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Analysis of time-domain scattering by periodic structures [☆]

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Abstract

This paper is devoted to the mathematical analysis of a time-domain electromagnetic scattering by periodic structures which are known as diffraction gratings. The scattering problem is reduced equivalently into an initial-boundary value problem in a bounded domain by using an exact transparent boundary condition. The well-posedness and stability of the solution are established for the reduced problem. Moreover, a priori energy estimates are obtained with minimum regularity requirement for the data and explicit dependence on the time.

Keywords: Time-domain Maxwell's equations; Diffraction gratings; Well-posedness and stability; A priori estimates

1. Introduction

This paper is concerned with the mathematical analysis of an electromagnetic scattering problem in periodic structures, where the wave propagation is governed by the time-domain Maxwell equations. The scattering theory in periodic diffractive structures, also known as diffraction grat-

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ings, has applications in many cutting-edge scientific areas including ultra-fast and high-energy lasers, space flight instruments, astronomy, and synchrotron spectrometers. A good introduction can be found in [10] to diffraction grating problems and various numerical approaches. The book [19] contains descriptions of several mathematical problems that arise in diffractive optics modeling in industry. Some more recent developments are addressed in [6] on theory, analysis, and computational techniques of diffractive optics.

The time-harmonic grating problems have been extensively studied by many researchers via either the integral equation methods or the variational methods [2–4,12,17]. A survey may be found in [7] for mathematical studies in rigorous grating theory. The general result may be stated as follows: The diffraction problem has a unique solution for all but a countable sequence of singular frequencies. Unique solvability for all frequencies can be obtained for gratings which have absorbing media or perfectly electrically conducting surfaces with Lipschitz profiles. Numerical methods are developed for both the two-dimensional Helmholtz equation (one-dimensional gratings) and the three-dimensional Maxwell equations (crossed or two-dimensional gratings) [5,8,9,15,30].

The time-domain scattering problems have attracted considerable attention due to their capability of capturing wide-band signals and modeling more general material and nonlinearity [11,22,23,25,29]. Comparing with the time-harmonic problems, the time-domain problems are much less studied due to the additional challenge of the temporal dependence. Rigorous mathematical analysis is very rare. The analysis can be found in [14,28] for the time-domain acoustic and electromagnetic obstacle scattering problems. We refer to [24] for the analysis of the time-dependent electromagnetic scattering from a three-dimensional open cavity. Numerical solutions can be found in [18,27] for the time-dependent wave scattering by periodic structures/surfaces. The theoretical analysis is still lacking for the time-domain scattering by periodic structures.

The goal of this work is to analyze mathematically the time-domain scattering problem which arises from the electromagnetic wave propagation in a periodic structure. Specifically, we consider an electromagnetic plane wave which is incident on a one-dimensional grating in $\mathbb{R}^3$. So the structure is assumed to be invariant in the y-direction and periodic in the x-direction. The three-dimensional Maxwell equations can be decomposed into two fundamental polarizations: transverse electric (TE) polarization and transverse magnetic (TM) polarization, where Maxwell's equations are reduced to the two-dimensional wave equation. We shall study the wave equation in two dimensions for both polarizations. The structure can also be characterized by the medium parameters: the electric permittivity and the magnetic permeability. They are periodic in x and assumed only to be bounded measurable functions. Hence our method works for very general gratings whose surfaces/interfaces are allowed to be Lipschitz profiles or even graphs of some Lipschitz continuous functions.

There are two challenges of the problem: time dependence and unbounded domain. In the frequency domain, various approaches have been developed to truncate unbounded domains into bounded ones, such as absorbing boundary conditions (ABCs), transparent boundary conditions (TBCs), and perfectly matched layer (PML) techniques. These effective boundary conditions are being extended to handle time-domain problems [1,13,20,21]. Utilizing the Laplace transform as a bridge between the time-domain and the frequency domain, we develop an exact time-domain TBC and reduce the problem equivalently into an initial boundary value problem in a bounded domain. Using the energy method with new energy functions, we show the well-posedness and stability of the time-dependent problem. The proofs are based on examining the well-posedness of the time-harmonic Helmholtz equations with complex wavenumbers and applying the abstract inversion theorem of the Laplace transform. Moreover, a priori estimates, featuring an explicit dependence on time and a minimum regularity requirement of the data, are established for the wave field by studying directly the time-domain wave equation.

The paper is organized as follows. In section 2, we introduce the model problem and develop a TBC to reduce it into an initial boundary value problem. Section 3 is devoted to the analysis of the reduced problem, where the well-posedness and stability are addressed and a priori estimates are provided. We conclude the paper with some remarks and directions for future work in section 4.

2. Problem formulation

In this section, we introduce the mathematical model of interest and develop an exact TBC to reduce the scattering problem from an unbounded domain into a bounded domain.

2.1. A model problem

Consider the system of time-domain Maxwell equations in $\mathbb{R}^3$ for $t > 0$:

$$\begin{cases} \nabla \times \boldsymbol{E}(x, y, z, t) + \mu \partial_t \boldsymbol{H}(x, y, z, t) = 0, \\ \nabla \times \boldsymbol{H}(x, y, z, t) - \varepsilon \partial_t \boldsymbol{E}(x, y, z, t) = 0, \end{cases} \tag{2.1}$$

where $\boldsymbol{E}$ is the electric field, $\boldsymbol{H}$ is the magnetic field, ε and μ are the dielectric permittivity and magnetic permeability, respectively, and satisfy

$$0 < \varepsilon_{\min} \le \varepsilon \le \varepsilon_{\max} < \infty, \quad 0 < \mu_{\min} \le \mu \le \mu_{\max} < \infty.$$

Here $\varepsilon_{\min}, \varepsilon_{\max}, \mu_{\min}, \mu_{\max}$ are constants. We assume that the structure is invariant in the y-direction and thus focus on the one-dimensional grating. The more complicated problem in biperiodic structures will be considered in a separate work. There are two fundamental polarizations for the one-dimensional structure:

(i) TE polarization. The electric and magnetic fields are

$$\boldsymbol{E}(x, y, z, t) = [0, E(x, z, t), 0]^\top, \quad \boldsymbol{H}(x, z, t) = [H_1(x, z, t), 0, H_3(x, z, t)]^\top.$$

Eliminating the magnetic field from (2.1), we get the wave equation for the electric field:

$$\varepsilon \partial_t^2 E(x, z, t) = \nabla \cdot (\mu^{-1} \nabla E(x, z, t)). \tag{2.2}$$

(ii) TM polarization. The electric and magnetic fields are

$$\boldsymbol{E}(x, y, z, t) = [E_1(x, z, t), 0, E_3(x, z, t)]^\top, \quad \boldsymbol{H}(x, y, z, t) = [0, H(x, z, t), 0]^\top.$$

We may eliminate the electric field from (2.1) and obtain the wave equation for the magnetic field:

$$\mu \partial_t^2 H(x, z, t) = \nabla \cdot (\varepsilon^{-1} \nabla H(x, z, t)). \tag{2.3}$$

It is clear to note from (2.2) and (2.3) that the TE and TM polarizations can be handled in a unified way by formally exchanging the roles of ε and μ. We will just present the results by using (2.2) as the model equation in the rest of the paper.

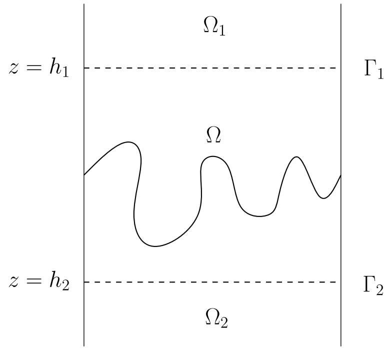

Fig. 1. Problem geometry of the time-domain scattering by a periodic structure.

Now let us specify the problem geometry, which is shown in Fig. 1. Since the structure and medium are assumed to be periodic in the x direction, there exists a period $\Lambda > 0$ such that

$$\varepsilon(x + n\Lambda, z) = \varepsilon(x, z), \quad \mu(x + n\Lambda, z) = \mu(x, z), \quad \forall (x, z) \in \mathbb{R}^2, n \in \mathbb{Z}.$$

We assume that ε and μ are constants away from the region $\Omega = \{(x, z) : 0 \le x \le \Lambda,\ h_2 \le z \le h_1\}$, where h_j are constants. Denote $\Omega_1 := \{(x, z) : 0 \le x \le \Lambda,\ z > h_1\}$ and $\Omega_2 := \{(x, z) : 0 \le x \le \Lambda,\ z < h_2\}$. There exist constants ε_j and μ_j such that

$$\varepsilon(x, z) = \varepsilon_j, \quad \mu(x, z) = \mu_j \quad \text{in } \Omega_j.$$

Throughout we also assume that $\varepsilon\mu \ge \varepsilon_1\mu_1$, which is usually satisfied since ε_1 and μ_1 are the electric permittivity and magnetic permeability in the free space Ω_1. Finally we define $\Gamma_1 = \{(x, z) : 0 \le x \le \Lambda,\ z = h_1\}$ and $\Gamma_2 := \{(x, z) : 0 \le x \le \Lambda,\ z = h_2\}$.

Consider an incoming plane wave E^{inc} which is incident on the structure from above. Explicitly we have

$$E^{\text{inc}}(x, z, t) = f(t - c_1 x - c_2 z),$$

where f is a smooth function and its regularity will be specified later, and $c_1 = \cos\theta/c$, $c_2 = \sin\theta/c$. Here θ, satisfying $0 < \theta < \pi$, is the incident angle, and $c = 1/\sqrt{\varepsilon_1\mu_1} > 0$ is the light speed in the free space. Clearly, the incident field $E^{\text{inc}}(x, z, t)$ satisfies the wave equation (2.2) when $\varepsilon = \varepsilon_1$, $\mu = \mu_1$.

