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Inverse obstacle scattering for elastic waves

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Abstract

Consider the scattering of a time-harmonic plane wave by a rigid obstacle which is embedded in an open space filled with a homogeneous and isotropic elastic medium. An exact transparent boundary condition is introduced to reduce the scattering problem into a boundary value problem in a bounded domain. Given the incident field, the direct problem is to determine the displacement of the wave field from the known obstacle; the inverse problem is to determine the obstacle's surface from the measurement of the displacement on an artificial boundary enclosing the obstacle. In this paper, we consider both the direct and inverse problems. The direct problem is shown to have a unique weak solution by examining its variational formulation. The domain derivative is derived for the displacement with respect to the variation of the surface. A continuation method with respect to the frequency is developed for the inverse problem. Numerical experiments are presented to demonstrate the effectiveness of the proposed method.

Keywords: elastic wave equation, inverse obstacle scattering, domain derivative

(Some figures may appear in colour only in the online journal)

1. Introduction

A basic problem in scattering theory is the scattering of a time-harmonic wave by an impenetrable medium, which is known as the obstacle scattering problem [7]. Given the

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incident field, the direct obstacle scattering problem is to determine the wave field from the known obstacle; the inverse obstacle scattering problem is to determine the shape of the obstacle from the measurement of the wave field. The obstacle scattering problems have been widely studied for acoustic and electromagnetic waves by numerous researchers [4, 19]. A large amount of information is available regarding their solutions for the governing Helmholtz and Maxwell equations [8, 26].

Recently, the elastic wave scattering problems have received ever increasing attention in both the engineering and mathematical communities for their significant applications in many scientific areas such as geophysics and seismology [21]. The elastic wave is governed by the Navier equation which is more complex due to coupling of the compressional and shear waves. The inverse elastic obstacle scattering problem is investigated mathematically in [12] for the uniqueness and numerically in [14, 24] for the shape reconstruction. We refer to [10, 15–17, 20, 22, 25] for some related direct and inverse scattering problems for elastic waves.

In this paper, we consider both the direct and inverse obstacle scattering problems for the two-dimensional elastic wave equation. The purpose of this work is fourfold:

(1) Develop an exact transparent boundary condition to reduce the scattering problem equivalently into a boundary value problem in a bounded domain.
(2) Establish the existence and uniqueness of the solution for the direct problem by studying its variational formulation.
(3) Characterize the domain derivative of the wave field with respect to the variation of the obstacle's surface.
(4) Propose a continuation method with respect to the frequency to reconstruct the obstacle's surface.

The obstacle is assumed to be an elastically rigid body which is embedded in an open space filled with a homogeneous and isotropic elastic medium. By introducing an exact transparent boundary condition on a circle enclosing the obstacle, we formulate the scattering model into a boundary value problem in a bounded domain. We show that the direct problem has a unique weak solution by studying its variational formulation. The proofs are based on asymptotic analysis of the boundary operator, the Helmholtz decomposition, and the Fredholm alternative theorem.

The calculation of domain derivatives, or more generally Fréchet derivatives of the wave field with respect to the perturbation of the boundary of the scatterer, is an essential step for inverse scattering problems. The domain derivatives of the inverse obstacle scattering problems have been discussed by many authors for the acoustic and electromagnetic waves [11, 18, 20, 27]. Recently, the domain derivative is studied in [5, 23] for the elastic wave by using boundary integral equations. Here we present a variational approach and give an explicit characterization of the domain derivative. We show that the domain derivative is the unique weak solution of some boundary value problem which shares essentially the same variational formulation as the direct scattering problem. Therefore, the domain derivative can be efficiently computed by solving a direct problem.

We propose a continuation method to solve the inverse problem. The method requires multi-frequency scattering data and proceed with respect to the frequency. At each step, we apply the steepest descent method with the starting point given by the output from the previous step, and create an approximation to the surface filtered at a higher frequency. Numerical experiments are presented to demonstrate the effectiveness of the method. We refer to [2, 3, 6, 29] for solving related inverse surface scattering problems by using multiple frequency data. A topic review can be found in [1] for solving general inverse scattering problems with multi-frequencies to gain increased stability.

The paper is organized as follows. Section 2 introduces the model problem and presents some notation and Sobolev spaces. The direct problem is discussed in section 3. Boundary value problems are formulated by using transparent boundary conditions. Uniqueness and existence are established for the variational problem of the direct scattering. Section 4 is devoted to the inverse problem. The domain derivative is derived for the wave field with respect to the surface. A continuation method is introduced for solving the inverse problem. Numerical experiments are presented in section 5. The paper is concluded with some remarks in section 6.

2. Problem formulation

Consider a two-dimensional elastically rigid obstacle, which is described by the bounded domain D with Lipschitz continuous boundary ∂D. Denote by $\boldsymbol{\nu} = (\nu_1, \nu_2)^\top$ and $\boldsymbol{\tau} = (\tau_1, \tau_2)^\top$ the unit normal and tangential vectors on ∂D, where $\tau_1 = \nu_2$ and $\tau_2 = -\nu_1$. Suppose that the infinite exterior domain $\mathbb{R}^2 \setminus \bar{D}$ is filled with a homogeneous and isotropic elastic medium with a unit mass density. Denote by $B = \{\boldsymbol{x} \in \mathbb{R}^2 : |\boldsymbol{x}| < R\}$ the ball with radius $R > 0$ such that $\bar{D} \subset B$. Let the sphere $\partial B = \{\boldsymbol{x} \in \mathbb{R}^2 : |\boldsymbol{x}| = R\}$ be the boundary of B. Denote by $\Omega = B \setminus \bar{D}$ the bounded domain enclosed by ∂D and ∂B.

