}|\xi_1\hat{u}_1 + \xi_2\hat{u}_2|^2\right] d\xi \ge 0.$$

Hence $\mathrm{Re}\,\langle T_j\mathbf{u},\mathbf{u}\rangle = 0$ implies

$$|\hat{u}_1|^2 + |\hat{u}_2|^2 = 0,$$

which yields $\mathbf{u} = 0$. ∎

LEMMA 3.3. *There exist two positive constants C_1 and C_2 such that*

$$\mathrm{Im}\,\langle T_j\mathbf{u},\mathbf{u}\rangle \geq \; C_1 \parallel \mathrm{curl}_{\Gamma_j}\mathbf{u} \parallel^2_{H^{-1/2}(\Gamma_j)} - C_2 \parallel \mathbf{u} \parallel^2_{H^{-1/2}(\Gamma_j)}$$

for all $\mathbf{u}$ in $H^{-1/2}_{\mathrm{curl}}(\Gamma_j)$.

Proof. By definitions (2.8), (3.6), and (3.8), we obtain

$$\langle T_j\mathbf{u},\mathbf{u}\rangle = \frac{1}{\omega\mu_j}\int_{\mathbb{R}^2}\left[\frac{\omega^2\varepsilon_j\mu_j}{\beta_j}(|\hat{u}_1|^2+|\hat{u}_2|^2) - \frac{1}{\beta_j}|\xi_1\hat{u}_2 - \xi_2\hat{u}_1|^2\right]d\xi.$$

Taking the imaginary part gives

$$\mathrm{Im}\langle T_j\mathbf{u},\mathbf{u}\rangle = \frac{1}{\omega\mu_j}\int_{\mathbb{R}^2}\left[\frac{b_j}{a_j^2+b_j^2}|\xi_1\hat{u}_2-\xi_2\hat{u}_1|^2 \right.$$
$$\left. + \frac{(a_j\psi_j - b_j\varphi_j)}{a_j^2+b_j^2}(|\hat{u}_1|^2+|\hat{u}_2|^2)\right]d\xi. \tag{3.14}$$

To prove the lemma, it is required to estimate

$$\frac{1}{\omega\mu_j}\frac{b_j(1+|\xi|^2)^{1/2}}{a_j^2+b_j^2} = \frac{1}{\sqrt{2}\omega\mu_j}\left[\frac{\left(\sqrt{\phi_j^2+\psi_j^2}-\phi_j\right)(1+\varphi_j-\phi_j)}{\phi_j^2+\psi_j^2}\right]^{1/2}.$$

Let

$$G_j(t) = \frac{\left(\sqrt{t^2+\psi_j^2}-t\right)(1+\varphi_j-t)}{t^2+\psi_j^2}.$$

It can be seen that G_j is a continuous positive function for $t \leq \varphi_j$ and $G_j(t) \to 2$ as $t \to -\infty$, and thus $G_j(t)$ can reach its minimum at some t_*. Therefore, we have

$$\frac{1}{\omega\mu_j}\frac{b_j(1+|\xi|^2)^{1/2}}{a_j^2+b_j^2} \geq \frac{\sqrt{G_j(t_*)}}{\sqrt{2}\omega\mu_j} = C_1.$$

Next we estimate

$$\frac{1}{\omega\mu_j}\left|\frac{(a_j\psi_j-b_j\varphi_j)}{a_j^2+b_j^2}(1+|\xi|^2)^{1/2}\right|$$
$$= \frac{1}{\omega\mu_j}\left[\frac{(1+\varphi_j-\phi_j)}{\phi_j^2+\psi_j^2}\left(\psi_j^2\frac{\sqrt{\phi_j^2+\psi_j^2}+\phi_j}{2} + \varphi_j^2\frac{\sqrt{\phi_j^2+\psi_j^2}-\phi_j}{2} - \varphi_j\psi_j^2\right)\right]^{1/2}.$$

Let

$$H_j(t) = \frac{(1+\varphi_j-t)}{t^2+\psi_j^2}\left(\psi_j^2\frac{\sqrt{t^2+\psi_j^2}+t}{2} + \varphi_j^2\frac{\sqrt{t^2+\psi_j^2}-t}{2} - \varphi_j\psi_j^2\right).$$

It can be seen that H_j is a continuous positive function for $t \le \varphi_j$ and $H_j(t) \to \varphi_j^2$ as $t \to -\infty$, and thus $H_j(t)$ can reach its maximum at some t^*. Therefore, we have

$$\frac{1}{\omega\mu_j}\left|\frac{(a_j\psi_j - b_j\varphi_j)}{a_j^2+b_j^2}(1+|\xi|^2)^{1/2}\right| \le \frac{\sqrt{H_j(t^*)}}{\omega\mu_j} = C_2.$$

The proof of the lemma follows from plugging the above estimates into (3.14). ∎

4. Variational problem. In this section, we shall introduce the variational formulation for the infinite rough surface scattering problem using transparent boundary conditions. The existence and uniqueness of a weak solution for the model problem are established by the variational approach.

We present a variational formulation of the Maxwell system in the space $H(\mathrm{curl},\Omega)$ and give a simple proof of the well-posedness for the boundary value problem. By eliminating the magnetic field from (2.1), we obtain the equation for the electric field:

$$\mathrm{curl}\,(\mu^{-1}\mathrm{curl}\,\mathbf{E}) - \omega^2\varepsilon\mathbf{E} = 0 \quad \text{in } \Omega. \tag{4.1}$$

Multiplying the complex conjugate of a test function $\mathbf{v} \in H(\mathrm{curl},\Omega)$, integrating over Ω, and using integration by parts (recall that the density Lemma 2.2 implies that the lateral boundary integrals vanish as $\rho \to \infty$), we arrive at the variational form for the scattering problem: find $\mathbf{E} \in H(\mathrm{curl},\Omega)$ such that

$$a(\mathbf{E},\mathbf{v}) = \langle \mathbf{f},\mathbf{v}\rangle_{\Gamma_1} \quad \text{for all } \mathbf{v} \in H(\mathrm{curl},\Omega), \tag{4.2}$$

with the sesquilinear form

$$a(\mathbf{u},\mathbf{v}) = \int_\Omega \mu^{-1}\mathrm{curl}\,\mathbf{u}\cdot\mathrm{curl}\,\overline{\mathbf{v}} - \int_\Omega \omega^2\varepsilon\mathbf{u}\cdot\overline{\mathbf{v}} - \mathrm{i}\omega\sum_{j=1}^{2}\int_{\Gamma_j} T_j\mathbf{u}_{\Gamma_j}\cdot\overline{\mathbf{v}}_{\Gamma_j} \tag{4.3}$$

and the linear functional

$$\langle \mathbf{f},\mathbf{v}\rangle_{\Gamma_1} = \int_{\Gamma_1}\mathbf{f}\cdot\overline{\mathbf{v}}_{\Gamma_1} \quad \text{with} \quad \mathbf{f} = \mathrm{i}\omega(\mathbf{H}^{\mathrm{inc}}\times\mathbf{n}_1 - T_1\mathbf{E}^{\mathrm{inc}}_{\Gamma_1}). \tag{4.4}$$

