An Overfilled Cavity Problem for Maxwell's Equations

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Abstract

This paper is concerned with the mathematical analysis of the scattering of a time-harmonic electromagnetic plane wave by an open and overfilled cavity which is embedded in a perfect electrically conducting infinite ground plane, where the electromagnetic wave propagation is governed by the Maxwell equations. Above the flat ground surface and the open aperture of the cavity, the space is assumed to be filled with a homogeneous medium with a constant permittivity and permeability; while the interior of the cavity is filled with some inhomogeneous medium with a variable permittivity and permeability. The scattering problem is modeled as a boundary value problem over a bounded domain, with transparent boundary condition proposed on the hemisphere enclosing the inhomogeneity represented by the cavity. The existence and uniqueness of the weak solution for the model problem are established by using a variational approach. The perfectly matched layer (PML) method is investigated to truncate the unbounded electromagnetic cavity scattering problem. It is shown that the truncated PML problem attains a unique solution. An explicit error estimate is given between the solution of the original scattering problem and that of the truncated PML problem. The error estimate implies that the PML solution converges exponentially to the original cavity scattering problem by increasing either the PML medium parameter or the PML layer thickness. The convergence result is expected to be useful for determining the PML medium parameter in the computational electromagnetic scattering problem.

Key words. Maxwell's equations, cavity scattering, variational formulation, perfectly matched layer, convergence

AMS subject classifications. 35Q61, 78A45, 78M30

1 Introduction

This paper is concerned with the mathematical analysis of an overfilled electromagnetic cavity scattering problem for Maxwell's equations. Broadly speaking, an overfilled cavity is referred to as an inhomogeneous medium with compact support whose partial boundary, called as the cavity wall, is a local perturbation of an infinite plane surface. So the cavity opening may protrude above

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the aperture on the infinite ground plane, which is in contrast with a regular cavity where the opening coincides with the aperture of the ground plane.

The analysis of the electromagnetic scattering properties of cavities in a conducting ground plane is of great interest to the engineering community due to its significant industrial and military applications, which include the design of cavity-backed conformal antennas and the characterization of radar cross section. The radar cross section is a measure of the detectability of a target by radar system. Deliberate control in the form of enhancement or reduction of the radar cross section of a target is of high importance in the aircraft or stealth design.

Time-harmonic analysis of cavity-backed apertures with penetrable material filling the cavity interior has been examined by numerous researchers in the engineering community; see e.g., Jin [30, 31], Jin and Volakis [32], Liu and Jin [42], Wood and Wood [50], and references cited therein. Mathematical treatment of the direct scattering problems involving cavities can be found in Ammari, Bao, and Wood [5, 6], Bao and Sun [14], Van and Wood [48], where a non-local transparent boundary condition, based on the Fourier transform, is proposed on the open aperture of the cavity. A closely related inverse cavity scattering problem, which is to determine the shape of the cavity, has been studied by Bao and Li [11], Feng and Ma [29], and Liu [43] for the two-dimensional Helmholtz equation, and Li [39] for the three-dimensional Maxwell equation in terms of uniqueness and stability. It is a common assumption that the cavity opening coincides with the aperture on an infinite ground plane, and hence simplifying the modeling of the exterior (to the cavity) domain. This limits the application of these methods since many cavity openings are not planar. Recently, Wood [49] has developed a technique for the two-dimensional Helmholtz equation that is capable of characterizing the scattering by over-filled cavities in the frequency domain, where an artificial boundary condition, based on Fourier series, is introduced on a semicircle enclosing the cavity. The solution domain is the cavity plus the interior region enclosed by the semicircle.

One of our goals in this paper is to generalize the results in Wood [49] to the three-dimensional Maxwell equation arising from the overfilled electromagnetic cavity scattering problem. We point out that the techniques completely differ from Wood [49] due to the more complicated model problem in the three-dimensional case. Specifically, we consider a time-harmonic electromagnetic plane wave incident at an open and overfilled cavity embedded in an infinite ground plane. The ground plane and the cavity wall are perfect electrical conductors. The interior of the cavity is filled with some inhomogeneous medium characterized by variable dielectric permittivity and magnetic permeability; while the space above the ground plane and the cavity opening is assumed to be filled with a homogeneous medium with a constant permittivity and permeability. Based on a transparent boundary condition proposed on a hemisphere enclosing the cavity, the scattering problem is modeled as a boundary value problem over a bounded domain. One of our main results for the overfilled cavity scattering problem indicates that it attains a unique weak solution for a general cavity medium. An important step of our approach is to introduce a Calderon operator and reduce the infinite nature of the scattering problem into a bounded domain via a transparent boundary condition on the hemisphere. The proofs rely on a combination of a variational approach, a Hodge decomposition, unique continuation, and the Fredholm alternative. Our method enjoys a great generality in the sense that it allows very general cavity structure and complex medium inside the cavity, i.e., the dielectric permittivity and magnetic permeability can be general spatially varying bounded measurable functions. Computationally, the variational approach reported here leads naturally to a class of finite edge element methods. Analysis and computation of the finite element methods for the scattering problem will be studied and reported elsewhere.

The PML technique, which was first proposed by Berenger [15, 16], is an important and popular mesh termination technique in computational wave propagation due to its effectiveness, simplicity, and flexibility; see e.g., Chen and Liu [24], Chen and Wu [25], Collino and Monk [26], Hohage, Schmidt, and Zschiedrich [33], Lassas and Somersalo [41], Teixera and Chew [46], and Turkel and Yefet [47]. Under the assumption that the exterior solution is composed of outgoing waves only, the basic idea of the PML technique is to surround the computational domain by a layer of finite thickness with specially designed model medium that either slows down or attenuates all the waves of any frequency propagating into the PML medium from inside the computational domain. In practice, the PML medium must be truncated and the truncation boundary generates reflective waves which can pollute the solution in the computational domain. Therefore, it is important to study the error estimate in the computational domain between the solution of the original scattering problem and that of the truncated PML problem. We refer to Bao and Wu [13], Bramble and Pasciak [17, 18], and Chen and Chen [23] for convergence analysis of the PML problems for three-dimensional electromagnetic obstacle scattering involving Maxwell's equations.

Another goal in this paper is to analyze the convergence of the PML solution for the overfilled electromagnetic cavity scattering problem. We shall use a PML to truncate the infinite half-space above the ground plane and the cavity into a bounded domain. Under a proper assumption on the PML medium parameter, we prove that the truncated PML problem attains a unique solution and obtain an error estimate between the solution of the scattering problem and the solution of the truncated PML problem in the computational domain. The error estimate implies particulary that the PML solution converges exponentially to the scattering problem when either the PML medium parameter or the thickness of the layer is increased. We refer to Zhang and Ma [52], Zhang, Ma, and Dong [53] for a finite element method with PML to solve the two-dimensional Helmholtz equation of the regular cavity scattering problem.

Related work for the scattering of acoustic and electromagnetic waves in a grating (periodic surface) structure (diffractive optics) and in an unbounded rough surface (with a nonlocal perturbation of a plane surface) have been extensively studied by either variational approaches or integral equation methods; see e.g., Ammari and Bao [2], Bao [10], Bao, Li, and Wu [12], Chandler-Wilde and Elschner [19], Chandler-Wilde and Monk [20, 21], Li, Wu, and Zheng [40], Li and Shen [41], and Lechleiter and Ritterbusch [38]. Recently, existence of the solution to the acoustic and electromagnetic scattering problems in an infinite periodic surface perturbed by a single inhomogeneous object placed inside the periodic structure is established via the integral equation methods by Ammari and Bao [3, 4]. One may consult Colton and Kress [27], Nédélec [45], and Monk [44] for extensive accounts of integral equation methods and finite element methods for general acoustic and electromagnetic scattering problems.

The outline of this paper is as follows. In Section 2, the Maxwell equations are introduced for the model problem. Section 3 discusses some of the properties of spherical harmonics, Sobolev spaces, and regularity of the trace operators on the hemisphere. The derivation of the transparent boundary condition and some estimates of the Calderon operator are presented in Section 4. Section 5 is devoted to the study of the variational problem over a bounded domain by using the transparent boundary condition. Based on a Hodge decomposition, compact embedding results, and the Fredholm alternative, the existence and uniqueness of the weak solution are established. The PML problem formulation and convergence analysis with an explicit error estimate for the truncated PML problem are presented in Section 6. The paper is concluded with some general remarks and directions for future research in Section 7.

2 Maxwell's equations

We shall introduce a mathematical model problem and define some notation for the electromagnetic scattering problems by an overfilled cavity. Let D be a cavity in the infinite ground plane

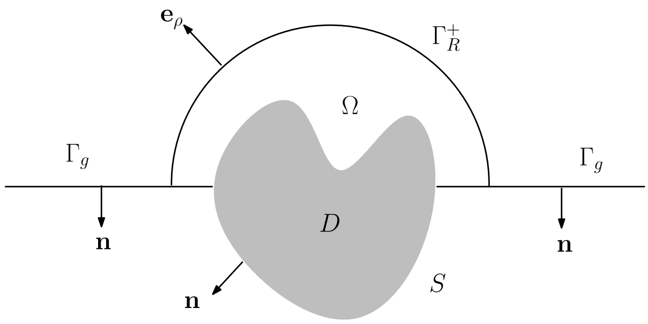

Figure 1: The problem geometry. An open and overfilled cavity D with boundary ∂D is placed on a perfect electrically conducting ground plane Γ_g. The cavity wall S is referred to as the part of the cavity boundary that lies below the ground plane.

with boundary ∂D, as seen in Figure 1. The cavity is filled with some inhomogeneous material, characterized by the dielectric permittivity ε and magnetic permeability μ, which protrudes above the ground plane. Denote by S the cavity wall, which is the part of the cavity boundary lying below the infinite ground plane and is assumed to be Lipschitz continuous. The infinite ground plane excluding the cavity opening is denoted as Γ_g. The infinite region above the cavity is denoted as $\mathbb{R}^3_+ \setminus D$, where it is filled with some homogeneous material represented by a constant dielectric permittivity ε_0 and a constant magnetic permeability μ_0. Throughout this paper, we assume for simplicity in exposition that $\varepsilon_0 = 1$ and $\mu_0 = 1$. Furthermore, let Γ_R^+ be a hemisphere of radius R large enough to completely enclose the overfilled portion of the cavity. We denote the region bounded by Γ_R^+ and the cavity wall S as Ω. Hence this region Ω consists of the cavity D and the homogeneous part between Γ_R^+ and the opening of the cavity. As we can see, the problem geometry is not only applied to the overfilled cavity scattering problems, but also to a broader class of scattering problems where the interface or the boundary is modeled as a local perturbation of an infinite plane.

The electromagnetic wave propagation is governed by the time-harmonic Maxwell equations (time dependence $e^{-\mathrm{i}\omega t}$):

$$\nabla \times \mathbf{E} = \mathrm{i}\omega\mu\mathbf{H}, \quad \nabla \times \mathbf{H} = -\mathrm{i}\omega\varepsilon\mathbf{E}, \tag{2.1}$$

where ω is the angular frequency, $\mathbf{E}$ and $\mathbf{H}$ are denoted as the electric field and the magnetic field, respectively. We assume that $\varepsilon \in L^\infty(D)$ and $\mu \in L^\infty(D)$ with

$$0 < \mathrm{Re}\,\varepsilon < \alpha, \quad \mathrm{Im}\,\varepsilon \geq 0, \quad \text{and} \quad 0 < \mu < \beta,$$

where α and β are two positive constants. The condition $\mathrm{Im}\,\varepsilon > 0$ corresponds for dispersive medium accounting for energy absorption. In this paper, we consider a general case with $\mathrm{Im}\,\varepsilon \geq 0$, which makes the analysis of the problem much more sophisticated. More regularity will be needed for ε and μ when proving the uniqueness via the unique continuation.

Due to the perfectly conducting material, the following homogenous Dirichlet boundary condition is satisfied for the tangential trace of the electric field on the infinite ground plane and the cavity wall:

$$\mathbf{n} \times \mathbf{E} = 0 \quad \text{on } \Gamma_g \cup S, \tag{2.2}$$

where $\mathbf{n}$ is the unit outward normal vector.

Let $(\mathbf{E}^{\mathrm{inc}}, \mathbf{H}^{\mathrm{inc}})$ be the plane waves that are incident upon the cavity from the above, where

$$\mathbf{E}^{\mathrm{inc}} = \mathbf{t}e^{\mathrm{i}\omega\mathbf{q}\cdot\mathbf{x}}, \quad \mathbf{H}^{\mathrm{inc}} = \mathbf{s}e^{\mathrm{i}\omega\mathbf{q}\cdot\mathbf{x}}, \quad \mathbf{s} = \frac{\mathbf{q} \times \mathbf{t}}{\omega}, \quad \mathbf{t} \cdot \mathbf{q} = 0.$$

Here $\mathbf{q} = (\alpha_1, \alpha_2, -\beta) = (\sin\theta_1\cos\theta_2, \sin\theta_1\sin\theta_2, -\cos\theta_1)$, and θ_1, θ_2 are incident angles satisfying $0 \le \theta_1 < \pi/2$, $0 \le \theta_2 < 2\pi$. Evidently, the incident electric field $\mathbf{E}^{\rm inc}$ and magnetic field $\mathbf{H}^{\rm inc}$ satisfy the Maxwell equation (2.1) for $\varepsilon = \varepsilon_0$ and $\mu = \mu_0$ in $\mathbb{R}^3_+ \setminus D$. The reflected fields $(\mathbf{E}^{\rm ref}, \mathbf{H}^{\rm ref})$ can be written as

$$\mathbf{E}^{\rm ref} = -\mathbf{t}e^{{\rm i}\omega\mathbf{q}^*\cdot\mathbf{x}} \quad \text{and} \quad \mathbf{H}^{\rm ref} = -\mathbf{t}\times\mathbf{q}^* e^{{\rm i}\omega\mathbf{q}^*\cdot\mathbf{x}},$$

where $\mathbf{q}^* = (\alpha_1, \alpha_2, \beta)$. Denote by

$$\mathbf{E}^{\rm b} = \mathbf{E}^{\rm inc} + \mathbf{E}^{\rm ref} \qquad \mathbf{H}^{\rm b} = \mathbf{H}^{\rm inc} + \mathbf{H}^{\rm ref}$$

the background electromagnetic fields from the unperturbed half-space, i.e., no cavity is present, then it is easy to check that

$$\mathbf{n}\times\mathbf{E}^{\rm b} = 0 \quad \text{on } \Gamma_g.$$

The total electric and magnetic fields can be decomposed as the summation of the background fields and the scattered fields:

$$\mathbf{E} = \mathbf{E}^{\rm b} + \mathbf{E}^{\rm s} \quad \text{and} \quad \mathbf{H} = \mathbf{H}^{\rm b} + \mathbf{H}^{\rm s}, \tag{2.3}$$

where $\mathbf{E}^{\rm s}, \mathbf{H}^{\rm s}$ are the scattered electric field and the magnetic field, respectively. In addition, the scattered fields are required to satisfy the Silver–Müller radiation condition:

$$\lim_{\rho\to\infty} \rho\left(\mathbf{E}^{\rm s} - \mathbf{H}^{\rm s}\times\mathbf{e}_\rho\right) = 0, \tag{2.4}$$

where $\rho = |\mathbf{x}|, \mathbf{e}_\rho = \mathbf{x}/|\mathbf{x}|$. Given the incident electromagnetic waves $(\mathbf{E}^{\rm inc}, \mathbf{H}^{\rm inc})$, the scattering problem is to determine the scattered fields $(\mathbf{E}^{\rm s}, \mathbf{H}^{\rm s})$ or equivalently the total fields $(\mathbf{E}, \mathbf{H})$ in the cavity D and in the homogeneous region above the cavity $\mathbb{R}^3_+ \setminus D$.

To analyze the problem, the open domain needs to be truncated into a bounded domain. Therefore, a suitable boundary condition has to be imposed on the boundary of the bounded domain so that no artificial wave reflection occurs. We shall present a transparent boundary condition on a hemisphere enclosing the inhomogeneous cavity. It is necessary to introduce some spherical harmonics in order to put the boundary operator in a suitable context.

3 Spherical harmonics on hemisphere

To describe the variational formulation of the cavity scattering problem, we introduce some of the properties of spherical harmonics on a hemisphere, which are relevant in scattering theory, and some Sobolev spaces for the boundary value problem. We refer to Lebedev [37] for a more detailed study of spherical harmonics on a whole sphere.

The spherical coordinates (ρ, θ, φ) are related to the Cartesian coordinates $\mathbf{x} = (x_1, x_2, x_3)$ by $x_1 = \rho\sin\theta\cos\varphi$, $x_2 = \rho\sin\theta\sin\varphi$, $x_3 = \rho\cos\theta$, with the local orthonormal basis $\{\mathbf{e}_\rho, \mathbf{e}_\theta, \mathbf{e}_\varphi\}$:

$$\begin{aligned}
\mathbf{e}_\rho &= (\sin\theta\cos\varphi,\ \sin\theta\sin\varphi,\ \cos\theta),\\
\mathbf{e}_\theta &= (\cos\theta\cos\varphi,\ \cos\theta\sin\varphi,\ -\sin\theta),\\
\mathbf{e}_\varphi &= (-\sin\varphi,\ \cos\varphi,\ 0),
\end{aligned}$$

where θ and φ are the Euler angles. Let $\Gamma = \{\mathbf{x} : \rho = 1\}$, $\Gamma^+ = \{\mathbf{x} : \rho = 1,\ x_3 \ge 0\}$, $\Gamma^- = \{\mathbf{x} : \rho = 1,\ x_3 \le 0\}$ be the unit sphere, upper unit hemisphere, and lower unit hemisphere, respectively. Denote by $\Gamma_R = \{\mathbf{x} : \rho = R\}$, $\Gamma_R^+ = \{\mathbf{x} : \rho = R,\ x_3 \ge 0\}$, $\Gamma_R^- = \{\mathbf{x} : \rho = R,\ x_3 \le 0\}$ the whole sphere, upper hemisphere, and lower hemisphere with radius R, respectively.