Let the obstacle be illuminated by a time-harmonic plane wave $\boldsymbol{u}^{\rm inc}$, which satisfies the two-dimensional Navier equation:

$$\mu\Delta\boldsymbol{u}^{\rm inc} + (\lambda + \mu)\nabla\nabla \cdot \boldsymbol{u}^{\rm inc} + \omega^2\boldsymbol{u}^{\rm inc} = 0 \quad \text{in } \mathbb{R}^2 \setminus \bar{D}, \tag{2.1}$$

where $\omega > 0$ is the angular frequency, λ and μ are the Lamé constants satisfying $\mu > 0$ and $\lambda + \mu > 0$. The incident field can be either a compressional plane wave:

$$\boldsymbol{u}^{\rm inc} = \boldsymbol{d}\,\mathrm{e}^{\mathrm{i}\kappa_1 \boldsymbol{x}\cdot\boldsymbol{d}}, \tag{2.2}$$

or a shear plane wave:

$$\boldsymbol{u}^{\rm inc} = \boldsymbol{d}^\perp \mathrm{e}^{\mathrm{i}\kappa_2 \boldsymbol{x}\cdot\boldsymbol{d}}, \tag{2.3}$$

where $\boldsymbol{d} = (\cos\varphi, \sin\varphi)^\top$, $\boldsymbol{d}^\perp = (-\sin\varphi, \cos\varphi)^\top$, $\varphi \in [0, 2\pi]$ is the incident angle

$$\kappa_1 = \frac{\omega}{\sqrt{\lambda + 2\mu}} \quad \text{and} \quad \kappa_2 = \frac{\omega}{\sqrt{\mu}}, \tag{2.4}$$

are the compressional wavenumber and the shear wavenumber, respectively.

The displacement of the total wave field $\boldsymbol{u}$ also satisfies the Navier equation:

$$\mu\Delta\boldsymbol{u} + (\lambda + \mu)\nabla\nabla \cdot \boldsymbol{u} + \omega^2\boldsymbol{u} = 0 \quad \text{in } \mathbb{R}^2 \setminus \bar{D}. \tag{2.5}$$

Since the obstacle is assumed to be elastically rigid, it holds that

$$\boldsymbol{u} = 0 \quad \text{on } \partial D. \tag{2.6}$$

The total wave $\boldsymbol{u}$ consists of the incident wave $\boldsymbol{u}^{\rm inc}$ and the scattered wave $\boldsymbol{v}$:

$$\boldsymbol{u} = \boldsymbol{u}^{\rm inc} + \boldsymbol{v}.$$

Subtracting (2.1) from (2.5) yields

$$\mu\Delta\boldsymbol{v} + (\lambda + \mu)\nabla\nabla \cdot \boldsymbol{v} + \omega^2\boldsymbol{v} = 0 \quad \text{in } \mathbb{R}^2 \setminus \bar{D}. \tag{2.7}$$

Let $\boldsymbol{w} = (w_1, w_2)^\top$ and w be a vector and scalar function, respectively. Introduce

$$\nabla \boldsymbol{w} = \begin{bmatrix} \partial_{x_1} w_1 & \partial_{x_2} w_1 \\ \partial_{x_1} w_2 & \partial_{x_2} w_2 \end{bmatrix}$$

and two curl operators

$$\operatorname{curl} \boldsymbol{w} = \partial_{x_1} w_2 - \partial_{x_2} w_1, \quad \mathbf{curl} w = (\partial_{x_2} w, -\partial_{x_1} w)^\top.$$

For any solution $\boldsymbol{v}$ of (2.7), the Helmholtz decomposition splits it into its compressional and shear parts:

$$\boldsymbol{v} = \nabla \phi_1 + \mathbf{curl} \phi_2, \tag{2.8}$$

where $\phi_j, j = 1, 2$ are scalar potential functions. Substituting (2.8) into (2.7) gives

$$\nabla[(\lambda + 2\mu)\Delta\phi_1 + \omega^2\phi_1] + \mathbf{curl}(\mu\Delta\phi_2 + \omega^2\phi_2) = 0,$$

which is fulfilled if ϕ_j satisfy the Helmholtz equation

$$\Delta\phi_j + \kappa_j^2\phi_j = 0. \tag{2.9}$$

In addition, ϕ_j are required to satisfy the Sommerfeld radiation condition

$$\lim_{\rho\to\infty} \rho^{1/2}(\partial_\rho \phi_j - \mathrm{i}\kappa_j \phi_j) = 0, \quad \rho = |\boldsymbol{x}|. \tag{2.10}$$

Given the incident field $\boldsymbol{u}^{\mathrm{inc}}$, the direct problem is to determine the displacement of the total field $\boldsymbol{u}$ for the known obstacle D. The inverse problem is to determine the obstacle's surface ∂D from the boundary measurement of the displacement $\boldsymbol{u}$ on ∂B, for the given incident field $\boldsymbol{u}^{\mathrm{inc}}$.