Although the incident field E^{inc} may not be a periodic function in the x-direction, we can verify that

$$E^{\text{inc}}(x + \Lambda, z, t) = E^{\text{inc}}(x, z, t - c_1\Lambda), \quad \forall (x, z) \in \mathbb{R}^2,\ t > 0.$$

Motivated by the uniqueness of the solution, we assume that the total field satisfies the same translation property, i.e.,

$$E(x + \Lambda, z, t) = E(x, z, t - c_1\Lambda), \quad (x, z) \in \mathbb{R}^2,\ t > 0.$$

We define

$$U(x,z,t)=E(x,z,t+c_1(x-\Lambda)),\quad U^{\rm inc}(x,z,t)=E^{\rm inc}(x,z,t+c_1(x-\Lambda)). \tag{2.4}$$

It follows from (2.4) that we get

$$U(x+\Lambda,z,t)=E(x+\Lambda,z,t+c_1x)=E(x,z,t+c_1x-c_1\Lambda)=U(x,z,t),$$

which shows that U is a periodic function in the x-direction with period Λ. Similarly, we can verify that the incident field $U^{\rm inc}$ is a trivially periodic function of x (independent of x) since

$$U^{\rm inc}(x,z,t)=E^{\rm inc}(x,z,t+c_1(x-\Lambda))=f(t-c_2z-c_1\Lambda).$$

Using the change of variables, we have

$$\partial_t E=\partial_t U,\quad \partial_x E=\partial_x U-c_1\partial_t U.$$

The equation (2.2) becomes

$$(\varepsilon-c_1^2\mu^{-1})\partial_t^2 U=\nabla\cdot(\mu^{-1}\nabla U)-c_1(\mu^{-1}\partial_{tx}U+\partial_x(\mu^{-1}\partial_t U)). \tag{2.5}$$

A simple calculation yields that

$$\begin{aligned}\varepsilon-c_1^2\mu^{-1}&=(\varepsilon\mu-\varepsilon_1\mu_1\cos^2\theta)\mu^{-1}\ge\varepsilon_1\mu_1(1-\cos^2\theta)\mu^{-1}\\&=\varepsilon_1\mu_1\mu^{-1}\sin^2\theta>0,\quad\forall\,\theta\in(0,\pi),\end{aligned}$$

which shows that the equation (2.5) is a well-defined wave equation.

It is easy to verify that the incident field $U^{\rm inc}$ satisfies (2.5) with $\varepsilon=\varepsilon_1$, $\mu=\mu_1$. To impose the initial conditions, we assume that the total field and the incident field vanish for $t<0$ so that the incident field $U^{\rm inc}=0$ and the scattered field $V=U-U^{\rm inc}=0$ for $t<0$. The initial conditions are

$$U|_{t=0}=\partial_t U|_{t=0}=0. \tag{2.6}$$

In addition U is Λ-periodic in the x-direction. This paper aims to study the well-posedness and stability of the scattering problem (2.5)–(2.6).

We introduce some notation. For any $s=s_1+{\rm i}s_2$ with $s_1,s_2\in\mathbb{R}$, $s_1>0$, define by $\breve{u}(s)$ the Laplace transform of the function $u(t)$, i.e.,

$$\breve{u}(s)=\mathscr{L}(u)(s)=\int_0^\infty e^{-st}u(t){\rm d}t.$$

Define a weighted periodic function space

$$H^1_{s,{\rm p}}(\Omega)=\{u\in H^1(\Omega):u(0,z)=u(\Lambda,z)\},$$

which is Sobolev space with the norm characterized by

$$\|u\|^2_{H^1_{s,\mathrm{p}}(\Omega)} = \int_\Omega \left(|\nabla u|^2 + |s|^2|u|^2\right)\mathrm{d}x\mathrm{d}z.$$

Given $u \in H^1_{s,\mathrm{p}}(\Omega)$, it has a Fourier expansion with respect to x:

$$u(x,z) = \sum_{n\in\mathbb{Z}} u_n(z)e^{\mathrm{i}\alpha_n x}, \quad \alpha_n = 2n\pi\Lambda^{-1}.$$

A simple calculation yields an equivalent norm in $H^1_{s,\mathrm{p}}(\Omega)$ via Fourier coefficients:

$$\|u\|^2_{H^1_{s,\mathrm{p}}(\Omega)} = \sum_{n\in\mathbb{Z}}\left(|s|^2+\alpha_n^2\right)\int_{h_2}^{h_1}|u_n(z)|^2\mathrm{d}z + \sum_{n\in\mathbb{Z}}\int_{h_2}^{h_1}|u_n'(z)|^2\mathrm{d}z. \tag{2.7}$$

For a periodic function u defined on Γ_j with Fourier coefficients u_n, we define a weighted trace functional space

$$H^\lambda_s(\Gamma_j) = \{u \in L^2(\Gamma_j) : \|u\|^2_{H^\lambda(\Gamma_j)} = \sum_{n\in\mathbb{Z}}\left(|s|^2+\alpha_n^2\right)^\lambda |u_n|^2 < \infty\}, \tag{2.8}$$

where $\lambda \in \mathbb{R}$. It is clear to note that the dual space of $H^{1/2}_s(\Gamma_j)$ is $H^{-1/2}_s(\Gamma_j)$ under the $L^2(\Gamma_j)$ inner product

$$\langle u, v\rangle_{\Gamma_j} = \int_{\Gamma_j} u\bar{v}\mathrm{d}\gamma_j.$$

The weighted Sobolev spaces $H^1_{s,\mathrm{p}}(\Omega)$ and $H^\nu_s(\Gamma_j)$ are equivalent to the standard Sobolev spaces $H^1_\mathrm{p}(\Omega)$ and $H^\lambda(\Gamma_j)$ since $|s| \neq 0$. Hereafter, the expression "$a \lesssim b$" stands for "$a \le Cb$", where C is a positive constant and its specific value is not required but should be always clear from the context.

2.2. *Transparent boundary condition*

We introduce a TBC to reformulate the scattering problem into an equivalent initial-boundary value problem in a bounded domain. The idea is to design a Dirichlet-to-Neumann (DtN) operator which maps the Dirichlet data to the Neumann data of the wave field.

Subtracting the incident field U^{inc} from the total field U in (2.5) and (2.6), we obtain the equation for the scattered field

$$(\varepsilon_1 - c_1^2\mu_1^{-1})\partial_t^2 V = \nabla\cdot(\mu_1^{-1}\nabla V) - c_1(\mu_1^{-1}\partial_{tx}V + \partial_x(\mu_1^{-1}\partial_t V)) \quad \text{in } \Omega_1,\ t>0, \tag{2.9}$$

and the initial conditions

$$V|_{t=0} = \partial_t V|_{t=0} = 0 \quad \text{in } \Omega_1. \tag{2.10}$$

Let $\breve{V}(x, z, s) = \mathscr{L}(V)$ be the Laplace transforms of $V(x, z, t)$ with respect to t. Recall that

$$\mathscr{L}(\partial_t V) = s\breve{V}(x, z, s) - V(x, z, 0),$$
$$\mathscr{L}(\partial_t^2 V) = s^2\breve{V}(x, z, s) - sV(x, z, 0) - \partial_t V(x, z, 0).$$

Taking the Laplace transform of (2.9) and using the initial conditions (2.10), we have

$$(\varepsilon_1 - c_1^2\mu_1^{-1})s^2\breve{V} = \nabla \cdot (\mu_1^{-1}\nabla\breve{V}) - c_1(\mu_1^{-1}s\partial_x\breve{V} + s\partial_x(\mu_1^{-1}\breve{V})),$$

which reduces to

$$(\varepsilon_1\mu_1 - c_1^2)s^2\breve{V} = \Delta\breve{V} - 2c_1 s\partial_x\breve{V} \quad \text{in } \Omega_1. \tag{2.11}$$

Since $\breve{V}$ is a periodic function in x, it has the Fourier expansion

$$\breve{V}(x, z) = \sum_{n\in\mathbb{Z}} \breve{V}_n(z)e^{\mathrm{i}\alpha_n x}, \quad z > h_1.$$

Substituting the Fourier expansion of $\breve{V}$ into (2.11), we obtain an ordinary differential equation for the Fourier coefficients:

$$\begin{cases} \partial_z^2\breve{V}_n(z) - (\beta_1^{(n)})^2\breve{V}_n(z) = 0, & z > h_1, \\ \breve{V}_n(z) = \breve{V}_n(h_1) \end{cases}$$

where

$$\beta_1^{(n)} = (\varepsilon_1\mu_1 s^2 + (\alpha_n + \mathrm{i}c_1 s)^2)^{1/2}, \quad \mathrm{Re}\beta_1^{(n)} < 0.$$

Using the outgoing radiation condition, we have

$$\breve{V}_n(z) = \breve{V}_n(h_1)e^{\beta_1^{(n)}(z-h_1)}.$$

Thus we get the Rayleigh expansion for the scattered field in Ω_1:

$$\breve{V}(x, z) = \sum_{n\in\mathbb{Z}} \breve{V}_n(h_1)e^{\mathrm{i}\alpha_n x}e^{\beta_1^{(n)}(z-h_1)}.$$

Taking the normal derivative of the above equation on Γ_1 yields

$$\partial_{\nu_1}\breve{V}(x, h_1) = \sum_{n\in\mathbb{Z}} \beta_1^{(n)}\breve{V}_n(h_1)e^{\mathrm{i}\alpha_n x},$$

where $\nu_1 = [0, 1]^\top$ is the unit normal vector on Γ_1.