Following Colton and Kress [27], let $\{Y_n^m(\theta,\varphi),\ |m|\le n,\ n=0,1,2,\dots\}$ be an orthonormal sequence of spherical harmonics of order n on the unit sphere Γ that satisfies

$$\Delta_\Gamma Y_n^m + n(n+1)Y_n^m = 0, \tag{3.1}$$

where

$$\Delta_\Gamma = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}$$

is the Laplace-Beltrami operator on Γ. Explicitly, the spherical harmonics of order n is written as

$$Y_n^m(\theta,\varphi) = \sqrt{\frac{2n+1}{4\pi}\frac{(n-|m|)!}{(n+|m|)!}}\,P_n^{|m|}(\cos\theta)e^{im\varphi}, \tag{3.2}$$

where the associated Legendre functions are

$$P_n^m(t) := (1-t^2)^{m/2}\frac{d^m P_n(t)}{dt^m}, \quad m = 0,1,\dots,n. \tag{3.3}$$

Here P_n is the Legendre polynomial of degree n, which is an even function if n is even and an odd function if n is odd.

Next we shall introduce the spherical harmonics defined on the hemisphere Γ_R^+. Define a sequence of rescaled spherical harmonics of order n:

$$X_n^m(\theta,\varphi) = \frac{\sqrt{2}}{R}\,Y_n^m(\theta,\varphi). \tag{3.4}$$

Denote by $L^2(\Gamma_R^+)$ the complex square integrable functions on the hemisphere Γ_R^+. For convenience of notation, we simply the following double summation

$$\sum_{|m|\le n} w_n^m := \sum_{n=1}^{\infty}\sum_{m=-n}^{n} w_n^m, \quad \sum_{|m|\le n}^{\text{odd}} w_n^m := \sum_{n=1}^{\infty}\sum_{\substack{m=-n\\ m+n=\text{odd}}}^{n} w_n^m, \quad \sum_{|m|\le n}^{\text{even}} w_n^m := \sum_{n=1}^{\infty}\sum_{\substack{m=-n\\ m+n=\text{even}}}^{n} w_n^m.$$

Lemma 3.1. *The spherical harmonics* $X_n^m(\theta,\varphi)$ *for* $|m|\le n$, $m+n=\text{odd}$, $n\in\mathbb{N}$, *form a complete orthonormal system in* $L^2(\Gamma_R^+)$.

Proof. First we prove the orthogonality. It can be verified from the definitions of the spherical harmonics (3.2) and (3.3) that

$$Y_n^m(\pi-\theta,\varphi) = (-1)^{m+n}\,Y_n^m(\theta,\varphi). \tag{3.5}$$

Using (3.4), (3.5), and the change of variables, we have

$$\int_{\Gamma_R^+} X_n^m(\theta,\varphi)\bar{X}_{n'}^{m'}(\theta,\varphi) = 2\int_{\Gamma^+} Y_n^m(\theta,\varphi)\bar{Y}_{n'}^{m'}(\theta,\varphi) = \int_{\Gamma} Y_n^m(\theta,\varphi)\bar{Y}_{n'}^{m'}(\theta,\varphi),$$

where the bar is denoted as the complex conjugate. It is proved (cf., Colton and Kress [27], Theorem 2.7) that the spherical harmonics $Y_n^m(\theta,\varphi)$ for $|m|\le n$, $n=0,1,2,\dots$, form a complete orthonormal system in $L^2(\Gamma)$, which yields the orthogonality of X_n^m on Γ_R^+, i.e.,

$$\int_{\Gamma_R^+} X_n^m(\theta,\varphi)\bar{X}_{n'}^{m'}(\theta,\varphi) = \delta_{nn'}\delta_{mm'}.$$

Next is to prove the completeness. For any $w \in L^2(\Gamma_R^+)$, we define an extended function $\tilde{w}$ to the whole sphere Γ_R by odd reflection:

$$\tilde{w}(\theta, \varphi) = \begin{cases} w(\theta, \varphi) & \text{if } \ 0 \le \theta \le \frac{\pi}{2}, \\ -w(\pi - \theta, \varphi) & \text{if } \ \frac{\pi}{2} < \theta \le \pi. \end{cases}$$

Evidently, the extension $\tilde{w} \in L^2(\Gamma)$ and thus can be written in a series expansion in terms of the orthonormal system Y_n^m in $L^2(\Gamma)$:

$$\tilde{w}(\theta, \varphi) = \sum_{|m| \le n} w_n^m Y_n^m(\theta, \varphi), \tag{3.6}$$

where the coefficient w_n^m is given by

$$\begin{aligned} w_n^m &= \int_\Gamma \tilde{w}(\theta, \varphi) \bar{Y}_n^m(\theta, \varphi) = \int_{\Gamma^+} w(\theta, \varphi) \bar{Y}_n^m(\theta, \varphi) - \int_{\Gamma^-} w(\pi - \theta, \varphi) \bar{Y}_n^m(\theta, \varphi) \\ &= \int_{\Gamma^+} w(\theta, \varphi) \bar{Y}_n^m(\theta, \varphi) - (-1)^{m+n} \int_{\Gamma^+} w(\theta, \varphi) \bar{Y}_n^m(\theta, \varphi). \end{aligned}$$

Therefore we have

$$w_n^m = 0 \quad \text{for } m + n = \text{even}$$

and

$$w_n^m = 2 \int_{\Gamma^+} w(\theta, \varphi) \bar{Y}_n^m(\theta, \varphi) \quad \text{for } m + n = \text{odd}.$$

Plugging the coefficient w_n^m into (3.6) and restricting $\tilde{w}$ to Γ_R^+, we obtain a series expansion of w in terms of the spherical harmonics X_n^m:

$$w(\theta, \varphi) = \sum_{|m| \le n}^{\text{odd}} w_n^m X_n^m(\theta, \varphi),$$

where

$$w_n^m = \int_{\Gamma_R^+} w(\theta, \varphi) \bar{X}_n^m(\theta, \varphi),$$

which completes the proof. □

To describe vector wave functions on the hemisphere, we introduce some boundary differential operators. For a smooth scalar function w defined on Γ_R^+, let

$$\nabla_\Gamma w = \frac{\partial w}{\partial \theta} \mathbf{e}_\theta + \frac{1}{\sin \theta} \frac{\partial w}{\partial \varphi} \mathbf{e}_\varphi \tag{3.7}$$

be the tangential gradient on Γ_R^+. The surface vector curl is defined by

$$\mathbf{curl}_\Gamma w = \nabla_\Gamma w \times \mathbf{e}_\rho. \tag{3.8}$$

Denote by div_Γ and curl_Γ the surface divergence and the surface scalar curl, respectively. For a smooth vector function $\mathbf{w}$ tangential to Γ_R^+, it can be represented by its coordinates in the local orthonormal basis:

$$\mathbf{w} = w_\theta \, \mathbf{e}_\theta + w_\varphi \, \mathbf{e}_\varphi,$$

where

$$w_\theta = \mathbf{w}\cdot\mathbf{e}_\theta \quad \text{and} \quad w_\varphi = \mathbf{w}\cdot\mathbf{e}_\varphi.$$

The surface divergence and the surface scalar curl can be defined as

$$\mathrm{div}_\Gamma \mathbf{w} = \frac{1}{\sin\theta}\left[\frac{\partial}{\partial\theta}(w_\theta \sin\theta) + \frac{\partial w_\varphi}{\partial\varphi}\right], \tag{3.9}$$

$$\mathrm{curl}_\Gamma \mathbf{w} = \frac{1}{\sin\theta}\left[\frac{\partial}{\partial\theta}(w_\varphi \sin\theta) - \frac{\partial w_\theta}{\partial\varphi}\right]. \tag{3.10}$$

It is known (cf., Nédélec [45]) that these boundary differential operators satisfy

$$\Delta_\Gamma = \mathrm{div}_\Gamma \nabla_\Gamma = -\mathrm{curl}_\Gamma\, \mathbf{curl}_\Gamma \quad \text{and} \quad \mathrm{curl}_\Gamma \nabla_\Gamma = \mathrm{div}_\Gamma \mathbf{curl}_\Gamma = 0. \tag{3.11}$$

Following Colton and Kress [27] (cf., Theorem 6.23), an orthonormal basis for $TL^2(\Gamma_R) = \{\mathbf{w} \in (L^2(\Gamma_R))^3 : \ \mathbf{e}_\rho \cdot \mathbf{w} = 0\}$, the tangential fields on Γ_R, consists of functions of the form

$$\mathbf{U}_n^m(\theta,\varphi) = \frac{1}{R\sqrt{n(n+1)}}\nabla_\Gamma Y_n^m(\theta,\varphi)$$

and

$$\mathbf{V}_n^m(\phi,\psi) = \mathbf{e}_\rho \times \mathbf{U}_n^m(\phi,\psi) = -\frac{1}{R\sqrt{n(n+1)}}\mathbf{curl}_\Gamma Y_n^m$$

for $|m| \le n$, $n \in \mathbb{N}$. It follows from (3.11) and (3.1) that

$$\mathrm{div}_\Gamma \mathbf{U}_n^m = -\frac{\sqrt{n(n+1)}}{R}Y_n^m, \quad \mathrm{curl}_\Gamma \mathbf{V}_n^m = -\frac{\sqrt{n(n+1)}}{R}Y_n^m,$$

and

$$\mathrm{curl}_\Gamma \mathbf{U}_n^m = \mathrm{div}_\Gamma \mathbf{V}_n^m = 0.$$

Define two sequences of tangential fields

$$\mathbf{X}_n^m(\theta,\varphi) = \frac{1}{\sqrt{n(n+1)}}\nabla_\Gamma X_n^m(\theta,\varphi) = \sqrt{2}\,\mathbf{U}_n^m(\theta,\varphi) \tag{3.12}$$

and

$$\mathbf{Y}_n^m(\theta,\varphi) = \mathbf{e}_\rho \times \mathbf{X}_n^m(\theta,\varphi) = \sqrt{2}\,\mathbf{V}_n^m(\theta,\varphi). \tag{3.13}$$

Using the definition of the tangential gradient (3.7), and noticing that $\mathbf{e}_\theta \times \mathbf{e}_\varphi = \mathbf{e}_\rho$, $\mathbf{e}_\varphi \times \mathbf{e}_\rho = \mathbf{e}_\theta$, $\mathbf{e}_\rho \times \mathbf{e}_\theta = \mathbf{e}_\varphi$, we get

$$\mathbf{e}_\theta \times \mathbf{X}_n^m(\frac{\pi}{2},\varphi) = 0 \quad \text{for } |m| \le n,\ m+n = \text{odd},\ n \in \mathbb{N}, \tag{3.14}$$

and

$$\mathbf{e}_\theta \times \mathbf{Y}_n^m(\frac{\pi}{2},\varphi) = 0 \quad \text{for } |m| \le n,\ m+n = \text{even},\ n \in \mathbb{N}. \tag{3.15}$$

Define a subspace of complex square integrable tangential fields functions on the hemisphere Γ_R^+:

$$TL^2(\Gamma_R^+) = \{\mathbf{w} \in (L^2(\Gamma_R^+))^3 : \ \mathbf{e}_\rho \cdot \mathbf{w} = 0\}.$$

Lemma 3.2. *The vector spherical harmonics* $\mathbf{X}_n^m$ $(m+n = \text{odd})$ *and* $\mathbf{Y}_n^m$ $(m+n = \text{even})$ *for* $|m| \le n$, $n \in \mathbb{N}$ *form a complete orthonormal system in* $TL^2(\Gamma_R^+)$.

Proof. First we prove the orthogonality. Following (3.7), we have

$$\nabla_\Gamma Y_n^m(\theta,\varphi) = \frac{\partial}{\partial\theta} Y_n^m(\theta,\varphi)\,\mathbf{e}_\theta + \frac{1}{\sin\theta}\frac{\partial}{\partial\varphi} Y_n^m(\theta,\varphi)\,\mathbf{e}_\varphi.$$

It can be verified from the spherical harmonics (3.2) and (3.3) that

$$\frac{\partial}{\partial\varphi} Y_n^m(\pi-\theta,\varphi) = (-1)^{m+n}\,\frac{\partial}{\partial\varphi} Y_n^m(\theta,\varphi) \tag{3.16}$$

and

$$\frac{\partial}{\partial\theta} Y_n^m(\pi-\theta,\varphi) = (-1)^{m+n+1}\,\frac{\partial}{\partial\theta} Y_n^m(\theta,\varphi). \tag{3.17}$$

Using (3.16) and (3.17), the change of variables, and the fact that the vector spherical harmonics $\mathbf{U}_n^m$ and $\mathbf{V}_n^m$ for $|m|\le n$, $n\in\mathbb{N}$ form a complete orthonormal system in $TL^2(\Gamma_R)$, we obtain

$$\int_{\Gamma_R^+} \mathbf{X}_n^m(\theta,\varphi)\cdot\bar{\mathbf{X}}_{n'}^{m'}(\theta,\varphi) = 2\int_{\Gamma_R^+} \mathbf{U}_n^m(\theta,\varphi)\cdot\bar{\mathbf{U}}_{n'}^{m'}(\theta,\varphi) = \int_{\Gamma_R} \mathbf{U}_n^m(\theta,\varphi)\cdot\bar{\mathbf{U}}_{n'}^{m'}(\theta,\varphi) = \delta_{nn'}\delta_{mm'}.$$

Similarly, it holds

$$\int_{\Gamma_R^+} \mathbf{Y}_n^m(\theta,\varphi)\cdot\bar{\mathbf{Y}}_{n'}^{m'}(\theta,\varphi) = \int_{\Gamma_R} \mathbf{V}_n^m(\theta,\varphi)\cdot\bar{\mathbf{V}}_{n'}^{m'}(\theta,\varphi) = \delta_{nn'}\delta_{mm'}.$$

Besides, it is easy to show that

$$\int_{\Gamma_R^+} \mathbf{X}_n^m(\theta,\varphi)\cdot\bar{\mathbf{Y}}_{n'}^{m'}(\theta,\varphi) = 2\int_{\Gamma_R^+} \mathbf{U}_n^m(\theta,\varphi)\cdot\bar{\mathbf{V}}_{n'}^{m'}(\theta,\varphi) = 0,$$

which completes the proof of the orthogonality.

Next is to prove the completeness. For any $\mathbf{w}\in TL^2(\Gamma_R^+)$, it can be represented by its coordinates in the local orthonormal basis

$$\mathbf{w}(\theta,\varphi) = w_\theta(\theta,\varphi)\,\mathbf{e}_\theta + w_\varphi(\theta,\varphi)\,\mathbf{e}_\varphi.$$

We define an extended tangential field to the whole sphere Γ_R by even reflection for w_θ and odd reflection for w_φ:

$$\tilde{\mathbf{w}}(\theta,\varphi) = \begin{cases} w_\theta(\theta,\varphi)\,\mathbf{e}_\theta + w_\varphi(\theta,\varphi)\,\mathbf{e}_\varphi & \text{if } 0\le\theta\le\frac{\pi}{2}, \\ w_\theta(\pi-\theta,\varphi)\,\mathbf{e}_\theta - w_\varphi(\pi-\theta,\varphi)\,\mathbf{e}_\varphi & \text{if } \frac{\pi}{2}<\theta\le\pi. \end{cases}$$

Evidently $\tilde{\mathbf{w}}\in TL^2(\Gamma_R)$ and thus it has the following series expansion

$$\tilde{\mathbf{w}} = \sum_{|m|\le n} w_{1n}^m \mathbf{U}_n^m + w_{2n}^m \mathbf{V}_n^m. \tag{3.18}$$

Here, using (3.16) and (3.17), we can compute the coefficient w_{1n} by

$$\begin{aligned} w_{1n}^m = \int_{\Gamma_R} \tilde{\mathbf{w}}\cdot\bar{\mathbf{U}}_n^m &= \frac{1}{R\sqrt{n(n+1)}}\int_{\Gamma_R^+}\left[w_\theta(\theta,\varphi)\frac{\partial}{\partial\theta}Y_n^m(\theta,\varphi) + \frac{w_\varphi(\theta,\varphi)}{\sin\theta}\frac{\partial}{\partial\varphi}Y_n^m(\theta,\varphi)\right] \\ &+ \frac{1}{R\sqrt{n(n+1)}}\int_{\Gamma_R^-}\left[w_\theta(\pi-\theta,\varphi)\frac{\partial}{\partial\theta}Y_n^m(\theta,\varphi) - \frac{w_\varphi(\pi-\theta,\varphi)}{\sin\theta}\frac{\partial}{\partial\varphi}Y_n^m(\theta,\varphi)\right] \\ &= \left[1-(-1)^{m+n}\right]\int_{\Gamma_R^+} \mathbf{w}\cdot\bar{\mathbf{U}}_n^m, \end{aligned}$$

and the coefficient w_{2n} by

$$
\begin{aligned}
w_{2n}^m &= \int_{\Gamma_R} \tilde{\mathbf{w}} \cdot \bar{\mathbf{V}}_n^m = \frac{1}{R\sqrt{n(n+1)}} \int_{\Gamma_R^+} \left[\frac{w_\theta(\theta,\varphi)}{\sin\theta} \frac{\partial}{\partial\varphi} Y_n^m(\theta,\varphi) - w_\varphi(\theta,\varphi) \frac{\partial}{\partial\theta} Y_n^m(\theta,\varphi) \right] \\
&\quad + \frac{1}{R\sqrt{n(n+1)}} \int_{\Gamma_R^-} \left[\frac{w_\theta(\pi-\theta,\varphi)}{\sin\theta} \frac{\partial}{\partial\varphi} Y_n^m(\theta,\varphi) + w_\varphi(\pi-\theta,\varphi) \frac{\partial}{\partial\theta} Y_n^m(\theta,\varphi) \right] \\
&= \left[1 + (-1)^{m+n}\right] \int_{\Gamma_R^+} \mathbf{w} \cdot \bar{\mathbf{V}}_n^m.
\end{aligned}
$$

Therefore, we have

$$
w_{1n}^m = 0 \quad \text{for } m+n = \text{even}, \quad w_{2n}^m = 0 \quad \text{for } m+n = \text{odd},
$$

and

$$
\begin{aligned}
w_{1n}^m &= 2 \int_{\Gamma_R^+} \mathbf{w} \cdot \bar{\mathbf{U}}_n^m \quad \text{for } m+n = \text{odd}, \\
w_{2n}^m &= 2 \int_{\Gamma_R^+} \mathbf{w} \cdot \bar{\mathbf{V}}_n^m \quad \text{for } m+n = \text{even}.
\end{aligned}
$$