THEOREM 4.1. *The variational problem* (4.2) *has a unique solution. Moreover, there exists a constant γ_2 depending only on ω,μ,ε such that*

$$|a(\mathbf{u},\mathbf{u})| \ge \gamma_2 \,\|\, \mathbf{u}\,\|^2_{H(\mathrm{curl},\Omega)} \quad \textit{for all } \mathbf{u} \in H(\mathrm{curl},\Omega). \tag{4.5}$$

Proof. We first prove the continuity and coercivity of the sesquilinear form a. The continuity follows directly from the Cauchy–Schwarz inequality and Lemma 3.1:

$$\begin{aligned}
|a(\mathbf{u},\mathbf{v})| &\le C_1 \,\|\, \mathbf{u}\,\|_{H(\mathrm{curl},\,\Omega)}\|\, \mathbf{v}\,\|_{H(\mathrm{curl},\,\Omega)} + C_2\sum_{j=1}^{2}\|\, T_j\mathbf{u}_{\Gamma_j}\,\|_{H^{-1/2}_{\mathrm{div}}(\Gamma_j)}\|\, \mathbf{v}_{\Gamma_j}\,\|_{H^{-1/2}_{\mathrm{curl}}(\Gamma_j)} \\
&\le C_1 \,\|\, \mathbf{u}\,\|_{H(\mathrm{curl},\,\Omega)}\|\, \mathbf{v}\,\|_{H(\mathrm{curl},\,\Omega)} + C_2 \,\|\, \mathbf{u}_{\Gamma_j}\,\|_{H^{-1/2}_{\mathrm{curl}}(\Gamma_j)}\|\, \mathbf{v}_{\Gamma_j}\,\|_{H^{-1/2}_{\mathrm{curl}}(\Gamma_j)} \\
&\le C \,\|\, \mathbf{u}\,\|_{H(\mathrm{curl},\,\Omega)}\|\, \mathbf{v}\,\|_{H(\mathrm{curl},\,\Omega)}.
\end{aligned}$$

Taking the real part of the sesquilinear form a and using Lemma 3.3 yields

$$\begin{aligned}
\mathrm{Re}[a(\mathbf{u},\mathbf{u})] &= \int_\Omega \mu^{-1}|\mathrm{curl}\,\mathbf{u}|^2 - \omega^2\int_\Omega \mathrm{Re}(\varepsilon)|\mathbf{u}|^2 + \omega\sum_{j=1}^{2}\mathrm{Im}\langle T_j\mathbf{u}_{\Gamma_j},\mathbf{u}_{\Gamma_j}\rangle \\
&\ge \int_\Omega \mu^{-1}|\mathrm{curl}\,\mathbf{u}|^2 - \omega^2\int_\Omega \mathrm{Re}(\varepsilon)|\mathbf{u}|^2 - \omega C_2\sum_{j=1}^{2}\|\, \mathbf{u}_{\Gamma_j}\,\|^2_{H^{-1/2}(\Gamma_j)}.
\end{aligned}$$

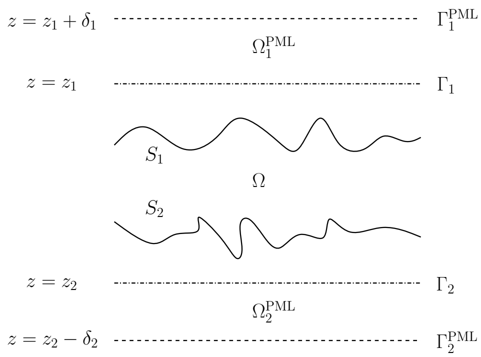

FIG. 2. *Geometry of the PML problem.*

Using Lemma 2.5 and setting η sufficiently small there, we find that

$$\mathrm{Re}[a(\mathbf{u},\mathbf{u})] \geq C_3 \parallel \mathrm{curl}\,\mathbf{u} \parallel_{0,\Omega}^2 - C_4 \parallel \mathbf{u} \parallel_{0,\Omega}^2,$$

where the constants $C_3 > 0$ and $C_4 \geq 0$. Taking the imaginary part of the sesquilinear form a and using Lemma 3.2 gives

$$\mathrm{Im}[a(\mathbf{u},\mathbf{u})] = -\omega^2 \int_\Omega \mathrm{Im}(\varepsilon)|\mathbf{u}|^2 - \omega \sum_{j=1}^{2} \mathrm{Re}\,\langle T_j \mathbf{u}_{\Gamma_j}, \mathbf{u}_{\Gamma_j}\rangle \leq -\omega^2 \int_\Omega \mathrm{Im}(\varepsilon)|\mathbf{u}|^2,$$

which yields with a constant $C_5 > 0$ that

$$|\mathrm{Im}[a(\mathbf{u},\mathbf{u})]| \geq C_5 \parallel \mathbf{u} \parallel_{0,\Omega}^2 .$$

Denote $\alpha = (1 + C_4)/C_5 > 0$. Then simple calculations yield

$$\begin{aligned}(1+\alpha)|a(\mathbf{u},\mathbf{u})| &\geq |\mathrm{Re}[a(\mathbf{u},\mathbf{u})]| + \alpha|\mathrm{Im}[a(\mathbf{u},\mathbf{u})]| \\ &\geq C_3 \parallel \mathrm{curl}\,\mathbf{u} \parallel_{0,\Omega}^2 + \parallel \mathbf{u} \parallel_{0,\Omega}^2 \geq C \parallel \mathbf{u} \parallel_{H(\mathrm{curl},\Omega)}^2 .\end{aligned}$$

This proves (4.5), namely, the coercivity of the sesquilinear form a in $H(\mathrm{curl},\Omega)$.

Now the Lax–Milgram lemma shows the existence and uniqueness of a solution of the variational problem (4.2). □

5. PML formulation and convergence. In this section, we introduce the variational formulation for the infinite rough surface scattering problem using the PML technique. Our goal is to prove the existence and uniqueness of the solution to the PML problem, and to derive an error estimate between the solution to the PML problem and the solution to the original rough surface scattering problem.

5.1. PML formulation. Now we turn to the introduction of absorbing PML layers. The domain Ω is surrounded with two PML layers of thicknesses δ_1 and δ_2 in Ω_1 and Ω_2, respectively. Figure 2 shows the geometry of the PML problem. The specially designed model medium in the PML layers should basically be chosen so that either the wave never reaches its external boundary or the amplitude of the reflected wave is so small that it does not essentially contaminate the solution in Ω. Let $s(\tau) = s_1(\tau) + \mathrm{i}s_2(\tau)$ be the model medium property which is continuous and satisfies

$$s_1 = 1,\ s_2 = 0 \quad \text{in } (z_2, z_1) \quad \text{and} \quad s_1 \geq 1,\ s_2 > 0 \quad \text{otherwise.} \tag{5.1}$$

We remark that, in contrast to the original PML condition which takes $s_1 = 1$ in the PML region, a variable s_1 is allowed in order to attenuate both the outgoing and evanescent waves. The advantage of this extension is that it makes our method insensitive to the distance of the PML region from the structure. Following the general idea in designing PML absorbing layers in Teixeira and Chew [47], we introduce the PML by complex coordinate stretching:

$$\tilde{z} = \int_0^z s(\tau) d\tau. \tag{5.2}$$

Define

$$\mathbf{E}_1^{\mathrm{inc}} = \mathbf{E}^{\mathrm{inc}}, \quad \mathbf{H}_1^{\mathrm{inc}} = \mathbf{H}^{\mathrm{inc}}, \quad \text{and} \quad \mathbf{E}_2^{\mathrm{inc}} = \mathbf{H}_2^{\mathrm{inc}} = 0.$$