Similarly, we can obtain the Rayleigh expansion for the total field in Ω_2:

$$\breve{U}(x,z) = \sum_{n\in\mathbb{Z}} \breve{U}_n(h_2) e^{\mathrm{i}\alpha_n x} e^{-\beta_2^{(n)}(z-h_2)},$$

where

$$\beta_2^{(n)} = (\varepsilon_2\mu_2 s^2 + (\alpha_n + \mathrm{i}c_1 s)^2)^{1/2}, \quad \mathrm{Re}\beta_2^{(n)} < 0.$$

Taking the normal derivative of $\breve{U}$ on Γ_2 gives

$$\partial_{\nu_2}\breve{U}(x,h_2) = \sum_{n\in\mathbb{Z}} \beta_2^{(n)} \breve{U}_n(h_2) e^{\mathrm{i}\alpha_n x},$$

where $\nu_2 = [0,-1]^\top$ is the normal vector on Γ_2. For any function $u(x,h_j)$ defined on Γ_j, we define the DtN operators

$$(\mathscr{B}_j u)(x,h_j) = \sum_{n\in\mathbb{Z}} \beta_j^{(n)} u_n(h_j) e^{\mathrm{i}\alpha_n x}, \quad u(x,h_j) = \sum_{n\in\mathbb{Z}} u_n(h_j) e^{\mathrm{i}\alpha_n x}. \tag{2.12}$$

Lemma 2.1. *There exists a positive constant C_1 such that*

$$\|u\|_{H_s^{1/2}(\Gamma_j)} \le C_1 \|u\|_{H^1_{s,\mathrm{p}}(\Omega)}, \quad \forall\, u \in H^1_{s,\mathrm{p}}(\Omega).$$

Proof. First we have

$$\begin{aligned}(h_1-h_2)|\zeta(h_j)|^2 &= \int_{h_2}^{h_1} |\zeta(z)|^2 \mathrm{d}z + \int_{h_2}^{h_1}\int_{z}^{h_j} \frac{\mathrm{d}}{\mathrm{d}t}|\zeta(t)|^2 \mathrm{d}t\,\mathrm{d}z \\ &\le \int_{h_2}^{h_1} |\zeta(z)|^2 \mathrm{d}z + (h_1-h_2)\int_{h_2}^{h_1} 2|\zeta(z)||\zeta'(z)|\mathrm{d}z,\end{aligned}$$

which gives

$$\begin{aligned}\big(|s|^2+\alpha_n^2\big)^{1/2}|\zeta(h_j)|^2 \le{}& (h_1-h_2)^{-1}\big(|s|^2+\alpha_n^2\big)^{1/2}\int_{h_2}^{h_1}|\zeta(z)|^2\mathrm{d}z \\ &+ \int_{h_2}^{h_1} 2\big(|s|^2+\alpha_n^2\big)^{1/2}|\zeta(z)||\zeta'(z)|\mathrm{d}z.\end{aligned}$$

It follows from the Cauchy–Schwarz inequality that

$$\big(|s|^2+\alpha_n^2\big)^{1/2}|\zeta(h_j)|^2 \le (h_1-h_2)^{-1}\big(|s|+|\alpha_n|\big)\int_{h_2}^{h_1}|\zeta(z)|^2\mathrm{d}z$$
$$+\big(|s|^2+\alpha_n^2\big)\int_{h_2}^{h_1}|\zeta(z)|^2\mathrm{d}z+\int_{h_2}^{h_1}|\zeta'(z)|^2\mathrm{d}z.$$

Using the fact that $s = s_1 + \mathrm{i}s_2$ with $s_1 > 0$, we have

$$|s| \le s_1^{-1}|s|^2, \quad |\alpha_n| \le (2\pi)^{-1}\Lambda\alpha_n^2.$$

Letting

$$C_1^2 = \max\{1+(h_1-h_2)^{-1}s_1^{-1},\ 1+(2\pi)^{-1}(h_1-h_2)^{-1}\Lambda\},$$

we can show that

$$\big(|s|^2+\alpha_n^2\big)^{1/2}|\zeta(h_j)|^2 \le C_1^2\Big(\big(|s|^2+\alpha_n^2\big)\int_{h_2}^{h_1}|\zeta(z)|^2\mathrm{d}z+\int_{h_2}^{h_1}|\zeta'(z)|^2\mathrm{d}z\Big).$$

The proof is completed by combing the above estimates and the definition (2.7). □

Lemma 2.2. *The DtN operator $\mathscr{B}_j : H^{1/2}_{s,\mathrm{p}}(\Gamma_j) \to H^{-1/2}_{s,\mathrm{p}}(\Gamma_j)$ is continuous, i.e.,*

$$\|\mathscr{B}_j u\|_{H^{-1/2}_{s,\mathrm{p}}(\Gamma_j)} \le C_2\|u\|_{H^{1/2}_{s,\mathrm{p}}(\Gamma_j)},$$

where $C_2 > 0$ is a constant.

Proof. For any $u \in H^{1/2}_{s,\mathrm{p}}(\Gamma_j)$, it follow form (2.8) that

$$\begin{aligned}\|\mathscr{B}_j u\|^2_{H^{-1/2}_{s,\mathrm{p}}(\Gamma_j)} &= \sum_{n\in\mathbb{Z}}\big(|s|^2+\alpha_n^2\big)^{-1/2}|\beta_j^{(n)}|^2|u_n(h_j)|^2\\ &= \sum_{n\in\mathbb{Z}}\big(|s|^2+\alpha_n^2\big)^{1/2}\big(|s|^2+\alpha_n^2\big)^{-1}|\beta_j^{(n)}|^2|u_n(h_j)|^2\\ &\le C_2^2\|u\|^2_{H^{1/2}_{s,\mathrm{p}}(\Gamma_j)},\end{aligned}$$

where we have used

$$|\beta_j^{(n)}|^2 = |\varepsilon_j\mu_j s^2+(\alpha_n+\mathrm{i}c_1 s)^2| \le \varepsilon_j\mu_j|s|^2+2(\alpha_n^2+c_1^2|s|^2) \le C_2^2\big(|s|^2+\alpha_n^2\big).$$

Here

$$C_2^2 = \max\{2, 2c_1^2+\varepsilon_{\max}\mu_{\max}\},$$

which completes the proof. □

Lemma 2.3. *We have the estimate*

$$\mathrm{Re}\langle (s\mu_j)^{-1}\mathscr{B}_j u, u\rangle_{\Gamma_j} \leq 0, \quad \forall\, u \in H^{1/2}_{s,\mathrm{p}}(\Gamma_j).$$

Proof. It follows from the definitions of (2.12) and (2.8) that we have

$$\langle (s\mu_j)^{-1}\mathscr{B}_j u, u\rangle_{\Gamma_j} = \sum_{n\in\mathbb{Z}} \frac{\bar{s}\beta_j^{(n)}}{|s|^2\mu_j}|u_n(h_j)|^2.$$

Let $\beta_j^{(n)} = a_j + \mathrm{i}b_j$, $s = s_1 + \mathrm{i}s_2$ with $s_1 > 0, a_j < 0$. Taking the real part of the above equation gives

$$\mathrm{Re}\langle (s\mu_j)^{-1}\mathscr{B}_j u, u\rangle_{\Gamma_j} = \sum_{n\in\mathbb{Z}} \frac{(s_1 a_j + s_2 b_j)}{|s|^2\mu_j}|u_n(h_j)|^2. \tag{2.13}$$

Recalling $(\beta_j^{(n)})^2 = \varepsilon_j\mu_j s^2 + (\alpha_n + \mathrm{i}c_1 s)^2$, we have

$$a_j^2 - b_j^2 = (\varepsilon_j\mu_j - c_1^2)(s_1^2 - s_2^2) + \alpha_n^2 - 2\alpha_n c_1 s_2 \tag{2.14}$$

and

$$a_j b_j = (\varepsilon_j\mu_j - c_1^2)s_1 s_2 + \alpha_n c_1 s_1. \tag{2.15}$$

Using (2.15), we get

$$s_1 a_j + s_2 b_j = \frac{s_1}{a_j}\big[a_j^2 + (\varepsilon_j\mu_j - c_1^2)s_2^2 + \alpha_n c_1 s_2\big]. \tag{2.16}$$

Plugging (2.14) into (2.16) gives

$$s_1 a_j + s_2 b_j = \frac{s_1}{a_j}\big[b_j^2 + (\varepsilon_j\mu_j - c_1^2)s_1^2 + \alpha_n^2 - \alpha_n c_1 s_2\big]. \tag{2.17}$$

Adding (2.16) and (2.17), we obtain

$$s_1 a_j + s_2 b_j = \frac{s_1}{2a_j}\big[a_j^2 + b_j^2 + (\varepsilon_j\mu_j - c_1^2)(s_1^2 + s_2^2) + \alpha_n^2\big]. \tag{2.18}$$

Substituting (2.18) into (2.13) yields

$$\begin{aligned}&\mathrm{Re}\langle (s\mu_j)^{-1}\mathscr{B}_j u, u\rangle_{\Gamma_j} \\ &\quad = \sum_{n\in\mathbb{Z}} \frac{s_1}{2a_j|s|^2\mu_j}\big[a_j^2 + b_j^2 + (\varepsilon_j\mu_j - c_1^2)(s_1^2 + s_2^2) + \alpha_n^2\big]|u_n(h_j)|^2 \leq 0,\end{aligned}$$

which completes the proof. □

Using the DtN operators (2.12), we obtain the following TBC in the s-domain:

$$\begin{cases} \partial_{\nu_1}\breve{U} = \mathscr{B}_1\breve{U} + \breve{\rho} & \text{on } \Gamma_1, \\ \partial_{\nu_2}\breve{U} = \mathscr{B}_2\breve{U} & \text{on } \Gamma_2, \end{cases} \tag{2.19}$$

where $\breve{\rho} = \partial_z\breve{U}^{\rm inc} - \mathscr{B}_1\breve{U}^{\rm inc}$. Taking the inverse Laplace transform of (2.19) yields the TBC in the time domain:

$$\begin{cases} \partial_{\nu_1}U = \mathscr{T}_1 U + \rho \quad \text{on } \Gamma_1, \\ \partial_{\nu_2}U = \mathscr{T}_2 U \quad \text{on } \Gamma_2, \end{cases} \tag{2.20}$$

where ρ is the inverse Laplace transform of $\breve{\rho}$, i.e., $\rho = \mathscr{L}^{-1}(\breve{\rho})$, and $\mathscr{T}_j = \mathscr{L}^{-1} \circ \mathscr{B}_j \circ \mathscr{L}$.

