MATHEMATICS OF COMPUTATION
Volume 83, Number 285, January 2014, Pages 215–233
S 0025-5718(2013)02730-5
Article electronically published on June 10, 2013

AN INVERSE RANDOM SOURCE PROBLEM FOR THE HELMHOLTZ EQUATION

GANG BAO, SHUI-NEE CHOW, PEIJUN LI, AND HAOMIN ZHOU

ABSTRACT. This paper is concerned with an inverse random source problem for the one-dimensional stochastic Helmholtz equation, which is to reconstruct the statistical properties of the random source function from boundary measurements of the radiating random electric field. Although the emphasis of the paper is on the inverse problem, we adapt a computationally more efficient approach to study the solution of the direct problem in the context of the scattering model. Specifically, the direct model problem is equivalently formulated into a two-point spatially stochastic boundary value problem, for which the existence and uniqueness of the pathwise solution is proved. In particular, an explicit formula is deduced for the solution from an integral representation by solving the two-point boundary value problem. Based on this formula, a novel and efficient strategy, which is entirely done by using the fast Fourier transform, is proposed to reconstruct the mean and the variance of the random source function from measurements at one boundary point, where the measurements are assumed to be available for many realizations of the source term. Numerical examples are presented to demonstrate the validity and effectiveness of the proposed method.

1. INTRODUCTION

The inverse source problem for wave propagation has been considered as a basic tool for the solution of reflection tomography, diffusion-based optical tomography [13], and more recently fluorescence microscopy [32], where the fluorescence in the specimen (such as green fluorescent protein) gives rise to emitted light which is focused to the detector by the same objective that is used for the excitation. This problem is largely motivated by medical applications in which it is desirable to use electric or magnetic measurements on the surface of the human body, such as the head, to infer the source currents inside of the body, such as the brain, that produced these measured data.

The problem has been extensively investigated in the literature both from the point of view of applied biomedical engineering and also as a mathematical problem.

Received by the editor June 24, 2010 and, in revised form, October 22, 2011.

2010 *Mathematics Subject Classification.* Primary 65N21, 78A46.

Key words and phrases. Inverse source problem, Helmholtz equation, stochastic differential equation.

The first author's research was supported in part by the NSF grants DMS-0908325, CCF-0830161, EAR-0724527, DMS-0968360, DMS-1211292, the ONR grant N00014-12-1-0319, a Key Project of the Major Research Plan of NSFC (No. 91130004), and a special research grant from Zhejiang University.

The third author's research was supported in part by NSF grants DMS-0914595 and DMS-1042958.

The fourth author's research was supported in part by NSF Faculty Early Career Development (CAREER) Award DMS-0645266 and DMS-1042998.

There are a number of works on the scalar and the full vector electromagnetic inverse source problem in the free space as well as in nonhomogeneous background media; see e.g., Devaney, Marengo, and Li [13], Eller and Valdivia [15], and the references cited therein. Most of the works make use of the fact that the radiation pattern determines the field everywhere outside the source volume. In other words, the inverse source problem is to determine a source function that generates a prescribed radiation pattern.

It is also known that there exist an infinite number of sources that radiate fields which vanish identically outside their support volumes so that the inverse source problem does not have a unique solution, i.e., an infinite number of solutions can be obtained by adding any one of these nonradiating sources to any given solution; see e.g. Devaney and Sherman [14]. Therefore, it is clear that the inverse source problem is ill-posed. In order to obtain a unique solution, it is necessary to give additional constraints that the source must satisfy. A typical choice of the constraint is to pick up the minimum energy solution, which represents the pseudo-inverse of the inverse source problem; see e.g. Marengo and Devaney [24]. Recently Bao *et al.* [3–5] investigated the multi-frequency inverse source problem in which the uniqueness is shown and some stability estimates are established from the radiated fields outside the source volume for a set of frequencies. We refer to Chen and Rokhlin [10] for an inverse medium scattering problem for the one-dimensional Helmholtz equation. See also Gelfand and Levitan [16] for a related inverse Sturm–Liouville problem.

In many applications the source and hence the radiating field may not be deterministic but rather are modeled by random processes, such as the Gaussian random field. Therefore, their governing equations are stochastic differential equations. In general, stochastic partial differential equations are known to be effective tools in modeling complex physical and engineering phenomena including the wave propagation; see e.g. Ishimaru [18], Keller [21], and Papanicolaou [29]. In this paper, we focus on the wave propagation governed by the one-dimensional stochastic Helmholtz equation with sources generated by a spatial Wiener process. Unlike deterministic differential equations, solutions of stochastic differential equations are random functions. Hence it is more important to study their statistical characteristics such as mean value, variance, and even higher order moments in many practical problems.

Stochastic inverse problems refer to the inverse problems that involve uncertainties, which are widely introduced to the mathematical models for three major reasons: (1) randomness directly appears in the studied systems; (2) incomplete knowledge of the systems must be modeled by uncertainties; (3) stochastic techniques are introduced to couple the interference between different scales more effectively, especially when the scale span is large. The first two reasons are commonly encountered and they can happen simultaneously for many different problems. Only recently has the third one been recognized as an effective tool for handling long range multiscale problems. It is our intention to study the inverse scattering with randomness and uncertainties which enter into the problem for all of these reasons.

In the context of the inverse random source problem, our goal is to deduce the statistical structure such as the mean value and standard deviation or variance of the source from physically realizable measurements of the radiated fields, such as the measurements taken on the boundaries. Although the deterministic counterpart has been extensively investigated from both the mathematical and numerical viewpoints, little is known for the stochastic case. To our best knowledge, the only available result is the so-called uniqueness result by Devaney [12], who showed that the auto-correlation function of the random source is uniquely determined everywhere outside the source region by the auto-correlation function of the radiated field. The computational result is at present completely open. Recently, a novel and efficient Wiener chaos expansion based technique has been developed for modeling and simulation of spatially incoherent sources in photonic crystals by Badieirostami *et al.* [1]. See Bao *et al.* [2] for a related inverse medium scattering problem with a stochastic source which is to reconstruct the refractive index of an inhomogeneous medium from the boundary measurements of the scattered random field. We refer to Cao *et al.* [9] for the finite element and discontinuous Galerkin method for solving the stochastic Helmholtz equation, Kloeden and Platen [22] for an account of various numerical methods and approximation schemes for general stochastic partial differential equations, and Calvetti and Somersalo [8] and Kaipio and Somersalo [19] for statistical inversion theory for general random inverse problems.

This work is devoted to the one-dimensional stochastic Helmholtz equation in a homogeneous background medium. The random source function, representing the electric current density, is assumed to have a compact support contained in a finite interval. The problem is modeled with an outgoing wave condition imposed on the lateral end points of the finite interval, which reduces the model to a second order stochastic two-point boundary value problem. We first convert this model problem into an equivalent first order stochastic two-point boundary value problem and show the pathwise existence and uniqueness of the solution for the direct source scattering problem. Then we explicitly deduce the solution from an integral representation by solving the two-point boundary value problem. The solution for the direct problem is given by a combination of a regular integral and an Itô integral. Furthermore, it connects the random wave field with the Fourier transform of the mean and variance of the random source function in an explicit manner. The boundary measurements of the radiating field are assumed to be available for many realizations of the source term. By studying the expectation and variance of the integral equation, we are able to develop an efficient algorithm to reconstruct the mean and variance, which is based on the fast Fourier transform (FFT). Our numerical examples, including the reconstructions of both smooth and non-smooth functions, demonstrate the validity and effectiveness of the proposed method. A related inverse random source scattering problem in inhomogeneous media may be found in [23], where Fredholm integral equations of the first kind are derived to reconstruct the mean and standard deviation of the random source function. We also refer to [7] and [25] for closely related works that address how the randomness affects the reconstruction of the deterministic part and how to improve the estimate of the deterministic part by introducing the uncertainty in a boundary value problem via a stochastic coefficient or source term.