Let $\widetilde{\mathbf{x}} = (x, y, \tilde{z})^\top$. It is clear that $\tilde{z} = z$ for $\mathbf{x} \in \Omega$ and

$$\operatorname{curl} \operatorname{diag}(1, 1, s(z))\mathbf{u} = \operatorname{diag}(s(z), s(z), 1) \operatorname{curl}_{\tilde{z}} \mathbf{u}.$$

Introduce the new fields $(\widetilde{\mathbf{E}}, \widetilde{\mathbf{H}})$:

$$\begin{aligned} \widetilde{\mathbf{E}}(\mathbf{x}) &= \mathbf{E}_j^{\mathrm{inc}}(\mathbf{x}) + \operatorname{diag}\,(1, 1, s(z)) \left(\mathbf{E}(\widetilde{\mathbf{x}}) - \mathbf{E}_j^{\mathrm{inc}}(\widetilde{\mathbf{x}})\right), \\ \widetilde{\mathbf{H}}(\mathbf{x}) &= \mathbf{H}_j^{\mathrm{inc}}(\mathbf{x}) + \operatorname{diag}\,(1, 1, s(z)) \left(\mathbf{H}(\widetilde{\mathbf{x}}) - \mathbf{H}_j^{\mathrm{inc}}(\widetilde{\mathbf{x}})\right). \end{aligned} \tag{5.3}$$

It follows from (2.1) and (5.2) that the fields $(\widetilde{\mathbf{E}}, \widetilde{\mathbf{H}})$ satisfy the following Maxwell equations:

$$\begin{gathered} \operatorname{curl} (\widetilde{\mathbf{E}} - \mathbf{E}^{\mathrm{inc}}) - \mathrm{i}\omega\tilde{\mu}(\widetilde{\mathbf{H}} - \mathbf{H}^{\mathrm{inc}}) = 0, \quad \operatorname{curl} (\widetilde{\mathbf{H}} - \mathbf{H}^{\mathrm{inc}}) + \mathrm{i}\omega\tilde{\varepsilon}(\widetilde{\mathbf{E}} - \mathbf{E}^{\mathrm{inc}}) \quad \text{in } \Omega_1, \\ \operatorname{curl} \widetilde{\mathbf{E}} - \mathrm{i}\omega\tilde{\mu}\widetilde{\mathbf{H}} = 0, \quad \operatorname{curl} \widetilde{\mathbf{H}} + \mathrm{i}\omega\tilde{\varepsilon}\,\widetilde{\mathbf{E}} = 0 \quad \text{otherwise,} \end{gathered} \tag{5.4}$$

where

$$\tilde{\varepsilon} = \operatorname{diag}\,(s(z), s(z), 1/s(z))\,\varepsilon, \qquad \tilde{\mu} = \operatorname{diag}\,(s(z), s(z), 1/s(z))\,\mu. \tag{5.5}$$

Define the PML regions

$$\Omega_1^{\mathrm{PML}} = \{\mathbf{x} \in \mathbb{R}^3 : z_1 < z < z_1 + \delta_1\} \quad \text{and} \quad \Omega_2^{\mathrm{PML}} = \{\mathbf{x} \in \mathbb{R}^3 : z_2 - \delta_2 < z < z_2\}.$$

The perfect conductor boundary condition can thus be imposed on

$$\Gamma_1^{\mathrm{PML}} = \{\mathbf{x} \in \mathbb{R}^3 : z = z_1 + \delta_1\} \quad \text{and} \quad \Gamma_2^{\mathrm{PML}} = \{\mathbf{x} \in \mathbb{R}^3 : z = z_2 - \delta_2\}$$

to truncate the PML media. We arrive at the following truncated PML problem: Find $(\mathbf{E}^{\mathrm{PML}}, \mathbf{H}^{\mathrm{PML}})$ such that

$$\begin{gathered} \operatorname{curl} \mathbf{E}^{\mathrm{PML}} - \mathrm{i}\omega\tilde{\mu}\mathbf{H}^{\mathrm{PML}} = \mathbf{g}_1, \quad \operatorname{curl} \mathbf{H}^{\mathrm{PML}} + \mathrm{i}\omega\tilde{\varepsilon}\mathbf{E}^{\mathrm{PML}} = \mathbf{g}_2, \\ \mathbf{E}^{\mathrm{PML}} \times \mathbf{n}_1 = \mathbf{E}^{\mathrm{inc}} \times \mathbf{n}_1 \quad \text{on } \Gamma_1^{\mathrm{PML}}, \quad \mathbf{E}^{\mathrm{PML}} \times \mathbf{n}_2 = 0 \quad \text{on } \Gamma_2^{\mathrm{PML}}, \end{gathered} \tag{5.6}$$

where

$$\begin{gathered} \mathbf{g}_1 = \operatorname{curl} \mathbf{E}^{\mathrm{inc}} - \mathrm{i}\omega\tilde{\mu}\mathbf{H}^{\mathrm{inc}}, \quad \mathbf{g}_2 = \operatorname{curl} \mathbf{H}^{\mathrm{inc}} + \mathrm{i}\omega\tilde{\varepsilon}\mathbf{E}^{\mathrm{inc}} \quad \text{in } \Omega_1^{\mathrm{PML}}, \\ \mathbf{g_1} = \mathbf{g}_2 = 0 \quad \text{otherwise.} \end{gathered} \tag{5.7}$$

We next present a weak formulation of the PML problem (5.6) in the domain

$$D = \{\mathbf{x} \in \mathbb{R}^3 : z_2 - \delta_2 < z < z_1 + \delta_1\}.$$

Eliminating the magnetic field from (5.6), we obtain the equation for the electric field:

$$\operatorname{curl}(\tilde{\mu}^{-1} \operatorname{curl} \mathbf{E}^{\mathrm{PML}}) - \omega^2 \tilde{\varepsilon}\, \mathbf{E}^{\mathrm{PML}} = \mathbf{h} \quad \text{in } D, \tag{5.8}$$

where

$$\begin{aligned} \mathbf{h} &= \operatorname{curl}(\tilde{\mu}^{-1} \operatorname{curl} \mathbf{E}^{\mathrm{inc}}) - \omega^2 \tilde{\varepsilon}\, \mathbf{E}^{\mathrm{inc}} \quad \text{in } \Omega_1^{\mathrm{PML}} \\ \mathbf{h} &= 0 \quad \text{otherwise}. \end{aligned} \tag{5.9}$$

Introduce the sesquilinear form $a_D : H(\operatorname{curl}, D) \times H(\operatorname{curl}, D) \to \mathbb{C}$ as

$$a_D(\mathbf{u}, \mathbf{v}) = \int_D \tilde{\mu}^{-1} \operatorname{curl} \mathbf{u} \cdot \operatorname{curl} \overline{\mathbf{v}} - \omega^2 \int_D \tilde{\varepsilon} \mathbf{u} \cdot \overline{\mathbf{v}}. \tag{5.10}$$