Plugging the coefficients w_{1n}^m and w_{2n}^m into (3.18) and restricting $\tilde{\mathbf{w}}$ to Γ_R^+, we obtain the expansion of $\mathbf{w}$ under the vector spherical harmonics $\mathbf{X}_n^m$ and $\mathbf{Y}_n^m$:

$$
\mathbf{w}(\theta,\varphi) = \sum_{|m|\le n}^{\text{odd}} w_{1n}^m \, \mathbf{X}_n^m(\theta,\varphi) + \sum_{|m|\le n}^{\text{even}} w_{2n}^m \, \mathbf{Y}_n^m(\theta,\varphi),
$$

where

$$
w_{1n}^m = \int_{\Gamma_R^+} \mathbf{w} \cdot \bar{\mathbf{X}}_n^m \quad \text{and} \quad w_{2n}^m = \int_{\Gamma_R^+} \mathbf{w} \cdot \bar{\mathbf{Y}}_n^m,
$$

which completes the proof. □

To describe the Calderon operator and transparent boundary condition in the formulation of the boundary value problem, we introduce some trace functional spaces. Denote by $H_0^s(\Gamma_R^+)$ the Sobolev space, the completion of $C_0^\infty(\Gamma_R^+)$ in the norm $\| \cdot \|_{H^s(\Gamma_R^+)}$ characterized by

$$
\| w \|_{H^s(\Gamma_R^+)}^2 = \sum_{|m|\le n}^{\text{odd}} (1 + n(n+1))^s \, |w_n^m|^2, \tag{3.19}
$$

where

$$
w(\theta,\varphi) = \sum_{|m|\le n}^{\text{odd}} w_n^m \, X_n^m(\theta,\varphi).
$$

Introduce the following spaces:

$$
\begin{aligned}
TH_0^s(\Gamma_R^+) &= \{\mathbf{w} \in (H_0^s(\Gamma_R^+))^3, \ \mathbf{e}_\rho \cdot \mathbf{w} = 0, \ \mathbf{e}_\theta \times \mathbf{w}(\frac{\pi}{2},\varphi) = 0\}, \\
TH_0^{-1/2}(\text{curl}, \Gamma_R^+) &= \{\mathbf{w} \in TH_0^{-1/2}(\Gamma_R^+), \ \text{curl}_\Gamma \mathbf{w} \in H^{-1/2}(\Gamma_R^+)\}, \\
TH_0^{-1/2}(\text{div}, \Gamma_R^+) &= \{\mathbf{w} \in TH_0^{-1/2}(\Gamma_R^+), \ \text{div}_\Gamma \mathbf{w} \in H^{-1/2}(\Gamma_R^+)\}.
\end{aligned}
$$

Due to Lemma 3.2, for any tangential field $\mathbf{w} \in TH_0^s(\Gamma_R^+)$, it can be represented in the series expansion

$$\mathbf{w} = \sum_{|m|\le n}^{\text{odd}} w_{1n}^m \, \mathbf{X}_n^m(\theta,\varphi) + \sum_{|m|\le n}^{\text{even}} w_{2n}^m \, \mathbf{Y}_n^m(\theta,\varphi).$$

Trace spaces of tangential vector fields on Γ_R^+ can be characterized by the weighted sums of the expansion coefficients. Using the series coefficients, the norm on the space $TH_0^s(\Gamma_R^+)$ can be characterized by

$$\| \mathbf{w} \|_{TH_0^s(\Gamma_R^+)}^2 = \sum_{|m|\le n}^{\text{odd}} (1+n(n+1))^s \, |w_{1n}^m|^2 + \sum_{|m|\le n}^{\text{even}} (1+n(n+1))^s \, |w_{2n}^m|^2; \tag{3.20}$$

the norm on the space $TH_0^{-1/2}(\text{curl}, \Gamma_R^+)$ can be characterized by

$$\| \mathbf{w} \|_{TH_0^{-1/2}(\text{curl}, \Gamma_R^+)}^2 = \sum_{|m|\le n}^{\text{odd}} \frac{1}{\sqrt{1+n(n+1)}} \, |w_{1n}^m|^2 + \sum_{|m|\le n}^{\text{even}} \sqrt{1+n(n+1)} \, |w_{2n}^m|^2; \tag{3.21}$$

and the norm on the space $TH_0^{-1/2}(\text{div}, \Gamma_R^+)$ can be characterized by

$$\| \mathbf{w} \|_{TH_0^{-1/2}(\text{div}, \Gamma_R^+)}^2 = \sum_{|m|\le n}^{\text{odd}} \sqrt{1+n(n+1)} \, |w_{1n}^m|^2 + \sum_{|m|\le n}^{\text{even}} \frac{1}{\sqrt{1+n(n+1)}} \, |w_{2n}^m|^2. \tag{3.22}$$

Define a dual pairing $\langle \cdot, \cdot \rangle_{\Gamma_R^+}$ by

$$\langle \mathbf{u}, \mathbf{v} \rangle_{\Gamma_R^+} = \sum_{|m|\le n}^{\text{odd}} u_{1n}^m \, \bar{v}_{1n}^m + \sum_{|m|\le n}^{\text{even}} u_{2n}^m \, \bar{v}_{2n}^m,$$

where

$$\mathbf{u} = \sum_{|m|\le n}^{\text{odd}} u_{1n}^m \, \mathbf{X}_n^m + \sum_{|m|\le n}^{\text{even}} u_{2n}^m \, \mathbf{Y}_n^m \quad \text{and} \quad \mathbf{v} = \sum_{|m|\le n}^{\text{odd}} v_{1n}^m \, \mathbf{X}_n^m + \sum_{|m|\le n}^{\text{even}} v_{2n}^m \, \mathbf{Y}_n^m.$$

Denote by $TH^{-1/2}(\text{div}, \Gamma_R^+)$ the dual space of $TH_0^{-1/2}(\text{curl}, \Gamma_R^+)$ and by $TH^{-1/2}(\text{curl}, \Gamma_R^+)$ the dual space of $TH_0^{-1/2}(\text{div}, \Gamma_R^+)$, i.e.,

$$TH^{-1/2}(\text{div}, \Gamma_R^+) = \left(TH_0^{-1/2}(\text{curl}, \Gamma_R^+)\right)' \quad \text{and} \quad TH^{-1/2}(\text{curl}, \Gamma_R^+) = \left(TH_0^{-1/2}(\text{div}, \Gamma_R^+)\right)'.$$

The norm on the space $TH^{-1/2}(\text{div}, \Gamma_R^+)$ is characterized by

$$\| \mathbf{u} \|_{TH^{-1/2}(\text{div}, \Gamma_R^+)} = \sup_{\mathbf{v} \in TH_0^{-1/2}(\text{curl}, \Gamma_R^+)} \frac{\langle \mathbf{u}, \mathbf{v} \rangle_{\Gamma_R^+}}{\| \mathbf{v} \|_{TH_0^{-1/2}(\text{curl}, \Gamma_R^+)}}, \tag{3.23}$$

and the norm on the space $TH^{-1/2}(\text{curl}, \Gamma_R^+)$ is characterized by

$$\| \mathbf{u} \|_{TH^{-1/2}(\text{curl}, \Gamma_R^+)} = \sup_{\mathbf{v} \in TH_0^{-1/2}(\text{div}, \Gamma_R^+)} \frac{\langle \mathbf{u}, \mathbf{v} \rangle_{\Gamma_R^+}}{\| \mathbf{v} \|_{TH_0^{-1/2}(\text{div}, \Gamma_R^+)}}. \tag{3.24}$$

Introduce the following space

$$H_S(\text{curl},\,\Omega) = \{\mathbf{u} \in (L^2(\Omega))^3,\ \nabla\times\mathbf{u} \in (L^2(\Omega))^3,\ \mathbf{n}\times\mathbf{u} = 0 \text{ on } S\},$$

which is clearly a Hilbert space for the norm:

$$\|\, \mathbf{u} \,\| = \Big(\|\, \mathbf{u} \,\|^2_{(L^2(\Omega))^3} + \|\, \nabla\times\mathbf{u} \,\|^2_{(L^2(\Omega))^3} \Big)^{1/2}. \tag{3.25}$$

Let Γ_ρ be the sphere of radius ρ and center the origin and let Γ_ρ^+ be the hemisphere $\{\mathbf{x} \in \Gamma_\rho,\ x_3 \geq 0\}$. For any vector field $\mathbf{u}$, denote the tangential component on the sphere Γ_ρ^+ by

$$\mathbf{u}_{\Gamma_\rho^+} = -\mathbf{e}_\rho \times (\mathbf{e}_\rho \times \mathbf{u}|_{\Gamma_\rho^+}).$$

To simply proofs, we shall employ positive constants C and C_i as generalized constants whose precise values are not required and may change line by line but should always be clear from the context.

The following trace regularity results in $TH_0^{-1/2}(\text{curl},\,\Gamma_R^+)$ and $TH_0^{-1/2}(\Gamma_R^+)$ are useful in subsequent analysis.

Lemma 3.3. *Let* $\gamma_1 = \sqrt{2\max\{R, 1/R\}}$. *The following estimate holds*

$$\|\, \mathbf{u}_{\Gamma_R^+} \,\|_{TH_0^{-1/2}(\text{curl},\,\Gamma_R^+)} \leq \gamma_1 \,\|\, \mathbf{u} \,\|_{H_S(\text{curl},\,\Omega)}$$

for any $\mathbf{u} \in H_S(\text{curl},\,\Omega)$.

Proof. Let B_R^+ be the half ball between Γ_R^+ and the ground. Noting that $\mathbf{u} = u_\rho\mathbf{e}_\rho + u_\theta\mathbf{e}_\theta + u_\varphi\mathbf{e}_\varphi = \mathbf{u}_{\Gamma_\rho} + u_r\mathbf{e}_\rho$, we have the following expansion in B_R^+ (cf. Lemma 3.1 and Lemma 3.2):

$$\mathbf{u} = \sum_{|m|\leq n}^{\text{odd}} u_{1n}^m(\rho)\,\mathbf{X}_n^m + \sum_{|m|\leq n}^{\text{even}} u_{2n}^m(\rho)\,\mathbf{Y}_n^m + \sum_{|m|\leq n}^{\text{odd}} u_{3n}^m(\rho) X_n^m \mathbf{e}_\rho. \tag{3.26}$$

From (3.21),

$$\|\, \mathbf{u}_{\Gamma_R^+} \,\|^2_{TH^{-1/2}(\text{curl},\,\Gamma_R^+)} = \sum_{|m|\leq n}^{\text{odd}} \frac{1}{\sqrt{1+n(n+1)}}\,|u_{1n}^m(R)|^2 + \sum_{|m|\leq n}^{\text{even}} \sqrt{1+n(n+1)}\,|u_{2n}^m(R)|^2. \tag{3.27}$$

Clearly,

$$\int_0^{\pi/2}\!\!\int_0^{2\pi} \mathbf{X}_n^m\cdot\bar{\mathbf{X}}_{n'}^{m'} R^2\sin\theta \mathrm{d}\varphi\mathrm{d}\theta = \int_{\Gamma_R^+} \mathbf{X}_n^m\cdot\bar{\mathbf{X}}_{n'}^{m'} = \delta_{nn'}\delta_{mm'},$$
$$\int_0^{\pi/2}\!\!\int_0^{2\pi} \mathbf{Y}_n^m\cdot\bar{\mathbf{Y}}_{n'}^{m'} R^2\sin\theta \mathrm{d}\varphi\mathrm{d}\theta = \delta_{nn'}\delta_{mm'},$$
$$\int_0^{\pi/2}\!\!\int_0^{2\pi} X_n^m\cdot\bar{X}_{n'}^{m'} R^2\sin\theta \mathrm{d}\varphi\mathrm{d}\theta = \delta_{nn'}\delta_{mm'}, \qquad \text{if } m+n+m'+n' = \text{ even}.$$
$$\int_0^{\pi/2}\!\!\int_0^{2\pi} \mathbf{X}_n^m\cdot\bar{\mathbf{Y}}_{n'}^{m'} R^2\sin\theta \mathrm{d}\varphi\mathrm{d}\theta = 0, \qquad \mathbf{X}_n^m\cdot\mathbf{e}_\rho = \mathbf{Y}_n^m\cdot\mathbf{e}_\rho = 0.$$

Therefore,

$$
\begin{aligned}
\| \mathbf{u} \|^2_{(L^2(B_R^+))^3} &= \int_0^R \int_0^{\pi/2} \int_0^{2\pi} \mathbf{u}\cdot\mathbf{u}\, \rho^2 \sin\theta \mathrm{d}\varphi \mathrm{d}\theta \mathrm{d}\rho \\
&= \frac{1}{R^2}\int_0^R \Big(\sum_{|m|\le n}^{\text{odd}} |u_{1n}^m(\rho)|^2 + \sum_{|m|\le n}^{\text{even}} |u_{2n}^m(\rho)|^2 + \sum_{|m|\le n}^{\text{odd}} |u_{3n}^m(\rho)|^2 \Big) \rho^2 \mathrm{d}\rho \qquad (3.28)
\end{aligned}
$$

On the other hand, noting that (cf. (3.12), (3.13), and (3.4))

$$
\mathrm{curl}_\Gamma \mathbf{X}_n^m = 0, \quad \mathrm{curl}_\Gamma \mathbf{Y}_n^m = -\sqrt{n(n+1)}\, X_n^m, \quad \text{and} \quad \nabla_\Gamma X_n^m \times \mathbf{e}_\rho = -\sqrt{n(n+1)}\, \mathbf{Y}_n^m,
$$

we have

$$
\begin{aligned}
\mathrm{curl}\, \mathbf{u} =& \frac{1}{\rho\sin\theta}\left(\frac{\partial(\sin\theta u_\varphi)}{\partial\theta} - \frac{\partial u_\theta}{\partial\varphi} \right)\mathbf{e}_\rho + \left(\frac{1}{\rho\sin\theta}\frac{\partial u_\rho}{\partial\varphi} - \frac{1}{\rho}\frac{\partial(\rho u_\varphi)}{\partial\rho} \right)\mathbf{e}_\theta \\
&+ \frac{1}{\rho}\left(\frac{\partial(\rho u_\theta)}{\partial\rho} - \frac{\partial u_\rho}{\partial\theta} \right)\mathbf{e}_\varphi \\
=& \frac{1}{\rho}\Big(\mathrm{curl}_\Gamma \mathbf{u}\, \mathbf{e}_\rho + \nabla_\Gamma u_\rho \times \mathbf{e}_\rho - \frac{\partial(\rho\mathbf{u})}{\partial\rho} \times \mathbf{e}_\rho \Big) \\
=& \frac{1}{\rho}\Big(- \sum_{|m|\le n}^{\text{even}} u_{2n}^m(\rho) \sqrt{n(n+1)}\, X_n^m\, \mathbf{e}_\rho - \sum_{|m|\le n}^{\text{odd}} u_{3n}^m(\rho) \sqrt{n(n+1)}\, \mathbf{Y}_n^m \\
&+ \sum_{|m|\le n}^{\text{odd}} \frac{\mathrm{d}(\rho u_{1n}^m(\rho))}{\mathrm{d}\rho} \mathbf{Y}_n^m - \sum_{|m|\le n}^{\text{even}} \frac{\mathrm{d}(\rho u_{2n}^m(\rho))}{\mathrm{d}\rho} \mathbf{X}_n^m \Big).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\| \mathrm{curl}\, \mathbf{u} \|^2_{(L^2(B_R^+))^3} = \frac{1}{R^2}\int_0^R \Bigg(& \sum_{|m|\le n}^{\text{even}} \Big(n(n+1)|u_{2n}^m(\rho)|^2 + \Big| \frac{\mathrm{d}(\rho u_{2n}^m(\rho))}{\mathrm{d}\rho} \Big|^2 \Big) \\
&+ \sum_{|m|\le n}^{\text{odd}} \Big| \frac{\mathrm{d}(\rho u_{1n}^m(\rho))}{\mathrm{d}\rho} - \sqrt{n(n+1)}\, u_{3n}^m(\rho) \Big|^2 \Bigg) \mathrm{d}\rho \qquad (3.29)
\end{aligned}
$$

By combining (3.28) and (3.29) we have

$$
\begin{aligned}
\| \mathbf{u} \|^2_{H(\mathrm{curl},\, B_R^+)} =& \frac{1}{R^2}\int_0^R \Bigg(\sum_{|m|\le n}^{\text{even}} \Big(\big(\rho^2 + n(n+1)\big)|u_{2n}^m(\rho)|^2 + \Big| \frac{\mathrm{d}(\rho u_{2n}^m(\rho))}{\mathrm{d}\rho} \Big|^2 \Big) \\
&+ \sum_{|m|\le n}^{\text{odd}} \Big(|u_{1n}^m(\rho)|^2\rho^2 + |u_{3n}^m(\rho)|^2\rho^2 \\
&+ \Big| \frac{\mathrm{d}(\rho u_{1n}^m(\rho))}{\mathrm{d}\rho} - \sqrt{n(n+1)}\, u_{3n}^m(\rho) \Big|^2 \Big) \Bigg) \mathrm{d}\rho \\
\ge& \frac{1}{R^2}\int_0^R \Bigg(\sum_{|m|\le n}^{\text{even}} \Big(\big(\rho^2 + n(n+1)\big)|u_{2n}^m(\rho)|^2 + \Big| \frac{\mathrm{d}(\rho u_{2n}^m(\rho))}{\mathrm{d}\rho} \Big|^2 \Big) \\
&+ \sum_{|m|\le n}^{\text{odd}} \Big(\rho^2 |u_{1n}^m(\rho)|^2 + \frac{\rho^2}{\rho^2 + n(n+1)} \Big| \frac{\mathrm{d}(\rho u_{1n}^m(\rho))}{\mathrm{d}\rho} \Big|^2 \Big) \Bigg) \mathrm{d}\rho \qquad (3.30)
\end{aligned}
$$

We have

$$\begin{aligned}
\frac{|u_{1n}^m(R)|^2}{\sqrt{1+n(n+1)}} &= \frac{1}{R^3\sqrt{1+n(n+1)}}\int_0^R \frac{\mathrm{d}\big(\rho|\rho u_{1n}^m(\rho)|^2\big)}{\mathrm{d}\rho}\mathrm{d}\rho \\
&= \frac{1}{R^3\sqrt{1+n(n+1)}}\int_0^R \Big(\rho^2|u_{1n}^m(\rho)|^2 + 2\rho\mathrm{Re}\big(\rho u_{1n}^m(\rho)\frac{\mathrm{d}(\rho\bar{u}_{1n}^m(\rho))}{\mathrm{d}\rho}\big)\Big)\mathrm{d}\rho \\
&\le \frac{1}{R^3}\int_0^R \Big(\rho^2|u_{1n}^m(\rho)|^2 + 2\max\{R,1\}\rho\big|u_{1n}^m(\rho)\big|\frac{\rho}{\sqrt{\rho^2+n(n+1)}}\Big|\frac{\mathrm{d}(\rho u_{1n}^m(\rho))}{\mathrm{d}\rho}\Big|\Big)\mathrm{d}\rho.
\end{aligned}$$

Thus,

$$\frac{|u_{1n}^m(R)|^2}{\sqrt{1+n(n+1)}} \le \max\Big\{1,\frac{1}{R}\Big\}\frac{1}{R^2}\int_0^R \Big(2\rho^2|u_{1n}^m(\rho)|^2 + \frac{\rho^2}{\rho^2+n(n+1)}\Big|\frac{\mathrm{d}(\rho u_{1n}^m(\rho))}{\mathrm{d}\rho}\Big|^2\Big)\mathrm{d}\rho. \tag{3.31}$$

Similarly,

$$\begin{aligned}
\sqrt{1+n(n+1)}\,|u_{2n}^m(R)|^2 &= \frac{\sqrt{1+n(n+1)}}{R^2}\int_0^R \frac{\mathrm{d}\big(|\rho u_{2n}^m(\rho)|^2\big)}{\mathrm{d}\rho}\mathrm{d}\rho \\
&\le \frac{\sqrt{1+n(n+1)}}{R^2}\int_0^R 2|\rho u_{2n}^m(\rho)|\Big|\frac{\mathrm{d}\big(\rho u_{2n}^m(\rho)\big)}{\mathrm{d}\rho}\Big|\mathrm{d}\rho \\
&\le \frac{\max\{R,1\}}{R^2}\int_0^R \Big((\rho^2+n(n+1))|u_{2n}^m(\rho)|^2 + \Big|\frac{\mathrm{d}(\rho u_{2n}^m(\rho))}{\mathrm{d}\rho}\Big|^2\Big)\mathrm{d}\rho. \qquad (3.32)
\end{aligned}$$

The proof of the lemma is completed by combining (3.27), (3.30), (3.31), and (3.32). □

Lemma 3.4. *For any $\eta>0$ there is a constant $C(\eta)$ such that the following estimate holds*

$$\|\,\mathbf{u}_{\Gamma_R^+}\|^2_{TH_0^{-1/2}(\Gamma_R^+)} \le \eta\,\|\,\nabla\times\mathbf{u}\,\|^2_{(L^2(\Omega))^3} + C(\eta)\,\|\,\mathbf{u}\,\|^2_{(L^2(\Omega))^3}$$

for any $\mathbf{u}\in H_S(\mathrm{curl},\Omega)$.