3. The reduced problem

In this section, we present the main results of this work, which include the well-posedness and stability of the scattering problem and related a priori estimates.

3.1. Well-posedness in the s-domain

Taking the Laplace transform of (2.5) and using the TBC (2.19), we may consider the following reduced boundary value problem:

$$\begin{cases} (\varepsilon - c_1^2\mu^{-1})s\breve{U} = \nabla\cdot((s\mu)^{-1}\nabla\breve{U}) - c_1(\mu^{-1}\partial_x\breve{U} + \partial_x(\mu^{-1}\breve{U})) & \text{in } \Omega, \\ \partial_{\nu_1}\breve{U} = \mathscr{B}_1\breve{U} + \breve{\rho} & \text{on } \Gamma_1, \\ \partial_{\nu_2}\breve{U} = \mathscr{B}_2\breve{U} & \text{on } \Gamma_2. \end{cases} \tag{3.1}$$

Next we introduce a variational formulation of the boundary value problem (3.1) and give a proof of its well-posedness in the space $H^1_{s,\rm p}(\Omega)$.

Multiplying (3.1) by the complex conjugate of a test function $v \in H^1_{s,\rm p}(\Omega)$, using the integration by parts and TBCs, we arrive at the variational problem: To find $\breve{U} \in H^1_{s,\rm p}(\Omega)$ such that

$$a(\breve{U}, v) = \langle (s\mu_1)^{-1}\breve{\rho}, v\rangle_{\Gamma_1}, \quad \forall\, v \in H^1_{s,\rm p}(\Omega), \tag{3.2}$$

where the sesquilinear form

$$a(\breve{U}, v) = \int_\Omega \big[(s\mu)^{-1}\nabla\breve{U}\cdot\nabla\bar{v} + (\varepsilon - c_1^2\mu^{-1})s\breve{U}\bar{v} + c_1(\mu^{-1}\partial_x\breve{U} + \partial_x(\mu^{-1}\breve{U}))\bar{v}\big]{\rm d}x{\rm d}z$$
$$- \sum_{j=1}^{2}\langle (s\mu_j)^{-1}\mathscr{B}_j\breve{U}, v\rangle_{\Gamma_j}. \tag{3.3}$$

Theorem 3.1. *The variational problem (3.2) has a unique solution $\breve{U} \in H^1_{s,\mathrm{p}}(\Omega)$, which satisfies*

$$\|\nabla \breve{U}\|_{L^2(\Omega)^2} + \|s\breve{U}\|_{L^2(\Omega)} \lesssim s_1^{-1}|s|\|\breve{\rho}\|_{H_s^{-1/2}(\Gamma_1)}.$$

Proof. It suffices to show the coercivity of the sesquilinear form of a, since the continuity follows directly from the Cauchy–Schwarz inequality, Lemma 2.1, and Lemma 2.2.

Letting $v = \breve{U}$ in (3.3), we get

$$a(\breve{U}, \breve{U}) = \int_\Omega \left[(s\mu)^{-1}|\nabla \breve{U}|^2 + (\varepsilon - c_1^2\mu^{-1})s|\breve{U}|^2 + c_1(\mu^{-1}\partial_x \breve{U} + \partial_x(\mu^{-1}\breve{U}))\bar{\breve{U}}\right]\mathrm{d}x\mathrm{d}z$$
$$- \sum_{j=1}^{2}\langle (s\mu_j)^{-1}\mathscr{B}_j \breve{U}, \breve{U}\rangle_{\Gamma_j}.$$

Taking the real part of the above equation yields

$$\mathrm{Re}\, a(\breve{U}, \breve{U}) = \int_\Omega \Big(\frac{s_1}{|s|^2\mu}|\nabla \breve{U}|^2 + (\varepsilon - c_1^2\mu^{-1})s_1|\breve{U}|^2\Big)\mathrm{d}x\mathrm{d}z - \mathrm{Re}\sum_{j=1}^{2}\langle (s\mu_j)^{-1}\mathscr{B}_j \breve{U}, \breve{U}\rangle_{\Gamma_j}$$
$$+ c_1 \mathrm{Re}\int_\Omega \big(\mu^{-1}\partial_x \breve{U}\bar{\breve{U}} + \partial_x(\mu^{-1}\breve{U})\bar{\breve{U}}\big)\mathrm{d}x\mathrm{d}z.$$

Since μ and $\breve{U}$ are periodic in x, we have from the integration by part that

$$\int_\Omega \big(\mu^{-1}\partial_x \breve{U}\bar{\breve{U}} + \partial_x(\mu^{-1}\breve{U})\bar{\breve{U}}\big)\mathrm{d}x\mathrm{d}z + \int_\Omega \big(\breve{U}\partial_x(\mu^{-1}\bar{\breve{U}}) + \mu^{-1}\breve{U}\partial_x\bar{\breve{U}}\big)\mathrm{d}x\mathrm{d}z = 0,$$

which gives

$$\mathrm{Re}\int_\Omega \big(\mu^{-1}\partial_x \breve{U}\bar{\breve{U}} + \partial_x(\mu^{-1}\breve{U})\bar{\breve{U}}\big)\mathrm{d}x\mathrm{d}z = 0.$$

Combining the above estimate and Lemma 2.3, we obtain

$$\mathrm{Re}\, a(\breve{U}, \breve{U}) \ge C\frac{s_1}{|s|^2}\int_\Omega \big(|\nabla \breve{U}|^2 + |s\breve{U}|^2\big)\mathrm{d}x\mathrm{d}z, \tag{3.4}$$

where $C = \mu_{\max}^{-1}\min\{1,\ \varepsilon_1\mu_1\sin^2\theta\}$.

It follows from the Lax–Milgram lemma that the variational problem (3.2) has a unique solution $\breve{U} \in H^1_{s,\mathrm{p}}(\Omega)$. Moreover, we have from (3.2) and Lemma 2.1 that

$$|a(\breve{U}, \breve{U})| \le (|s|\mu_1)^{-1}\|\breve{\rho}\|_{H_s^{-1/2}(\Gamma_1)}\|\breve{U}\|_{H_s^{1/2}(\Gamma_1)} \le C_1(|s|\mu_1)^{-1}\|\breve{\rho}\|_{H_s^{-1/2}(\Gamma_1)}\|\breve{U}\|_{H^1_{s,\mathrm{p}}(\Omega)}. \tag{3.5}$$

Combing (3.4) and (3.5) leads to

$$\|\nabla\breve{U}\|^2_{L^2(\Omega)^2} + \|s\breve{U}\|^2_{L^2(\Omega)} \lesssim s_1^{-1}|s|\|\breve{\rho}\|_{H_s^{-1/2}(\Gamma_1)}\|\breve{U}\|_{H^1_{s,\mathrm{p}}(\Omega)},$$

which completes the proof after applying the Cauchy–Schwarz inequality. □

3.2. Well-posedness in the time-domain

Using the time-domain TBC (2.20), we consider the reduced initial-boundary value problem:

$$\begin{cases} (\varepsilon - c_1^2\mu^{-1})\partial_t^2 U = \nabla\cdot(\mu^{-1}\nabla U) - c_1(\mu^{-1}\partial_{tx}U + \partial_x(\mu^{-1}\partial_t U)) & \text{in } \Omega,\ t>0,\\ U|_{t=0} = \partial_t U|_{t=0} = 0 & \text{in } \Omega,\\ \partial_{\nu_1} U = \mathscr{T}_1 U + \rho & \text{on } \Gamma_1,\ t>0,\\ \partial_{\nu_2} U = \mathscr{T}_2 U & \text{on } \Gamma_2,\ t>0. \end{cases} \tag{3.6}$$

The following lemma (cf. [26, Theorem 43.1]) is an analogue of Paley–Wiener–Schwarz theorem for Fourier transform of the distributions with compact support in the case of Laplace transform.

Lemma 3.2. *Let $\breve{h}(s)$ denote a holomorphic function in the half-plane $s_1 > \sigma_0$, valued in the Banach space $\mathbb{E}$. The two following conditions are equivalent:*

(1) *there is a distribution $h \in \mathcal{D}'_+(\mathbb{E})$ whose Laplace transform is equal to $\breve{h}(s)$;*
(2) *there is a real σ_1 with $\sigma_0 \le \sigma_1 < \infty$ and an integer $m \ge 0$ such that for all complex numbers s with $\mathrm{Re}s = s_1 > \sigma_1$, it holds that $\|\breve{h}(s)\|_{\mathbb{E}} \lesssim (1+|s|)^m$,*

where $\mathcal{D}'_+(\mathbb{E})$ is the space of distributions on the real line which vanish identically in the open negative half line.