Define $H_0(\operatorname{curl}, D) = \{\mathbf{u} \in H(\operatorname{curl}, D),\ \mathbf{u} \times \mathbf{n}_j = 0 \text{ on } \Gamma_j^{\mathrm{PML}},\ j = 1, 2\}$. The weak formulation on the PML model (5.6) reads as follows: Find $\mathbf{E}^{\mathrm{PML}} \in H(\operatorname{curl}, D)$ such that $\mathbf{E}^{\mathrm{PML}} \times \mathbf{n}_1 = \mathbf{E}^{\mathrm{inc}} \times \mathbf{n}_1$ on Γ_1^{PML}, $\mathbf{E}^{\mathrm{PML}} \times \mathbf{n}_2 = 0$ on Γ_2^{PML}, and

$$a_D(\mathbf{E}^{\mathrm{PML}}, \mathbf{F}) = \int_D \mathbf{h} \cdot \overline{\mathbf{F}} \quad \text{for all } \mathbf{F} \in H_0(\operatorname{curl}, D). \tag{5.11}$$

We will reformulate the variational problem (5.11) in the domain D into an equivalent variational formulation in the domain Ω and discuss the existence and convergence of the weak solution to the equivalent weak formulation. To do so, we need to introduce the transparent boundary condition for the PML problem.

5.2. Transparent boundary condition for the PML problem. We introduce transparent boundary conditions for the PML problem by using capacity operators. Similarly, we will show only how to deduce the transparent boundary condition on Γ_1, and state the corresponding transparent boundary condition on Γ_2 without derivation.

In Ω_1^{PML}, the scattered electric field $\widetilde{\mathbf{E}}^{\mathrm{sc}} = \widetilde{\mathbf{E}} - \mathbf{E}^{\mathrm{inc}}$ and the scattered magnetic field $\widetilde{\mathbf{H}}^{\mathrm{sc}} = \widetilde{\mathbf{H}} - \mathbf{H}^{\mathrm{inc}}$ satisfy

$$\operatorname{curl} \widetilde{\mathbf{E}}^{\mathrm{sc}} - \mathrm{i}\omega \tilde{\mu}_1 \widetilde{\mathbf{H}}^{\mathrm{sc}} = 0, \qquad \operatorname{curl} \widetilde{\mathbf{H}}^{\mathrm{sc}} + \mathrm{i}\omega \tilde{\varepsilon}_1 \widetilde{\mathbf{E}}^{\mathrm{sc}} = 0, \tag{5.12}$$

where

$$\tilde{\varepsilon}_1 = \operatorname{diag}\,(s(z), s(z), 1/s(z))\, \varepsilon_1, \qquad \tilde{\mu}_1 = \operatorname{diag}\,(s(z), s(z), 1/s(z))\, \mu_1.$$

Let $\widetilde{\mathbf{E}}^{\mathrm{sc}} = (E_1, E_2, E_3)^\top$. Denote by $\widetilde{\mathbf{E}}^{\mathrm{sc}}_{\Gamma_1} = (E_1(\rho, z_1), E_2(\rho, z_1), 0)^\top$ the tangential component of the electric field on Γ_1. Denote by $\widetilde{\mathbf{H}}^{\mathrm{sc}} \times \mathbf{n}_1 = (H_2(\rho, z_1), -H_1(\rho, z_1), 0)^\top$ the tangential trace of the magnetic field on Γ_1, where

$$\begin{aligned} H_2(\rho, z_1) &= \frac{1}{\mathrm{i}\omega\mu_1} \left[\partial_z E_1(\rho, z_1) - \partial_x E_3(\rho, z_1)\right], \\ -H_1(\rho, z_1) &= \frac{1}{\mathrm{i}\omega\mu_1} \left[\partial_z E_2(\rho, z_1) - \partial_y E_3(\rho, z_1)\right]. \end{aligned}$$

Eliminating the magnetic field from (5.12), we arrive at the equation for the electric field,

$$\operatorname{curl}(\tilde{\mu}_1^{-1} \operatorname{curl} \widetilde{\mathbf{E}}^{\mathrm{sc}}) - \omega^2 \tilde{\varepsilon}_1 \widetilde{\mathbf{E}}^{\mathrm{sc}} = 0,$$

which has the components

$$s^{-1}\partial_z(s^{-1}\partial_z E_1) + \partial_y^2 E_1 - \partial_x[\partial_y E_2 + s^{-1}\partial_z(s^{-1}E_3)] + \omega^2\varepsilon_1\mu_1 E_1 = 0, \tag{5.13a}$$

$$s^{-1}\partial_z(s^{-1}\partial_z E_2) + \partial_x^2 E_2 - \partial_y[\partial_x E_1 + s^{-1}\partial_z(s^{-1}E_3)] + \omega^2\varepsilon_1\mu_1 E_2 = 0, \tag{5.13b}$$

$$\partial_z(\partial_x E_1 + \partial_y E_2) - \partial_x^2 E_3 - \partial_y^2 E_3 - \omega^2\varepsilon_1\mu_1 E_3 = 0. \tag{5.13c}$$

Noting $\nabla \cdot (\tilde{\varepsilon}_1 \widetilde{\mathbf{E}}^{\mathrm{sc}}) = 0$, we have

$$\partial_x E_1 + \partial_y E_2 + s^{-1}\partial_z(s^{-1}E_3) = 0. \tag{5.14}$$

Plugging (5.14) into (5.13) yields

$$s^{-1}\partial_z(s^{-1}\partial_z E_1) + \partial_y^2 E_1 + \partial_x^2 E_1 + \omega^2\varepsilon_1\mu_1 E_1 = 0, \tag{5.15a}$$

$$s^{-1}\partial_z(s^{-1}\partial_z E_2) + \partial_x^2 E_2 + \partial_y^2 E_2 + \omega^2\varepsilon_1\mu_1 E_2 = 0, \tag{5.15b}$$

$$\partial_z[s^{-1}\partial_z(s^{-1}E_3)] + \partial_x^2 E_3 + \partial_y^2 E_3 + \omega^2\varepsilon_1\mu_1 E_3 = 0. \tag{5.15c}$$

Following the complex coordinate stretching (5.2), using the perfect conductor boundary condition, and taking the Fourier transform with respect to ρ, we get the two-point boundary value problem to (5.15a) and (5.15b) for $j = 1, 2$:

$$\partial_{\tilde{z}}^2 \hat{E}_j(\xi, z) + (\omega^2\varepsilon_1\mu_1 - |\xi|^2)\hat{E}_j(\xi, z) = 0 \quad \text{in } \Omega_1^{\mathrm{PML}}, \tag{5.16a}$$

$$\hat{E}_j(\xi, z) = \hat{E}_j(\xi, z_1) \quad \text{on } \Gamma_1, \tag{5.16b}$$

$$\hat{E}_j(\xi, z) = 0 \quad \text{on } \Gamma_1^{\mathrm{PML}}. \tag{5.16c}$$

The general solution to (5.16) can be expressed in the variable z as

$$\hat{E}_j(\xi, z) = A_j e^{\mathrm{i}\beta_1(\xi)\int_0^z s(\tau)d\tau} + B_j e^{-\mathrm{i}\beta_1(\xi)\int_0^z s(\tau)d\tau}.$$