Proof. We have

$$\|\,\mathbf{u}_{\Gamma_R^+}\|^2_{H^{-1/2}(\Gamma_R^+)} = \sum_{|m|\le n}^{\mathrm{odd}} \frac{1}{\sqrt{1+n(n+1)}}\,|u_{1n}^m(R)|^2 + \sum_{|m|\le n}^{\mathrm{even}} \frac{1}{\sqrt{1+n(n+1)}}\,|u_{2n}^m(R)|^2. \tag{3.33}$$

From (3.28) and (3.29),

$$\begin{aligned}
&\eta\,\|\,\nabla\times\mathbf{u}\,\|^2_{(L^2(\Omega))^3} + C(\eta)\,\|\,\mathbf{u}\,\|^2_{(L^2(\Omega))^3} \\
&= \frac{1}{R^2}\int_0^R \Bigg(\sum_{|m|\le n}^{\mathrm{even}} \Big(\big(C(\eta)\rho^2+\eta\,n(n+1)\big)|u_{2n}^m(\rho)|^2 + \eta\Big|\frac{\mathrm{d}(\rho u_{2n}^m(\rho))}{\mathrm{d}\rho}\Big|^2\Big) \\
&\qquad + \sum_{|m|\le n}^{\mathrm{odd}} \Big(C(\eta)|u_{1n}^m(\rho)|^2\rho^2 + C(\eta)|u_{3n}^m(\rho)|^2\rho^2 \\
&\qquad\qquad + \eta\Big|\frac{\mathrm{d}(\rho u_{1n}^m(\rho))}{\mathrm{d}\rho} - \sqrt{n(n+1)}\,u_{3n}^m(\rho)\Big|^2\Big)\Bigg)\mathrm{d}\rho \\
&\ge \frac{1}{R^2}\int_0^R \Bigg(\sum_{|m|\le n}^{\mathrm{even}} \Big(\big(C(\eta)\rho^2+\eta\,n(n+1)\big)|u_{2n}^m(\rho)|^2 + \eta\Big|\frac{\mathrm{d}(\rho u_{2n}^m(\rho))}{\mathrm{d}\rho}\Big|^2\Big) \\
&\qquad + \sum_{|m|\le n}^{\mathrm{odd}} \Big(C(\eta)\rho^2|u_{1n}^m(\rho)|^2 + \frac{\eta\,C(\eta)\rho^2}{C(\eta)\rho^2+\eta\,n(n+1)}\Big|\frac{\mathrm{d}(\rho u_{1n}^m(\rho))}{\mathrm{d}\rho}\Big|^2\Big)\Bigg)\mathrm{d}\rho \qquad (3.34)
\end{aligned}$$

All we need is to choose $C(\eta)$ properly to bound the right hand side of (3.33) by that of (3.34). We have

$$
\begin{aligned}
\frac{|u_{1n}^m(R)|^2}{\sqrt{1+n(n+1)}} &= \frac{1}{R^3\sqrt{1+n(n+1)}}\int_0^R \frac{\mathrm{d}\big(\rho|\rho u_{1n}^m(\rho)|^2\big)}{\mathrm{d}\rho}\mathrm{d}\rho \\
&= \frac{1}{R^3\sqrt{1+n(n+1)}}\int_0^R \Big(\rho^2|u_{1n}^m(\rho)|^2 + 2\rho\mathrm{Re}\big(\rho u_{1n}^m(\rho)\frac{\mathrm{d}(\rho\bar{u}_{1n}^m(\rho))}{\mathrm{d}\rho}\big)\Big)\mathrm{d}\rho \\
&\le \frac{1}{R^3}\int_0^R \Big(\rho^2|u_{1n}^m(\rho)|^2 + 2\rho\big|u_{1n}^m(\rho)\big|\frac{\rho}{\sqrt{1+n(n+1)}}\Big|\frac{\mathrm{d}(\rho u_{1n}^m(\rho))}{\mathrm{d}\rho}\Big|\Big)\mathrm{d}\rho \\
&\le \frac{1}{R^3}\int_0^R \Big(\Big(1+\frac{C(\eta)\rho^2+\eta\, n(n+1)}{R\,\eta\, C(\eta)(1+n(n+1))}\Big)\rho^2|u_{1n}^m(\rho)|^2 \\
&\qquad + \frac{R\,\eta\, C(\eta)\rho^2}{C(\eta)\rho^2+\eta\, n(n+1)}\Big|\frac{\mathrm{d}(\rho u_{1n}^m(\rho))}{\mathrm{d}\rho}\Big|^2\Big)\mathrm{d}\rho.
\end{aligned}
$$

Let

$$
C(\eta) = \frac{1}{\eta} + \frac{2}{R}. \tag{3.35}
$$

Then, for any integer n,

$$
1+\frac{C(\eta)\rho^2+\eta\, n(n+1)}{R\,\eta\, C(\eta)(1+n(n+1))} \le 1+\frac{R}{\eta}+\frac{1}{R\,C(\eta)} \le R\,C(\eta).
$$

Thus,

$$
\frac{|u_{1n}^m(R)|^2}{\sqrt{1+n(n+1)}} \le \frac{1}{R^2}\int_0^R \Big(C(\eta)\rho^2|u_{1n}^m(\rho)|^2 + \frac{\eta\, C(\eta)\rho^2}{C(\eta)\rho^2+\eta\, n(n+1)}\Big|\frac{\mathrm{d}(\rho u_{1n}^m(\rho))}{\mathrm{d}\rho}\Big|^2\Big)\mathrm{d}\rho. \tag{3.36}
$$

Similarly,

$$
\begin{aligned}
\frac{|u_{2n}^m(R)|^2}{\sqrt{1+n(n+1)}} &\le \frac{1}{R^2}\int_0^R \Big(C(\eta)\rho^2|u_{2n}^m(\rho)|^2 + \frac{\eta\, C(\eta)\rho^2}{C(\eta)\rho^2+\eta\, n(n+1)}\Big|\frac{\mathrm{d}(\rho u_{2n}^m(\rho))}{\mathrm{d}\rho}\Big|^2\Big)\mathrm{d}\rho \\
&\le \frac{1}{R^2}\int_0^R \Big(\big(C(\eta)\rho^2+\eta\, n(n+1)\big)|u_{2n}^m(\rho)|^2 + \eta\Big|\frac{\mathrm{d}(\rho u_{2n}^m(\rho))}{\mathrm{d}\rho}\Big|^2\Big)\mathrm{d}\rho. \qquad (3.37)
\end{aligned}
$$

Now the proof of the lemma is completed by combining (3.33)–(3.37). □

4 Transparent boundary condition on hemisphere

In this section, we introduce a Calderon operator which induces a transparent boundary condition and maps the tangential component of the electric field to the value of the tangential trace of the magnetic field on the hemisphere.

Let $h_n^{(1)}(z)$ be the spherical Hankel function of the first kind of order n. We introduce the vector wave functions

$$
\mathbf{M}_n^m(\rho,\theta,\varphi) = \nabla\times\big(\mathbf{x}h_n^{(1)}(\omega\rho)X_n^m(\theta,\varphi)\big), \quad \mathbf{N}_n^m(\rho,\theta,\varphi) = \frac{1}{\mathrm{i}\omega}\nabla\times\mathbf{M}_n^m(\theta,\varphi), \tag{4.1}
$$

which are the radiation solutions of the Maxwell equations in $\mathbb{R}^3\setminus\{0\}$, i.e.,

$$
\nabla\times\mathbf{M}_n^m(\rho,\theta,\varphi) = \mathrm{i}\omega\mathbf{N}_n^m(\rho,\theta,\varphi), \quad \nabla\times\mathbf{N}_n^m(\rho,\theta,\varphi) = -\mathrm{i}\omega\mathbf{M}_n^m(\rho,\theta,\varphi). \tag{4.2}
$$

It can be verified from (4.1) that the vector wave functions satisfy

$$\mathbf{M}_n^m(\rho,\theta,\varphi) = h_n^{(1)}(\omega\rho)\nabla_\Gamma X_n^m(\theta,\varphi)\times\mathbf{e}_\rho \tag{4.3}$$

and

$$\mathbf{N}_n^m(\rho,\theta,\varphi) = \frac{\sqrt{n(n+1)}}{\mathrm{i}\omega\rho}\left(h_n^{(1)}(\omega\rho)+\omega\rho(h_n^{(1)})'(\omega\rho)\right)\mathbf{X}_n^m(\theta,\varphi) + \frac{n(n+1)}{\mathrm{i}\omega\rho}h_n^{(1)}(\omega\rho)Y_n^m(\theta,\varphi)\mathbf{e}_\rho. \tag{4.4}$$

Simple calculation yields

$$\begin{aligned}
\mathbf{e}_\theta\times\mathbf{M}_n^m(\rho,\frac{\pi}{2},\varphi) &= 0 \quad \text{for } |m|\le n,\ m+n=\text{even},\ n\in\mathbb{N},\\
\mathbf{e}_\theta\times\mathbf{N}_n^m(\rho,\frac{\pi}{2},\varphi) &= 0 \quad \text{for } |m|\le n,\ m+n=\text{odd},\ n\in\mathbb{N}.
\end{aligned}$$

Therefore, in the domain $\mathbb{R}^3_+\setminus\Omega$, the solution of the scattered field $\mathbf{E}^s(\rho,\theta,\varphi)$, which satisfies the perfectly electric conducting condition $\mathbf{n}\times\mathbf{E}^{\mathrm{s}}=0$ on Γ_g, i.e., $\mathbf{e}_\theta\times\mathbf{E}^{\mathrm{s}}(\rho,\frac{\pi}{2},\varphi)=0$, can be written in the series expansion

$$\mathbf{E}^{\mathrm{s}}(\rho,\theta,\varphi) = \sum_{|m|\le n}^{\text{odd}}\alpha_n^m\,\mathbf{N}_n^m(\rho,\theta,\varphi) + \sum_{|m|\le n}^{\text{even}}\beta_n^m\,\mathbf{M}_n^m(\rho,\theta,\varphi) \tag{4.5}$$

with uniform convergence on compact subsets in $\mathbb{R}^3_+\setminus\Omega$. The corresponding scattered magnetic field $\mathbf{H}^{\mathrm{s}}$ is given by

$$\mathbf{H}^{\mathrm{s}} = \frac{1}{\mathrm{i}\omega}\nabla\times\mathbf{E}^{\mathrm{s}} = \sum_{|m|\le n}^{\text{odd}}-\alpha_n^m\,\mathbf{M}_n^m(\rho,\theta,\varphi) + \sum_{|m|\le n}^{\text{even}}\beta_n^m\,\mathbf{N}_n^m(\rho,\theta,\varphi). \tag{4.6}$$

To deduce an explicit representation of the Calderon operator, we need to express $\mathbf{E}^{\mathrm{s}}_{\Gamma_R^+} = -\mathbf{e}_\rho\times(\mathbf{e}_\rho\times\mathbf{E}^{\mathrm{s}})$ and $\mathbf{H}^{\mathrm{s}}\times\mathbf{e}_\rho$ on Γ_R^+ in terms of the coefficients α_n^m and β_n^m.

From the definition (4.3), it can be verified that

$$\begin{aligned}
-\mathbf{e}_\rho\times(\mathbf{e}_\rho\times\mathbf{M}_n^m(\rho,\theta,\varphi)) &= -\sqrt{n(n+1)}\,h_n^{(1)}(\omega\rho)\mathbf{Y}_n^m(\theta,\varphi),\\
-\mathbf{e}_\rho\times(\mathbf{e}_\rho\times\mathbf{N}_n^m(\rho,\theta,\varphi)) &= \frac{\sqrt{n(n+1)}}{\mathrm{i}\omega\rho}\left(h_n^{(1)}(\omega\rho)+\omega\rho(h_n^{(1)})'(\omega\rho)\right)\mathbf{X}_n^m(\theta,\varphi),
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{e}_\rho\times\mathbf{M}_n^m(\rho,\theta,\varphi) &= \sqrt{n(n+1)}\,h_n^{(1)}(\omega\rho)\mathbf{X}_n^m(\theta,\varphi),\\
\mathbf{e}_\rho\times\mathbf{N}_n^m(\rho,\theta,\varphi) &= \frac{\sqrt{n(n+1)}}{\mathrm{i}\omega\rho}\left(h_n^{(1)}(\omega\rho)+\omega\rho(h_n^{(1)})'(\omega\rho)\right)\mathbf{Y}_n^m(\theta,\varphi).
\end{aligned}$$

Therefore the tangential component of the scattered electric field is

$$\mathbf{E}^{\mathrm{s}}_{\Gamma_R^+} = \sum_{|m|\le n}^{\text{odd}}\frac{\sqrt{n(n+1)}}{\mathrm{i}\omega R}\left(h_n^{(1)}(\omega R)+\omega R(h_n^{(1)})'(\omega R)\right)\alpha_n^m\,\mathbf{X}_n^m(\theta,\varphi) - \sum_{|m|\le n}^{\text{even}}\sqrt{n(n+1)}\,h_n^{(1)}(\omega R)\beta_n^m\,\mathbf{Y}_n^m(\theta,\varphi),$$

and the tangential trace of the scattered magnetic field is

$$\mathbf{H}^{\mathrm{s}} \times \mathbf{e}_\rho = \sum_{|m|\le n}^{\text{odd}} \sqrt{n(n+1)}\, h_n^{(1)}(\omega R)\alpha_n^m\, \mathbf{X}_n^m(\theta,\varphi) - \sum_{|m|\le n}^{\text{even}} \frac{\sqrt{n(n+1)}}{\mathrm{i}\omega R} \left(h_n^{(1)}(\omega R) + \omega R (h_n^{(1)})'(\omega R)\right) \beta_n^m\, \mathbf{Y}_n^m(\theta,\varphi).$$

Therefore we have the following explicit representation of the Calderon operator T: for any tangential component of the electric field

$$\mathbf{u} = \sum_{|m|\le n}^{\text{odd}} \alpha_n^m\, \mathbf{X}_n^m + \sum_{|m|\le n}^{\text{even}} \beta_n^m\, \mathbf{Y}_n^m,$$

the tangential trace of the magnetic field is

$$T\mathbf{u} = \sum_{|m|\le n}^{\text{odd}} \frac{\mathrm{i}\omega R\, \alpha_n^m}{1 + z_n^{(1)}(\omega R)}\, \mathbf{X}_n^m + \sum_{|m|\le n}^{\text{even}} \frac{(1 + z_n^{(1)}(\omega R))\beta_n^m}{\mathrm{i}\omega R}\, \mathbf{Y}_n^m, \tag{4.7}$$

where

$$z_n^{(1)}(t) = \frac{t\,(h_n^{(1)})'(t)}{h_n^{(1)}(t)}.$$

Lemma 4.1. *The Calderon operator* $T : TH_0^{-1/2}(\mathrm{curl},\,\Gamma_R^+) \to TH^{-1/2}(\mathrm{div},\,\Gamma_R^+)$ *is continuous, i.e., there exists a positive constant* C *such that*

$$\| T\mathbf{u} \|_{TH^{-1/2}(\mathrm{div},\Gamma_R^+)} \le C \| \mathbf{u} \|_{TH_0^{-1/2}(\mathrm{curl},\Gamma_R^+)}$$

for any $\mathbf{u} \in TH_0^{-1/2}(\mathrm{curl},\,\Gamma_R^+)$.