Theorem 3.3. *The initial-boundary value problem (3.6) has a unique solution $U(x,z,t)$, which satisfies*

$$U(x,z,t) \in L^2(0,T; H^1_{\mathrm{p}}(\Omega)) \cap H^1(0,T; L^2(\Omega))$$

and the stability estimate

$$\max_{t\in[t,T]}\left(\|\partial_t U\|_{L^2(\Omega)} + \|\partial_t(\nabla U)\|_{L^2(\Omega)^2}\right)$$
$$\lesssim \left(\|\rho\|_{L^1(0,T;H^{-1/2}(\Gamma_1))} + \max_{t\in[0,T]}\|\partial_t\rho\|_{H^{-1/2}(\Gamma_1)} + \|\partial_t^2\rho\|_{L^1(0,T;H^{-1/2}(\Gamma_1))}\right). \tag{3.7}$$

Proof. First we have

$$\int_0^T \left(\|\nabla U\|^2_{L^2(\Omega)^2} + \|\partial_t U\|^2_{L^2(\Omega)}\right)\mathrm{d}t$$

$$\leq \int_0^T e^{-2s_1(t-T)} \left(\|\nabla U\|^2_{L^2(\Omega)^2} + \|\partial_t U\|^2_{L^2(\Omega)} \right) \mathrm{d}t$$

$$= e^{2s_1 T} \int_0^T e^{-2s_1 t} \left(\|\nabla U\|^2_{L^2(\Omega)^2} + \|\partial_t U\|^2_{L^2(\Omega)} \right) \mathrm{d}t$$

$$\lesssim \int_0^\infty e^{-2s_1 t} \left(\|\nabla U\|^2_{L^2(\Omega)^2} + \|\partial_t U\|^2_{L^2(\Omega)} \right) \mathrm{d}t.$$

Hence it suffices to estimate the integral

$$\int_0^\infty e^{-2s_1 t} \left(\|\nabla U\|^2_{L^2(\Omega)^2} + \|\partial_t U\|^2_{L^2(\Omega)} \right) \mathrm{d}t.$$

Taking the Laplace transform of (3.6) yields

$$\begin{cases} (\varepsilon - c_1^2\mu^{-1}) s\breve{U} = \nabla \cdot ((s\mu)^{-1}\nabla\breve{U}) - c_1(\mu^{-1}\partial_x\breve{U} + \partial_x(\mu^{-1}\breve{U})) & \text{in } \Omega, \\ \partial_{\nu_1}\breve{U} = \mathscr{B}_1\breve{U} + \breve{\rho} & \text{on } \Gamma_1, \\ \partial_{\nu_2}\breve{U} = \mathscr{B}_2\breve{U} & \text{on } \Gamma_2. \end{cases}$$

The well-posedness of $\breve{U} \in H^1_{s,\mathrm{p}}(\Omega)$ follows directly from Theorem 3.1. By the trace theorem in Lemma 2.1, we get

$$\|\nabla\breve{U}\|^2_{L^2(\Omega)^2} + \|s\breve{U}\|^2_{L^2(\Omega)} \lesssim s_1^{-2}|s|^2\|\breve{\rho}\|^2_{H^{-1/2}(\Gamma_1)} \lesssim s_1^{-2}|s|^2\|\breve{U}^{\mathrm{inc}}\|^2_{H^1_{\mathrm{p}}(\Omega)}.$$

It follows from [26, Lemma 44.1] that $\breve{U}$ is a holomorphic function of s on the half plane $s_1 > \bar{\gamma} > 0$, where $\bar{\gamma}$ is any positive constant. Hence we have from Lemma 3.2 that the inverse Laplace transform of $\breve{U}$ exists and is supported in $[0, \infty]$.

One may verify from the inverse Laplace transform that

$$\breve{U} = \mathscr{L}(U) = \mathscr{F}(e^{-s_1 t}U),$$

where $\mathscr{F}$ is the Fourier transform with respect to s_2. Recall the Plancherel or Parseval identity for the Laplace transform (cf. [16, (2.46)])

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \breve{u}(s)\breve{v}(s)\mathrm{d}s_2 = \int_0^\infty e^{-2s_1 t}u(t)v(t)\mathrm{d}t, \quad \forall\, s_1 > \lambda, \tag{3.8}$$

where $\breve{u} = \mathscr{L}(u)$, $\breve{v} = \mathscr{L}(v)$ and λ is abscissa of convergence for the Laplace transform of u and v.

Using (3.8), we have

$$\int_0^\infty e^{-2s_1 t}\left(\|\nabla U\|^2_{L^2(\Omega)^2}+\|\partial_t U\|^2_{L^2(\Omega)}\right)\mathrm{d}t = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\|\nabla \breve{U}\|^2_{L^2(\Omega)^2}+\|s\breve{U}\|^2_{L^2(\Omega)}\right)\mathrm{d}s_2$$

$$\lesssim s_1^{-2}\int_{-\infty}^{\infty}|s|^2\left(\|\breve{U}^{\mathrm{inc}}\|^2_{L^2(\Omega)}+\|\nabla\breve{U}^{\mathrm{inc}}\|^2_{L^2(\Omega)^2}\right)\mathrm{d}s_2.$$

Since $U^{\mathrm{inc}}|_{t=0}=\partial_t U^{\mathrm{inc}}|_{t=0}=0$ in Ω, we have $\mathscr{L}(\partial_t U^{\mathrm{inc}})=s\breve{U}^{\mathrm{inc}}$ in Ω. It is easy to note that

$$|s|^2\breve{U}^{\mathrm{inc}}=(2s_1-s)s\breve{U}^{\mathrm{inc}}=2s_1\mathscr{L}(\partial_t U^{\mathrm{inc}})-\mathscr{L}(\partial_t^2 U^{\mathrm{inc}}),$$
$$|s|^2\nabla\breve{U}^{\mathrm{inc}}=(2s_1-s)s\nabla\breve{U}^{\mathrm{inc}}=2s_1\mathscr{L}(\partial_t \nabla U^{\mathrm{inc}})-\mathscr{L}(\partial_t^2 \nabla U^{\mathrm{inc}}).$$

Hence we have

$$\int_0^\infty e^{-2s_1 t}\left(\|\nabla U\|^2_{L^2(\Omega)^2}+\|\partial_t U\|^2_{L^2(\Omega)}\right)\mathrm{d}t$$

$$\lesssim \int_{-\infty}^{\infty}\|\mathscr{L}(\partial_t U^{\mathrm{inc}})\|^2_{L^2(\Omega)}\mathrm{d}s_2 + s_1^{-2}\int_{-\infty}^{\infty}\|\mathscr{L}(\partial_t^2 U^{\mathrm{inc}})\|^2_{L^2(\Omega)}\mathrm{d}s_2$$

$$+\int_{-\infty}^{\infty}\|\mathscr{L}(\partial_t \nabla U^{\mathrm{inc}})\|^2_{L^2(\Omega)^2}\mathrm{d}s_2 + s_1^{-2}\int_{-\infty}^{\infty}\|\mathscr{L}(\partial_t^2 \nabla U^{\mathrm{inc}})\|^2_{L^2(\Omega)^2}\mathrm{d}s_2.$$

Using the Parseval identity (3.8) again gives

$$\int_0^\infty e^{-2s_1 t}\left(\|\nabla U\|^2_{L^2(\Omega)^2}+\|\partial_t U\|^2_{L^2(\Omega)}\right)\mathrm{d}t$$

$$\lesssim \int_0^\infty e^{-2s_1 t}\|\partial_t U^{\mathrm{inc}}\|^2_{H^1_{\mathrm{p}}(\Omega)}\mathrm{d}t + s_1^{-2}\int_0^\infty e^{-2s_1 t}\|\partial_t^2 U^{\mathrm{inc}}\|^2_{H^1_{\mathrm{p}}(\Omega)}\mathrm{d}t,$$

which shows that

$$U(x,z,t)\in L^2(0,T;H^1_{\mathrm{p}}(\Omega)\cap H^1(0,T;L^2(\Omega)).$$

Next we prove the stability. Let $\tilde{U}(x,z,t)$ be the extension of $U(x,z,t)$ with respect to t in $\mathbb{R}$ such that $\tilde{U}(x,z,t)=0$ outside the interval $[0,t]$. By the Parseval identity (3.8), we follow the proof of Lemma 2.3 and get

$$\operatorname{Re}\int_0^t e^{-2s_1 t}\langle \mathscr{T}_j U, \partial_t U\rangle_{\Gamma_j} \mathrm{d}t = \operatorname{Re}\int_{\Gamma_j}\int_0^\infty e^{-2s_1 t}\langle \mathscr{T}_j \tilde{U}, \partial_t \tilde{U}\rangle_{\Gamma_j} \mathrm{d}t$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \operatorname{Re}\langle \mathscr{B}_j \breve{\tilde{U}}, s\breve{\tilde{U}}\rangle_{\Gamma_j} \mathrm{d}s_2 \le 0,$$

which yields after taking $s_1 \to 0$ that

$$\operatorname{Re}\int_0^t \langle \mathscr{T}_j U, \partial_t U\rangle_{\Gamma_j} \mathrm{d}t \le 0. \tag{3.9}$$

For any $0 < t < T$, consider the energy function

$$e_1(t) = \|(\varepsilon - c_1^2\mu^{-1})^{1/2}\partial_t U(\cdot, t)\|^2_{L^2(\Omega)} + \|\mu^{-1/2}\nabla U(\cdot, t)\|^2_{L^2(\Omega)^2}.$$