We introduce some functional spaces. Let $L^2(\Omega)^2 = L^2(\Omega) \times L^2(\Omega)$ be the product space of $L^2(\Omega)$ equipped with the inner product and norm:

$$(\boldsymbol{u}, \boldsymbol{v}) = \int_\Omega \boldsymbol{u} \cdot \bar{\boldsymbol{v}} \,\mathrm{d}\boldsymbol{x}, \quad \|\boldsymbol{u}\|_{0,\Omega} = (\boldsymbol{u}, \boldsymbol{u})^{1/2}.$$

For a given function w on ∂B, it has the Fourier series expansion

$$w(R, \theta) = \sum_{n\in\mathbb{Z}} w^{(n)} \mathrm{e}^{\mathrm{i}n\theta}, \quad w^{(n)} = \frac{1}{2\pi}\int_0^{2\pi} w(R, \theta)\mathrm{e}^{-\mathrm{i}n\theta}\mathrm{d}\theta.$$

Let $H^s(\Omega)$ and $H^s(\partial B)$ be the standard Sobolev spaces with the norms given by

$$\|w\|_{s,\Omega}^2 = \sum_{|\alpha|\leqslant s} \int_\Omega |D^\alpha w(\boldsymbol{x})|^2 \,\mathrm{d}\boldsymbol{x}, \quad \|w\|_{s,\partial B}^2 = 2\pi \sum_{n\in\mathbb{Z}} (1 + n^2)^s |w^{(n)}(R)|^2.$$

Define $H^1_{\partial D}(\Omega) = \{w \in H^1(\Omega) : w = 0 \text{ on } \partial D\}$. Let $H^1_{\partial D}(\Omega)^2 = H^1_{\partial D}(\Omega) \times H^1_{\partial D}(\Omega)$ and $H^s(\partial B)^2 = H^s(\partial B) \times H^s(\partial B)$ be the Cartesian product spaces equipped with the corresponding 2-norms of $H^1_{\partial D}(\Omega)$ and $H^s(\partial B)$, respectively. It is known that $H^{-s}(\partial B)^2$ is the dual space of $H^s(\partial B)^2$ with respect to the inner product

$$\langle \boldsymbol{u}, \boldsymbol{v} \rangle_{\partial B} = \int_{\partial B} \boldsymbol{u} \cdot \bar{\boldsymbol{v}} \,\mathrm{d}s.$$

Define the inner product in $\mathbb{C}^2$:

$$\langle \boldsymbol{u}, \boldsymbol{v} \rangle = \boldsymbol{v}^* \boldsymbol{u}, \quad \forall \boldsymbol{u}, \boldsymbol{v} \in \mathbb{C}^2,$$

where $\boldsymbol{v}^*$ is the conjugate transpose of $\boldsymbol{v}$. Throughout the paper, we take the notation of $a \lesssim b$ or $a \gtrsim b$ to stand for $a \leqslant Cb$ or $a \geqslant Cb$, where C is a positive constant.

3. Direct scattering

In this section, we consider the direct scattering problem and show that it has a unique weak solution by studying its variational formulation.

3.1. Transparent boundary conditions

In the exterior domain $\mathbb{R}^2 \backslash \bar{B}$, it follows from the radiation condition (2.10) that the solutions of (2.9) have Fourier series expansions in the polar coordinates

$$\phi_j(\rho, \theta) = \sum_{n\in\mathbb{Z}} \frac{H_n^{(1)}(\kappa_j \rho)}{H_n^{(1)}(\kappa_j R)} \phi_j^{(n)}(R) \mathrm{e}^{\mathrm{i}n\theta}, \tag{3.1}$$

where $H_n^{(1)}$ is the Hankel function of the first kind with order n.

We define two boundary operators

$$(\mathscr{T}_j w)(R, \theta) = \frac{1}{R} \sum_{n\in\mathbb{Z}} h_n(\kappa_j R) w^{(n)} \mathrm{e}^{\mathrm{i}n\theta}, \tag{3.2}$$

where $h_n(z) = z H_n^{(1)'}(z)/H_n^{(1)}(z)$. It can be shown from the properties of the Hankel functions in [30] that

$$\mathrm{Re}\, h_n(z) < 0, \quad \mathrm{Im}\, h_n(z) > 0, \quad \forall z > 0. \tag{3.3}$$

Taking the derivative of (3.1) with respect to ρ, evaluating it at $\rho = R$, and using the boundary operators (3.2), we get the transparent boundary conditions

$$\partial_\rho \phi_j = \mathscr{T}_j \phi_j \quad \text{on } \partial B. \tag{3.4}$$

It is shown in [9] that $\mathscr{T}_j$ is a bounded linear operator from $H^{1/2}(\partial B)$ to $H^{-1/2}(\partial B)$. Furthermore, the following result can be easily proved by using (3.3). The proof is omitted for simplicity.

Lemma 3.1. *It holds that*

$$\mathrm{Re}\langle \mathscr{T}_j w, w\rangle_{\partial B} \leqslant 0, \quad \mathrm{Im}\langle \mathscr{T}_j w, w\rangle_{\partial B} \geqslant 0, \quad \forall\ w \in H^{1/2}(\partial B).$$

If $\mathrm{Re}\langle \mathscr{T}_j w, w\rangle_{\partial B} = 0$ *or* $\mathrm{Im}\langle \mathscr{T}_j w, w\rangle_{\partial B} = 0$, *then* $w = 0$ *on* ∂B.

In the frequency domain, the transparent boundary conditions (3.4) become

$$(\partial_\rho \phi_j)^{(n)}(R) = \alpha_j^{(n)} \phi_j^{(n)}(R), \quad \alpha_j^{(n)} = \frac{\kappa_j H_n^{(1)'}(\kappa_j R)}{H_n^{(1)}(\kappa_j R)}. \tag{3.5}$$

Taking $\partial_{\rho\rho}$ of (3.1) and evaluating them at $\rho = R$ yields

$$(\partial_{\rho\rho} \phi_j)^{(n)}(R) = \beta_j^{(n)} \phi_j^{(n)}(R), \quad \beta_j^{(n)} = \frac{\kappa_j^2 H_n^{(1)''}(\kappa_j R)}{H_n^{(1)}(\kappa_j R)}. \tag{3.6}$$

The polar coordinates (ρ, θ) are related to the Cartesian coordinates $\boldsymbol{x} = (x_1, x_2)$ by $x_1 = \rho \cos\theta$, $x_2 = \rho \sin\theta$, with the local orthonormal basis $\boldsymbol{e}_\rho$, $\boldsymbol{e}_\theta$:

$$\boldsymbol{e}_\rho = (\cos\theta, \sin\theta)^\top, \quad \boldsymbol{e}_\theta = (-\sin\theta, \cos\theta)^\top.$$