The paper is organized as follows. In Section 2, we present the model problem and formulate it as a first order two-point stochastic boundary value problem. The existence and uniqueness of the direct problem are established, and the solution formula is explicitly derived from the integral representation for the solution of the two-point boundary value problem. Based on the solution, we propose an inversion method for the reconstruction of the mean and variance of the random source. In Section 3, we discuss numerical implementation of the method and present three numerical examples to demonstrate the validity and effectiveness of the proposed approach. The paper is concluded with general remarks and directions for future research in Section 4.

2. Inverse source problem

In this section, we introduce a mathematical model for the inverse random source problem in wave propagation. Although the emphasis of the paper is on the inverse problem, we present a computationally more efficient approach to study the solution of the direct problem in the context of the scattering model. The model problem is first converted into a two-point stochastic boundary value problem. A theoretical framework for the direct model problem is established via the integral representation for the solution of the boundary value problem, which allows us to derive explicit reconstruction formulas for the solution of the inverse random source problem.

2.1. **The model problem.** Consider the one-dimensional Helmholtz equation in homogeneous background medium

$$u''(x,\omega) + \omega^2 u(x,\omega) = f(x), \tag{2.1}$$

where the magnetic permeability and the electric permittivity of the vacuum are assumed to be the unity for simplicity, $\omega > 0$ is the angular frequency, and f, representing the electric current density, is a stochastic source function assumed to have the form

$$f(x) = g(x) + h(x)dW_x.$$

Here g and h are deterministic real functions with compact supports contained in $[0,1]$, and W_x is a one-dimensional spatial Wiener process, and dW_x is its stochastic differential in the Itô sense which is commonly used as a model for the white noise, i.e, a spatial Gaussian random field. Following from the standard stochastic theory on the white noise, we have

$$\mathbb{E}[f(x)] = g(x) \quad \text{and} \quad \mathbb{V}[f(x)] = h^2(x),$$

where $\mathbb{E}$ and $\mathbb{V}$ are the expectation and variance operators, respectively. Obviously, because of the random source, the solution u, the radiating field, is also a random function. Typical boundary conditions imposed on u are the so-called outgoing radiation boundary conditions, which are equivalent to the boundary conditions

$$u'(0,\omega) + \mathrm{i}\omega u(0,\omega) = 0, \tag{2.2}$$

which accounts for the left-going wave at $x = 0$, and

$$u'(1,\omega) - \mathrm{i}\omega u(1,\omega) = 0, \tag{2.3}$$

which accounts for the right-going wave at $x = 1$.

There are two types of source scattering problems posed for the above equations: the direct (forward) random source scattering problem and the inverse random source scattering problem. Given the mean g and the standard deviation h of the random source function f, the direct problem is to determine the random wave field u. On the contrary, the inverse problem is to determine the mean value g and the standard deviation h or the variance h^2 of the random source from the boundary measurements of the random wave field $u(0,\omega)$, which are assumed to be available for many realizations of the source term over a range of frequencies. Our main goal is to investigate both the direct and inverse problems and propose a novel and efficient numerical algorithm to solve the inverse source problem. We remark that we use the left boundary point $x = 0$ in our discussion, all of the results are still true if the measurements are taken at the right boundary point $x = 1$.

First, we show that the direct problem has a unique pathwise solution for each realization of the random field dW_x, and the solution is given by an explicit formula, which serves as the foundation of our numerical algorithm for the inverse problem. To begin with, we convert the second order wave equation in the direct problem into a first order two-point stochastic boundary value problem.

Let $u_1(x,\omega) = u(x,\omega)$ and $u_2(x,\omega) = u'(x,\omega)$, the second order stochastic boundary value problem (2.1)–(2.3) can be equivalently written as the first order stochastic two-point boundary value problem

$$d\mathbf{u} = (M\mathbf{u} + \mathbf{g})dx + \mathbf{h}dW_x, \tag{2.4}$$

together with the boundary conditions given in the form of the linear equations

$$A_0\mathbf{u}_0 = 0, \tag{2.5}$$

$$B_1\mathbf{u}_1 = 0, \tag{2.6}$$

where

$$\mathbf{u}(x,\omega) = \begin{bmatrix} u_1(x,\omega) \\ u_2(x,\omega) \end{bmatrix}, \quad \mathbf{g}(x) = \begin{bmatrix} 0 \\ g(x) \end{bmatrix}, \quad \mathbf{h}(x) = \begin{bmatrix} 0 \\ h(x) \end{bmatrix}, \quad M = \begin{bmatrix} 0 & 1 \\ -\omega^2 & 0 \end{bmatrix},$$

and

$$A_0 = [\mathrm{i}\omega \ 1], \quad B_1 = [-\mathrm{i}\omega \ 1].$$

Here we have set

$$\mathbf{u}_0 = \mathbf{u}(0,\omega) \quad \text{and} \quad \mathbf{u}_1 = \mathbf{u}(1,\omega).$$

We will study this stochastic two-point boundary value problem to establish our analysis for the direct and inverse source scattering problems in the rest of this section. Before presenting the results for the stochastic source scattering problem, we first briefly describe the solutions for the direct and inverse deterministic source scattering problems, i.e., when the standard deviation h is zero, in the following section.

2.2. **Inverse deterministic source problem.** Consider the one-dimensional deterministic Helmholtz equation in the interval $[0, 1]$

$$u''(x,\omega) + \omega^2 u(x,\omega) = g(x), \tag{2.7}$$

together with the outgoing radiation boundary conditions

$$u'(0,\omega) + \mathrm{i}\omega u(0,\omega) = 0 \quad \text{and} \quad u'(1,\omega) - \mathrm{i}\omega u(1,\omega) = 0. \tag{2.8}$$

Given the deterministic real source function g, the direct source scattering problem is to determine the wave field u; the inverse source scattering problem is to determine the function g from the boundary measurements of the wave field $u(0,\omega)$ over a range of frequencies.

It is known that the two-point boundary value problem (2.7) and (2.8) has a unique solution [20]. More precisely, let

$$G(x,y,\omega) = \frac{e^{\mathrm{i}\omega|x-y|}}{2\mathrm{i}\omega},$$

then the solution to the direct source scattering problem for (2.7) and (2.8) can be written explicitly as

$$u(x,\omega) = \int_0^1 G(x,y,\omega)g(y)dy = \frac{1}{2\mathrm{i}\omega}\int_0^1 e^{\mathrm{i}\omega|x-y|}g(y)dy. \tag{2.9}$$

So G is the Green's function for the one-dimensional Helmholtz equation (2.7) with the outgoing radiation condition (2.8).

Next we derive reconstruction formulas for the source function g. Evaluating (2.9) on both sides at $x = 0$ yields

$$2\mathrm{i}\omega u(0,\omega) = \int_0^1 e^{\mathrm{i}\omega y}g(y)dy. \tag{2.10}$$

Therefore the source function g can be reconstructed by taking the inverse Fourier transform on both sides of (2.10). Let

$$u(0,\omega) = \mathrm{Re}u(0,\omega) + \mathrm{i}\mathrm{Im}u(0,\omega).$$

We may split (2.11) into real and imaginary parts:

$$2\omega\mathrm{Re}u(0,\omega) = \int_0^1 \sin(\omega y)g(y)dy, \tag{2.11}$$

$$2\omega\mathrm{Im}u(0,\omega) = -\int_0^1 \cos(\omega y)g(y)dy. \tag{2.12}$$

Equivalently, the source function g can be recovered from either the inverse sine transform (2.11) or the inverse cosine transform (2.12).