It follows from (3.6) that we have

$$\int_0^t e'(t)dt = 2\operatorname{Re}\int_0^t\int_\Omega \big((\varepsilon - c_1^2\mu^{-1})\partial_t^2 U \partial_t \bar{U} + \mu^{-1}\partial_t(\nabla U)\cdot\nabla\bar{U}\big)\mathrm{d}x\mathrm{d}z\mathrm{d}t$$

$$= 2\operatorname{Re}\int_0^t\int_\Omega \big(\nabla\cdot(\mu^{-1}\nabla U)\partial_t\bar{U} + \mu^{-1}\partial_t(\nabla U)\cdot\nabla\bar{U}\big)\mathrm{d}x\mathrm{d}z\mathrm{d}t$$

$$- 2\operatorname{Re}\int_0^t\int_\Omega \big(c_1(\mu^{-1}\partial_{tx}U + \partial_x(\mu^{-1}\partial_t U))\partial_t\bar{U}\big)\mathrm{d}x\mathrm{d}z\mathrm{d}t.$$

Since μ and U are periodic functions in x, integrating by parts yields

$$\int_0^t\int_\Omega \big(\mu^{-1}\partial_{tx}U\partial_t\bar{U} + \mu^{-1}\partial_{tx}\bar{U}\partial_t U + \partial_x(\mu^{-1}\partial_t U)\partial_t\bar{U} + \partial_x(\mu^{-1}\partial_t\bar{U})\partial_t U\big)\mathrm{d}x\mathrm{d}z\mathrm{d}t = 0,$$

which gives

$$\operatorname{Re}\int_0^t\int_\Omega \big(c_1(\mu^{-1}\partial_{tx}U + \partial_x(\mu^{-1}\partial_t U))\partial_t\bar{U}\big)\mathrm{d}x\mathrm{d}z\mathrm{d}t = 0.$$

Since $e_1(0) = 0$, we obtain from (3.9) that

$$e_1(t) = \int_0^t e'(t)\mathrm{d}t = 2\operatorname{Re}\int_0^t\int_\Omega \big(-\mu^{-1}\nabla U\cdot\partial_t(\nabla\bar{U}) + \mu^{-1}\partial_t(\nabla U)\cdot\nabla\bar{U}\big)\mathrm{d}x\mathrm{d}z\mathrm{d}t$$

$$
\begin{aligned}
&+ 2\mathrm{Re}\int_0^t \sum_{j=1}^{2}\int_{\Gamma_j} \mu_j^{-1}\partial_\nu U \partial_t \bar{U} \mathrm{d}\gamma_j \mathrm{d}t \\
&= 2\mathrm{Re}\int_0^t \sum_{j=1}^{2} \mu_j^{-1}\langle \mathscr{T}_j U, \partial_t U\rangle_{\Gamma_j}\mathrm{d}t + 2\mathrm{Re}\int_0^t \langle \rho, \partial_t U\rangle_{\Gamma_1}\mathrm{d}t \\
&\le 2\mathrm{Re}\int_0^t \big(\|\rho\|_{H^{-1/2}(\Gamma_1)}\|\partial_t U\|_{H^{1/2}(\Gamma_1)}\big)\mathrm{d}t \\
&\lesssim 2\mathrm{Re}\int_0^t \big(\|\rho\|_{H^{-1/2}(\Gamma_1)}\|\partial_t U\|_{H^1_{\mathrm{p}}(\Omega)}\big)\mathrm{d}t \\
&\le 2\big(\max_{t\in[0,T]}\|\partial_t U\|_{H^1_{\mathrm{p}}(\Omega)}\big)\|\rho\|_{L^1(0,T;H^{-1/2}(\Gamma_1))}.
\end{aligned}
$$

Taking the derivative of (3.6) with respect to t, we know that $\partial_t U$ also satisfies the same equations with ρ replaced by $\partial_t\rho$. Hence, we may consider the similar energy function

$$
e_2(t) = \|(\varepsilon - c_1^2\mu^{-1})^{1/2}\partial_t^2 U(\cdot,t)\|^2_{L^2(\Omega)} + \|\mu^{-1/2}\partial_t(\nabla U(\cdot,t))\|^2_{L^2(\Omega)^2}
$$

and get the estimate

$$
\begin{aligned}
e_2(t) &\le 2\mathrm{Re}\int_0^t\int_{\Gamma_1} \partial_t\rho\, \partial_t^2\bar{U}\mathrm{d}\gamma_1\mathrm{d}t \\
&= 2\mathrm{Re}\int_{\Gamma_1} \partial_t\rho\,\partial_t\bar{U}\,|_0^t\,\mathrm{d}\gamma_1 - 2\mathrm{Re}\int_0^t\int_{\Gamma_1}\partial_t^2\rho\,\partial_t\bar{U}\mathrm{d}\gamma_1\mathrm{d}t \\
&\le 2\big(\max_{t\in[0,T]}\|\partial_t U\|_{H^1_{\mathrm{p}}(\Omega)}\big)\big(\max_{t\in[0,T]}\|\partial_t\rho\|_{H^{-1/2}(\Gamma_1)} + \|\partial_t^2\rho\|_{L^1(0,T;H^{-1/2}(\Gamma_1))}\big).
\end{aligned}
$$

Combing the above estimates, we can obtain

$$
\begin{aligned}
\max_{t\in[0,T]}\|\partial_t U\|^2_{H^1_{\mathrm{p}}(\Omega)} &\lesssim \max_{t\in[0,T]} e_1(t) + e_2(t) \\
&\lesssim \big(\|\rho\|_{L^1(0,T;H^{-1/2}(\Gamma_1))} + \max_{t\in[0,T]}\|\partial_t\rho\|_{H^{-1/2}(\Gamma_1)} + \|\partial_t^2\rho\|_{L^1(0,T;H^{-1/2}(\Gamma_1))}\big)\|\partial_t U\|_{H^1_p(\Omega)},
\end{aligned}
$$

which give the estimate (3.7) after applying the Cauchy–Schwarz inequality. □

3.3. A priori estimates

In this section, we derive a priori estimates for the total field with a minimum regularity requirement for the data and an explicit dependence on the time.

The variation problem of (3.6) in time domain is to find $U \in H^1_{\rm p}(\Omega)$ for all $t > 0$ such that

$$\int_\Omega (\varepsilon - c_1^2\mu^{-1})\partial_t^2 U\bar{w}\mathrm{d}x\mathrm{d}z = -\int_\Omega \mu^{-1}\nabla U\cdot\nabla\bar{w}\mathrm{d}x\mathrm{d}z + \sum_{j=1}^{2}\int_{\Gamma_j}\mu_j^{-1}\mathscr{T}_j U\bar{w}\mathrm{d}\gamma_j$$
$$+\int_{\Gamma_1}\rho\bar{w}\mathrm{d}\gamma_1 - c_1\int_\Omega(\mu^{-1}\partial_{tx}U + \partial_x(\mu^{-1}\partial_t U))\bar{w}\mathrm{d}x\mathrm{d}z, \quad \forall\, w \in H^1_{\rm p}(\Omega). \tag{3.10}$$

To show the stability of its solution, we follow the argument in [26] but with a careful study of the TBC.

Theorem 3.4. *Let $U \in H^1_{\rm p}(\Omega)$ be the solution of (3.6). Given $\rho \in L^1(0, T; H^{-1/2}(\Gamma_1))$, we have for any $T > 0$ that*

$$\|U\|_{L^\infty(0,T;L^2(\Omega))} + \|\nabla U\|_{L^\infty(0,T;L^2(\Omega))} \lesssim T\|\rho\|_{L^1(0,T;H^{-1/2}(\Gamma_1))} + \|\partial_t\rho\|_{L^1(0,T;H^{-1/2}(\Gamma_1))}, \tag{3.11}$$

and

$$\|U\|_{L^2(0,T;L^2(\Omega))} + \|\nabla U\|_{L^2(0,T;L^2(\Omega))}$$
$$\lesssim T^{3/2}\|\rho\|_{L^1(0,T;H^{-1/2}(\Gamma_1))} + T^{1/2}\|\partial_t\rho\|_{L^1(0,T;H^{-1/2}(\Gamma_1))}. \tag{3.12}$$

Proof. Let $0 < \xi < T$ and define an auxiliary function

$$\psi_1(x,z,t) = \int_t^\xi U(x,z,\tau)\mathrm{d}\tau, \quad (x,z)\in\Omega,\ 0 \le t \le \xi.$$

It is clear that

$$\psi_1(x,z,\xi) = 0, \quad \partial_t\psi_1(x,z,t) = -U(x,z,t). \tag{3.13}$$

For any $\phi(x,z,t) \in L^2(0,\xi; L^2(\Omega))$, we have

$$\int_0^\xi \phi(x,z,t)\bar{\psi}_1(x,z,t)\mathrm{d}t = \int_0^\xi \Big(\int_0^t \phi(x,z,\tau)\mathrm{d}\tau\Big)\bar{U}(x,z,t)\mathrm{d}t. \tag{3.14}$$