For any vector $\boldsymbol{w} = (w_1, w_2)^\top$ given in the Cartesian coordinates, it has a representation in the polar coordinates $\boldsymbol{w} = \hat{w}_1 \boldsymbol{e}_\rho + \hat{w}_2 \boldsymbol{e}_\theta$, which will be still denoted as $\boldsymbol{w} = (\hat{w}_1, \hat{w}_2)^\top$ for simplicity. For any function w, it is easy to verify that

$$\nabla w = \left(\partial_\rho w,\ \frac{1}{\rho}\partial_\theta w\right)^\top, \quad \mathbf{curl} w = \left(\frac{1}{\rho}\partial_\theta w,\ -\partial_\rho w\right)^\top$$

and

$$\Delta w = \left(\partial_{\rho\rho} + \frac{1}{\rho}\partial_\rho + \frac{1}{\rho^2}\partial_{\theta\theta}\right)w.$$

In the new coordinates, the Helmholtz decomposition (2.8) takes the form

$$\boldsymbol{v} = \left(\partial_\rho\phi_1 + \frac{1}{\rho}\partial_\theta\phi_2,\ \frac{1}{\rho}\partial_\theta\phi_1 - \partial_\rho\phi_2\right)^\top. \tag{3.7}$$

Taking the Fourier transform of (3.7) at $\rho = R$ and applying the boundary condition (3.5), we obtain

$$\boldsymbol{v}^{(n)}(R) = \begin{bmatrix} \alpha_1^{(n)} & \frac{\mathrm{i}n}{R} \\ \frac{\mathrm{i}n}{R} & -\alpha_2^{(n)} \end{bmatrix} \begin{bmatrix} \phi_1^{(n)}(R) \\ \phi_2^{(n)}(R) \end{bmatrix}. \tag{3.8}$$

Taking ∂_ρ of (3.7), applying the Fourier transform, and using (3.5) and (3.6), we have

$$(\partial_\rho\boldsymbol{v})^{(n)}(R) = \begin{bmatrix} \beta_1^{(n)} & -\frac{\mathrm{i}n}{R^2} + \frac{\mathrm{i}n}{R}\alpha_2^{(n)} \\ -\frac{\mathrm{i}n}{R^2} + \frac{\mathrm{i}n}{R}\alpha_1^{(n)} & -\beta_2^{(n)} \end{bmatrix} \begin{bmatrix} \phi_1^{(n)}(R) \\ \phi_2^{(n)}(R) \end{bmatrix}. \tag{3.9}$$

It follows from (2.8) that

$$\nabla\cdot\boldsymbol{v} = \Delta\phi_1 = \left(\partial_{\rho\rho} + \frac{1}{\rho}\partial_\rho + \frac{1}{\rho^2}\partial_{\theta\theta}\right)\phi_1,$$

which yields after taking the Fourier transform that

$$(\nabla\cdot\boldsymbol{v})^{(n)}(R) = \left(\beta_1^{(n)} + \frac{1}{R}\alpha_1^{(n)} - \left(\frac{n}{R}\right)^2\right)\phi_1^{(n)}(R). \tag{3.10}$$

Define a boundary operator for the displacement of the scattered wave

$$\mathscr{B}\boldsymbol{v} = \mu\partial_\rho\boldsymbol{v} + (\lambda + \mu)(\nabla\cdot\boldsymbol{v})\boldsymbol{e}_\rho \quad \text{on } \partial B,$$

which gives after taking the Fourier transform and using (3.9) and (3.10) that

$$\begin{aligned}(\mathscr{B}\boldsymbol{v})^{(n)}(R) = \mu &\begin{bmatrix} \beta_1^{(n)} & -\frac{\mathrm{i}n}{R^2} + \frac{\mathrm{i}n}{R}\alpha_2^{(n)} \\ -\frac{\mathrm{i}n}{R^2} + \frac{\mathrm{i}n}{R}\alpha_1^{(n)} & -\beta_2^{(n)} \end{bmatrix} \begin{bmatrix} \phi_1^{(n)}(R) \\ \phi_2^{(n)}(R) \end{bmatrix} \\ &+ (\lambda + \mu) \begin{bmatrix} \beta_1^{(n)} + \frac{1}{R}\alpha_1^{(n)} - \frac{n^2}{R^2} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \phi_1^{(n)}(R) \\ \phi_2^{(n)}(R) \end{bmatrix}. \end{aligned} \tag{3.11}$$

Combining (3.8) and (3.11) yields the transparent boundary condition:

$$\mathscr{B}\boldsymbol{v} = \mathscr{T}\boldsymbol{v} := \sum_{n\in\mathbb{Z}} M_n \boldsymbol{v}^{(n)} \mathrm{e}^{\mathrm{i}n\theta} \quad \text{on } \partial B, \tag{3.12}$$

where

$$M_n = \begin{bmatrix} M_{11}^{(n)} & M_{12}^{(n)} \\ M_{21}^{(n)} & M_{22}^{(n)} \end{bmatrix} = \Lambda_n^{-1} \begin{bmatrix} N_{11}^{(n)} & N_{12}^{(n)} \\ N_{21}^{(n)} & N_{22}^{(n)} \end{bmatrix}.$$