Proof. For any $\mathbf{u}, \mathbf{v} \in TH_0^{-1/2}(\mathrm{curl},\,\Gamma_R^+)$, they have the following expansion

$$\mathbf{u} = \sum_{|m|\le n}^{\text{odd}} u_{1n}^m\, \mathbf{X}_n^m + \sum_{|m|\le n}^{\text{even}} u_{2n}^m\, \mathbf{Y}_n^m \quad \text{and} \quad \mathbf{v} = \sum_{|m|\le n}^{\text{odd}} v_{1n}^m\, \mathbf{X}_n^m + \sum_{|m|\le n}^{\text{even}} v_{2n}^m\, \mathbf{Y}_n^m.$$

Following the definition of the Calderon operator (4.7), we have

$$T\mathbf{u} = \sum_{|m|\le n}^{\text{odd}} \frac{\mathrm{i}\omega R\, u_{1n}^m}{1 + z_n^{(1)}(\omega R)}\, \mathbf{X}_n^m + \sum_{|m|\le n}^{\text{even}} \frac{(1 + z_n^{(1)}(\omega R))u_{2n}^m}{\mathrm{i}\omega R}\, \mathbf{Y}_n^m.$$

It follows from definition (3.23) that

$$\| T\mathbf{u} \|_{TH^{-1/2}(\mathrm{div},\Gamma_R^+)} = \sup_{\mathbf{v}\in TH_0^{-1/2}(\mathrm{curl},\Gamma_R^+)} \frac{\langle T\mathbf{u}, \mathbf{v}\rangle_{\Gamma_R^+}}{\| \mathbf{v} \|_{TH_0^{-1/2}(\mathrm{curl},\Gamma_R^+)}}.$$

To prove the lemma, it is required to estimate

$$\langle T\mathbf{u}, \mathbf{v}\rangle_{\Gamma_R^+} = \sum_{|m|\le n}^{\text{odd}} \frac{\mathrm{i}\omega R}{1 + z_n^{(1)}(\omega R)}\, u_{1n}^m\, \bar{v}_{1n}^m + \sum_{|m|\le n}^{\text{even}} \frac{(1 + z_n^{(1)}(\omega R))}{\mathrm{i}\omega R}\, u_{2n}^m\, \bar{v}_{2n}^m.$$

It follows from the Cauchy-Schwarz inequality that

$$\langle T\mathbf{u},\mathbf{v}\rangle_{\Gamma_R^+} \le \left[\sum_{|m|\le n}^{\text{odd}} \frac{\sqrt{1+n(n+1)}}{|1+z_n^{(1)}(\omega R)|^2}\,|\omega R|^2\,|u_{1n}^m|^2 + \sum_{|m|\le n}^{\text{even}} \frac{|1+z_n^{(1)}(\omega R)|^2}{\sqrt{1+n(n+1)}}\frac{|u_{2n}^m|^2}{|\omega R|^2}\right]^{1/2}$$
$$\times\left[\sum_{|m|\le n}^{\text{odd}} \frac{1}{\sqrt{1+n(n+1)}}|v_{1n}^m|^2 + \sum_{|m|\le n}^{\text{even}} \sqrt{1+n(n+1)}|v_{2n}^m|^2\right]^{1/2}.$$

It is proved in Kirsch and Monk [34] (cf., Lemma 3.1) that there exist positive constants C_1 and C_2 such that

$$C_1\,n \le |1+z_n^{(1)}(\omega R)| \le C_2\,n \quad \text{for all } n.$$

Thus, we have

$$\begin{aligned}\frac{\sqrt{1+n(n+1)}}{|1+z_n^{(1)}(\omega R)|^2}\,|\omega R|^2\,|u_{1n}^m|^2 &= \frac{1}{\sqrt{1+n(n+1)}}\frac{1+n(n+1)}{|1+z_n^{(1)}(\omega R)|^2}|\omega R|^2|u_{1n}^m|^2\\ &\le C_3\frac{1}{\sqrt{1+n(n+1)}}\,|u_{1n}^m|^2\end{aligned}$$

and

$$\begin{aligned}\frac{|1+z_n^{(1)}(\omega R)|^2}{\sqrt{1+n(n+1)}}\frac{|u_{2n}^m|^2}{|\omega R|^2} &= \sqrt{1+n(n+1)}\frac{|1+z_n^{(1)}(\omega R)|^2}{1+n(n+1)}\frac{|u_{2n}^m|^2}{|\omega R|^2}\\ &\le C_4\sqrt{1+n(n+1)}\,|u_{2n}^m|^2.\end{aligned}$$

Combining above estimates yields

$$\langle T\mathbf{u},\mathbf{v}\rangle_{\Gamma_R^+} \le C \parallel \mathbf{u} \parallel_{TH_0^{-1/2}(\operatorname{curl}\Gamma_R^+)} \parallel \mathbf{v} \parallel_{TH_0^{-1/2}(\operatorname{curl}\Gamma_R^+)}$$

where the constant C depends on $\omega R, C_1, C_2, C_3$, and C_4. □

Lemma 4.2. *Let* $\mathbf{u}$ *be in* $TH_0^{-1/2}(\mathrm{curl},\Gamma_R^+)$. *It holds* $\mathrm{Re}\,\langle T\mathbf{u},\mathbf{u}\rangle_{\Gamma_R^+}\ge 0$. *If* $\mathrm{Re}\,\langle T\mathbf{u},\mathbf{u}\rangle_{\Gamma_R^+}=0$, *then* $\mathbf{u}=0$ *on* Γ_R^+.

Proof. Be definitions, we obtain

$$\langle T\mathbf{u},\mathbf{u}\rangle_{\Gamma_R^+} = \sum_{|m|\le n}^{\text{odd}} \frac{\mathrm{i}\omega R}{1+z_n^{(1)}(\omega R)}\,|u_{1n}^m|^2 + \sum_{|m|\le n}^{\text{even}} \frac{1+z_n^{(1)}(\omega R)}{\mathrm{i}\omega R}\,|u_{2n}^m|^2.$$

Taking the real part of the above identity gives

$$\mathrm{Re}\,\langle T\mathbf{u},\mathbf{u}\rangle_{\Gamma_R^+} = \sum_{|m|\le n}^{\text{odd}} \frac{\omega R\,\mathrm{Im}\,z_n^{(1)}(\omega R)}{|1+z_n^{(1)}(\omega R)|^2}\,|u_{1n}^m|^2 + \sum_{|m|\le n}^{\text{even}} \frac{\mathrm{Im}\,z_n^{(1)}(\omega R)}{\omega R}|u_{2n}^m|^2.$$

It is known (cf., Nédélec [45], Lemma 2.6.1) that

$$0 < \mathrm{Im}\,z_n^{(1)}(\omega R) \le \omega R.$$

Thus

$$\mathrm{Re}\,\langle T\mathbf{u},\mathbf{u}\rangle_{\Gamma_R^+} \ge 0.$$

If $\mathrm{Re}\,\langle T\mathbf{u},\mathbf{u}\rangle_{\Gamma_R^+}=0$ implies that $u_{1n}^m=u_{2n}^m=0$, which yields $\mathbf{u}=0$ on Γ_R^+. □

Lemma 4.3. *There exist a positive constant C such that*

$$\mathrm{Im}\langle T\mathbf{u},\mathbf{u}\rangle_{\Gamma_R^+} \geq -C \parallel \mathbf{u} \parallel^2_{TH_0^{-1/2}(\Gamma_R^+)},$$

for any $\mathbf{u}$ *in* $TH_0^{-1/2}(\mathrm{curl},\,\Gamma_R^+)$.

Proof. For any $\mathbf{u} \in TH_0^{-1/2}(\mathrm{curl},\,\Gamma_R^+)$, it has the following expansion

$$\mathbf{u} = \sum_{|m|\leq n}^{\text{odd}} u_{1n}^m\,\mathbf{X}_n^m + \sum_{|m|\leq n}^{\text{even}} u_{2n}^m\,\mathbf{Y}_n^m.$$

Be definitions, we obtain

$$\langle T\mathbf{u},\mathbf{u}\rangle_{\Gamma_R^+} = \sum_{|m|\leq n}^{\text{odd}} \frac{\mathrm{i}\omega R}{1+z_n^{(1)}(\omega R)}\,|u_{1n}^m|^2 + \sum_{|m|\leq n}^{\text{even}} \frac{1+z_n^{(1)}(\omega R)}{\mathrm{i}\omega R}\,|u_{2n}^m|^2.$$

Taking the imaginary part of the above identity gives

$$\mathrm{Im}\,\langle T\mathbf{u},\mathbf{u}\rangle_{\Gamma_R^+} = \sum_{|m|\leq n}^{\text{odd}} \frac{\omega R\,(1+\mathrm{Re}\,z_n^{(1)}(\omega R))}{|1+z_n^{(1)}(\omega R)|^2}\,|u_{1n}^m|^2 - \sum_{|m|\leq n}^{\text{even}} \frac{1+\mathrm{Re}\,z_n^{(1)}(\omega R)}{\omega R}|u_{2n}^m|^2.$$

It is known (cf., Nédélec [45], Lemma 2.6.1) that

$$-(n+1) \leq \mathrm{Re}\,z_n^{(1)}(\omega R) \leq -1.$$

Thus

$$\begin{aligned}\mathrm{Im}\,\langle T\mathbf{u},\mathbf{u}\rangle_{\Gamma_R^+} &\geq \omega R \sum_{|m|\leq n}^{\text{odd}} \frac{(1+\mathrm{Re}\,z_n^{(1)}(\omega R))}{|1+z_n^{(1)}(\omega R)|^2}\,|u_{1n}^m|^2 \geq -\omega R \sum_{|m|\leq n}^{\text{odd}} \frac{n}{|1+z_n^{(1)}(\omega R)|^2}\,|u_{1n}^m|^2 \\ &\geq -C \sum_{|m|\leq n}^{\text{odd}} \frac{1}{\sqrt{1+n(n+1)}}\,|u_{1n}^m|^2 \geq -C \parallel \mathbf{u} \parallel^2_{TH_0^{-1/2}(\Gamma_R^+)},\end{aligned}$$

which completes the proof. □

Lemma 4.4. *There exists a positive constant C such that*

$$|\langle T\nabla_\Gamma u, \nabla_\Gamma v\rangle_{\Gamma_R^+}| \leq C \parallel u \parallel_{H^1(\Omega)} \parallel v \parallel_{H^1(\Omega)},$$

for any $\nabla_\Gamma u$ *and* $\nabla_\Gamma v$ *in* $\in TH_0^{-1/2}(\mathrm{curl},\,\Gamma_R^+)$.

Proof. Following the expansion

$$u = \sum_{|m|\leq n}^{\text{odd}} u_n^m X_n^m \quad \text{and} \quad v = \sum_{|m|\leq n}^{\text{odd}} v_n^m X_n^m,$$

we have

$$\nabla_\Gamma u = \sum_{|m|\leq n}^{\text{odd}} \sqrt{n(n+1)}\,u_n^m\,\mathbf{X}_n^m \quad \text{and} \quad \nabla_\Gamma v = \sum_{|m|\leq n}^{\text{odd}} \sqrt{n(n+1)}\,v_n^m\,\mathbf{X}_n^m$$

and

$$T\nabla_\Gamma u = \mathrm{i}\omega R \sum_{|m|\le n}^{\text{odd}} \frac{\sqrt{n(n+1)}}{1+z_n^{(1)}(\omega R)}\, u_n^m\, \mathbf{X}_n^m.$$

By definition, we have

$$\langle T\nabla_\Gamma u, \nabla_\Gamma v\rangle_{\Gamma_R^+} = \mathrm{i}\omega R \sum_{|m|\le n}^{\text{odd}} \frac{n(n+1)}{1+z_n^{(1)}(\omega R)}\, u_n^m\, \bar{v}_n^m.$$

It follows from the Cauchy–Schwarz inequality that

$$\begin{aligned}
|\langle T\nabla_\Gamma u, \nabla_\Gamma v\rangle_{\Gamma_R^+}| &\le \omega R \sum_{|m|\le n}^{\text{odd}} \frac{n(n+1)}{|1+z_n^{(1)}(\omega R)|} |u_n^m|\, |v_n^m| \\
&\le C \sum_{|m|\le n}^{\text{odd}} (1+n(n+1))^{1/2} |u_n^m|\, |v_n^m| \\
&\le C \parallel u \parallel_{H^{1/2}(\Gamma_R^+)} \parallel v \parallel_{H^{1/2}(\Gamma_R^+)} \le C \parallel u \parallel_{H^1(\Omega)} \parallel v \parallel_{H^1(\Omega)},
\end{aligned}$$

where the last inequality follows from the trace theorem of standard elliptic boundary value problems.

□

Lemma 4.5. *There exists a positive constant C such that*

$$|\langle T\mathbf{u}_{\Gamma_R^+}, \nabla_\Gamma v\rangle_{\Gamma_R^+}| \le C \parallel v \parallel_{H^1(\Omega)} \parallel \mathbf{u} \parallel_{H(\mathrm{curl},\,\Omega)},$$

for any $\mathbf{u}$ *in* $H_S(\mathrm{curl},\,\Omega)$ *and* $\nabla_\Gamma v$ *in* $\in TH_0^{-1/2}(\mathrm{curl},\,\Gamma_R^+)$.

Proof. It follows from the expansions that

$$\mathbf{u} = \sum_{|m|\le n}^{\text{odd}} u_{1n}^m\, \mathbf{X}_n^m + \sum_{|m|\le n}^{\text{even}} u_{2n}^m\, \mathbf{Y}_n^m \quad \text{and} \quad \nabla_\Gamma v = \sum_{|m|\le n}^{\text{odd}} \sqrt{n(n+1)}\, v_n^m\, \mathbf{X}_n^m$$

By definition

$$\langle T\mathbf{u}_{\Gamma_R^+}, \nabla_\Gamma v\rangle_{\Gamma_R^+} = \mathrm{i}\omega R \sum_{|m|\le n}^{\text{odd}} \frac{\sqrt{n(n+1)}}{1+z_n^{(1)}(\omega R)} u_{1n}^m\, \bar{v}_n^m,$$

which gives

$$\begin{aligned}
|\langle T\mathbf{u}_{\Gamma_R^+}, \nabla_\Gamma v\rangle_{\Gamma_R^+}| &\le C \sum_{|m|\le n}^{\text{odd}} |u_{1n}^m|\, |v_n^m| \\
&\le C \left(\sum_{|m|\le n}^{\text{odd}} \frac{1}{\sqrt{1+n(n+1)}}\, |u_{1n}^m|^2 \right)^{1/2} \left(\sum_{|m|\le n}^{\text{odd}} \sqrt{1+n(n+1)}\, |v_n^m|^2 \right)^{1/2} \\
&\le C \parallel \mathbf{u}_{\Gamma_R^+} \parallel_{TH_0^{-1/2}(\mathrm{curl},\,\Gamma_R^+)} \parallel v \parallel_{H^{1/2}(\Gamma_R^+)} .
\end{aligned}$$

The proof is completed by applying the trace property and Lemma 3.3.

□

Lemma 4.6. *For any* $\nabla_\Gamma u$ *in* $TH_0^{-1/2}(\mathrm{curl},\,\Gamma_R^+)$*, it holds that*

$$\mathrm{Im}\,\langle T\nabla_\Gamma u, \nabla_\Gamma u\rangle_{\Gamma_R^+} \leq 0.$$

Proof. By definition, we have

$$\langle T\nabla_\Gamma u, \nabla_\Gamma u\rangle_{\Gamma_R^+} = \sum_{|m|\leq n}^{\mathrm{odd}} n(n+1)\frac{\mathrm{i}\omega R}{1+z_n^{(1)}(\omega R)}\,|u_n^m|^2.$$

Taking the imaginary part gives and using $-(n+1) \leq \mathrm{Re}z_n^{(1)} \leq -1$ (cf. Nédélec [45], Lemma 2.6.1), we obtain

$$\mathrm{Im}\,\langle T\nabla_\Gamma u, \nabla_\Gamma u\rangle_{\Gamma_R^+} = \sum_{|m|\leq n}^{\mathrm{odd}} n(n+1)\frac{\omega R(1+\mathrm{Re}z_n^{(1)}(\omega R))}{|1+z_n^{(1)}(\omega R)|^2}\,|u_n^m|^2 \leq 0,$$

which completes the proof. □

Using the Calderon operator, the following transparent boundary condition may be proposed on the hemisphere Γ_R^+:

$$T(\mathbf{E}_{\Gamma_R^+} - \mathbf{E}^{\mathrm{inc}}_{\Gamma_R^+}) = (\mathbf{H} - \mathbf{H}^{\mathrm{inc}}) \times \mathbf{e}_\rho,$$

which maps the tangential component of the scattered electric field to the tangential trace of the scattered magnetic field. Equivalently, it can be written as

$$(\nabla \times \mathbf{E}) \times \mathbf{e}_\rho = \mathrm{i}\omega T\mathbf{E}_{\Gamma_R^+} + \mathbf{f}, \tag{4.8}$$

where

$$\mathbf{f} = \mathrm{i}\omega(\mathbf{H}^{\mathrm{inc}} \times \mathbf{e}_\rho - T\mathbf{H}^{\mathrm{inc}}).$$

5 Variational problem

In this section, we shall introduce the variational formulation for the overfilled cavity scattering problem using the transparent boundary condition. Based on a Hodge decomposition, compact embedding results, and the Fredholm alternative, the existence and uniqueness of the solution for the model problem are established by a variational approach.

5.1 Uniqueness

We present a variational formulation of the Maxwell equations in the space $H_S(\mathrm{curl},\,\Omega)$ and give a proof of the uniqueness for the boundary value problem.

By eliminating the magnetic field from (2.1), we obtain the equation for the electric field:

$$\nabla \times (\mu^{-1}\nabla \times \mathbf{E}) - \omega^2\varepsilon\mathbf{E} = 0 \quad \text{in } \Omega. \tag{5.1}$$

Multiplying the complex conjugate of a test function $\mathbf{w}$ in $H_S(\mathrm{curl},\,\Omega)$, integrating over Ω, and using integration by parts, we arrive at the variational form for the scattering problem: find $\mathbf{E} \in H_S(\mathrm{curl},\,\Omega)$ such that

$$a(\mathbf{E}, \mathbf{w}) = \langle \mathbf{f}, \mathbf{w}\rangle_{\Gamma_R^+} \quad \text{for all } \mathbf{w} \in H_S(\mathrm{curl},\,\Omega), \tag{5.2}$$

where the sesquilinear form

$$a(\mathbf{E}, \mathbf{w}) = \int_\Omega \mu^{-1}(\nabla \times \mathbf{E}) \cdot (\nabla \times \bar{\mathbf{w}}) - \omega^2 \int_\Omega \varepsilon \mathbf{E} \cdot \bar{\mathbf{w}} - \mathrm{i}\omega \int_{\Gamma_R^+} T\mathbf{E}_{\Gamma_R^+} \cdot \bar{\mathbf{w}}_{\Gamma_R^+}, \tag{5.3}$$

and

$$\mathbf{f} = \mathrm{i}\omega \left(\mathbf{H}^{\mathrm{inc}} \times \mathbf{e}_\rho - T\mathbf{E}_{\Gamma_R^+} \right).$$

The following uniqueness proof is based on a unique continuation result to Maxwell's equations. We refer to Eller, Isakov, Nakamura, and Tataru [28] for a proof of the unique continuation result.