Applying the boundary conditions in (5.16) yields

$$A_j(\xi) = a_1(\xi)\hat{E}_j(\xi, z_1), \quad B_j(\xi) = b_1(\xi)\hat{E}_j(\xi, z_1),$$

where

$$a_1(\xi) = \frac{e^{-\mathrm{i}\beta_1(\xi)z_1}}{1 - e^{2\mathrm{i}\beta_1(\xi)\varpi_1}}, \qquad b_1(\xi) = \frac{e^{\mathrm{i}\beta_1(\xi)z_1}}{1 - e^{-2\mathrm{i}\beta_1(\xi)\varpi_1}},$$

and

$$\varpi_1 = \int_{z_1}^{z_1+\delta_1} s(\tau)d\tau. \tag{5.17}$$

Taking the inverse Fourier transform yields

$$E_j(\rho, z) = \int_{\mathbb{R}^2} \left[a_1(\xi)e^{\mathrm{i}\beta_1(\xi)\int_0^z s(\tau)d\tau} + b_1(\xi)e^{-\mathrm{i}\beta_1(\xi)\int_0^z s(\tau)d\tau}\right] \hat{E}_j(\xi, z_1)e^{\mathrm{i}\rho\cdot\xi}d\xi,$$

which gives after a simple calculation

$$\partial_z E_j(\rho, z_1) = \int_{\mathbb{R}^2} \mathrm{i}\beta_1(\xi) \coth(-\mathrm{i}\beta_1(\xi)\varpi_1) \hat{E}_j(\xi, z_1) e^{\mathrm{i}\rho\cdot\xi} d\xi.$$

Here $\coth(t) = (e^t + e^{-t})/(e^t - e^{-t})$.

Next we consider (5.15c). Let $F = s^{-1}E_3$. It follows from (5.14) and the perfect conductor boundary condition that

$$\partial_z F(\rho, z) = 0 \quad \text{on } \Gamma_1^{\mathrm{PML}}.$$

Following the complex coordinate stretching (5.2) and taking the Fourier transform with respect to ρ again, we get the two-point boundary value problem for F_3 with constant coefficients

$$\begin{aligned} \partial_{\tilde{z}}^2 \widehat{F}(\xi, z) + (\omega^2\varepsilon_1\mu_1 - |\xi|^2)\widehat{F}(\xi, z) &= 0 \quad \text{in } \Omega_1^{\mathrm{PML}}, \\ \widehat{F}(\xi, z) &= \hat{E}_3(\xi, z_1) \quad \text{on } \Gamma_1, \\ \partial_{\tilde{z}} \widehat{F}(\xi, z) &= 0 \quad \text{on } \Gamma_1^{\mathrm{PML}}. \end{aligned} \tag{5.18}$$

The general solution to (5.18) is

$$\widehat{F}(\xi, z) = s^{-1}\widehat{E}_3(\xi, z) = Ae^{\mathrm{i}\beta_1(\xi)\int_0^z s(\tau)d\tau} + Be^{-\mathrm{i}\beta_1(\xi)\int_0^z s(\tau)d\tau}. \tag{5.19}$$

Applying the boundary conditions in (5.18) yields

$$A(\xi) = a(\xi)\hat{E}_3(\xi, z_1), \quad B(\xi) = b(\xi)\hat{E}_3(\xi, z_1),$$

where

$$a(\xi) = \frac{e^{-\mathrm{i}\beta_1(\xi)z_1}}{1 + e^{2\mathrm{i}\beta_1(\xi)\varpi_1}}, \qquad b(\xi) = \frac{e^{\mathrm{i}\beta_1(\xi)z_1}}{1 + e^{-2\mathrm{i}\beta_1(\xi)\varpi_1}}.$$

Taking the partial derivative of (5.19) with respect to z gives

$$\partial_z(s^{-1}\widehat{E}_3(\xi, z)) = \mathrm{i}\beta_1(\xi)s\left(Ae^{\mathrm{i}\beta_1(\xi)\int_0^z s(\tau)d\tau} - Be^{-\mathrm{i}\beta_1(\xi)\int_0^z s(\tau)d\tau}\right).$$

Evaluating the above equation at z_1 yields

$$\partial_z(s^{-1}\widehat{E}_3)(\xi, z_1) = \mathrm{i}\beta_1(\xi)\left(Ae^{\mathrm{i}\beta_1(\xi)z_1} - Be^{-\mathrm{i}\beta_1(\xi)z_1}\right) = \frac{\mathrm{i}\beta_1(\xi)}{\coth(-\mathrm{i}\beta_1(\xi)\varpi_1)}\widehat{E}_3(\xi, z_1).$$

Recalling $\beta_1(\xi) \neq 0$ for all ξ, we have

$$\widehat{E}_3(\xi, z_1) = \frac{\coth(-\mathrm{i}\beta_1(\xi)\zeta_1)}{\mathrm{i}\beta_1(\xi)}\partial_z(s^{-1}\widehat{E}_3)(\xi, z_1).$$

Noting (5.14), we deduce that

$$\begin{aligned} E_3(\rho, z_1) &= \int_{\mathbb{R}^2} \frac{\coth(-\mathrm{i}\beta_1(\xi)\varpi_1)}{\mathrm{i}\beta_1(\xi)}\partial_z(s^{-1}\widehat{E}_3(\xi, z_1))e^{\mathrm{i}\rho\cdot\xi}d\xi \\ &= -\int_{\mathbb{R}^2} \frac{\coth(-\mathrm{i}\beta_1(\xi)\varpi_1)}{\beta_1(\xi)}[\xi_1\widehat{E}_1(\xi, z_1) + \xi_2\widehat{E}_2(\xi, z_1)]e^{\mathrm{i}\rho\cdot\xi}d\xi. \end{aligned}$$

Taking the partial derivative with respect to x and y gives

$$\partial_x E_3(\rho, z_1) = -\int_{\mathbb{R}^2} \frac{\mathrm{i}\xi_1}{\beta_1(\xi)} \left[\xi_1 \widehat{E}_1(\xi, z_1) + \xi_2 \widehat{E}_2(\xi, z_1)\right] \coth(-\mathrm{i}\beta_1(\xi)\varpi_1) e^{\mathrm{i}\rho\cdot\xi} d\xi,$$

$$\partial_y E_3(\rho, z_1) = -\int_{\mathbb{R}^2} \frac{\mathrm{i}\xi_2}{\beta_1(\xi)} \left[\xi_1 \widehat{E}_1(\xi, z_1) + \xi_2 \widehat{E}_2(\xi, z_1)\right] \coth(-\mathrm{i}\beta_1(\xi)\varpi_1) e^{\mathrm{i}\rho\cdot\xi} d\xi.$$