Here

$$\Lambda_n = \left(\frac{n}{R}\right)^2 - \alpha_1^{(n)}\alpha_2^{(n)}$$

and

$$N_{11}^{(n)} = -\alpha_2^{(n)}\xi^{(n)} + \mu\left(\frac{n}{R}\right)^2 \eta_2^{(n)}, \quad N_{12}^{(n)} = -\frac{\mathrm{i}n}{R}\xi^{(n)} + \mu\frac{\mathrm{i}n}{R}\alpha_1^{(n)}\eta_2^{(n)},$$

$$N_{21}^{(n)} = -\mu\frac{\mathrm{i}n}{R}\alpha_2^{(n)}\eta_1^{(n)} + \mu\frac{\mathrm{i}n}{R}\beta_2^{(n)}, \quad N_{22}^{(n)} = \mu\left(\frac{n}{R}\right)^2 \eta_1^{(n)} - \mu\alpha_1^{(n)}\beta_2^{(n)},$$

where

$$\xi^{(n)} = (\lambda + 2\mu)\beta_1^{(n)} + (\lambda + \mu)\left(\frac{1}{R}\alpha_1^{(n)} - \left(\frac{n}{R}\right)^2\right), \quad \eta_j^{(n)} = \alpha_j^{(n)} - \frac{1}{R}.$$

Remark 3.2. It is clear to note from (3.3) that $\alpha_j^{(n)} = h_n(\kappa_j R)\big/R$ and

$$\mathrm{Im}\Lambda_n = -\mathrm{Re}\,\alpha_1^{(n)}\mathrm{Im}\,\alpha_2^{(n)} - \mathrm{Re}\,\alpha_2^{(n)}\mathrm{Im}\,\alpha_1^{(n)} > 0, \quad \forall\, n \in \mathbb{Z}.$$

Thus we have $\Lambda_n \neq 0$, $\forall\, n \in \mathbb{Z}$.

Using the asymptotic expansions of the Hankel functions

$$\frac{H_n^{(1)'}(z)}{H_n^{(1)}(z)} = -\frac{|n|}{z} + \frac{z}{2|n|} + O\left(\frac{1}{|n|^2}\right)$$

and

$$\frac{H_n^{(1)''}(z)}{H_n^{(1)}(z)} = \frac{|n|(|n|+1)}{z^2} - 1 + O\left(\frac{1}{|n|}\right),$$

we obtain

$$\Lambda_n = \frac{\kappa_1^2 + \kappa_2^2}{2}\left(1 + O\left(\frac{1}{|n|}\right)\right),$$

$$\alpha_j^{(n)} = -\frac{|n|}{R}\left(1 - \frac{(\kappa_j R)^2}{2|n|^2} + O\left(\frac{1}{|n|^3}\right)\right),$$

$$\beta_j^{(n)} = \frac{|n|(|n|+1)}{R^2}\left(1 - \frac{(\kappa_j R)^2}{|n|^2} + O\left(\frac{1}{|n|^3}\right)\right).$$

Noting (2.4) and combining the above estimates, we may verify that

$$M_{11}^{(n)} = -\left[\frac{2\mu(\lambda+2\mu)}{R(\lambda+3\mu)}\right]|n| + O(1), \quad M_{12}^{(n)} = \mathrm{i}\left[\frac{\mu(\lambda+\mu)}{R(\lambda+3\mu)}\right]n + O(1),$$
$$M_{21}^{(n)} = -\mathrm{i}\left[\frac{\mu(\lambda+\mu)}{R(\lambda+3\mu)}\right]n + O(1), \quad M_{22}^{(n)} = -\left[\frac{2\mu(\lambda+2\mu)}{R(\lambda+3\mu)}\right]|n| + O(1).$$

Lemma 3.3. *The matrix* $\hat{M}_n = -\frac{1}{2}(M_n + M_n^*)$ *is positive definite for sufficiently large* $|n|$.

Proof. A simple calculation yields

$$\hat{M}_n = \begin{bmatrix} \hat{M}_{11}^{(n)} & \hat{M}_{11}^{(n)} \\ \hat{M}_{21}^{(n)} & \hat{M}_{11}^{(n)} \end{bmatrix},$$

where

$$\hat{M}_n^{(11)} = \left[\frac{2\mu(\lambda+2\mu)}{R(\lambda+3\mu)}\right]|n| + O(1), \quad \hat{M}_n^{(12)} = -\mathrm{i}\left[\frac{\mu(\lambda+\mu)}{R(\lambda+3\mu)}\right]n + O(1),$$
$$\hat{M}_n^{(21)} = \mathrm{i}\left[\frac{\mu(\lambda+\mu)}{R(\lambda+3\mu)}\right]n + O(1), \quad \hat{M}_n^{(22)} = \left[\frac{2\mu(\lambda+2\mu)}{R(\lambda+3\mu)}\right]|n| + O(1).$$

For sufficiently large $|n|$, we have $\hat{M}_{11}^{(n)} > 0$ and

$$\det(\hat{M}_n) = \frac{\mu^2[4(\lambda+2\mu)^2 - (\lambda+\mu)^2]}{[R(\lambda+3\mu)]^2}|n|^2 + O(|n|) > 0,$$

which completes the proof by applying Sylvester's criterion. □

Lemma 3.4. *The boundary operator* $\mathscr{T} : H^{1/2}(\partial B)^2 \to H^{-1/2}(\partial B)^2$ *is continuous, i.e.,*

$$\|\mathscr{T}\boldsymbol{u}\|_{-1/2,\partial B} \lesssim \|\boldsymbol{u}\|_{1/2,\partial B}, \quad \forall \boldsymbol{u} \in H^{1/2}(\partial B)^2.$$