2.3. **Inverse random source problem.** In this section, we discuss the solution for the stochastic two-point boundary value problem (2.4)–(2.6) in terms of the source scattering problem. Using the theory presented in Appendix, we first obtain the existence and uniqueness for the direct random source problem for Eqs. (2.4)–(2.6). Then we deduce the integral equations to recover the mean and the variance of the random source function based on the constructive proof in Appendix A. We refer to Bal [6] for an alternative approach for the proof of the existence of the solution for the direct random source scattering problem from the point of the classical central limit result.

Corollary 2.1. *The two-point stochastic boundary value problem* (2.4)–(2.6) *attains a unique solution.*

Proof. Following Theorem A.1, it suffices to prove

$$\det \begin{bmatrix} A_0 \\ B_1 e^M \end{bmatrix} \neq 0$$

for $A_0 = [\mathrm{i}\omega \ 1], B_1 = [-\mathrm{i}\omega \ 1]$, and

$$M = \begin{bmatrix} 0 & 1 \\ -\omega^2 & 0 \end{bmatrix}.$$

Since M is a non-singular matrix, it can be verified that there exists a non-singular matrix Q such that

$$Q^{-1}MQ = \Lambda,$$

where

$$\Lambda = \begin{bmatrix} \mathrm{i}\omega & \\ & -\mathrm{i}\omega \end{bmatrix}, \quad Q = \begin{bmatrix} 1 & 1 \\ \mathrm{i}\omega & -\mathrm{i}\omega \end{bmatrix}, \quad \text{and} \quad Q^{-1} = \frac{1}{2\mathrm{i}\omega} \begin{bmatrix} \mathrm{i}\omega & 1 \\ \mathrm{i}\omega & -1 \end{bmatrix}.$$

A simple calculation yields

$$\det \begin{bmatrix} A_0 \\ B_1 e^M \end{bmatrix} = \begin{vmatrix} \mathrm{i}\omega & 1 \\ -\mathrm{i}\omega e^{-\mathrm{i}\omega} & e^{-\mathrm{i}\omega} \end{vmatrix} = 2\mathrm{i}\omega e^{-\mathrm{i}\omega} \neq 0.$$

It follows from Theorem A.1 that the two-point boundary value problem (2.4)–(2.6) has a unique solution. □

It is known that the solution for the initial value problem of the stochastic differential equation (2.4) is given by

$$\mathbf{u}(x,\omega) = e^{Mx}\left[\mathbf{u}_0 + \int_0^x e^{-My}\mathbf{g}(y)dy + \int_0^x e^{-My}\mathbf{h}(y)dW_y\right], \tag{2.13}$$

where

$$\mathbf{u}(x,\omega) = \begin{bmatrix} u_1(x,\omega) \\ u_2(x,\omega) \end{bmatrix}, \quad \mathbf{g}(x) = \begin{bmatrix} 0 \\ g(x) \end{bmatrix}, \quad \mathbf{h}(x) = \begin{bmatrix} 0 \\ h(x) \end{bmatrix}, \quad M = \begin{bmatrix} 0 & 1 \\ -\omega^2 & 0 \end{bmatrix}.$$

Here we have set

$$\mathbf{u}_0 = \mathbf{u}(0,\omega) = \begin{bmatrix} u_1(0,\omega) \\ u_2(0,\omega) \end{bmatrix}.$$

It follows from the radiation condition (2.5) that

$$u_1(0,\omega) = u(0,\omega) \quad \text{and} \quad u_2(0,\omega) = u'(0,\omega) = -\mathrm{i}\omega u(0,\omega).$$

Evaluating the solution (2.13) at $x = 1$ yields

$$\mathbf{u}_1 = e^{M}\left[\mathbf{u}_0 + \int_0^1 e^{-My}\mathbf{g}(y)dy + \int_0^1 e^{-My}\mathbf{h}(y)dW_y\right], \tag{2.14}$$

where

$$\mathbf{u}_1 = \mathbf{u}(1,\omega) = \begin{bmatrix} u_1(1,\omega) \\ u_2(1,\omega) \end{bmatrix}.$$

Explicitly, we can compute

$$e^M = \begin{bmatrix} \cos\omega & \frac{1}{\omega}\sin\omega \\ -\omega\sin\omega & \cos\omega \end{bmatrix}, \quad e^M\mathbf{u}_0 = u(0,\omega)e^{-\mathrm{i}\omega}\begin{bmatrix} 1 \\ -\mathrm{i}\omega \end{bmatrix},$$

and

$$e^{M(1-y)} = \begin{bmatrix} \cos[(1-y)\omega] & \frac{1}{\omega}\sin[(1-y)\omega] \\ -\omega\sin[(1-y)\omega] & \cos[(1-y)\omega] \end{bmatrix}.$$

Substituting the above expressions into (2.14), we get the two components for the vector $\mathbf{u}_1$:

$$\begin{aligned} u_1(1,\omega) = u(0,\omega)e^{-\mathrm{i}\omega} &+ \frac{1}{\omega}\int_0^1 \sin[(1-y)\omega]g(y)dy \\ &+ \frac{1}{\omega}\int_0^1 \sin[(1-y)\omega]h(y)dW_y, \end{aligned} \tag{2.15}$$

$$\begin{aligned} u_2(1,\omega) = -\mathrm{i}\omega u(0,\omega)e^{-\mathrm{i}\omega} &+ \int_0^1 \cos[(1-y)\omega]g(y)dy \\ &+ \int_0^1 \cos[(1-y)\omega]h(y)dW_y. \end{aligned} \tag{2.16}$$

It follows from the radiation condition (2.6) that

$$u_2(1,\omega) = u'(1,\omega) = \mathrm{i}\omega u(1,\omega) = \mathrm{i}\omega u_1(1,\omega).$$

Combining the above equation with (2.15) and (2.16), we arrive at the integral representation for the radiating field at $x = 0$:

$$u(0,\omega) = \frac{1}{2\mathrm{i}\omega}\int_0^1 e^{\mathrm{i}\omega y} g(y)dy + \frac{1}{2\mathrm{i}\omega}\int_0^1 e^{\mathrm{i}\omega y} h(y)dW_y. \tag{2.17}$$

Once $u(0,\omega)$ is available, we may plug it into (2.13) and derive the explicit expression of the solution for the direct source scattering problem:

$$u(x,\omega) = \frac{1}{2\mathrm{i}\omega}\int_0^1 e^{\mathrm{i}\omega|x-y|} g(y)dy + \frac{1}{2\mathrm{i}\omega}\int_0^1 e^{\mathrm{i}\omega|x-y|} h(y)dW_y. \tag{2.18}$$

Remark 2.1. It is readily seen that the solution (2.18) for the direct stochastic source scattering problem will reduce to the solution (2.9) for the deterministic source scattering problem when the standard deviation h is zero. Therefore the solution formula (2.18) can be seen as a generalization of the solution formula (2.9) from the deterministic problem to the stochastic problem.

Next we derive the formulas to reconstruct the mean and variance of the random source function. Evaluating at $x = 0$ and multiplying $2\mathrm{i}\omega$ on both sides of (2.18) gives

$$2\mathrm{i}\omega u(0,\omega) = \int_0^1 e^{\mathrm{i}\omega y} g(y)dy + \int_0^1 e^{\mathrm{i}\omega y} h(y)dW_y. \tag{2.19}$$

We easily obtain the relation between the data $u(0,\omega)$ and the mean value g after taking the expectation on both sides of (2.19):

$$2\mathrm{i}\omega\mathbb{E}[u(0,\omega)] = \int_0^1 e^{\mathrm{i}\omega y} g(y)dy, \tag{2.20}$$

where the following basic property for the Itô integral is used

$$\mathbb{E}\left[\int_0^1 e^{\mathrm{i}\omega y} h(y)dW_y\right] = 0.$$

Remark 2.2. Comparing the analogous reconstruction formulas (2.20) and (2.10) for the mean g, it is clear that the solution formula (2.20) is a generalization of the solution formula (2.10) from the deterministic problem to its stochastic counterpart.