Therefore we have the components of tangential trace of the magnetic field on Γ_1:

$$\begin{aligned} H_2(\rho, b_1) = \frac{1}{\omega\mu_1} \int_{\mathbb{R}^2} &\left[\beta_1(\xi)\hat{E}_1(\xi, b_1) + \frac{\xi_1}{\beta_1(\xi)}(\xi_1 \hat{E}_1(\xi, b_1) + \xi_2 \hat{E}_2(\xi, b_1))\right] \\ &\times \coth(-\mathrm{i}\beta_1(\xi)\varpi_1) e^{\mathrm{i}\rho\cdot\xi} d\xi, \end{aligned}$$

$$\begin{aligned} -H_1(\rho, b_1) = \frac{1}{\omega\mu_1} \int_{\mathbb{R}^2} &\left[\beta_1(\xi)\hat{E}_2(\xi, b_1) + \frac{\xi_2}{\beta_1(\xi)}(\xi_1 \hat{E}_1(\xi, b_1) + \xi_2 \hat{E}_2(\xi, b_1))\right] \\ &\times \coth(-\mathrm{i}\beta_1(\xi)\varpi_1) e^{\mathrm{i}\rho\cdot\xi} d\xi, \end{aligned}$$

or equivalently

$$\begin{aligned} H_2(\rho, b_1) = \frac{1}{\omega\mu_1} \int_{\mathbb{R}^2} \frac{1}{\beta_1} &\left[\omega^2\varepsilon_1\mu_1 \hat{E}_1(\xi, b_1) + \xi_2(\xi_1 \hat{E}_2(\xi, b_1) - \xi_2 \hat{E}_1(\xi, b_1))\right] \\ &\times \coth(-\mathrm{i}\beta_1(\xi)\varpi_1) e^{\mathrm{i}\rho\cdot\xi} d\xi, \end{aligned}$$

$$\begin{aligned} -H_1(\rho, b_1) = \frac{1}{\omega\mu_1} \int_{\mathbb{R}^2} \frac{1}{\beta_1} &\left[\omega^2\varepsilon_1\mu_1 \hat{E}_1(\xi, b_1) - \xi_1(\xi_1 \hat{E}_2(\xi, b_1) - \xi_2 \hat{E}_1(\xi, b_1))\right] \\ &\times \coth(-\mathrm{i}\beta_1(\xi)\varpi_1) e^{\mathrm{i}\rho\cdot\xi} d\xi. \end{aligned}$$

Therefore, for any tangential vector $\mathbf{u} = (u_1, u_2, 0)^\top$ on Γ_1 whose Fourier transform is $\hat{\mathbf{u}}$, we can define the following capacity operator:

$$T_1^{\mathrm{PML}}\mathbf{u} = (v_1, v_2, 0)^\top,$$

where v_1, v_2 are, respectively, the inverse Fourier transforms of $\hat{v}_1, \hat{v}_2$, which are defined by

$$\begin{aligned} \hat{v}_1 &= \frac{1}{\omega\mu_1}\left[\beta_1\hat{u}_1 + \frac{\xi_1}{\beta_1}(\xi_1\hat{u}_1 + \xi_2\hat{u}_2)\right]\coth(-\mathrm{i}\beta_1(\xi)\varpi_1), \\ \hat{v}_2 &= \frac{1}{\omega\mu_1}\left[\beta_1\hat{u}_2 + \frac{\xi_2}{\beta_1}(\xi_1\hat{u}_1 + \xi_2\hat{u}_2)\right]\coth(-\mathrm{i}\beta_1(\xi)\varpi_1), \end{aligned} \tag{5.20}$$

or equivalently

$$\begin{aligned} \hat{v}_1 &= \frac{1}{\omega\mu_1\beta_1}\left[\omega^2\varepsilon_1\mu_1\hat{u}_1 + \xi_2(\xi_1\hat{u}_2 - \xi_2\hat{u}_1)\right]\coth(-\mathrm{i}\beta_1(\xi)\varpi_1), \\ \hat{v}_2 &= \frac{1}{\omega\mu_1\beta_1}\left[\omega^2\varepsilon_1\mu_1\hat{u}_2 - \xi_1(\xi_1\hat{u}_2 - \xi_2\hat{u}_1)\right]\coth(-\mathrm{i}\beta_1(\xi)\varpi_1). \end{aligned} \tag{5.21}$$

Similarly, for any tangential vector $\mathbf{u} = (u_1, u_2, 0)^\top$ on Γ_2, define the capacity operator T_2^{PML} as

$$T_2^{\mathrm{PML}}\mathbf{u} = (v_1, v_2, 0)^\top,$$

where

$$
\begin{aligned}
\hat v_1 &= \frac{1}{\omega\mu_2}\left[\beta_2 \hat u_1 + \frac{\xi_1}{\beta_2}(\xi_1\hat u_1 + \xi_2 \hat u_2)\right]\coth(-\mathrm{i}\beta_2(\xi)\varpi_2),\\
\hat v_2 &= \frac{1}{\omega\mu_2}\left[\beta_2 \hat u_2 + \frac{\xi_2}{\beta_2}(\xi_1\hat u_1 + \xi_2 \hat u_2)\right]\coth(-\mathrm{i}\beta_2(\xi)\varpi_2),
\end{aligned}
\tag{5.22}
$$

or equivalently

$$
\begin{aligned}
\hat v_1 &= \frac{1}{\omega\mu_2\beta_2}\left[\omega^2\varepsilon_2\mu_2\hat u_1 + \xi_2(\xi_1\hat u_2 - \xi_2\hat u_1)\right]\coth(-\mathrm{i}\beta_2(\xi)\varpi_2),\\
\hat v_2 &= \frac{1}{\omega\mu_2\beta_2}\left[\omega^2\varepsilon_2\mu_2\hat u_2 - \xi_1(\xi_1\hat u_2 - \xi_2\hat u_1)\right]\coth(-\mathrm{i}\beta_2(\xi)\varpi_2).
\end{aligned}
\tag{5.23}
$$

Here

$$
\beta_2^2(\xi) = \omega^2\varepsilon_2\mu_2 - |\xi|^2 \quad \text{with } \mathrm{Im}\beta_2(\xi) > 0
$$

and

$$
\varpi_2 = \int_{z_2-\delta_2}^{z_2} s(\tau)d\tau. \tag{5.24}
$$

Using the capacity operator, we may propose the following transparent boundary condition:

$$
\begin{aligned}
T_1^{\mathrm{PML}}(\mathbf{E}_{\Gamma_1}^{\mathrm{PML}} - \mathbf{E}_{\Gamma_1}^{\mathrm{inc}}) &= (\mathbf{H}^{\mathrm{PML}} - \mathbf{H}^{\mathrm{inc}})\times\mathbf{n}_1 \quad \text{on } \Gamma_1,\\
T_2^{\mathrm{PML}}(\mathbf{E}_{\Gamma_2}^{\mathrm{PML}}) &= \mathbf{H}^{\mathrm{PML}}\times\mathbf{n}_2 \quad \text{on } \Gamma_2,
\end{aligned}
\tag{5.25}
$$

which maps the tangential components of the electric fields to the tangential traces of the magnetic fields.

5.3. Convergence of the PML solution. We shall prove the existence and uniqueness of the solution of the above PML problem (5.11) and derive an error estimate between $\mathbf{E}^{\mathrm{PML}}$ and $\mathbf{E}$, the solution of the original infinite rough surface scattering problem in Ω. To achieve this goal, we first find an equivalent formulation of (5.11) in the domain Ω.