Proof. It follows from (3.12) and the asymptotic expansions of $M_{ij}^{(n)}$ that

$$\begin{aligned}\|\mathscr{T}\boldsymbol{u}\|_{-1/2,\partial B}^2 &= \sum_{n\in\mathbb{Z}} (1+n^2)^{-1/2} |M_n \boldsymbol{u}^{(n)}(R)|^2 \\ &\lesssim \sum_{n\in\mathbb{Z}} (1+n^2)^{1/2} |\boldsymbol{u}^{(n)}(R)|^2 = \|\boldsymbol{u}\|_{1/2,\partial B}^2,\end{aligned}$$

which completes the proof. □

3.2. Uniqueness

It follows from the Dirichlet boundary condition (2.6) and the Helmholtz decomposition (2.8) that

$$\boldsymbol{v} = \nabla\phi_1 + \mathbf{curl}\phi_2 = -\boldsymbol{u}^{\mathrm{inc}} \quad \text{on } \partial D.$$

Taking the dot product of the above equation with $\boldsymbol{\nu}$ and $\boldsymbol{\tau}$, respectively, we have

$$\partial_\nu \phi_1 - \partial_\tau \phi_2 = u, \quad \partial_\nu \phi_2 + \partial_\tau \phi_1 = v,$$

where

$$u = -\boldsymbol{\nu} \cdot \boldsymbol{u}^{\rm inc}, \quad v = -\boldsymbol{\tau} \cdot \boldsymbol{u}^{\rm inc}.$$

Using the transparent boundary conditions (3.4), we obtain a coupled boundary value problem for the scalar potentials ϕ_j:

$$\begin{cases} \Delta\phi_j + \kappa_j^2 \phi_j = 0 & \text{in } \Omega, \\ \partial_\nu \phi_1 - \partial_\tau \phi_2 = u & \text{on } \partial D, \\ \partial_\nu \phi_2 + \partial_\tau \phi_1 = v & \text{on } \partial D, \\ \partial_\rho \phi_j - \mathscr{T}_j \phi_j = 0 & \text{on } \partial B. \end{cases} \tag{3.13}$$

The weak formulation of (3.13) is to find $(\phi_1, \phi_2) \in H^1(\Omega)^2$ such that

$$a(\phi_1, \phi_2; \psi_1, \psi_2) = \langle u, \psi_1 \rangle_{\partial D} + \langle v, \psi_2 \rangle_{\partial D}, \quad \forall (\psi_1, \psi_2) \in H^1(\Omega)^2, \tag{3.14}$$

where the sesquilinear form $a : H^1(\Omega)^2 \times H^1(\Omega)^2 \to \mathbb{C}$ is defined by

$$\begin{aligned} a(\phi_1, \phi_2; \psi_1, \psi_2) = {} & (\nabla\phi_1, \nabla\psi_1) + (\nabla\phi_2, \nabla\psi_2) - \kappa_1^2(\phi_1, \psi_1) - \kappa_2^2(\phi_2, \psi_2) \\ & - \langle \partial_\tau \phi_2, \psi_1 \rangle_{\partial D} + \langle \partial_\tau \phi_1, \psi_2 \rangle_{\partial D} - \langle \mathscr{T}_1 \phi_1, \psi_1 \rangle_{\partial B} - \langle \mathscr{T}_2 \phi_2, \psi_2 \rangle_{\partial B}. \end{aligned}$$

Theorem 3.5. *The variational problem (3.14) has at most one solution.*

Proof. It suffices to show that $\phi_j = 0$ in Ω if $u = v = 0$. If (ϕ_1, ϕ_2) satisfy the homogeneous variational problem (3.14), then we have

$$\begin{aligned} & (\nabla\phi_1, \nabla\phi_1) + (\nabla\phi_2, \nabla\phi_2) - \kappa_1^2(\phi_1, \phi_1) - \kappa_2^2(\phi_2, \phi_2) \\ & - \langle \partial_\tau \phi_2, \phi_1 \rangle_{\partial D} + \langle \partial_\tau \phi_1, \phi_2 \rangle_{\partial D} - \langle \mathscr{T}_1 \phi_1, \phi_1 \rangle_{\partial B} - \langle \mathscr{T}_2 \phi_2, \phi_2 \rangle_{\partial B} = 0. \end{aligned} \tag{3.15}$$

Noting $\langle \partial_\tau \phi_1, \phi_2 \rangle_{\partial D} = -\langle \phi_1, \partial_\tau \phi_2 \rangle_{\partial D}$, we get

$$\langle \partial_\tau \phi_1, \phi_2 \rangle_{\partial D} - \langle \partial_\tau \phi_2, \phi_1 \rangle_{\partial D} = -2\,\mathrm{Re}\langle \partial_\tau \phi_2, \phi_1 \rangle_{\partial D}. \tag{3.16}$$

Taking the imaginary part of (3.15) and using (3.16), we obtain

$$\mathrm{Im}\langle \mathscr{T}_1 \phi_1, \phi_1 \rangle_{\partial B} + \mathrm{Im}\langle \mathscr{T}_2 \phi_2, \phi_2 \rangle_{\partial B} = 0,$$

which gives $\phi_j^{(n)} = 0$, $n \in \mathbb{Z}$, due to lemma 3.1. Thus we have $\phi_j = 0$ and $\partial_\rho \phi_j = 0$ on ∂B. By the Holmgren uniqueness theorem, we have $\phi_j = 0$ in $\mathbb{R}^2 \backslash \bar{B}$. A unique continuation result concludes that $\phi_j = 0$ in Ω. □