Indeed, using integration by parts and (3.13), we have

$$\int_0^\xi \phi(x,z,t)\bar{\psi}_1(x,z,t)\mathrm{d}t = \int_0^\xi \Big(\phi(x,z,t)\int_t^\xi \bar{U}(x,z,\tau)\mathrm{d}\tau\Big)\mathrm{d}t$$

$$= \int_0^\xi \int_t^\xi \bar{U}(x,z,\tau)\mathrm{d}\tau \mathrm{d}\Big(\int_0^t \phi(x,z,\varsigma)\mathrm{d}\varsigma\Big)$$

$$= \int_t^\xi \bar{U}(x,z,\tau)\mathrm{d}\tau \int_0^t \phi(x,z,\varsigma)\mathrm{d}\varsigma \,\Big|_0^\xi + \int_0^\xi \Big(\int_0^t \phi(x,z,\varsigma)\mathrm{d}\varsigma\Big)\bar{U}(x,z,t)\mathrm{d}t$$

$$= \int_0^\xi \Big(\int_0^t \phi(x,z,\tau)\mathrm{d}\tau\Big)\bar{U}(x,z,t)\mathrm{d}t.$$

Next, we take the test function $w = \psi_1$ in (3.10) and get

$$\int_\Omega (\varepsilon - c_1^2\mu^{-1})\partial_t^2 U \bar{\psi}_1 \mathrm{d}x\mathrm{d}z = -\int_\Omega \mu^{-1}\nabla U \cdot \nabla\bar{\psi}_1 \mathrm{d}x\mathrm{d}z + \sum_{j=1}^2 \int_{\Gamma_j} \mu_j^{-1}\mathscr{T}_j U \bar{\psi}_1 \mathrm{d}\gamma_j$$
$$+ \int_{\Gamma_1} \rho\bar{\psi}_1 \mathrm{d}\gamma_1 - c_1 \int_\Omega (\mu^{-1}\partial_{tx}U + \partial_x(\mu^{-1}\partial_t U))\bar{\psi}_1 \mathrm{d}x\mathrm{d}z. \tag{3.15}$$

It follows from (3.13) and the initial conditions in (3.6) that

$$\mathrm{Re}\int_0^\xi \int_\Omega (\varepsilon - c_1^2\mu^{-1})\partial_t^2 U \bar{\psi}_1 \mathrm{d}x\mathrm{d}z\mathrm{d}t$$

$$= \mathrm{Re}\int_\Omega \int_0^\xi \Big(\partial_t((\varepsilon - c_1^2\mu^{-1})\partial_t U\bar{\psi}_1) + (\varepsilon - c_1^2\mu^{-1})\partial_t U\bar{U}\Big)\mathrm{d}t\mathrm{d}x\mathrm{d}z$$

$$= \mathrm{Re}\int_\Omega \Big((\varepsilon - c_1^2\mu^{-1})\partial_t U\bar{\psi}_1)\,\Big|_0^\xi + \frac{1}{2}(\varepsilon - c_1^2\mu^{-1})|U|^2 \,\Big|_0^\xi\Big)\mathrm{d}x\mathrm{d}z$$

$$= \frac{1}{2}\|(\varepsilon - c_1^2\mu^{-1})^{1/2}U(\cdot,\xi)\|_{L^2(\Omega)}^2.$$

Integrating (3.15) from $t = 0$ to $t = \xi$ and taking the real part yield

$$\frac{1}{2}\|(\varepsilon - c_1^2\mu^{-1})^{1/2}U(\cdot,\xi)\|_{L^2(\Omega)}^2 + \mathrm{Re}\int_0^\xi \int_\Omega \mu^{-1}\nabla U \cdot \nabla\bar{\psi}_1 \mathrm{d}x\mathrm{d}z\mathrm{d}t$$

$$= \frac{1}{2}\|(\varepsilon - c_1^2\mu^{-1})^{1/2}U(\cdot,\xi)\|_{L^2(\Omega)}^2 + \frac{1}{2}\int_\Omega \mu^{-1}\Big|\int_0^\xi \nabla U(\cdot,t)\mathrm{d}t\Big|^2 \mathrm{d}x\mathrm{d}z$$

$$= \mathrm{Re} \int_0^{\xi} \sum_{j=1}^{2} \int_{\Gamma_j} \mu_j^{-1} \mathscr{T}_j U \bar{\psi}_1 \mathrm{d}\gamma_j \mathrm{d}t + \mathrm{Re} \int_0^{\xi} \int_{\Gamma_1} \rho \bar{\psi}_1 \mathrm{d}\gamma_1 \mathrm{d}t$$

$$- c_1 \mathrm{Re} \int_0^{\xi} \int_{\Omega} (\mu^{-1} \partial_{tx} U + \partial_x (\mu^{-1} \partial_t U)) \bar{\psi}_1 \mathrm{d}x \mathrm{d}z \mathrm{d}t. \tag{3.16}$$

In what follows, we estimate the three terms of the right-hand side of (3.16) separately.

By the property (3.14), we have

$$\mathrm{Re} \int_0^{\xi} \int_{\Gamma_j} \mu_j^{-1} \mathscr{T}_j U \bar{\psi}_1 \mathrm{d}\gamma_j \mathrm{d}t = \mathrm{Re} \int_0^{\xi} \int_0^{t} \big(\int_{\Gamma_j} \mu_j^{-1} \mathscr{T}_j U(\cdot, \tau) \mathrm{d}\gamma_j \big) \mathrm{d}\tau \bar{U}(\cdot, t) \mathrm{d}t.$$

Let $\tilde{U}$ be the extension of U with respect to t in $\mathbb{R}$ such that $\tilde{U} = 0$ outside the interval $[0, \xi]$. We obtain from the Parseval identity and Lemma 2.3 that

$$\mathrm{Re} \int_0^{\xi} e^{-2s_1 t} \int_0^{t} \big(\int_{\Gamma_j} \mu_j^{-1} \mathscr{T}_j U(\cdot, \tau) \mathrm{d}\gamma_j \big) \mathrm{d}\tau \bar{U}(\cdot, t) \mathrm{d}t$$

$$= \mathrm{Re} \int_{\Gamma_j} \int_0^{\infty} e^{-2s_1 t} \big(\int_0^{t} \mu_j^{-1} \mathscr{T}_j \tilde{U}(\cdot, \tau) \mathrm{d}\tau \big) \bar{\tilde{U}}(\cdot, t) \mathrm{d}t d\gamma_j$$

$$= \mathrm{Re} \int_{\Gamma_j} \int_0^{\infty} e^{-2s_1 t} \big(\int_0^{t} \mathscr{L}^{-1} \circ \mu_j^{-1} \mathscr{B}_j \circ \mathscr{L} \tilde{U}(\cdot, \tau) \mathrm{d}\tau \big) \bar{\tilde{U}}(\cdot, t) \mathrm{d}\gamma_j \mathrm{d}t$$

$$= \mathrm{Re} \int_{\Gamma_j} \int_0^{\infty} e^{-2s_1 t} \big(\mathscr{L}^{-1} \circ (s\mu_j)^{-1} \mathscr{B}_j \circ \mathscr{L} \tilde{U}(\cdot, t) \big) \bar{\tilde{U}}(\cdot, t) \mathrm{d}\gamma_j \mathrm{d}t$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{Re} \langle (s\mu_j)^{-1} \mathscr{B}_j \breve{\tilde{U}}, \breve{\tilde{U}} \rangle_{\Gamma_j} \mathrm{d}s_2 \leq 0,$$

where we have used the fact that

$$\int_0^{t} u(\tau) \mathrm{d}\tau = \mathscr{L}^{-1}(s^{-1} \breve{u}(s)).$$

After taking $s_1 \to 0$, we obtain that

$$\operatorname{Re} \int_0^{\xi} \sum_{j=1}^{2} \int_{\Gamma_j} \mu_j^{-1} \mathscr{T}_j U \bar{\psi}_1 \mathrm{d}\gamma_j \mathrm{d}t \le 0. \tag{3.17}$$

For $0 \le t \le \xi \le T$, we have from (3.14) that

$$\begin{aligned}
\operatorname{Re} \int_0^{\xi} \int_{\Gamma_1} \rho \bar{\psi}_1 \mathrm{d}\gamma_1 \mathrm{d}t &= \int_0^{\xi} \Big(\int_0^{t} \int_{\Gamma_1} \rho(\tau) \mathrm{d}\gamma_1 \mathrm{d}\tau \Big) \bar{U} \mathrm{d}t \\
&\le \int_0^{\xi} \int_0^{t} \|\rho(\cdot, \tau)\|_{H^{-1/2}(\Gamma_1)} \|U(\cdot, t)\|_{H^{1/2}(\Gamma_1)} \mathrm{d}\tau \mathrm{d}t \\
&\lesssim \int_0^{\xi} \int_0^{t} \|\rho(\cdot, \tau)\|_{H^{-1/2}(\Gamma_1)} \|U(\cdot, t)\|_{H^1_{\mathrm{p}}(\Omega)} \mathrm{d}\tau \mathrm{d}t \\
&\le \Big(\int_0^{\xi} \|\rho(\cdot, t)\|_{H^{-1/2}(\Gamma_1)} \mathrm{d}t \Big) \Big(\int_0^{\xi} \|U(\cdot, t)\|_{H^1_{\mathrm{p}}(\Omega)} \mathrm{d}t \Big).
\end{aligned} \tag{3.18}$$