Theorem 5.1. *Assume that $\varepsilon, \mu \in C^2(\Omega)$. Then the variational problem (5.2) has at most one solution.*

Proof. It suffices to show that $\mathbf{E} = 0$ in Ω if $\mathbf{E}^{\mathrm{inc}} = \mathbf{H}^{\mathrm{inc}} = 0$. If $\mathbf{E}$ satisfies the homogeneous variational problem in Ω, then

$$\int_\Omega \mu^{-1} |\nabla \times \mathbf{E}|^2 - \omega^2 \int_\Omega \varepsilon |\mathbf{E}|^2 - \mathrm{i}\omega \int_{\Gamma_R^+} T\mathbf{E}_{\Gamma_R^+} \cdot \bar{\mathbf{E}}_{\Gamma_R^+} = 0.$$

Taking the imaginary part yields

$$\omega^2 \int_\Omega \mathrm{Im}(\varepsilon) |\mathbf{E}|^2 + \omega \mathrm{Re} \int_{\Gamma_R^+} T\mathbf{E}_{\Gamma_R^+} \cdot \bar{\mathbf{E}}_{\Gamma_R^+} = 0. \tag{5.4}$$

It follows from $\mathrm{Im}\,\varepsilon \geq 0$ and Lemma 4.2 that $\mathbf{E} \times \mathbf{e}_\rho = 0$ on Γ_R^+, i.e., $\mathbf{H} \times \mathbf{e}_\rho = 0$ on Γ_R^+. Noting the transparent boundary condition (4.8), we have $(\nabla \times \mathbf{E}) \times \mathbf{e}_\rho = 0$ on Γ_R^+. An application of Holmgrens uniqueness theorem in Abbound and Nédélec [8] yields $\mathbf{E} = 0$ in $\mathbb{R}^3_+ \setminus \Omega$. By unique continuation in [28], we get $\mathbf{E} = 0$ in Ω. □

Remark 5.1. *From [28], the unique continuation holds for $H^1(\Omega)$ solution of the Maxwell equations, which requires the regularity of the permittivity ε and permeability μ. When ε has a positive imaginary part, i.e., $\mathrm{Im}\,\varepsilon > 0$, the uniqueness result is obvious from (5.4) even for $\varepsilon, \mu \in L^\infty(\Omega)$.*

5.2 Hodge decomposition

We present a version of Hodge decomposition and a compactness lemma. The results are crucial in the proof of our theorem on the existence. Let us begin with a technical lemma.

Lemma 5.1. *Given $h \in L^2(\Omega)$ and $g \in H^{-1/2}(\Gamma_R^+)$, the boundary value problem*

$$\nabla \cdot (\varepsilon \nabla u) = h \quad \text{in } \Omega, \tag{5.5}$$

$$\partial_\rho u = \frac{\mathrm{i}}{\omega} \mathrm{div}_\Gamma T \nabla_\Gamma u + g \quad \text{on } \Gamma_R^+, \tag{5.6}$$

has a unique solution in $H^1_S(\Omega) = \{w \in H^1(\Omega) : w = 0 \text{ on } S\}$.

Proof. We examine the weak form of the boundary value problem. Multiplying the complex conjugate of the test function v on both sides of (5.5) and integrating over Ω, we have

$$\int_\Omega \nabla \cdot (\varepsilon \nabla u) \cdot \bar{v} = \int_\Omega h \bar{v} \quad \text{for all}\, v \in H^1_S(\Omega).$$

Integration by parts gives after using the boundary condition (5.6) that

$$\int_\Omega \varepsilon\nabla u\cdot\nabla\bar{v} - \frac{\mathrm{i}}{\omega}\int_{\Gamma_R^+} (\mathrm{div}_\Gamma T\nabla_\Gamma u)\,\bar{v} = \int_{\Gamma_R^+} g\,\bar{v} - \int_\Omega h\,\bar{v}.$$

It follows from the divergence theorem on the boundary that

$$\int_\Omega \varepsilon\nabla u\cdot\nabla\bar{v} + \frac{\mathrm{i}}{\omega}\int_{\Gamma_R^+} T\nabla_\Gamma u\cdot\nabla_\Gamma\bar{v} = \int_{\Gamma_R^+} g\,\bar{v} - \int_\Omega h\,\bar{v}.$$

Denote a sesquilinear form

$$b(u,v) = \int_\Omega \varepsilon\nabla u\cdot\nabla\bar{v} + \frac{\mathrm{i}}{\omega}\int_{\Gamma_R^+} T\nabla_\Gamma u\cdot\nabla_\Gamma\bar{v}.$$

The variational problem takes the form: find $u\in H_S^1(\Omega)$, such that

$$b(u,v) = \int_{\Gamma_R^+} g\,\bar{v} - \int_\Omega h\,\bar{v} \quad \text{for all}\, v\in H_S^1(\Omega).$$

From the Cauchy–Schwarz inequality and Lemma 4.4 we have

$$\begin{aligned}|b(u,v)| \le & C_1 \parallel \nabla u \parallel_{(L^2(\Omega))^3} \parallel \nabla v \parallel_{(L^2(\Omega))^3} + C_2 \parallel u \parallel_{H^1(\Omega)} \parallel v \parallel_{H^1(\Omega)} \\ \le & \, C \parallel u \parallel_{H^1(\Omega)} \parallel v \parallel_{H^1(\Omega)} .\end{aligned}$$

Next is to prove the coercivity. Using Lemma 4.6 gives

$$\mathrm{Re}\, b(u,u) = \int_\Omega \mathrm{Re}(\varepsilon)|\nabla u|^2 - \frac{1}{\omega}\,\mathrm{Im}\int_{\Gamma_R^+} T\nabla_\Gamma u\cdot\nabla_\Gamma\bar{u} \ge \int_\Omega \mathrm{Re}(\varepsilon)|\nabla u|^2.$$

By Poincaré's inequality, we obtain

$$|b(u,u)| \ge \mathrm{Re}\, b(u,u) \ge C \parallel u \parallel^2_{H^1(\Omega)} \quad \text{for all}\, u\in H_S^1(\Omega).$$

The proof is completed by a direct application of the Lax–Milgram lemma. □

Before giving a compact imbedding result, we introduce a Hodge decomposition of $H_S(\mathrm{curl},\Omega)$ by using the spaces

$$\mathbb{H} = \{\mathbf{u}\in H_S(\mathrm{curl},\Omega) : \nabla\cdot(\varepsilon\mathbf{u}) = 0 \text{ in } \Omega,\ \mathbf{u}\cdot\mathbf{e}_\rho = \frac{\mathrm{i}}{\omega}\mathrm{div}_\Gamma T\mathbf{u}_{\Gamma_R^+} \text{ on } \Gamma_R^+\}$$

and

$$\mathbb{H}^\perp = \{\mathbf{u} :\ \mathbf{u} = \nabla u,\ u\in H_S^1(\Omega)\}.$$

Lemma 5.2. *The spaces* $\mathbb{H}$ *and* $\mathbb{H}^\perp$ *are closed subspaces of* $H_S(\mathrm{curl},\,\Omega)$*, which is the direct sum of the spaces* $\mathbb{H}$ *and* $\mathbb{H}^\perp$*, i.e.,*

$$H_S(\mathrm{curl},\,\Omega) = \mathbb{H}\oplus\mathbb{H}^\perp.$$

Proof. Take $\mathbf{u}_n = \nabla u_n\in\mathbb{H}^\perp$, and it follows from $\mathbf{u}_n\to\mathbf{u}$ in $H_S(\mathrm{curl},\Omega)$ that

$$\begin{aligned}\parallel \mathbf{u}_n - \mathbf{u} \parallel^2_{H(\mathrm{curl},\,\Omega)} =& \parallel \mathbf{u}_n - \mathbf{u} \parallel^2_{(L^2(\Omega))^3} + \parallel \nabla\times\mathbf{u}_n - \nabla\times\mathbf{u} \parallel^2_{(L^2(\Omega))^3} \\ =& \parallel \nabla u_n - \mathbf{u} \parallel^2_{(L^2(\Omega))^3} + \parallel \nabla\times\mathbf{u} \parallel^2_{(L^2(\Omega))^3} \to 0 \quad \text{as } n\to\infty.\end{aligned}$$

The closedness of $\mathbb{H}^\perp$ follows from $\| \nabla \times \mathbf{u} \|_{(L^2(\Omega))^3} = 0$.

Equivalently, the subspace $\mathbb{H}$ can be represented as

$$\mathbb{H} = \{\mathbf{u} \in H_S(\mathrm{curl}, \Omega) : \int_\Omega \varepsilon \mathbf{u} \cdot \nabla \bar{v} - \frac{\mathrm{i}}{\omega} \int_{\Gamma_R^+} \mathrm{div}_\Gamma T \mathbf{u}_{\Gamma_R^+} \bar{v} = 0 \quad \text{for all } v \in H_S^1(\Omega)\}.$$

For fixed $v \in H_S^1(\Omega)$, define the linear functional:

$$\begin{aligned} f(\mathbf{u}) &= \int_\Omega \varepsilon \mathbf{u} \cdot \nabla \bar{v} - \frac{\mathrm{i}}{\omega} \int_{\Gamma_R^+} \mathrm{div}_\Gamma T \mathbf{u}_{\Gamma_R^+} \bar{v} \\ &= \int_\Omega \varepsilon \mathbf{u} \cdot \nabla \bar{v} + \frac{\mathrm{i}}{\omega} \int_{\Gamma_R^+} T \mathbf{u}_{\Gamma_R^+} \cdot \nabla_\Gamma \bar{v}. \end{aligned}$$

Applying the Cauchy–Schwarz inequality and Lemma 4.5 yield

$$|f(\mathbf{u})| \leq C \| v \|_{H^1(\Omega)} \| \mathbf{u} \|_{H(\mathrm{curl}, \Omega)},$$

which gives

$$|f(\mathbf{u})| \leq C \| \mathbf{u} \|_{H(\mathrm{curl}, \Omega)} .$$

Let $\mathbf{u}_n \in \mathbb{H}$, and $\mathbf{u}_n \to \mathbf{u}$ in $H_S(\mathrm{curl}, \Omega)$. We have

$$|f(\mathbf{u})| = |f(\mathbf{u} - \mathbf{u}_n) + f(\mathbf{u}_n)| = |f(\mathbf{u} - \mathbf{u}_n)| \leq C \| \mathbf{u} - \mathbf{u}_n \|_{H(\mathrm{curl}, \Omega)} \to 0 \quad \text{as } n \to \infty,$$

which implies $\mathbf{u} \in \mathbb{H}$, and thus the closedness of the space $\mathbb{H}$.

For any $\mathbf{u} \in H_S(\mathrm{curl}, \Omega)$, define $u \in H_S^1(\Omega)$ by the solution of

$$a(\nabla u, \nabla v) = a(\mathbf{u}, \nabla v) \quad \text{for all } v \in H_S^1(\Omega),$$

which gives in differential form

$$\begin{aligned} \nabla \cdot (\varepsilon \nabla u) &= \nabla \cdot (\varepsilon \mathbf{u}) \quad \text{in } \Omega \\ \partial_\rho u &= \frac{\mathrm{i}}{\omega} \mathrm{div}_\Gamma T \nabla_\Gamma u + g \quad \text{on } \Gamma_R^+, \end{aligned}$$

where

$$g = \mathbf{u} \cdot \mathbf{e}_\rho - \frac{\mathrm{i}}{\omega} \mathrm{div}_\Gamma T \nabla_\Gamma \mathbf{u}_{\Gamma_R^+}.$$

Following Lemma 5.1, there exists a unique solution u in $H_S^1(\Omega)$.

Denote

$$\mathbf{v} := \mathbf{u} - \nabla u.$$

Then

$$a(\mathbf{v}, \nabla v) = 0 \quad \text{for all } v \in H_S^1(\Omega).$$

By noting the sesquilinear form of a, integration by parts yields

$$\int_\Omega \nabla \cdot (\varepsilon \mathbf{v}) \bar{v} - \int_{\Gamma_R^+} \left(\mathbf{v} \cdot \mathbf{e}_\rho - \frac{\mathrm{i}}{\omega} \mathrm{div}_\Gamma T \mathbf{v}_{\Gamma_R^+} \right) \bar{v} = 0 \quad \text{for all } v \in H_S^1(\Omega),$$

which gives that $\mathbf{v} \in \mathbb{H}$. Finally, we show that $\mathbb{H} \cap \mathbb{H}^\perp$ consists of the trivial function only. Indeed, if $\mathbf{u} = \nabla u \in \mathbb{H} \cap \mathbb{H}^\perp$, then

$$\begin{aligned} \nabla \cdot (\varepsilon \nabla u) &= 0 \quad \text{in } \Omega, \\ \partial_\rho u &= \frac{\mathrm{i}}{\omega} \mathrm{div}_\Gamma T \nabla_\Gamma u \quad \text{on } \Gamma_R^+, \end{aligned}$$

which implies that $\mathbf{u} = \nabla u = 0$ from Lemma 5.1. □

Lemma 5.3. *The space $\mathbb{H}$ is compactly embedded into the space $(L^2(\Omega))^3$.*

Proof. Consider a bounded set of function $\{\mathbf{u}_n\}_{n=1}^{\infty} \subset \mathbb{H}$. Each function $\mathbf{u}_n \in \mathbb{H}$ can be extended to all of $\mathbb{R}^3_+ \cup \Omega$ by solving the exterior Maxwell equation

$$\begin{aligned} \nabla \times (\nabla \times \mathbf{v}_n) - \omega^2 \mathbf{v}_n = 0 \quad & \text{in } \mathbb{R}^3_+ \setminus \Omega, \\ \mathbf{e}_\rho \times \mathbf{v}_n = \mathbf{e}_\rho \times \mathbf{u}_n \quad & \text{on } \Gamma_R^+, \\ \mathbf{n} \times \mathbf{v}_n = 0 \quad & \text{on } \Gamma_g, \end{aligned}$$

together with the Silver–Müller radiation condition at infinity. The function $\mathbf{w}_n$ defined by

$$\mathbf{w}_n = \begin{cases} \mathbf{u}_n & \text{in } \Omega, \\ \mathbf{v}_n & \text{in } \mathbb{R}^3_+ \setminus \Omega, \end{cases}$$

is in $H_{\rm loc}(\text{curl}, \mathbb{R}^3_+ \setminus \Omega)$ since the tangential components are continuous across Γ_R^+. Furthermore, since $\mathbf{u}_n \in \mathbb{H}$, we have the constraint that

$$\mathbf{u}_n \cdot \mathbf{e}_\rho = \frac{\mathrm{i}}{\omega} \mathrm{div}_\Gamma T \mathbf{u}_{n \Gamma_R^+} \quad \text{on } \Gamma_R^+.$$

It follows from the continuity of the tangential components that

$$T \mathbf{u}_{n \Gamma_R^+} = T \mathbf{v}_{n \Gamma_R^+} = \frac{\mathrm{i}}{\omega} (\nabla \times \mathbf{v}_n) \times \mathbf{e}_\rho.$$

Using the identity

$$\mathrm{div}_\Gamma ((\nabla \times \mathbf{v}_n) \times \mathbf{e}_\rho) = (\nabla \times (\nabla \times \mathbf{v}_n)) \cdot \mathbf{e}_\rho$$

and Maxwell's equations, we obtain

$$\frac{\mathrm{i}}{\omega} \mathrm{div}_\Gamma T \mathbf{v}_{n \Gamma_R^+} = \mathbf{v}_n \cdot \mathbf{e}_\rho,$$

which yields

$$\mathbf{u}_n \cdot \mathbf{e}_\rho = \mathbf{v}_n \cdot \mathbf{e}_\rho \quad \text{on } \Gamma_R^+.$$

Therefore, the normal component of $\mathbf{w}_n$ is also continuous and this extended function has a well-defined divergence. The divergence free conditions inside Ω and in the complement of $\mathbb{R}^3_+ \setminus \Omega$ show that $\nabla \cdot (\varepsilon \mathbf{w}_n) = 0$ in $\mathbb{R}^3_+ \cup \Omega$.

Now we choose a cutoff function $\chi \in C_0^\infty(\mathbb{R}^3_+ \cup \Omega)$ such that $\chi = 1$ in $\overline{\Omega}$. We can apply the general compactness result (cf., Theorem 4.7 in Monk [44]) to the sequence $\{\chi \mathbf{w}_n\}$ and extract a subsequence converging strongly in $(L^2(\Omega))^3$, which completes the proof.

□

The compact embedding results are similar in spirit to the one used for solving the grating problem by Ammari and Bao [2] and the cavity problem by Ammari, Bao, and Wood [7]. We refer to Weber [51] for a compact embedding result on bounded domain. The next result verifies that we have indeed removed the null-space of the curl from $\mathbb{H}$ and is referred to as Friedriches inequality, showing that the curl-curl sesquilinear form is coercive on $\mathbb{H}$.

Lemma 5.4. *There exists a positive constant C such that*

$$\| \mathbf{u} \|_{(L^2(\Omega))^3} \leq C \| \nabla \times \mathbf{u} \|_{(L^2(\Omega))^3}$$

for all $\mathbf{u} \in \mathbb{H}$.

Proof. Suppose that $\mathbf{u} \in \mathbb{H}$ and $\nabla \times \mathbf{u} = 0$. Since $\mathbf{u}$ is curl free, there is a function $u \in H^1_S(\Omega)$ such that $\mathbf{u} = \nabla u$. Hence $\mathbf{u} \in \mathbb{H}^\perp$. It follows from Lemma 5.2 that $\mathbf{u} = 0$.

Assume that the estimate is false. For any n, there exists $\mathbf{u}_n \in \mathbb{H}$ with $\| \mathbf{u}_n \|_{H(\mathrm{curl},\, \Omega)} = 1$ such that

$$\| \nabla \times \mathbf{u}_n \|_{(L^2(\Omega))^3} < \frac{1}{n} \| \mathbf{u}_n \|_{(L^2(\Omega))^3} .$$

We have $\nabla \times \mathbf{u}_n \to 0$ as $n \to \infty$. Since $\mathbf{u}_n \rightharpoonup \mathbf{u} \in \mathbb{H}$, we have

$$\int_\Omega \nabla \times \mathbf{u}_n \cdot \nabla \times \bar{\mathbf{v}} + \int_\Omega \mathbf{u}_n \cdot \bar{\mathbf{v}} \to \int_\Omega \nabla \times \mathbf{u} \cdot \nabla \times \bar{\mathbf{v}} + \int_\Omega \mathbf{u} \cdot \bar{\mathbf{v}} \quad \text{for all } \mathbf{v} \in \mathbb{H}. \tag{5.7}$$

Using Lemma 5.3, there is a subsequence $\mathbf{u}_{n_j}$ such that $\mathbf{u}_{n_j} \to \mathbf{u}$ in $(L^2(\Omega))^3$ as $n_j \to \infty$. It follows from (5.7) by noting $\nabla \times \mathbf{u}_{n_j} \to 0$ in $(L^2(\Omega))^3$ as $n_j \to \infty$ that

$$\int_\Omega \nabla \times \mathbf{u} \cdot \nabla \times \bar{\mathbf{v}} = 0 \quad \text{for all } \mathbf{v} \in \mathbb{H},$$

which implies $\nabla \times \mathbf{u} = 0$. Hence $\mathbf{u} = 0$, which is a contradiction. □

5.3 Existence

We investigate the well-posedness of the variational problem by examining the sesquilinear form. It is clear that the sesquilinear form a is not coercive in $H_S(\mathrm{curl},\, \Omega)$. We will use the argument of Fredholm alternative since the Lax-Milgram lemma does not apply.