We split all the complex functions into the sum of real part and imaginary part in order to derive the connection between the boundary measurements $u(0,\omega)$ and the standard deviation h.

Denote

$$u(0,\omega) = \mathrm{Re}u(0,\omega) + \mathrm{i}\,\mathrm{Im}u(0,\omega).$$

Then (2.19) can be decomposed into two equations corresponding to the real part and the imaginary part:

$$2\omega\mathrm{Re}u(0,\omega) = \int_0^1 \sin(\omega y)g(y)dy + \int_0^1 \sin(\omega y)h(y)dW_y, \tag{2.21}$$

$$2\omega\mathrm{Im}u(0,\omega) = -\int_0^1 \cos(\omega y)g(y)dy - \int_0^1 \cos(\omega y)h(y)dW_y. \tag{2.22}$$

Recalling the basic property for the Itô integrals

$$\mathbb{E}\left[\int_0^1 \sin(\omega y)h(y)dW_y\right] = \mathbb{E}\left[\int_0^1 \cos(\omega y)h(y)dW_y\right] = 0,$$

we take the expectation on both sides of (2.21) and (2.22) and obtain

$$2\omega\mathbb{E}\left[\mathrm{Re}u(0,\omega)\right] = \int_0^1 \sin(\omega y)g(y)dy, \tag{2.23}$$

$$2\omega\mathbb{E}\left[\mathrm{Im}u(0,\omega)\right] = -\int_0^1 \cos(\omega y)g(y)dy. \tag{2.24}$$

Therefore the mean value g can be recovered from either the inverse sine transform from (2.23) or the inverse cosine transform from (2.24).

Remark 2.3. Clearly, the reconstruction formulas (2.23) and (2.24) for the stochastic problem are analogous to the reconstruction formulas (2.11) and (2.12) for the deterministic problem.

Both (2.23) and (2.24) are only valid for positive angular frequency $\omega > 0$. The zero Fourier mode is missing which leads to the non-uniqueness of the reconstruction, i.e., any vertical shift of the reconstructed function will give the same non-zero Fourier modes corresponding to the positive angular frequencies. In practice, the zero Fourier mode is set to be zero. After the inverse sine or cosine transform, the reconstructed function can be artificially shifted in the vertical direction to make the value vanish at the lateral point $x = 0$ or $x = 1$ since the function g is assumed to have a compact support contained in the interval $[0, 1]$.

Using the Itô isometry, we have

$$\mathbb{E}\left[\left(\int_0^1 \sin(\omega y)h(y)dW_y\right)^2\right] = \int_0^1 \sin^2(\omega y)h^2(y)dy = \frac{1}{2}\int_0^1 [1-\cos(2\omega y)]h^2(y)dy,$$

$$\mathbb{E}\left[\left(\int_0^1 \cos(\omega y)h(y)dW_y\right)^2\right] = \int_0^1 \cos^2(\omega y)h^2(y)dy = \frac{1}{2}\int_0^1 [1+\cos(2\omega y)]h^2(y)dy.$$

Taking the variance on both sides of (2.21) and (2.22) and using the Itô isometry, we get

$$4\omega^2\mathbb{V}\left[\mathrm{Re}u(0,\omega)\right] = \frac{1}{2}\int_0^1 [1-\cos(2\omega y)]h^2(y)dy,$$

$$4\omega^2\mathbb{V}\left[\mathrm{Im}u(0,\omega)\right] = \frac{1}{2}\int_0^1 [1+\cos(2\omega y)]h^2(y)dy.$$

Subtracting the two equations above we deduce

$$4\omega^2\{\mathbb{V}\left[\mathrm{Im}u(0,\omega)\right] - \mathbb{V}\left[\mathrm{Re}u(0,\omega)\right]\} = \int_0^1 \cos(2\omega y)h^2(y)dy. \tag{2.25}$$

The variance h^2 or the standard deviation h of the random source function can thus be retrieved from taking the inverse cosine transform on both sides of (2.25). Furthermore, the zero Fourier mode can also be recovered by adding the two equations above

$$4\omega^2\{\mathbb{V}\left[\mathrm{Im}u(0,\omega)\right] + \mathbb{V}\left[\mathrm{Re}u(0,\omega)\right]\} = \int_0^1 h^2(y)dy. \tag{2.26}$$

Following equations (2.21), (2.22), and (2.25), we may conclude that the inverse problem has a unique solution, i.e., the mean value g and the standard deviation h can be uniquely determined, if the data $\mathbb{E}[\text{Re}u(0,\omega)]$, $\mathbb{E}[\text{Im}u(0,\omega)]$, $\mathbb{V}[\text{Re}u(0,\omega)]$, and $\mathbb{V}[\text{Im}u(0,\omega)]$ are given for all frequencies $\omega > 0$. This is certainly an ideal situation since the data may only be available at a finite number of a discrete set of frequencies in practice. The uniqueness is still valid as long as the data covers all the Fourier modes of the mean value g and the standard deviation h. Otherwise the uniqueness will not hold if some Fourier coefficients of the functions g and h are missing. In the reconstruction, we adopt so-called filtered backprojection algorithm which assumes that the Fourier coefficients at all of the unobserved frequencies are zero. This algorithm produces the reconstruction with minimal energy solution under the observation constraints where the Fourier coefficients at all of the unobserved frequencies are set to be zero. Finally, we comment that the inverse sine or cosine transform is realized by the fast Fourier transform (FFT) [30].

3. Numerical experiments

In this section, we discuss the algorithmic implementation and present three numerical examples to demonstrate the validity and effectiveness of the proposed method.

The scattering data $u(0,\omega)$ is obtained from two different approaches to avoid the so-called inverse crime. One is based on an integral equation and another is based on a differential equation. Both approaches are numerically implemented and give the same performance of the reconstructions. We briefly introduce how we obtain the scattering data in the following.

In the integral equation approach, we use the solution representation in (2.18). We evaluate both sides at $x = 0$:

$$u(0,\omega) = \frac{1}{2\mathrm{i}\omega}\int_0^1 e^{\mathrm{i}\omega y} g(y)dy + \frac{1}{2\mathrm{i}\omega}\int_0^1 e^{\mathrm{i}\omega y} h(y)dW_y.$$

Numerically the integrals are approximated by the trapezoidal rule

$$u(0,\omega) \approx \frac{1}{2\mathrm{i}\omega}\left[\Delta y \sum_{m=0}^{M-1} e^{\mathrm{i}\omega y_m} g(y_m) + \sum_{n=0}^{N-1} e^{\mathrm{i}\omega y_n} h(y_n) dW_n\right],$$

where $\Delta y = 1/M, y_m = m\Delta y = m/M, y_n = n/N$, and the spatial Brownian motion $dW_n = \xi_n/\sqrt{N}$, in which $\xi_n \in N(0,1)$ is a random variable in the standard Gaussian distribution with zero mean and unit variance. We generate ξ_n by a random number generator in FORTRAN90. In the following examples, M and N are taken as $M = N = 256$.

Another approach is based on solving a stochastic initial value problem. It can be verified that

$$u(0,\omega) = -\frac{v(0)}{2\mathrm{i}\omega}.$$

To obtain the data $u(0,\omega)$, it suffices to solve the stochastic ordinary differential equation

$$dv = (g - \mathrm{i}\omega v)dx + h dW_x,$$
$$v(1) = 0.$$

Then we apply a numerical method over $[0,1]$ to compute the solution. We first discretize the interval. Let $\Delta x = 1/N$ for some positive integer N, and $x_i = i\Delta x =$

i/N. Denote the numerical approximation to $v_1(x_n)$ by v_1^n. The Euler–Maruyama method takes the form

$$v_1^{n+1} = v_1^n + \left[g(x_{n+1}) - \mathrm{i}\omega v_1^{n+1}\right] \Delta x + h(x_{n+1}) \left[W(x_{n+1}) - W(x_n)\right]$$

for $n = N-1, N-2, \ldots, 0$. We refer to [17] for an introduction to numerical simulation of stochastic differential equations.