Using integration by parts and (3.13), we have

$$\begin{aligned}
&\int_0^{\xi} \int_{\Omega} \mu^{-1} \partial_t (\partial_x U) \bar{\psi}_1 \mathrm{d}x \mathrm{d}z \mathrm{d}t + \int_0^{\xi} \int_{\Omega} \partial_x (\mu^{-1} \partial_t U) \bar{\psi}_1 \mathrm{d}x \mathrm{d}z \mathrm{d}t \\
&= \int_{\Omega} \big(\mu^{-1} \partial_x U \bar{\psi}_1 \big) \big|_0^{\xi} \mathrm{d}x \mathrm{d}z - \int_0^{\xi} \mu^{-1} \partial_x U \partial_t \bar{\psi}_1 \mathrm{d}t \mathrm{d}x \mathrm{d}z \\
&\quad + \int_{\Omega} \partial_x (\mu^{-1} U) \cdot \bar{\psi}_1 \big|_0^{\xi} \mathrm{d}x \mathrm{d}z - \int_0^{\xi} \partial_x (\mu^{-1} U) \cdot \partial_t \bar{\psi}_1 \mathrm{d}x \mathrm{d}z \mathrm{d}t \\
&= \int_0^{\xi} \int_{\Omega} \big(\mu^{-1} \partial_x U + \partial_x (\mu^{-1} U) \big) \cdot \bar{U} \mathrm{d}x \mathrm{d}z \mathrm{d}t.
\end{aligned}$$

By the periodicity of μ and U in x, it yields that

$$\int_0^{\xi} \int_{\Omega} \big(\mu^{-1} \partial_x U + \partial_x (\mu^{-1} U) \big) \bar{U} \mathrm{d}x \mathrm{d}z \mathrm{d}t + \int_0^{\xi} \int_{\Omega} \big(\mu^{-1} \partial_x \bar{U} + \partial_x (\mu^{-1} \bar{U}) \big) U \mathrm{d}x \mathrm{d}z \mathrm{d}t = 0.$$

Thus

$$\mathrm{Re}\int_0^\xi\int_\Omega(\mu^{-1}\partial_{tx}U+\partial_x(\mu^{-1}\partial_t U))\bar{\psi}_1\mathrm{d}x\mathrm{d}z\mathrm{d}t=0. \tag{3.19}$$

Substituting (3.17)–(3.19) into (3.16), we have for any $\xi\in[0,T]$ that

$$\frac{1}{2}\|(\varepsilon-c_1^2\mu^{-1})^{1/2}U(\cdot,\xi)\|^2_{L^2(\Omega)}+\frac{1}{2}\int_\Omega\mu^{-1}\Big|\int_0^\xi\nabla U(\cdot,t)\mathrm{d}t\Big|^2\mathrm{d}x\mathrm{d}z$$
$$\le\Big(\int_0^\xi\|\rho(\cdot,t)\|_{H^{-1/2}(\Gamma_1)}\mathrm{d}t\Big)\Big(\int_0^\xi\|U(\cdot,t)\|_{H^1_{\mathrm{p}}(\Omega)}\Big)\mathrm{d}t. \tag{3.20}$$

Taking the derivative of (3.6) with respect to t, we know that $\partial_t U$ satisfies the same equation with ρ replaced by $\partial_t\rho$. Define

$$\psi_2(x,z,t)=\int_t^\xi\partial_t U(x,z,\tau)\mathrm{d}\tau,\quad (x,z)\in\Omega,\ 0\le t\le\xi.$$

We may follow the same steps as those for ψ_1 to obtain

$$\frac{1}{2}\|(\varepsilon-c_1^2\mu^{-1})^{1/2}\partial_t U(\cdot,\xi)\|^2_{L^2(\Omega)}+\frac{1}{2}\int_\Omega\mu^{-1}\Big|\int_0^\xi\partial_t(\nabla U(\cdot,t))\mathrm{d}t\Big|^2\mathrm{d}x\mathrm{d}z$$
$$=\mathrm{Re}\int_0^\xi\sum_{j=1}^2\int_{\Gamma_j}\mu_j^{-1}\mathscr{T}_j\partial_t U\bar{\psi}_2\mathrm{d}\gamma_j\mathrm{d}t+\mathrm{Re}\int_0^\xi\int_{\Gamma_1}\partial_t\rho\bar{\psi}_2\mathrm{d}\gamma_1\mathrm{d}t$$
$$-c_1\mathrm{Re}\int_0^\xi\int_\Omega(\mu^{-1}\partial_{ttx}U+\partial_x(\mu^{-1}\partial_t^2U))\bar{\psi}_2\mathrm{d}x\mathrm{d}z\mathrm{d}t. \tag{3.21}$$

Integrating by parts yields that

$$\frac{1}{2}\int_\Omega\mu^{-1}\Big|\int_0^\xi\partial_t(\nabla U(\cdot,t))\mathrm{d}t\Big|^2\mathrm{d}x\mathrm{d}z=\frac{1}{2}\|\mu^{-1/2}\nabla U(\cdot,\xi)\|^2_{L^2(\Omega)}. \tag{3.22}$$

The first and the third terms on the right-hand side of (3.21) are discussed as above. We only have to consider the second term. By (3.13), Lemma 2.1, and Lemma 2.2, we get

$$\int_0^\xi \int_{\Gamma_1} \partial_t \rho \bar{\psi}_2 \mathrm{d}\gamma_1 \mathrm{d}t = \int_0^\xi \int_0^t \Big(\int_{\Gamma_1} \partial_\tau \rho(\cdot,\tau)\mathrm{d}\gamma_1\Big)\mathrm{d}\tau\, \partial_t \bar{U}(\cdot,t)\mathrm{d}t$$

$$= \int_{\Gamma_1} \Big(\int_0^t \partial_\tau \rho(\cdot,\tau)\mathrm{d}\tau\Big)\bar{U}(\cdot,t)\,\Big|_0^\xi\, \mathrm{d}\gamma_1 - \int_0^\xi \int_{\Gamma_1} \partial_t \rho(\cdot,t) U(\cdot,t)\mathrm{d}\gamma_1 \mathrm{d}t$$

$$\lesssim \int_0^\xi \|\partial_t \rho(\cdot,t)\|_{H^{-1/2}(\Gamma_1)} \|U(\cdot,t)\|_{H^{1/2}(\Gamma_1)}\mathrm{d}t$$

$$\lesssim \int_0^\xi \|\partial_t \rho(\cdot,t)\|_{H^{-1/2}(\Gamma_1)} \|U(\cdot,t)\|_{H^1_{\mathrm{p}}(\Omega)}\mathrm{d}t. \tag{3.23}$$

Substituting (3.22) and (3.23) into (3.21), we have for any $\xi \in [0, T]$ that

$$\frac{1}{2}\|(\varepsilon - c_1^2\mu^{-1})^{1/2}\partial_t U(\cdot,\xi)\|^2_{L^2(\Omega)} + \frac{1}{2}\|\mu^{-1/2}\nabla U(\cdot,\xi)\|^2_{L^2(\Omega)}$$

$$\lesssim \int_0^\xi \|\partial_t \rho(\cdot,t)\|_{H^{-1/2}(\Gamma_1)} \|U(\cdot,t)\|_{H^1_{\mathrm{p}}(\Omega)}\mathrm{d}t. \tag{3.24}$$

Combing the estimates (3.20) and (3.24), we obtain

$$\|U(\cdot,\xi)\|^2_{L^2(\Omega)} + \|\nabla U(\cdot,\xi)\|^2_{L^2(\Omega)^2} \lesssim \Big(\int_0^\xi \|\rho(\cdot,t)\|_{H^{-1/2}(\Gamma_1)}\mathrm{d}t\Big)\Big(\int_0^\xi \|U(\cdot,t)\|_{H^1_{\mathrm{p}}(\Omega)}\mathrm{d}t\Big)$$

$$+ \int_0^\xi \|\partial_t \rho(\cdot,t)\|_{H^{-1/2}(\Gamma_1)} \|U(\cdot,t)\|_{H^1_{\mathrm{p}}(\Omega)}\mathrm{d}t. \tag{3.25}$$

Taking the L^∞-norm with respect to ξ on both side of (3.25) yields

$$\|U\|^2_{L^\infty(0,T;\, L^2(\Omega))} + \|\nabla U\|^2_{L^\infty(0,T;L^2(\Omega)^2)} \lesssim T\|\rho\|_{L^1(0,T;H^{-1/2}(\Gamma_1))}\|U\|_{L^\infty(0,T;H^1_{\mathrm{p}}(\Omega))}$$
$$+ \|\partial_t \rho\|_{L^1(0,T;H^{-1/2}(\Gamma_1))}\|U\|_{L^\infty(0,T;H^1_{\mathrm{p}}(\Omega))},$$

which gives the estimate (3.11) after applying the Young inequality.

Integrating (3.25) with respect to ξ from 0 to T and using the Cauchy–Schwarz inequality, we obtain

$$\|U\|^2_{L^2(0,T;\, L^2(\Omega))} + \|\nabla U\|^2_{L^2(0,T;\, L^2(\Omega)^2)} \lesssim T^{3/2}\|\rho\|_{L^1(0,T;H^{-1/2}(\Gamma_1))}\|U\|_{L^2(0,T;H^1_{\mathrm{p}}(\Omega))}$$
$$+ T^{1/2}\|\partial_t \rho\|_{L^1(0,T;H^{-1/2}(\Gamma_1))}\|U\|_{L^2(0,T;H^1_{\mathrm{p}}(\Omega))},$$

which implies the estimate (3.12) by using the Young inequality again. □

4. Conclusion

In this paper, we studied the time-domain scattering problem in a one-dimensional grating. The TE and TM cases were considered in a unified approach. The scattering problem was reduced equivalently into an initial-boundary value problem in a bounded domain by using the exact time-domain DtN map. The reduced problem was shown to have a unique solution by using the energy method. The stability was also presented. The main ingredients of the proofs were the Laplace transform, the Lax–Milgram lemma, and the Parseval identity. Moreover, by directly considering the variational problem of the time-domain wave equation, we obtained a priori estimates with explicit dependence on time. In the future, we plan to investigate the time-domain scattering by biperiodic structures where the full three-dimensional Maxwell's equations should be considered. The progress will be reported elsewhere.

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