Theorem 5.2. *The variational problem* (5.2) *has a unique weak solution in* $H_S(\mathrm{curl},\, \Omega)$ *given by* $\mathbf{E} = \mathbf{u} + \nabla u$, *where* $\mathbf{u} \in \mathbb{H},\ u \in H^1_S(\Omega)$.

Proof. Using the Hodge decomposition, we take $\mathbf{E} = \mathbf{u} + \nabla u$ and $\mathbf{F} = \mathbf{v} + \nabla v$ for any $\mathbf{v} \in \mathbb{H}, v \in H^1_S(\Omega)$. Observe that for $\mathbf{u} \in \mathbb{H}$ and $v \in H^1_S(\Omega)$, we have

$$\begin{aligned} a(\mathbf{u}, \nabla v) &= -\omega^2 \int_\Omega \varepsilon \mathbf{u} \cdot \nabla \bar{v} - \mathrm{i}\omega \int_{\Gamma_R^+} T\mathbf{u}_\Gamma \cdot \nabla_\Gamma \bar{v} \\ &= \omega^2 \int_\Omega \bar{v}\, \nabla \cdot (\varepsilon \mathbf{u}) - \omega^2 \int_{\Gamma_R^+} \bar{v}\, (\mathbf{u} \cdot \mathbf{e}_\rho) + \mathrm{i}\omega \int_{\Gamma_R^+} \bar{v}\, \mathrm{div}_\Gamma T \mathbf{u}_{\Gamma_R^+} \\ &= \omega^2 \int_\Omega \bar{v}\, \nabla \cdot (\varepsilon \mathbf{u}) - \omega^2 \int_{\Gamma_R^+} \bar{v} \left(\mathbf{u} \cdot \mathbf{e}_\rho - \frac{\mathrm{i}}{\omega} \mathrm{div}_\Gamma T \mathbf{u}_{\Gamma_R^+} \right) = 0. \end{aligned}$$

The variational equation (5.2) can be decomposed into the form

$$a(\mathbf{u}, \mathbf{v}) + a(\nabla u, \mathbf{v}) + a(\nabla u, \nabla v) = \langle \mathbf{f}, \mathbf{v} \rangle_{\Gamma_R^+} + \langle \mathbf{f}, \nabla v \rangle_{\Gamma_R^+} \quad \text{for all } \mathbf{v} \in \mathbb{H},\ v \in H^1_S(\Omega). \tag{5.8}$$

First, we determine $u \in H^1_S(\Omega)$ by the solution of

$$a(\nabla u, \nabla v) = \langle \mathbf{f}, \nabla v \rangle_{\Gamma_R^+} \quad \text{for all } v \in H^1_S(\Omega),$$

which gives explicitly

$$-\omega^2 \int_\Omega \varepsilon \nabla u \cdot \nabla \bar{v} - \mathrm{i}\omega \int_{\Gamma_R^+} T \nabla_\Gamma u \cdot \nabla_\Gamma \bar{v} = \int_\Gamma \mathbf{f} \cdot \nabla_{\Gamma_R^+} \bar{v} \quad \text{for all } v \in H^1_S(\Omega).$$

Integration by parts yields

$$\omega^2 \int_{\Omega} \bar{v}\, \nabla \cdot (\varepsilon \nabla u) - \omega^2 \int_{\Gamma_R^+} \bar{v}\, (\nabla_\Gamma u \cdot \mathbf{e}_\rho) + \mathrm{i}\omega \int_{\Gamma_R^+} \bar{v}\, \mathrm{div}_\Gamma T \nabla_\Gamma u$$
$$= - \int_{\Gamma_R^+} \bar{v}\, \mathrm{div}_\Gamma \mathbf{f} \quad \text{for all } v \in H_S^1(\Omega),$$

which is equivalent to the boundary value problem: find $u \in H_S(\Omega)$ such that

$$\nabla \cdot (\varepsilon \nabla u) = 0 \quad \text{in } \Omega,$$
$$\partial_\rho u = \frac{\mathrm{i}}{\omega}\, \mathrm{div}_\Gamma T \nabla_\Gamma u + \frac{1}{\omega^2} \mathrm{div}_\Gamma \mathbf{f} \quad \text{on } \Gamma_R^+.$$

By Lemma 5.1, it has a unique solution in $H_S^1(\Omega)$.

The variational problem (5.8) can be formulated: find $\mathbf{u} \in \mathbb{H}$ such that

$$a(\mathbf{u}, \mathbf{v}) = \langle \mathbf{f}, \mathbf{v} \rangle_{\Gamma_R^+} - a(\nabla u, \mathbf{v}) \quad \text{for all } \mathbf{v} \in \mathbb{H}. \tag{5.9}$$

The continuity of the sesquilinear form a follows from the Cauchy–Schwarz inequality, Lemma 4.1, and Lemma 3.3:

$$\begin{aligned} |a(\mathbf{u}, \mathbf{v})| &\le C_1 \parallel \mathbf{u} \parallel_{H(\mathrm{curl},\,\Omega)} \parallel \mathbf{v} \parallel_{H(\mathrm{curl},\,\Omega)} + C_2 \parallel T \mathbf{u}_{\Gamma_R^+} \parallel_{TH^{-1/2}(\mathrm{div},\,\Gamma_R^+)} \parallel \mathbf{v}_{\Gamma_R^+} \parallel_{TH_0^{-1/2}(\mathrm{curl},\,\Gamma_R^+)} \\ &\le C_1 \parallel \mathbf{u} \parallel_{H(\mathrm{curl},\,\Omega)} \parallel \mathbf{v} \parallel_{H(\mathrm{curl},\,\Omega)} + C_2 \parallel \mathbf{u}_{\Gamma_R^+} \parallel_{TH_0^{-1/2}(\mathrm{curl},\,\Gamma_R^+)} \parallel \mathbf{v}_{\Gamma_R^+} \parallel_{TH_0^{-1/2}(\mathrm{curl},\,\Gamma_R^+)} \\ &\le C \parallel \mathbf{u} \parallel_{H(\mathrm{curl},\,\Omega)} \parallel \mathbf{v} \parallel_{H(\mathrm{curl},\,\Omega)} . \end{aligned}$$

Taking the real part of the sesquilinear form a yields

$$\mathrm{Re}\, a(\mathbf{u}, \mathbf{u}) = \int_{\Omega} \mu^{-1} |\nabla \times \mathbf{u}|^2 - \omega^2 \int_{\Omega} \mathrm{Re}(\varepsilon) |\mathbf{u}|^2 + \omega \mathrm{Im} \langle T \mathbf{u}_{\Gamma_R^+}, \mathbf{u}_{\Gamma_R^+} \rangle_{\Gamma_R^+}.$$

It follows from Lemma 3.4 for sufficiently small η and Lemma 4.3 that

$$\begin{aligned} \mathrm{Re}\, a(\mathbf{u}, \mathbf{u}) &\ge C_1 \parallel \mathbf{u} \parallel^2_{H(\mathrm{curl},\Omega)} - C_2 \parallel \mathbf{u} \parallel^2_{(L^2(\Omega))^3} - C_3 \parallel \mathbf{u}_{\Gamma_R^+} \parallel^2_{TH_0^{-1/2}(\Gamma_R^+)} \\ &\ge C_1 \parallel \mathbf{u} \parallel^2_{H(\mathrm{curl},\Omega)} - C_2 \parallel \mathbf{u} \parallel^2_{(L^2(\Omega))^3} . \end{aligned}$$

Noting the compact imbedding of $\mathbb{H}$ into $(L^2(\Omega))^3$ from Lemma 5.3, the proof is complete by a direct application of the Fredholm alternative.

The existence of the solution follows from the Fredholm alternative and uniqueness result Theorem 5.1, which completes the prove of the well-posedness of the variational problem (5.2).

□

Since the variational problem (5.2) attains a unique weak solution in $H(\mathrm{curl}, \Omega)$, then the general theory in Babuška and Aziz [9] implies that there exists a constant γ_2 such that the following inf-sup condition holds:

$$\sup_{0 \ne \mathbf{v} \in H(\mathrm{curl},\Omega)} \frac{|a(\mathbf{u}, \mathbf{v})|}{\parallel \mathbf{v} \parallel_{H(\mathrm{curl},\Omega)}} \ge \gamma_2 \parallel \mathbf{u} \parallel_{H(\mathrm{curl},\Omega)} \quad \text{for all } \mathbf{u} \in H_S(\mathrm{curl}, \Omega). \tag{5.10}$$

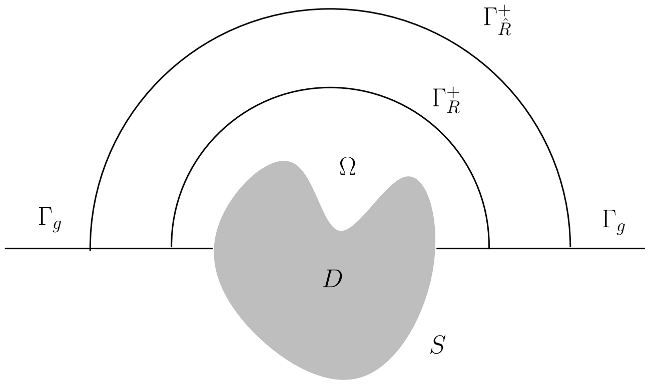

Figure 2: The PML problem geometry

6 The PML problem

In this section, we introduce the variational formulation for the cavity scattering using the PML technique. The goal is to prove the existence and uniqueness of the solution to the PML problem, and derive an error estimate between the solution to the PML problem and the solution to the original scattering problem.

6.1 PML formulation

Now we turn to the introduction of absorbing PML layer. The domain Ω is surrounded with a PML layer in $\mathbb{R}^3_+ \setminus \Omega$. Figure 2 shows the geometry of the PML problem. The specially designed model medium in the PML layer should basically be chosen so that either the wave never reaches its external boundary or the amplitude of the reflected wave is so small that it does not essentially contaminate the solution in Ω. Following the general idea in designing PML absorbing layer in Teixeira and Chew [46], we introduce the PML by a change of variables

$$\rho \to \hat\rho + \mathrm{i}\int_0^{\hat\rho} s(\tau)d\tau, \tag{6.1}$$

where $s(\tau)$ is a continuous function satisfying $s(\tau) \geq 0$ and $s(\tau) = 0$ for $0 \leq \tau \leq R$. In the Cartesian coordinates, the change of variables is equivalent to

$$\mathbf{x} \to \hat{\mathbf{x}} = (\hat\rho \sin\theta\cos\varphi,\ \hat\rho\sin\theta\sin\varphi,\ \hat\rho\cos\theta).$$

It is clear that $s(\tau) = 0$, $\rho = \hat\rho$, and $\hat{\mathbf{x}} = \mathbf{x}$ for $0 \leq \hat\rho \leq R$.

Let $Q = (\mathbf{e}_\rho, \mathbf{e}_\theta, \mathbf{e}_\varphi)$ be a 3×3 matrix composed of $\mathbf{e}_\rho, \mathbf{e}_\theta$, and $\mathbf{e}_\varphi$. The Maxwell equations for the PML medium in the Cartesian coordinates can be written as

$$\nabla_{\hat{\mathbf{x}}} \times \mathbf{E}^{\mathrm{PML}} = \mathrm{i}\omega\hat\mu\mathbf{H}^{\mathrm{PML}}, \quad \nabla_{\hat{\mathbf{x}}} \times \mathbf{H}^{\mathrm{PML}} = -\mathrm{i}\omega\hat\varepsilon\mathbf{E}^{\mathrm{PML}}, \tag{6.2}$$

where

$$\hat\varepsilon = \hat Q\varepsilon, \quad \hat\mu = \hat Q\mu, \quad \hat Q = Q\,\mathrm{diag}\left[(\rho/\hat\rho)^2/(1+\mathrm{i}s(\hat\rho)),\ (1+\mathrm{i}s(\hat\rho)), (1+\mathrm{i}s(\hat\rho))\right] Q^\top. \tag{6.3}$$

In practical computation, the PML medium is truncated by a perfect conductor boundary condition on $\Gamma^+_{\hat R} = \{\hat\rho = \hat R\}$ for some $\hat R > R$. Denote the region bounded by $\Gamma^+_{\hat R}$ and the cavity wall S as $\hat{\Omega}$. The scattering problem with a truncated PML takes the form: Find $(\hat{\mathbf{E}}^{\rm PML}, \hat{\mathbf{H}}^{\rm PML})$ such that

$$\nabla_{\hat{\mathbf{x}}} \times \hat{\mathbf{E}}^{\rm PML} = {\rm i}\omega\hat{\mu}\hat{\mathbf{H}}^{\rm PML}, \quad \nabla_{\hat{\mathbf{x}}} \times \hat{\mathbf{H}}^{\rm PML} = -{\rm i}\hat{\varepsilon}\hat{\mathbf{E}}^{\rm PML}, \quad \text{in } \hat{\Omega} \tag{6.4}$$

with boundary conditions

$$\begin{aligned} \mathbf{e}_{\hat{\rho}} \times \hat{\mathbf{E}}^{\rm PML} &= \mathbf{e}_{\hat{\rho}} \times \mathbf{E}^{\rm b} \quad \text{on } \Gamma_{\hat{R}}^{+}, \\ \mathbf{n} \times \hat{\mathbf{E}}^{\rm PML} &= 0 \quad \text{on } S. \end{aligned}$$

The goal of this section is to estimate the error between $(\hat{\mathbf{E}}^{\rm PML}, \hat{\mathbf{H}}^{\rm PML})$ and $(\mathbf{E}, \mathbf{H})$.

6.2 Transparent boundary condition for the PML problem

Note that the expressions (4.5) and (4.6) form a class of solutions for the Maxwell equations, and if we replace the first kind of spherical Hankel functions in (4.5) and (4.6) with the second kind of spherical Hankel functions, then we get another class of solutions for the Maxwell equations. Let

$$\tilde{R} = \hat{R} + {\rm i}\int_0^{\hat{R}} s(\tau) d\tau.$$

By choosing properly a linear combination of the two classes of solutions, we get the following solution for the electric field

$$\begin{aligned} \mathbf{E}^{\rm s,PML} &= \sum_{|m|\le n}^{\rm odd} \frac{\alpha_n^m}{{\rm i}\omega\rho} \left(\frac{h_n^{(1)}(\omega\rho)(1 + z_n^{(1)}(\omega\rho))}{h_n^{(1)}(\omega\tilde{R})(1 + z_n^{(1)}(\omega\tilde{R}))} - \frac{h_n^{(2)}(\omega\rho)(1 + z_n^{(2)}(\omega\rho))}{h_n^{(2)}(\omega\tilde{R})(1 + z_n^{(2)}(\omega\tilde{R}))} \right) \nabla_\Gamma X_n^m \\ &+ \sum_{|m|\le n}^{\rm odd} \frac{\alpha_n^m}{{\rm i}\omega\rho} \left(\frac{n(n+1)h_n^{(1)}(\omega\rho)}{h_n^{(1)}(\omega\tilde{R})(1 + z_n^{(1)}(\omega\tilde{R}))} - \frac{n(n+1)h_n^{(2)}(\omega\rho)}{h_n^{(2)}(\omega\tilde{R})(1 + z_n^{(2)}(\omega\tilde{R}))} \right) X_n^m \mathbf{e}_\rho \\ &+ \sum_{|m|\le n}^{\rm even} \beta_n^m \left(\frac{h_n^{(1)}(\omega\rho)}{h_n^{(1)}(\omega\tilde{R})} - \frac{h_n^{(2)}(\omega\rho)}{h_n^{(2)}(\omega\tilde{R})} \right) \nabla_\Gamma X_n^m \times \mathbf{e}_\rho \end{aligned}$$

and the magnetic field

$$\begin{aligned} \mathbf{H}^{\rm s,PML} &= \sum_{|m|\le n}^{\rm odd} -\alpha_n^m \left(\frac{h_n^{(1)}(\omega\rho)}{h_n^{(1)}(\omega\tilde{R})(1 + z_n^{(1)}(\omega\tilde{R}))} - \frac{h_n^{(2)}(\omega\rho)}{h_n^{(2)}(\omega\tilde{R})(1 + z_n^{(2)}(\omega\tilde{R}))} \right) \nabla_\Gamma X_n^m \times \mathbf{e}_\rho \\ &+ \sum_{|m|\le n}^{\rm even} \frac{\beta_n^m}{{\rm i}\omega\rho} \left(\frac{h_n^{(1)}(\omega\rho)(1 + z_n^{(1)}(\omega\rho))}{h_n^{(1)}(\omega\tilde{R})} - \frac{h_n^{(2)}(\omega\rho)(1 + z_n^{(2)}(\omega\rho))}{h_n^{(2)}(\omega\tilde{R})} \right) \nabla_\Gamma X_n^m \\ &+ \sum_{|m|\le n}^{\rm even} \frac{\beta_n^m}{{\rm i}\omega\rho} \left(\frac{n(n+1)h_n^{(1)}(\omega\rho)}{h_n^{(1)}(\omega\tilde{R})} - \frac{n(n+1)h_n^{(2)}(\omega\rho)}{h_n^{(2)}(\omega\tilde{R})} \right) X_n^m \mathbf{e}_\rho, \end{aligned}$$

where $h_n^{(2)}(t)$ is the spherical Hankel function of the second kind of order n and

$$z_n^{(2)}(t) = \frac{t\,(h_n^{(2)})'(t)}{h_n^{(2)}(t)}.$$

Furthermore, it can be verified that

$$\mathbf{e}_{\hat{\rho}} \times \mathbf{E}^{\rm s,PML} = 0 \quad \text{on } \Gamma_{\hat{R}}^{+}.$$