Example 1. Let

$$\begin{aligned} g(x) &= 0.3 \left[(1-\cos(2x)) - \frac{16}{21}(1-\cos(3x)) + \frac{5}{28}(1-\cos(4x))\right], \\ h(x) &= 0.6 - 0.3\cos(x) - 0.3\cos(2x), \end{aligned}$$

reconstruct the mean value and the standard deviation given by

$$g_1(x) = g(2\pi x) \quad \text{and} \quad h_1(x) = h(2\pi x)$$

inside the interval $[0, 1]$. This is a relatively simple example as both functions g_1 and h_1 contain few low frequency Fourier modes. For the reconstruction of the mean value g_1, the scattering data $u(0, \omega_k)$ is computed at discrete frequencies $\omega_k = k\pi, k = 1, 2, \ldots, 8$; while the scattering data $u(0, \omega_k)$ is computed at frequencies $\omega_k = k\pi/2, k = 1, 2, \ldots, 8$, for the reconstruction of the standard deviation h_1. The data covers all the frequency coefficients of this example. To test the stability of the method, we reconstruct the mean value and the standard deviation or the variance using different numbers of realization. This is equivalent to using data with different level of error. Figure 1 shows the reconstructed mean value and variance and the exact ones with different numbers of realizations. As expected, the relative $L^2([0, 1])$ error "err" decreases from $\text{err} = 3.05 \times 10^{-1}$ to $\text{err} = 6.44 \times 10^{-3}$ and from $\text{err} = 1.16 \times 10^{-1}$ to $\text{err} = 4.00 \times 10^{-3}$ for the mean value and the variance, respectively, as the number of realization "nr" increases from $\text{nr} = 10^3$ to $\text{nr} = 10^6$. It is obvious that the better reconstruction may be obtained when the more accurate data is used. In fact, the reconstruction corresponding to the number of realizations $\text{nr} = 10^6$ is actually indistinguishable from the exact functions from the graphs.

Example 2. Let

$$\begin{aligned} g(x) &= 0.4 \left[(1-\cos(3x)) - \frac{1215}{2783}(1-\cos(11x)) + \frac{7}{23}(1-\cos(12x))\right], \\ h(x) &= 0.5e^1 - 0.3e^{\cos(2x)} - 0.3e^{\cos(3x)}, \end{aligned}$$

reconstruct the mean value and the standard deviation given by

$$g_2(x) = g(2\pi x) \quad \text{and} \quad h_2(x) = h(2\pi x)$$

inside the interval $[0, 1]$. This example is more complicated than Example 1 since both functions contain more higher frequency modes. Correspondingly, the data at high frequencies should be computed to recover the mean value g_2 and the standard deviation h_2. For the reconstruction of the mean value g_2, the scattering data $u(0, \omega_k)$ is computed at discrete frequencies $\omega_k = k\pi, k = 1, 2, \ldots, 16$; while the scattering data $u(0, \omega_k)$ is computed at frequencies $\omega_k = k\pi/2, k = 1, 2, \ldots, 16$, for the reconstruction of the standard deviation h_2. Figure 2 shows the reconstructed mean value and variance and the exact ones with different numbers of realizations. Not surprisingly, the relative error decreases from $\text{err} = 3.84 \times 10^{-1}$ to $\text{err} = 9.15 \times 10^{-3}$ and from $\text{err} = 1.67 \times 10^{-1}$ to $\text{err} = 5.03 \times 10^{-3}$ for the mean value and the variance, respectively, as the number of realization increases from $\mathrm{nr} = 10^3$ to $\mathrm{nr} = 10^6$.

Example 3. Reconstruct the mean value

$$g_3(x) = \begin{cases} 0.5 & \text{for } 0.15 < x < 0.35, \\ 0.5 & \text{for } 0.65 < x < 0.85, \\ 0 & \text{otherwise}, \end{cases}$$

and the standard deviation

$$h_3(x) = \begin{cases} 0.5 & \text{for } 0.3 < x < 0.7, \\ 0 & \text{otherwise}, \end{cases}$$

inside the interval $[0, 1]$. In this example, the functions are discontinuous. It is well known that the piecewise constant function contains infinitely many Fourier coefficients that decay slowly. To show the effect of the maximum frequency on the reconstruction, we use the number of realization $\mathrm{nr} = 10^6$ to generate the data which is intended to reduce the effect of the data error. For the reconstruction of the mean value g_3, the scattering data $u(0, \omega_k)$ is computed at frequencies $\omega_k = k\pi, k = 1, 2, \ldots, \mathrm{nw}$; while the scattering data $u(0, \omega_k)$ is computed at $\omega_k = k\pi/2, k = 1, 2, \ldots, \mathrm{nw}$, for the reconstruction of the standard deviation h_2, where "nw" is the maximum number of frequency. Figure 3 shows the reconstructed mean value and variance and the exact ones with different numbers of frequencies. As one can see, the relative error decreases from $\mathrm{err} = 3.13 \times 10^{-1}$ to $\mathrm{err} = 1.90 \times 10^{-1}$ and from $\mathrm{err} = 2.32 \times 10^{-1}$ to $\mathrm{err} = 1.23 \times 10^{-1}$ for the mean value and the variance, respectively, as the number of frequency increases from $\mathrm{nw} = 8$ to $\mathrm{nw} = 32$.

In summary, the following observations can be made from Figure 1 to Figure 3. When the functions contain few low Fourier modes or fast decaying Fourier coefficients, accurate and stable reconstructions can be obtained easily. When the functions are discontinuous, the oscillatory behavior near the discontinuities displays the well-known Gibbs phenomenon. To encounter this challenge, we have also implemented an alternative approach of the ℓ_1-minimization based method together with the Bregman iteration [31] for all presented three examples. Generally speaking, the alternative approach produces similar results for smooth functions and may reduce the oscillations for discontinuous functions if appropriate parameters are chosen in the iteration. However, we feel the alternative approach is beyond the scope of the current paper, since our main intention is to report the novel numerical methods to reconstruct random source functions from boundary measurements. Therefore we decide to show only the numerical results based on filtered backprojection algorithm which involves just one FFT in the computation. Finally, we comment on the reconstructed variances, which should be positive functions. Due to the error or the Gibbs phenomenon, it is possible that negative values may appear for the reconstructed variances. In practice, the variance could be artificially set to be zero wherever it is negative.