3.3. Well-posedness

Using the transparent boundary condition (3.12), we obtain a boundary value problem for the displacement of the total wave $\boldsymbol{u}$:

$$\begin{cases} \mu\Delta \boldsymbol{u} + (\lambda + \mu)\nabla\nabla \cdot \boldsymbol{u} + \omega^2 \boldsymbol{u} = 0 & \text{in } \Omega, \\ \boldsymbol{u} = 0 & \text{on } \partial D, \\ \mathscr{B}\boldsymbol{u} = \mathscr{T}\boldsymbol{u} + \boldsymbol{g} & \text{on } \partial B, \end{cases} \tag{3.17}$$

where $\boldsymbol{g} = (\mathscr{B} - \mathscr{T})\boldsymbol{u}^{\text{inc}}$. The variational problem of (3.17) is to find $\boldsymbol{u} \in H^1_{\partial D}(\Omega)^2$ such that

$$b(\boldsymbol{u}, \boldsymbol{v}) = g(\boldsymbol{v}), \quad \forall\, \boldsymbol{v} \in H^1_{\partial D}(\Omega)^2, \tag{3.18}$$

where the sesquilinear form $b : H^1_{\partial D}(\Omega)^2 \times H^1_{\partial D}(\Omega)^2 \to \mathbb{C}$ is defined by

$$\begin{aligned} b(\boldsymbol{u}, \boldsymbol{v}) \;=\; & \mu \int_\Omega \nabla \boldsymbol{u} : \nabla \bar{\boldsymbol{v}}\, \mathrm{d}\boldsymbol{x} + (\lambda + \mu) \int_\Omega (\nabla \cdot \boldsymbol{u})(\nabla \cdot \bar{\boldsymbol{v}}) \mathrm{d}\boldsymbol{x} \\ & - \omega^2 \int_\Omega \boldsymbol{u} \cdot \bar{\boldsymbol{v}}\, \mathrm{d}\boldsymbol{x} - \langle \mathscr{T}\boldsymbol{u}, \boldsymbol{v}\rangle_{\partial B} \end{aligned}$$

and the linear functional $g : H^1_{\partial D}(\Omega)^2 \to \mathbb{C}$ is defined by

$$g(\boldsymbol{v}) = \langle \boldsymbol{g}, \boldsymbol{v}\rangle_{\partial B}.$$

Here $A : B = \operatorname{tr}(AB^\top)$ is the Frobenius inner product of square matrices A, B.

It follows from the standard trace theorem of the Sobolev spaces that we have

Lemma 3.6. *It holds the estimate*

$$\|\boldsymbol{u}\|_{1/2,\partial B} \lesssim \|\boldsymbol{u}\|_{1,\Omega}, \quad \forall \boldsymbol{u} \in H^1_{\partial D}(\Omega)^2.$$

Lemma 3.7. *For any $\varepsilon > 0$, there exists a positive constant $C(\varepsilon)$ such that*

$$\|\boldsymbol{u}\|_{0,\partial B} \leqslant \varepsilon \|\boldsymbol{u}\|_{1,\Omega} + C(\varepsilon)\|\boldsymbol{u}\|_{0,\Omega}, \quad \forall\, \boldsymbol{u} \in H^1_{\partial D}(\Omega)^2.$$

Proof. Let $\tilde{\boldsymbol{u}}$ be the zero extension of $\boldsymbol{u}$ as defined in lemma 3.6. For any given $\varepsilon > 0$, it follows from Young's inequality that

$$\begin{aligned} |\tilde{\boldsymbol{u}}^{(n)}(R)|^2 = \int_{R'}^{R} \frac{\mathrm{d}}{\mathrm{d}\rho} |\tilde{\boldsymbol{u}}^{(n)}(\rho)|^2 \mathrm{d}\rho \leqslant \int_{R'}^{R} 2|\tilde{\boldsymbol{u}}^{(n)}(\rho)| \left| \frac{\mathrm{d}}{\mathrm{d}\rho} \tilde{\boldsymbol{u}}^{(n)}(\rho) \right| \mathrm{d}\rho \\ \leqslant (R'\varepsilon^2)^{-1} \int_{R'}^{R} |\tilde{\boldsymbol{u}}^{(n)}(\rho)|^2\, \mathrm{d}\rho + (R'\varepsilon^2) \int_{R'}^{R} \left| \frac{\mathrm{d}}{\mathrm{d}\rho} \tilde{\boldsymbol{u}}^{(n)}(\rho) \right|^2 \mathrm{d}\rho, \end{aligned}$$

which gives

$$|\tilde{\boldsymbol{u}}^{(n)}(R)|^2 \leqslant C(\varepsilon) \int_{R'}^{R} |\tilde{\boldsymbol{u}}^{(n)}(\rho)|^2 \rho\, \mathrm{d}\rho + \varepsilon^2 \int_{R'}^{R} \left| \frac{\mathrm{d}}{\mathrm{d}\rho} \tilde{\boldsymbol{u}}^{(n)}(\rho) \right|^2 \rho \mathrm{d}\rho.$$

The proof is completed by noting that $\|\tilde{\boldsymbol{u}}\|_{0,\partial B} = \|\boldsymbol{u}\|_{0,\partial B}$, $\|\tilde{\boldsymbol{u}}\|_{0,\Omega} = \|\boldsymbol{u}\|_{0,\Omega}$, and $\|\tilde{\boldsymbol{u}}\|_{1,\Omega} = \|\boldsymbol{u}\|_{1,\Omega}$. □

Lemma 3.8. *It holds the estimate*

$$\|\boldsymbol{u}\|_{1,\Omega} \lesssim \|\nabla\boldsymbol{u}\|_{0,\Omega}, \quad \forall\ \boldsymbol{u} \in H^1_{\partial D}(\Omega)^2.$$

Proof. Let $\tilde{\boldsymbol{u}}$ be the zero extension of $\boldsymbol{u}$ as defined in lemma 3.6. It follows from the Cauchy–Schwarz inequality that

$$|\tilde{\boldsymbol{u}}(\rho)|^2 = \left|\int_{R'}^{\rho} \partial_\rho \tilde{\boldsymbol{u}}\ \mathrm{d}\rho\right|^2 \lesssim \int_{R'}^{R} |\partial_\rho \tilde{\boldsymbol{u}}|^2\ \mathrm{d}\rho.$$