Simple calculation yields the tangential component of the electric field is

$$\mathbf{E}^{\mathrm{s,PML}}_{\Gamma_R^+} = \sum_{|m|\le n}^{\mathrm{odd}} \frac{\alpha_n^m}{\mathrm{i}\omega R}\left(\frac{h_n^{(1)}(\omega R)(1+z_n^{(1)}(\omega R))}{h_n^{(1)}(\omega\tilde{R})(1+z_n^{(1)}(\omega\tilde{R}))} - \frac{h_n^{(2)}(\omega R)(1+z_n^{(2)}(\omega R))}{h_n^{(2)}(\omega\tilde{R})(1+z_n^{(2)}(\omega\tilde{R}))}\right)\nabla_\Gamma X_n^m$$
$$+ \sum_{|m|\le n}^{\mathrm{even}} \beta_n^m \left(\frac{h_n^{(1)}(\omega\rho)}{h_n^{(1)}(\omega\tilde{R})} - \frac{h_n^{(2)}(\omega\rho)}{h_n^{(2)}(\omega\tilde{R})}\right)\nabla_\Gamma X_n^m \times \mathbf{e}_\rho$$

and the tangential trace of the magnetic field is

$$\mathbf{H}^{\mathrm{s,PML}} \times \mathbf{e}_\rho = \sum_{|m|\le n}^{\mathrm{odd}} \alpha_n^m \left(\frac{h_n^{(1)}(\omega R)}{h_n^{(1)}(\omega\tilde{R})(1+z_n^{(1)}(\omega\tilde{R}))} - \frac{h_n^{(2)}(\omega R)}{h_n^{(2)}(\omega\tilde{R})(1+z_n^{(2)}(\omega\tilde{R}))}\right)\nabla_\Gamma X_n^m$$
$$+ \sum_{|m|\le n}^{\mathrm{even}} \frac{\beta_n^m}{\mathrm{i}\omega R}\left(\frac{h_n^{(1)}(\omega R)(1+z_n^{(1)}(\omega R))}{h_n^{(1)}(\omega\tilde{R})} - \frac{h_n^{(2)}(\omega R)(1+z_n^{(2)}(\omega R))}{h_n^{(2)}(\omega\tilde{R})}\right)\nabla_\Gamma X_n^m \times \mathbf{e}_\rho.$$

Thus we obtain an explicit representation for the PML Calderon operator T^{PML}: for any tangential component of the electric field

$$\mathbf{u} = \sum_{|m|\le n}^{\mathrm{odd}} \alpha_n^m \mathbf{X}_n^m + \sum_{|m|\le n}^{\mathrm{even}} \beta_n^m \mathbf{Y}_n^m,$$

the tangential trace of the magnetic field is

$$T^{\mathrm{PML}}\mathbf{u} = \sum_{|m|\le n}^{\mathrm{odd}} \mathrm{i}\omega R\, \alpha_n^m \left(\frac{r_{1n}}{t_{1n}}\right)\mathbf{X}_n^m + \sum_{|m|\le n}^{\mathrm{even}} \frac{\beta_n^m}{\mathrm{i}\omega R}\left(\frac{r_{2n}}{t_{2n}}\right)\mathbf{Y}_n^m, \tag{6.5}$$

where

$$\begin{aligned}
r_{1n} =& h_n^{(1)}(\omega R)h_n^{(2)}(\omega\tilde{R})(1+z_n^{(2)}(\omega\tilde{R})) - h_n^{(2)}(\omega R)h_n^{(1)}(\omega\tilde{R})(1+z_n^{(1)}(\omega\tilde{R})),\\
t_{1n} =& h_n^{(1)}(\omega R)(1+z_n^{(1)}(\omega R))h_n^{(2)}(\omega\tilde{R})(1+z_n^{(2)}(\omega\tilde{R}))\\
&- h_n^{(1)}(\omega R)(1+z_n^{(1)}(\omega R))h_n^{(2)}(\omega\tilde{R})(1+z_n^{(2)}(\omega\tilde{R})),\\
r_{2n} =& h_n^{(1)}(\omega R)h_n^{(2)}(\omega\tilde{R})(1+z_n^{(1)}(\omega\tilde{R})) - h_n^{(1)}(\omega\tilde{R})h_n^{(2)}(\omega R)(1+z_n^{(2)}(\omega R)),\\
t_{2n} =& h_n^{(1)}(\omega R)h_n^{(2)}(\omega\tilde{R}) - h_n^{(1)}(\omega\tilde{R})h_n^{(2)}(\omega R).
\end{aligned}$$

Using the Calderon operator for the PML problem, we may propose the following transparent boundary condition:

$$T^{\mathrm{PML}}(\hat{\mathbf{E}}^{\mathrm{PML}}_{\Gamma_R^+} - \mathbf{E}^{\mathrm{b}}_{\Gamma_R^+}) = (\hat{\mathbf{H}}^{\mathrm{PML}} - \mathbf{H}^{\mathrm{b}}) \times \mathbf{e}_\rho \quad \text{on } \Gamma_R^+, \tag{6.6}$$

which maps the tangential components of the scattered electric fields to the tangential trace of the scattered magnetic field.

6.3 Convergence of the PML solution

We shall prove the existence and uniqueness of the solution of the PML problem and derive an error estimate between $\hat{\mathbf{E}}^{\mathrm{PML}}$ and $\mathbf{E}$, the solution of the original cavity scattering problem in Ω. To achieve this goal, we first find an equivalent formulation in the domain Ω.

We obtain the equation for the electric field after eliminating the magnetic field

$$\nabla \times (\mu^{-1} \nabla \times \hat{\mathbf{E}}^{\text{PML}}) - \omega^2 \varepsilon \hat{\mathbf{E}}^{\text{PML}} = 0 \quad \text{in } \Omega. \tag{6.7}$$

Multiplying the complex conjugate of a test function $\mathbf{F}$ in $H_S(\text{curl}, \Omega)$, integrating over Ω, and using integration by parts, we arrive at the variational form for the PML problem: Find $\hat{\mathbf{E}}^{\text{PML}} \in H_0(\text{curl}, \Omega)$ such that

$$a^{\text{PML}}(\hat{\mathbf{E}}^{\text{PML}}, \mathbf{F}) = \langle \mathbf{f}^{\text{PML}}, \mathbf{F} \rangle_{\Gamma_R^+} \quad \text{for all } \mathbf{F} \in H_0(\text{curl}, \Omega), \tag{6.8}$$

where the sesquilinear form

$$a^{\text{PML}}(\mathbf{u}, \mathbf{v}) = \int_\Omega \mu^{-1} \nabla \times \mathbf{u} \cdot \nabla \times \bar{\mathbf{v}} - \omega^2 \int_\Omega \varepsilon \mathbf{u} \cdot \bar{v} - \mathrm{i}\omega \int_{\Gamma_R^+} T^{\text{PML}} \mathbf{u}_{\Gamma_R^+} \cdot \bar{v}_{\Gamma_R^+}, \tag{6.9}$$

the linear functional

$$\langle \mathbf{f}^{\text{PML}}, \mathbf{v} \rangle_{\Gamma_R^+} = \mathrm{i}\omega \int_{\Gamma_R^+} (\mathbf{H}^{\text{b}} \times \mathbf{e}_\rho - T^{\text{PML}} \mathbf{E}^{\text{b}}_{\Gamma_R^+}) \cdot \bar{v}_{\Gamma_R^+}, \tag{6.10}$$

and

$$\mathbf{f}^{\text{PML}} = \mathrm{i}\omega (\mathbf{H}^{\text{b}} \times \mathbf{e}_\rho - T^{\text{PML}} \mathbf{E}^{\text{b}}_{\Gamma_R^+}).$$

By using the representation of T and T^{PML}, we have

$$(T^{\text{PML}} - T)\mathbf{u} = \sum_{|m| \le n}^{\text{odd}} \xi_n \alpha_n^m \mathbf{X}_n^m + \sum_{|m| \le n}^{\text{even}} \zeta_n \beta_n^m \mathbf{Y}_n^m \tag{6.11}$$

where

$$\xi_n = \frac{\mathrm{i}\omega R (z_n^{(2)}(\omega R) - z_n^{(1)}(\omega R))(1 + z_n^{(1)}(\omega R))^{-2} h_n^{(2)}(\omega R)(h_n^{(1)}(\omega R))^{-1}}{\dfrac{h_n^{(2)}(\omega \tilde{R})(1 + z_n^{(2)}(\omega \tilde{R}))}{h_n^{(1)}(\omega \tilde{R})(1 + z_n^{(1)}(\omega \tilde{R}))} - \dfrac{h_n^{(2)}(\omega R)(1 + z_n^{(2)}(\omega R))}{h_n^{(1)}(\omega R)(1 + z_n^{(1)}(\omega R))}} \tag{6.12}$$

and

$$\zeta_n = \frac{(\mathrm{i}\omega R)^{-1}(z_n^{(1)}(\omega R) - z_n^{(2)}(\omega R)) h_n^{(2)}(\omega R)(h_n^{(1)}(\omega R))^{-1}}{h_n^{(2)}(\omega \tilde{R})(h_n^{(1)}(\omega \tilde{R}))^{-1} - h_n^{(2)}(\omega R)(h_n^{(1)}(\omega R))^{-1}}. \tag{6.13}$$

Therefore it is essential to derive upper bounds for ξ_n and ζ_n in order to estimate the error between T and T^{PML}.

Let

$$\eta = \int_R^{\hat{R}} s(\tau) d\tau \quad \text{and} \quad a = \min\left\{\frac{1}{2}, \frac{\omega R}{5}\right\}. \tag{6.14}$$

Suppose

$$\eta \ge \max\left\{\frac{7R}{5}, \hat{R}, \frac{17}{\omega}\right\}. \tag{6.15}$$

The following lemma plays a key role in the subsequent analysis and the proof can be found in Bao and Wu [13] for solving an obstacle scattering problem.

Lemma 6.1. *Under the assumption* (6.15), *the following estimates holds for all* $n \in \mathbb{N}$*:*

$$\omega |\xi_n| \le \frac{M}{\sqrt{1 + n(n+1)}} \quad \text{and} \quad \omega |\zeta_n| \le M \sqrt{1 + n(n+1)}, \tag{6.16}$$

where

$$M = \frac{4\omega a^{-1} \max\{(\omega R)^2 (3\omega R + 3/2)^2, 1\}}{e^{\omega\eta[2 - (a\eta/R)^{-2} + (a\eta/R)^{-4}/19]} - 10}.$$

Lemma 6.2. *For any* $\mathbf{u}$ *and* $\mathbf{v}$ *in* $H_S(\mathrm{curl}, \Omega)$, *it holds*

$$\Big|\omega \int_{\Gamma_R^+} (T^{\mathrm{PML}} - T)\mathbf{u}_{\Gamma_R^+} \cdot \bar{\mathbf{v}}_{\Gamma_R^+}\Big| \leq M \parallel \mathbf{u}_{\Gamma_R^+} \parallel_{TH_0^{-1/2}(\mathrm{curl},\, \Gamma_R^+)} \parallel \mathbf{v}_{\Gamma_R^+} \parallel_{TH_0^{-1/2}(\mathrm{curl},\, \Gamma_R^+)} .$$

Proof. For any $\mathbf{u}, \mathbf{v} \in H_S(\mathrm{curl},\, \Omega)$, it follows from Lemma 3.3 that $\mathbf{u}_{\Gamma_R^+}, \mathbf{v}_{\Gamma_R^+} \in TH^{-1/2}(\mathrm{curl},\, \Gamma_R^+)$, which have the expansion

$$\mathbf{u}_{\Gamma_R^+} = \sum_{|m|\leq n}^{\mathrm{odd}} u_{1n} \mathbf{X}_n^m + \sum_{|m|\leq n}^{\mathrm{even}} u_{2n} \mathbf{Y}_n^m \quad \text{and} \quad \mathbf{v}_{\Gamma_R^+} = \sum_{|m|\leq n}^{\mathrm{odd}} v_{1n} \mathbf{X}_n^m + \sum_{|m|\leq n}^{\mathrm{even}} v_{2n} \mathbf{Y}_n^m.$$

It follows from the definition of T and T^{PML}, the orthogonality of the basis functions, and the estimates for ξ_n and ζ_n in (6.16) that

$$\begin{aligned}\Big|\omega \int_{\Gamma_R^+} \left(T^{\mathrm{PML}} - T\right) \mathbf{u}_{\Gamma_R^+} \cdot \bar{\mathbf{v}}_{\Gamma_R^+}\Big| &= \omega \Big| \sum_{|m|\leq n}^{\mathrm{odd}} \xi_n u_{1n}^m \bar{v}_{1n}^m + \sum_{|m|\leq n}^{\mathrm{even}} \zeta_n u_{2n}^m \bar{v}_{2n}^m \Big| \\ &\leq M \left[\sum_{|m|\leq n}^{\mathrm{odd}} \sqrt{1 + n(n+1)} |u_{1n}|\, |v_{1n}| + \sum_{|m|\leq n}^{\mathrm{even}} \frac{1}{\sqrt{1+n(n+1)}} |u_{2n}|\, |v_{2n}| \right].\end{aligned}$$

The proof is completed by the Cauchy–Schwarz inequality and the definition of the norm for $TH_0^{-1/2}(\mathrm{curl},\, \Gamma_R^+)$. □

Theorem 6.1. *Let* γ_1 *and* γ_2 *be the costants in Lemma 3.3 and inf-sup condition* (5.10), *respectively. Suppose* $M\gamma_1^2 < \gamma_2$. *Then the PML problem* (6.8) *has a unique solution* $\hat{E}^{\mathrm{PML}}$. *Moreover, it has the error estimate*

$$\parallel \hat{\mathbf{E}}^{\mathrm{PML}} - \mathbf{E} \parallel_{\Omega} := \sup_{0 \neq \mathbf{F} \in H_S(\mathrm{curl}, \Omega)} \frac{|a(\hat{\mathbf{E}}^{\mathrm{PML}} - \mathbf{E}, \mathbf{F})|}{\parallel \mathbf{F} \parallel_{H(\mathrm{curl}, \Omega)}} \leq \gamma_1 M \parallel \hat{\mathbf{E}}^{\mathrm{PML}}_{\Gamma_R^+} - \mathbf{E}^{\mathrm{b}}_{\Gamma_R^+} \parallel_{TH^{-1/2}(\mathrm{curl},\, \Gamma_R^+)} . \quad (6.17)$$

Proof. To prove the existence of a unique solution of the PML problem, it suffices to show the coercivity for the sesquilinear form $a^{\mathrm{PML}} : H_S(\mathrm{curl}, \Omega) \times H_S(\mathrm{curl}, \Omega) \to \mathbb{C}$ defined in (6.9). Due to Lemma 3.3 and Lemma 6.2 and the assumption $M\gamma_1^2 < \gamma_2$, it is clear that for any $\mathbf{u}$ and $\mathbf{v}$ in $H_S(\mathrm{curl}, \Omega)$

$$\begin{aligned}|a^{\mathrm{PML}}(\mathbf{u}, \mathbf{v})| &\geq |a(\mathbf{u}, \mathbf{v})| - \Big|\omega \int_{\Gamma_R^+} \left(T^{\mathrm{PML}} - T\right) \mathbf{u}_{\Gamma_R^+} \cdot \bar{\mathbf{v}}_{\Gamma_R^+}\Big| \\ &\geq |a(\mathbf{u}, \mathbf{v})| - M\gamma_1^2 \parallel \mathbf{u} \parallel_{H(\mathrm{curl},\, \Omega)} \parallel \mathbf{v} \parallel_{H(\mathrm{curl},\, \Omega)} \\ &\geq (\gamma_2 - M\gamma_1^2) \parallel \mathbf{u} \parallel_{H(\mathrm{curl},\, \Omega)} \parallel \mathbf{v} \parallel_{H(\mathrm{curl},\, \Omega)} .\end{aligned}$$

It remains to prove the estimate (6.17). By (5.2)–(5.3) and (6.8)–(6.9), we conclude that

$$\begin{aligned}a(\hat{\mathbf{E}}^{\mathrm{PML}} - \mathbf{E}, \mathbf{F}) &= -\mathrm{i}\omega \int_{\Gamma_R^+} (T^{\mathrm{PML}} - T)\mathbf{E}^{\mathrm{b}}_{\Gamma_R^+} \cdot \bar{\mathbf{F}}_{\Gamma_R^+} + a^{\mathrm{PML}}(\mathbf{E}^{\mathrm{PML}}, \mathbf{F}) - a(\mathbf{E}^{\mathrm{PML}}, \mathbf{F}) \\ &= \mathrm{i}\omega \int_{\Gamma_1} (T^{\mathrm{PML}} - T)(\hat{\mathbf{E}}^{\mathrm{PML}}_{\Gamma_R^+} - \mathbf{E}^{\mathrm{b}}_{\Gamma_R^+}) \cdot \bar{\mathbf{F}}_{\Gamma_R^+}\end{aligned}$$

for any $\mathbf{F} \in H_S(\mathrm{curl}, \Omega)$. The proof is complete after using Lemmas 6.2 and 3.3.

□

The constant η is known as the PML parameter. Here, we examine the structure of the constant M which controls the modeling error between the solution of the PML problem and that of the original scattering problem. Obviously, the constant M approaches zero exponentially as the PML parameter η goes to infinity. From the definition (6.14), η may be calculated by the medium property $s(\tau)$, which is usually taken as a power function

$$s(\tau) = \delta \left(\frac{\theta - R}{\hat{R} - R} \right)^m \quad \text{for } \tau \geq R, \quad m \geq 1.$$

Thus we have

$$\eta = \delta(\hat{R} - R)/(m + 1).$$

It is evident that the PML approximation error is reduced by either enlarging the medium parameter δ or increasing the layer thickness $\hat{R} - R$.

7 Concluding remarks

In this paper we have proposed a variational formulation for the overfilled electromagnetic scattering problem for Maxwell's equations and studied the use of the PML to truncate the scattering problem into a bounded domain. The scattering problem is reduced to a boundary value problem by using a transparent boundary condition on a hemisphere enclosing the inhomogeneous cavity. We have shown the uniqueness and existence of the weak solution for the variational problem. Under some proper assumptions on the PML medium parameter, it is shown that the truncated PML problem attains a unique solution in $H_S(\mathrm{curl}, \Omega)$. An explicit error estimate between the solution of the original scattering problem and that of the truncated PML problem in the computational domain is obtained. The error estimate implies particularly that the PML solution converges exponentially to the original scattering solution by increasing either the PML medium parameter or the PML layer thickness. Computationally, the variational approach reported here leads naturally to a class of finite element methods. Analysis and computation of an adaptive finite edge element method with a posterior error estimate for the scattering problem will be studied and reported elsewhere.

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