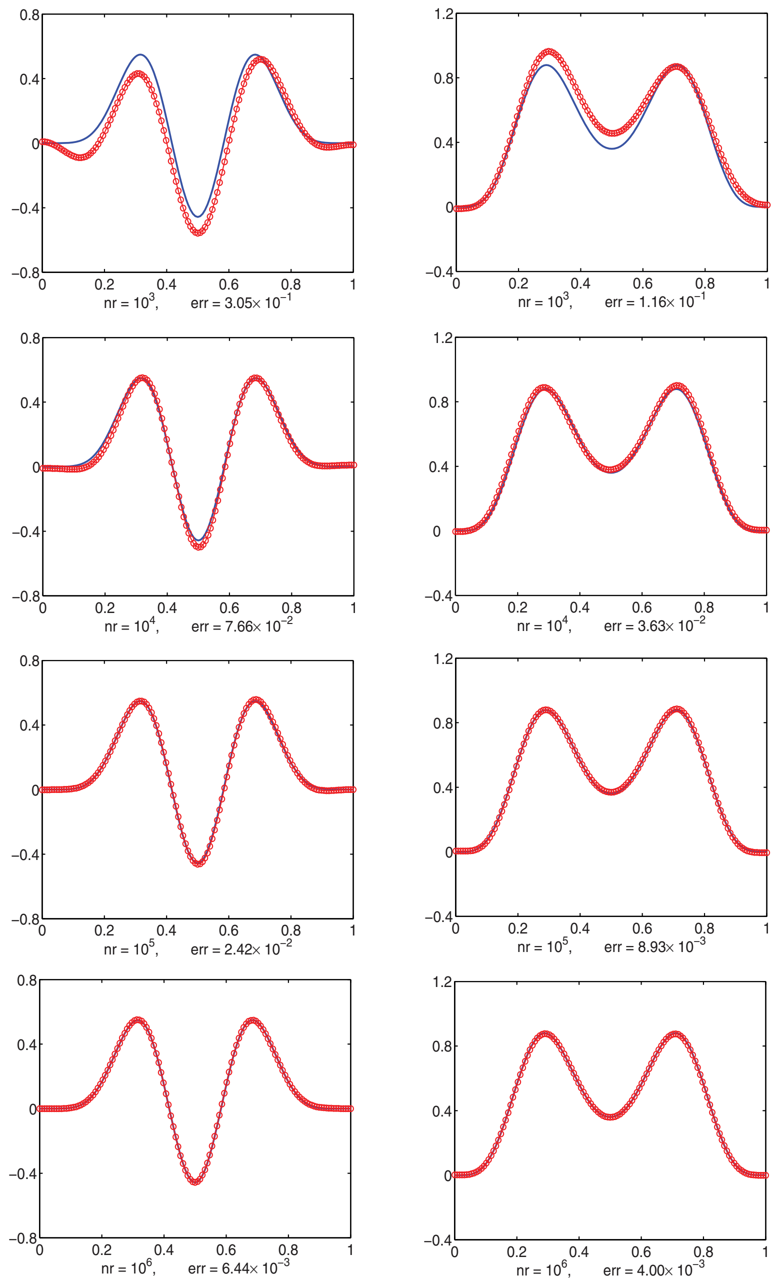

FIGURE 1. Example 1. Solid blue line: exact solutions; circled red line: reconstructed solutions. (left column) reconstruction of the mean g. The reconstruction error (err) decreases from 3.05×10^{-1} to 6.44×10^{-3} as the number of realization (nr) increases from 10^3 to 10^6 from top to bottom; (right column) reconstruction of the variance h^2. The reconstruction error (err) decreases from 1.16×10^{-1} to 4.00×10^{-3} as the number of realization (nr) increases from 10^3 to 10^6 from top to bottom.

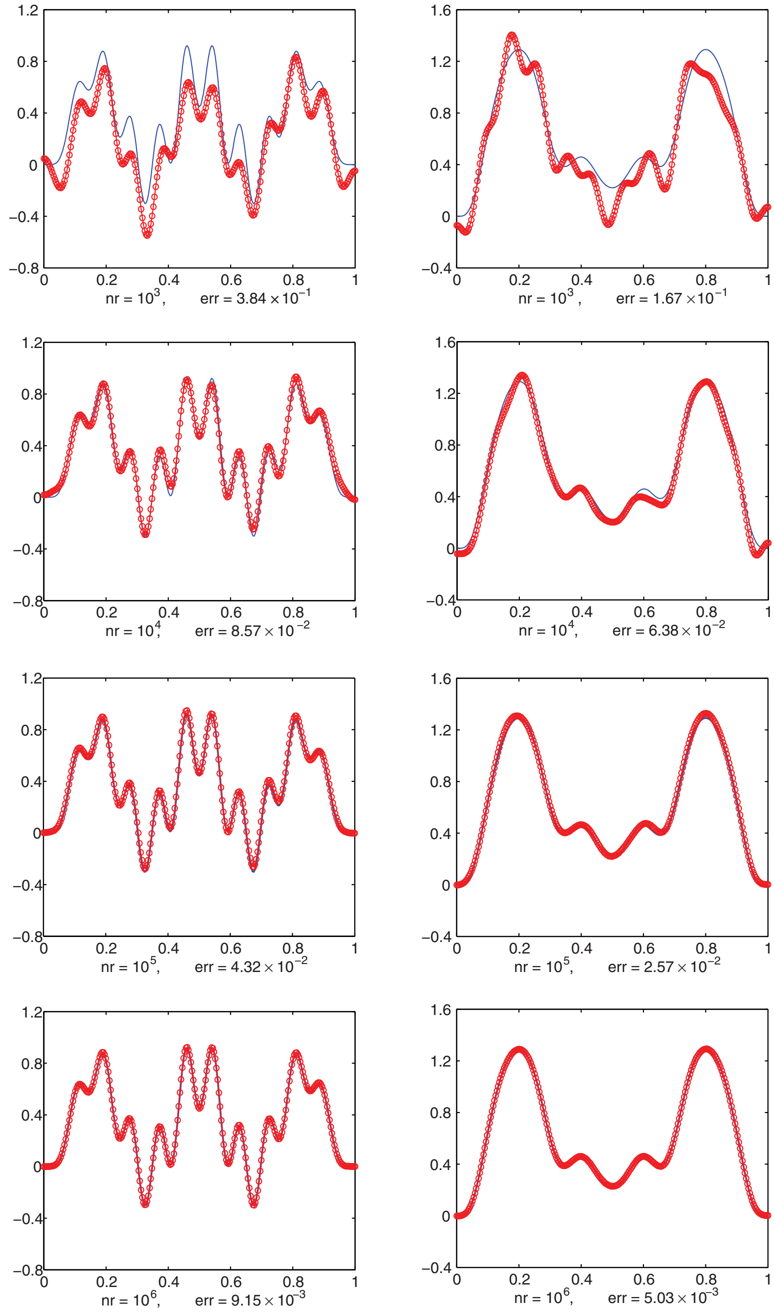

FIGURE 2. Example 2. Solid blue line: exact solutions; circled red line: reconstructed solutions. (left column) reconstruction of the mean g. The reconstruction error (err) decreases from 3.84×10^{-1} to 9.15×10^{-3} as the number of realization (nr) increases from 10^3 to 10^6 from top to bottom; (right column) reconstruction of the variance h^2. The reconstruction error (err) decreases from 1.67×10^{-1} to 5.03×10^{-3} as the number of realization (nr) increases from 10^3 to 10^6 from top to bottom.