Noting $\mathbf{f} = \mathbf{g} = 0$ in Ω, we obtain the equation for the electric field after eliminating the magnetic field from (5.6):

$$
\mathrm{curl}\,(\mu^{-1}\mathrm{curl}\,\mathbf{E}^{\mathrm{PML}}) - \omega^2\varepsilon\mathbf{E}^{\mathrm{PML}} = 0 \quad \text{in } \Omega. \tag{5.26}
$$

Multiplying the complex conjugate of a test function $\mathbf{F}$ in $H(\mathrm{curl},\Omega)$, integrating over Ω, and using integration by parts, we arrive at the variational form for the PML problem: Find $\mathbf{E}^{\mathrm{PML}} \in H(\mathrm{curl},\Omega)$ such that

$$
a^{\mathrm{PML}}(\mathbf{E}^{\mathrm{PML}}, \mathbf{F}) = \langle \mathbf{f}^{\mathrm{PML}}, \mathbf{F}\rangle_{\Gamma_1} \quad \text{for all } \mathbf{F}\in H(\mathrm{curl},\Omega), \tag{5.27}
$$

where the sesquilinear form

$$
a^{\mathrm{PML}}(\mathbf{u},\mathbf{v}) = \int_\Omega \mu^{-1}\mathrm{curl}\,\mathbf{u}\cdot\mathrm{curl}\,\overline{\mathbf{v}} - \omega^2\int_\Omega \varepsilon\mathbf{u}\cdot\overline{\mathbf{v}} - \mathrm{i}\omega\sum_{j=1}^{2}\int_{\Gamma_j} T_j^{\mathrm{PML}}\mathbf{u}_{\Gamma_j}\cdot\overline{\mathbf{v}}_{\Gamma_j}, \tag{5.28}
$$

the linear functional

$$\langle \mathbf{f}^{\mathrm{PML}}, \mathbf{v}\rangle_{\Gamma_1} = \mathrm{i}\omega \int_{\Gamma_1} (\mathbf{H}^{\mathrm{inc}} \times \mathbf{n}_1 - T_1^{\mathrm{PML}} \mathbf{E}^{\mathrm{inc}}_{\Gamma_1}) \cdot \overline{\mathbf{v}}_{\Gamma_1}, \tag{5.29}$$

and

$$\mathbf{f}^{\mathrm{PML}} = \mathrm{i}\omega(\mathbf{H}^{\mathrm{inc}} \times \mathbf{n}_1 - T_1^{\mathrm{PML}} \mathbf{E}^{\mathrm{inc}}_{\Gamma_1}).$$

The following lemma establishes the relationship between this variational problem and the weak formulation (5.11) straightforwardly from the above derivation, and hence the details of its proof are omitted.

LEMMA 5.1. *Any solution of the problem* (5.11) *restricted to* Ω *is a solution of* (5.27). *Conversely, any solution of the problem* (5.27) *can be uniquely extended to the whole domain* D *as a solution of* (5.11).

Next we turn to estimating the error between $\mathbf{E}$ and $\mathbf{E}^{\mathrm{PML}}$. Clearly, it suffices to estimate the error between the capacity operator T_j and T_j^{PML}.

LEMMA 5.2. *Let* $s(\tau) = s_1(\tau) + \mathrm{i}s_2(\tau)$ *be the model medium property satisfying* (5.1). *We have*

$$\mathrm{Re}\,[-\mathrm{i}\beta_j s(\tau)] \geq \frac{s_1}{\sqrt{2}} \left(\sqrt{\varphi_j^2 + \psi_j^2} - \varphi_j\right)^{1/2},$$

where β_j *is defined in* (3.11) *and* $\varphi_j = \omega^2 \mathrm{Re}(\varepsilon_j \mu_j)$, $\psi_j = \omega^2 \mathrm{Im}(\varepsilon_j \mu_j)$.

Proof. A simple calculation yields

$$\mathrm{Re}\,[-\mathrm{i}\beta_j s(\tau)] = \mathrm{Re}\beta_j s_2(\tau) + \mathrm{Im}\beta_j s_1(\tau).$$

Recalling the definitions of $\mathrm{Re}\beta_j$ and $\mathrm{Im}\beta_j$ in (3.13), we easily obtain $\mathrm{Re}\beta_j \geq 0$ and

$$\mathrm{Im}\beta_j \geq \frac{1}{\sqrt{2}} \left(\sqrt{\varphi_j^2 + \psi_j^2} - \varphi_j\right)^{1/2},$$

where we have used (3.12) and the fact that $\mathrm{Im}\beta_j$ is a monotonically decreasing function of ϕ_j in (3.13). The proof is completed by combining the above estimates. ∎

The following lemma plays a key role in the subsequent analysis.

LEMMA 5.3. *For any* $\mathbf{u}$ *and* $\mathbf{v}$ *in* $H(\mathrm{curl}, \Omega)$, *we have*

$$\left|\omega \int_{\Gamma_j} (T_j - T_j^{\mathrm{PML}}) \mathbf{u}_{\Gamma_j} \cdot \overline{\mathbf{v}}_{\Gamma_j}\right| \leq M_j \parallel \mathbf{u}_{\Gamma_j} \parallel_{H^{-1/2}_{\mathrm{curl}}(\Gamma_j)} \parallel \mathbf{v}_{\Gamma_j} \parallel_{H^{-1/2}_{\mathrm{curl}}(\Gamma_j)},$$

where

$$M_j = \frac{2}{\mu_j(e^{\Lambda_j} - 1)} \times \left[1 + \frac{(1+\varphi_j)^2}{\psi_j^2}\right]^{1/4} \times \max\left\{\sqrt{\varphi_j^2 + \psi_j^2}, 1\right\},$$
$$\Lambda_j = \sqrt{2}\mathrm{Re}(\varpi_j) \left[(\varphi_j^2 + \psi_j^2)^{1/2} - \varphi_j\right]^{1/2}.$$

Proof. For any $\mathbf{u} = (u_1, u_2, u_3)^\top$, $\mathbf{v} = (v_1, v_2, v_3)^\top \in H(\mathrm{curl}, \Omega)$, it follows from the definition of T_j and T_j^{PML} that

$$\omega \int_{\Gamma_j} \left(T_j - T_j^{\mathrm{PML}}\right) \mathbf{u}_{\Gamma_j} \cdot \overline{\mathbf{v}}_{\Gamma_j} \tag{5.30}$$
$$= \int_{\mathbb{R}^2} \frac{1 - \coth(-\mathrm{i}\beta_j \varpi_j)}{\mu_j \beta_j} \left[\omega^2 \varepsilon_j \mu_j (\hat{u}_1 \overline{\hat{v}}_1 + \hat{u}_2 \overline{\hat{v}}_2) - (\xi_1 \hat{u}_2 - \xi_2 \hat{u}_1)(\xi_1 \overline{\hat{v}}_2 - \xi_2 \overline{\hat{v}}_1)\right] d\xi.$$

To prove the lemma, it is required to estimate

$$\frac{|1-\coth(-\mathrm{i}\beta_j\varpi_j)|(1+|\xi|^2)^{1/2}}{\mu_j|\beta_j|} = \frac{2}{\mu_j|e^{-2\mathrm{i}\beta_j\varpi_j}-1|}\left[\frac{(1+\varphi_j-\phi_j)^2}{\psi_j^2+\phi_j^2}\right]^{1/4}.$$

From Lemma 5.2,

$$|e^{-2\mathrm{i}\beta_j\varpi_j}-1| \geq |e^{-2\mathrm{i}\beta_j\varpi_j}|-1 \geq e^{\Lambda_j}-1.$$