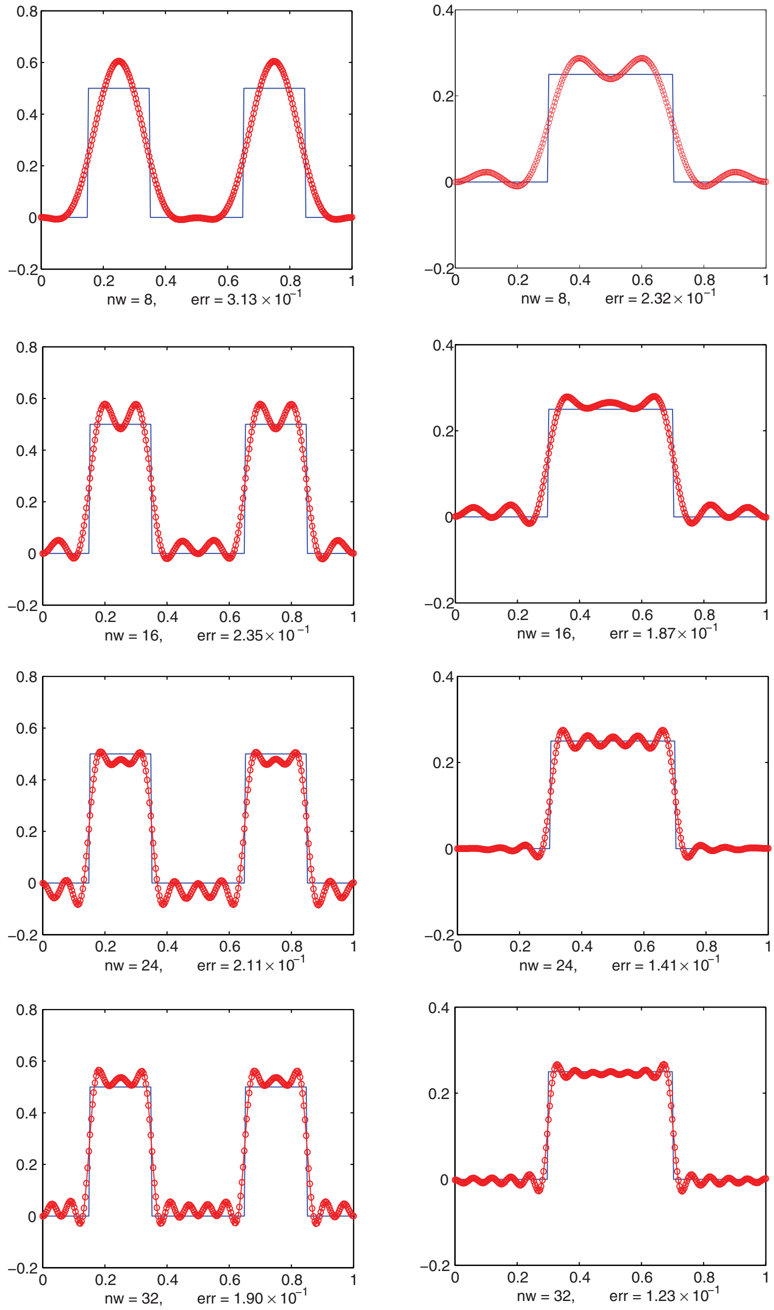

FIGURE 3. Example 3. Solid blue line: exact solutions; circled red line: reconstructed solutions. (left column) reconstruction of the mean g. The reconstruction error (err) decreases from 3.13×10^{-1} to 1.90×10^{-1} as the number of frequency (nw) increases from 8 to 32 from top to bottom; (right column) reconstruction of the variance h^2. The reconstruction error (err) decreases from 2.32×10^{-1} to 1.23×10^{-1} as the number of frequency (nw) increases from 8 to 32 from top to bottom.

4. Concluding remarks

We have studied an inverse scattering problem for the stochastic Helmholtz equation in one dimension with a random source function. The problem is to reconstruct the mean and the standard deviation of the random source function from boundary measurements of the radiating wave field. The scattering model problem was formulated as a two-point stochastic boundary value problem and an integral equation was derived for the solution, which sets up the relation between the data and the target functions through FFT. The method is extremely efficient and accurate for smooth functions. Although we only consider the Gaussian random field in this paper, the strategy can be extended to other types of randomness in the source function with minor modification. The inverse random source scattering problem in higher dimensions and in inhomogeneous background medium are being investigated, and we will report the progress elsewhere in the future.

Appendix A. Two-point stochastic boundary value problem

In this appendix, we establish a criterion for the existence and uniqueness of the pathwise solution for a general two-point stochastic boundary value problem. To solve the two-point stochastic boundary value problem, we first treat it as a standard initial value problem at the left boundary point $x = 0$, and then enforce the solution to satisfy the boundary condition at the right end boundary point $x = 1$. We refer to Nualart and Pardoux [26], and Ocone and Pardoux [27] for discussions on general boundary value problems for stochastic differential equations.

Consider the general first order stochastic differential equation

$$d\mathbf{u} = (M\mathbf{u} + \mathbf{g})dx + \mathbf{h}dW_x \tag{A.1}$$

together with the boundary conditions given in the form of the linear equations

$$A_0\mathbf{u}_0 = \mathbf{v}_0, \tag{A.2}$$

$$B_1\mathbf{u}_1 = \mathbf{v}_1, \tag{A.3}$$

where $\mathbf{u}(x) \in \mathbb{C}^n, \mathbf{g}(x) \in \mathbb{C}^n$, and $\mathbf{h}(x) \in \mathbb{C}^n$ are n-dimensional vector fields, $\mathbf{v}_0 \in \mathbb{C}^{n_1}$ $(n_1 < n)$ is a given n_1-dimensional vector field, $M \in \mathbb{C}^{n\times n}$ is a constant matrix, $A_0 \in \mathbb{C}^{n_1\times n}$ matrix, and $B_1 \in \mathbb{C}^{n_2\times n}$ and $\mathbf{v}_1 \in \mathbb{C}^{n_2}$ with $n_1 + n_2 = n$. Here we have set $\mathbf{u}_0 = \mathbf{u}(0)$ and $\mathbf{u}_1 = \mathbf{u}(1)$.

For the general first order two-point stochastic boundary value problem (A.1)–(A.3), we give a necessary and sufficient condition for the pathwise existence and uniqueness of the solution for any fixed realization of the Wiener process W_x.

Assume that $\mathbf{u}_0$ is known, then the solution for the initial value problem of the first order stochastic differential equation (A.1) can be explicitly written as

$$\mathbf{u}(x) = e^{Mx}\left[\mathbf{u}_0 + \int_0^x e^{-My}\mathbf{g}(y)dy + \int_0^x e^{-My}\mathbf{h}(y)dW_y\right], \tag{A.4}$$

where the last expression is given in the sense of the Itô integral. Certainly, it is only an assumption that $\mathbf{u}_0$ is given. In fact, we need to determine $\mathbf{u}_0$ by enforcing the solution (A.4) to satisfy the boundary conditions (A.2) and (A.3).

Evaluating the solution (A.4) at $x = 1$ yields

$$\mathbf{u}_1 = e^{M}\left[\mathbf{u}_0 + \int_0^1 e^{-My}\mathbf{g}(y)dy + \int_0^1 e^{-My}\mathbf{h}(y)dW_y\right],$$

which is required to satisfy the boundary condition (A.3) at the right end point $x = 1$:

$$B_1\mathbf{u}_1 = B_1 e^M \left[\mathbf{u}_0 + \int_0^1 e^{-My}\mathbf{g}(y)dy + \int_0^1 e^{-My}\mathbf{h}(y)dW_y\right] = \mathbf{v}_1.$$

We denote the random vector

$$B_1 e^M \int_0^1 e^{-My}\mathbf{h}(y)dW_y = \mathbf{w} \in \mathbb{C}^n.$$

The well-posedness of the two-point stochastic boundary value problem (A.1)–(A.3) can be equivalently formulated as follows: Given $\mathbf{v}_0$ and $\mathbf{v}_1$, for any random process $\mathbf{w}$, there exists a unique solution $\mathbf{u}_0$ to the linear equations

$$\begin{aligned} A_0\mathbf{u}_0 &= \mathbf{v}_0, \\ B_1 e^M \mathbf{u}_0 &= \mathbf{v}_1 - \mathbf{w} - B_1 e^M \int_0^1 e^{-My}\mathbf{g}(y)dy. \end{aligned}$$

It follows from the linear algebra that the unique solvability of the above linear system can be obtained if the coefficient matrix is non-singular. Therefore we obtain the necessary and sufficient condition for the well-posedness of the two-point stochastic boundary value problem.

Theorem A.1. *The two-point stochastic boundary value problem* (A.1)–(A.3) *has a unique solution if and only if*

$$\det \begin{bmatrix} A_0 \\ B_1 e^M \end{bmatrix} \neq 0. \tag{A.5}$$

Acknowledgment

The authors wish to thank the reviewers for comments and suggestions that helped us to improve the presentation of the paper.

References

[1] M. Badieirostami, A. Adibi, H. Zhou, and S. Chow, Model for efficient simulation of spatially incoherent light using the Wiener chaos expansion method, Opt. Lett., 32 (2007), 3188–3190.

[2] G. Bao, S.-N. Chow, P. Li, and H. Zhou, Numerical solution of an inverse medium scattering problem with a stochastic source, Inverse Problems, 26 (2010), 074014.