Let

$$F_j(t) = \frac{(1+\varphi_j-t)^2}{\psi_j^2+t^2}.$$

We have proved in Lemma 3.1 that

$$\frac{(1+\varphi_j-\phi_j)^2}{\psi_j^2+\phi_j^2} = F_j(\phi_j) \leq F_j(K_j) = 1+\frac{(1+\varphi_j)^2}{\psi_j^2}.$$

Combining the above estimates yields

$$\frac{|1-\coth(-\mathrm{i}\beta_j\varpi_j)|(1+|\xi|^2)^{1/2}}{\mu_j|\beta_j|} \leq \frac{2(F_j(K_j))^{1/4}}{\mu_j(e^{\Lambda_j}-1)}.$$

Recall $|\omega^2\varepsilon_j\mu_j| = \sqrt{\varphi_j^2+\psi_j^2}$. The proof of the lemma follows from plugging the above estimate into (5.30) and using the Cauchy–Schwarz inequality. ◻

THEOREM 5.4. *Let γ_1 and γ_2 be the constants in Lemma* 2.4 *and in* (4.5)*, respectively. Suppose $(M_1+M_2)\gamma_1^2<\gamma_2$. Then the PML problem* (5.27) *has a unique solution $\mathbf{E}^{\mathrm{PML}}$. Moreover, it has the error estimate*

$$\| \mathbf{E}-\mathbf{E}^{\mathrm{PML}} \|_{\Omega} := \sup_{0\neq\mathbf{F}\in H(\mathrm{curl},\Omega)} \frac{|a(\mathbf{E}-\mathbf{E}^{\mathrm{PML}},\mathbf{F})|}{\| \mathbf{F} \|_{H(\mathrm{curl},\Omega)}}$$

$$\leq \gamma_1 M_1 \| \mathbf{E}^{\mathrm{PML}}-\mathbf{E}^{\mathrm{inc}} \|_{H^{-1/2}_{\mathrm{curl}}(\Gamma_1)} + \gamma_1 M_2 \| \mathbf{E}^{\mathrm{PML}} \|_{H^{-1/2}_{\mathrm{curl}}(\Gamma_2)}. \tag{5.31}$$

Proof. By Lemma 5.1, it suffices to show that the variational problem (5.27) has a unique solution. The key point is to show the coercivity for the sesquilinear form $a^{\mathrm{PML}}: H(\mathrm{curl},\Omega)\times H(\mathrm{curl},\Omega)\to\mathbb{C}$, defined in (5.28). Due to Lemmas 2.4 and 5.3 and the assumption $(M_1+M_2)\gamma_1^2<\gamma_2$, it is clear that for any $\mathbf{u}$ and $\mathbf{v}$ in $H(\mathrm{curl},\Omega)$

$$\begin{aligned}|a^{\mathrm{PML}}(\mathbf{u},\mathbf{v})| &\geq |a(\mathbf{u},\mathbf{v})| - \sum_{j=1}^{2}\left|\omega\int_{\Gamma_j}\left(T_j-T_j^{\mathrm{PML}}\right)\mathbf{u}_{\Gamma_j}\cdot\overline{\mathbf{v}}_{\Gamma_j}\right| \\ &\geq |a(\mathbf{u},\mathbf{v})| - (M_1+M_2)\gamma_1^2 \| \mathbf{u} \|_{H(\mathrm{curl},\Omega)} \| \mathbf{v} \|_{H(\mathrm{curl},\Omega)}.\end{aligned}$$

It remains to prove the estimate (5.31). By (4.2)–(4.4) and (5.27)–(5.29), we conclude that

$$\begin{aligned}a(\mathbf{E}-\mathbf{E}^{\mathrm{PML}},\mathbf{F}) &= -\mathrm{i}\omega\int_{\Gamma_1}(T_1-T_1^{\mathrm{PML}})\mathbf{E}^{\mathrm{inc}}_{\Gamma_1}\cdot\overline{\mathbf{F}}_{\Gamma_1} + a^{\mathrm{PML}}(\mathbf{E}^{\mathrm{PML}},\mathbf{F}) - a(\mathbf{E}^{\mathrm{PML}},\mathbf{F}) \\ &= \mathrm{i}\omega\int_{\Gamma_1}(T_1-T_1^{\mathrm{PML}})(\mathbf{E}^{\mathrm{PML}}_{\Gamma_1}-\mathbf{E}^{\mathrm{inc}}_{\Gamma_1})\cdot\overline{\mathbf{F}}_{\Gamma_1} \\ &\qquad + \mathrm{i}\omega\int_{\Gamma_2}(T_2-T_2^{\mathrm{PML}})\mathbf{E}^{\mathrm{PML}}_{\Gamma_2}\cdot\overline{\mathbf{F}}_{\Gamma_2}\end{aligned}$$

for any $\mathbf{F} \in H(\mathrm{curl}, \Omega)$. The proof is complete after using Lemmas 5.3 and 2.4. $\square$

Now let us take a closer look at the structure of constant M_j, which controls the modeling error of the PML equation towards the original grating problem. The constant M_j approaches zero exponentially as the PML parameters $\mathrm{Re}(\varpi_j)$ tend to infinity. From the definition (5.17) and (5.24), the quantities $\mathrm{Re}(\varpi_j)$ can be calculated by the medium property $s(\tau)$, which is usually taken as a power function:

$$s(\tau) = \begin{cases} 1 + \sigma_1 \left(\dfrac{\tau - z_1}{\delta_1} \right)^m & \text{if } \tau \geq z_1, \\ 1 + \sigma_2 \left(\dfrac{z_2 - \tau}{\delta_2} \right)^m & \text{if } \tau \leq z_2, \end{cases} \qquad m \geq 1.$$

Thus we have

$$\mathrm{Re}(\varpi_j) = \left[1 + \frac{\mathrm{Re}(\sigma_j)}{m+1} \right] \delta_j. \tag{5.32}$$

It is obvious that either enlarging the thickness δ_j of the PML layers or enlarging the medium parameters $\mathrm{Re}(\sigma_j)$ will reduce the PML approximation error.

6. Concluding remarks. In this paper we have proposed a variational formulation for the infinite rough surface scattering problem for Maxwell's equations and studied the use of the PML to truncate the scattering problem in the direction vertically away from the rough surfaces. The scattering problem is reduced to a boundary value problem by using transparent boundary conditions. We have shown the uniqueness and existence of the weak solution for the variational problem. Under some proper assumptions on the PML medium parameter, it is shown that the truncated PML problem attains a unique solution in $H(\mathrm{curl}, \Omega)$. An explicit error estimate between the solution of the scattering problem and that of the truncated PML problem in the computational domain is obtained. The error estimate implies particularly that the PML solution converges exponentially to the scattering solution by increasing either the PML medium parameter or the PML layer thickness. Computationally, the variational approach reported here leads naturally to a class of finite element methods. Analysis and computation of the finite element methods for the infinite rough surface scattering problem will be studied and reported elsewhere. Another closely related but more challenging problem is to study the infinite rough surface scattering problem in lossless medium, i.e., $\mathrm{Im}\,\varepsilon = 0$. Without energy decay, the capacity operator introduced in this paper is unbounded, and the proposed method cannot be directly applied to this case. We have not investigated the effect of the vertical side boundary truncation either, which is certainly an issue for the numerical computation. We are going to examine the lossless case and hope to be able to address these issues by studying the weighted Sobolev spaces and limiting absorption principle in the future.

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