[3] Gang Bao, Junshan Lin, and Faouzi Triki, *A multi-frequency inverse source problem*, J. Differential Equations **249** (2010), no. 12, 3443–3465, DOI 10.1016/j.jde.2010.08.013. MR2737437 (2012c:35464)

[4] Gang Bao, Junshan Lin, and Faouzi Triki, *Numerical solution of the inverse source problem for the Helmholtz equation with multiple frequency data*, Mathematical and statistical methods for imaging, Contemp. Math., vol. 548, Amer. Math. Soc., Providence, RI, 2011, pp. 45–60, DOI 10.1090/conm/548/10835. MR2868487 (2012j:65375)

[5] Gang Bao, Junshan Lin, and Faouzi Triki, *An inverse source problem with multiple frequency data*, C. R. Math. Acad. Sci. Paris **349** (2011), no. 15-16, 855–859, DOI 10.1016/j.crma.2011.07.009 (English, with English and French summaries). MR2835891 (2012h:35372)

[6] Guillaume Bal, *Central limits and homogenization in random media*, Multiscale Model. Simul. **7** (2008), no. 2, 677–702, DOI 10.1137/070709311. MR2443008 (2010b:35508)

[7] Guillaume Bal and Kui Ren, *Physics-based models for measurement correlations: application to an inverse Sturm-Liouville problem*, Inverse Problems **25** (2009), no. 5, 055006, 13, DOI 10.1088/0266-5611/25/5/055006. MR2501024 (2010d:81278)

[8] Daniela Calvetti and Erkki Somersalo, *Introduction to Bayesian scientific computing*, Surveys and Tutorials in the Applied Mathematical Sciences, vol. 2, Springer, New York, 2007. Ten lectures on subjective computing. MR2351679 (2008g:62002)

[9] Y.-Z. Cao, R. Zhang, and K. Zhang, Finite element and discontinuous Galerkin method for stochastic Helmholtz equation in two- and three-dimensions, J. Comput. Math., 26 (2008), 701–715. MR2444727

[10] Y. Chen and V. Rokhlin, *On the inverse scattering problem for the Helmholtz equation in one dimension*, Inverse Problems **8** (1992), no. 3, 365–391. MR1166487 (93c:34034)

[11] David Colton and Rainer Kress, *Inverse acoustic and electromagnetic scattering theory*, 2nd ed., Applied Mathematical Sciences, vol. 93, Springer-Verlag, Berlin, 1998. MR1635980 (99c:35181)

[12] A. Devaney, The inverse problem for random sources, J. Math. Phys., 20 (1979), 1687–1691.

[13] Anthony J. Devaney, Edwin A. Marengo, and Mei Li, *Inverse source problem in nonhomogeneous background media*, SIAM J. Appl. Math. **67** (2007), no. 5, 1353–1378 (electronic), DOI 10.1137/060658618. MR2341753 (2008m:45024)

[14] A. J. Devaney and George C. Sherman, *Nonuniqueness in inverse source and scattering problems*, IEEE Trans. Antennas and Propagation **30** (1982), no. 5, 1034–1042, DOI 10.1109/TAP.1982.1142902. With comments by Norbert N. Bojarski and by W. Ross Stone, and with replies by the authors. MR674190 (83k:78015)

[15] Matthias Eller and Nicolas P. Valdivia, *Acoustic source identification using multiple frequency information*, Inverse Problems **25** (2009), no. 11, 115005, 20, DOI 10.1088/0266-5611/25/11/115005. MR2546000

[16] I. M. Gel′fand and B. M. Levitan, *On the determination of a differential equation from its spectral function*, Amer. Math. Soc. Transl. (2) **1** (1955), 253–304. MR0073805 (17,489c)

[17] Desmond J. Higham, *An algorithmic introduction to numerical simulation of stochastic differential equations*, SIAM Rev. **43** (2001), no. 3, 525–546 (electronic), DOI 10.1137/S0036144500378302. MR1872387 (2002j:65008)

[18] A. Ishimaru, Wave Propagation and Scattering in Random Media, New York: Academic, 1978.

[19] Jari Kaipio and Erkki Somersalo, *Statistical and computational inverse problems*, Applied Mathematical Sciences, vol. 160, Springer-Verlag, New York, 2005. MR2102218 (2005g:65001)

[20] Herbert B. Keller, *Numerical solution of two point boundary value problems*, Society for Industrial and Applied Mathematics, Philadelphia, Pa., 1976. Regional Conference Series in Applied Mathematics, No. 24. MR0433897 (55 #6868)

[21] Joseph B. Keller, *Wave propagation in random media*, Proc. Sympos. Appl. Math., Vol. XIII, American Mathematical Society, Providence, R.I., 1962, pp. 227–246. MR0140261 (25 #3683)

[22] Peter E. Kloeden and Eckhard Platen, *Numerical solution of stochastic differential equations*, Applications of Mathematics (New York), vol. 23, Springer-Verlag, Berlin, 1992. MR1214374 (94b:60069)

[23] Peijun Li, *An inverse random source scattering problem in inhomogeneous media*, Inverse Problems **27** (2011), no. 3, 035004, 22, DOI 10.1088/0266-5611/27/3/035004. MR2772523 (2011j:78017)

[24] Edwin A. Marengo and Anthony J. Devaney, *The inverse source problem of electromagnetics: linear inversion formulation and minimum energy solution*, IEEE Trans. Antennas and Propagation **47** (1999), no. 2, 410–412, DOI 10.1109/8.761085. MR1686310 (2000a:78015)

[25] James Nolen and George Papanicolaou, *Fine scale uncertainty in parameter estimation for elliptic equations*, Inverse Problems **25** (2009), no. 11, 115021, 22, DOI 10.1088/0266-5611/25/11/115021. MR2558681 (2011b:35568)

[26] D. Nualart and É. Pardoux, *Boundary value problems for stochastic differential equations*, Ann. Probab. **19** (1991), no. 3, 1118–1144. MR1112409 (92j:60072)

[27] Daniel Ocone and Étienne Pardoux, *Linear stochastic differential equations with boundary conditions*, Probab. Theory Related Fields **82** (1989), no. 4, 489–526, DOI 10.1007/BF00341281. MR1002898 (91a:60154)

[28] B. Øksendal, Stochastic Differential Equations, 6th ed., Springer-Verlag, Berlin, 2005.

[29] George C. Papanicolaou, *Wave propagation in a one-dimensional random medium*, SIAM J. Appl. Math. **21** (1971), 13–18. MR0290655 (44 #7835)

[30] William H. Press, Saul A. Teukolsky, William T. Vetterling, and Brian P. Flannery, *Numerical recipes in Fortran 90*, 2nd ed., Fortran Numerical Recipes, vol. 2, Cambridge University Press, Cambridge, 1996. The art of parallel scientific computing; With a foreword by Michael Metcalf; With separately available software. MR1414681 (98a:65001a)

[31] Wotao Yin, Stanley Osher, Donald Goldfarb, and Jerome Darbon, *Bregman iterative algorithms for l_1-minimization with applications to compressed sensing*, SIAM J. Imaging Sci. **1** (2008), no. 1, 143–168, DOI 10.1137/070703983. MR2475828 (2010f:90170)

[32] R. Yuste, Fluorescence microscopy today, Nat. Methods, 2 (2005), 902–904.

Department of Mathematics, Zhejiang University, Hangzhou 310027, China — and — Department of Mathematics, Michigan State University, East Lansing, Michigan 48824
E-mail address: bao@math.msu.edu

School of Mathematics, Georgia Institute of Technology, Atlanta, Georgia 30332
E-mail address: chow@math.gatech.edu

Department of Mathematics, Purdue University, West Lafayette, Indiana 47907
E-mail address: lipeijun@math.purdue.edu

School of Mathematics, Georgia Institute of Technology, Atlanta, Georgia 30332
E-mail address: hmzhou@math.gatech.edu