Hence we have

$$\begin{aligned}\|\tilde{\boldsymbol{u}}\|^2_{0,\tilde{\Omega}} &= \int_{R'}^{R}\int_0^{2\pi} |\tilde{\boldsymbol{u}}|^2 \rho\ \mathrm{d}\theta\mathrm{d}\rho \lesssim \int_{R'}^{R}\int_0^{2\pi}\int_{R'}^{R} |\partial_\rho\tilde{\boldsymbol{u}}|^2\ \mathrm{d}\rho\mathrm{d}\theta\mathrm{d}\rho\\ &\lesssim \int_0^{2\pi}\int_{R'}^{R} |\partial_\rho\tilde{\boldsymbol{u}}|^2\mathrm{d}\rho\mathrm{d}\theta \lesssim \|\nabla\tilde{\boldsymbol{u}}\|^2_{0,\tilde{\Omega}}.\end{aligned}$$

It is easy to note that

$$\|\boldsymbol{u}\|_{0,\Omega} = \|\tilde{\boldsymbol{u}}\|_{0,\tilde{\Omega}}, \quad \|\nabla\boldsymbol{u}\|_{0,\Omega} = \|\nabla\tilde{\boldsymbol{u}}\|_{0,\tilde{\Omega}}, \quad \|\boldsymbol{u}\|^2_{1,\Omega} = \|\boldsymbol{u}\|^2_{0,\Omega} + \|\nabla\boldsymbol{u}\|^2_{0,\Omega},$$

which completes the proof. □

Theorem 3.9. *The linear functional g is bounded in $H^1_{\partial D}(\Omega)^2$.*

Proof. It is only needed to show that g is bounded in $H^1_{\partial D}(\Omega)^2$ for the compressional plane wave (2.2) since the proof is similar for the shear plane wave (2.3). In the polar coordinates, we have

$$\boldsymbol{u}^{\mathrm{inc}} = \boldsymbol{d}\mathrm{e}^{\mathrm{i}\kappa_1\boldsymbol{x}\cdot\boldsymbol{d}} = \mathrm{e}^{\mathrm{i}\kappa_1\rho\cos(\theta-\varphi)}(\cos(\theta-\varphi), -\sin(\theta-\varphi))^\top. \tag{3.19}$$

Following the Jacobi–Anger expansion

$$\mathrm{e}^{\mathrm{i}\kappa_1\rho\cos(\theta-\varphi)} = \sum_{n\in\mathbb{Z}} \mathrm{i}^n J_n(\kappa_1\rho)\mathrm{e}^{\mathrm{i}n(\theta-\varphi)}, \tag{3.20}$$

we get

$$\begin{aligned}\mathrm{e}^{\mathrm{i}\kappa_1\rho\cos(\theta-\varphi)}\cos(\theta-\varphi) &= \frac{1}{2}\mathrm{e}^{\mathrm{i}\kappa_1\rho\cos(\theta-\varphi)}(\mathrm{e}^{\mathrm{i}(\theta-\varphi)} + \mathrm{e}^{-\mathrm{i}(\theta-\varphi)})\\ &= \sum_{n\in\mathbb{Z}} \frac{\mathrm{i}^{n-1}}{2}(J_{n-1}(\kappa_1\rho) - J_{n+1}(\kappa_1\rho))\mathrm{e}^{\mathrm{i}n(\theta-\varphi)}\\ &= \sum_{n\in\mathbb{Z}} \mathrm{i}^{n-1}J_n'(\kappa_1\rho)\mathrm{e}^{-\mathrm{i}n\varphi}\mathrm{e}^{\mathrm{i}n\theta}\end{aligned} \tag{3.21}$$

and

$$\begin{aligned}\mathrm{e}^{\mathrm{i}\kappa_1\rho\cos(\theta-\varphi)}\sin(\theta-\varphi) &= \frac{1}{2\mathrm{i}}\mathrm{e}^{\mathrm{i}\kappa_1\rho\cos(\theta-\varphi)}(\mathrm{e}^{\mathrm{i}(\theta-\varphi)} - \mathrm{e}^{-\mathrm{i}(\theta-\varphi)})\\ &= -\sum_{n\in\mathbb{Z}} \frac{\mathrm{i}^n}{2}(J_{n-1}(\kappa_1\rho) + J_{n+1}(\kappa_1\rho))\mathrm{e}^{\mathrm{i}n(\theta-\varphi)}\\ &= -\sum_{n\in\mathbb{Z}} \mathrm{i}^n\left(\frac{n}{\kappa_1\rho}\right)J_n(\kappa_1\rho)\mathrm{e}^{-\mathrm{i}n\varphi}\mathrm{e}^{\mathrm{i}n\theta}.\end{aligned} \tag{3.22}$$

Substituting (3.21) and (3.22) into (3.19) yields

$$(\boldsymbol{u}^{\mathrm{inc}})^{(n)}(R) = \mathrm{e}^{-\mathrm{i}n\varphi}\left(\mathrm{i}^{n-1}J_n'(\kappa_1 R),\ -\mathrm{i}^n\left(\frac{n}{\kappa_1 R}\right)J_n(\kappa_1 R)\right)^\top.$$

It is easy to verify from (3.12) that $\|\mathscr{T}\boldsymbol{u}^{\mathrm{inc}}\|_{-1/2,\partial B} < \infty$.

On the other hand, a simple calculation yields

$$\begin{aligned}\mathscr{B}\boldsymbol{u}^{\mathrm{inc}} &= \mu\partial_\rho\boldsymbol{u}^{\mathrm{inc}} + (\lambda+\mu)(\nabla\cdot\boldsymbol{u}^{\mathrm{inc}})\boldsymbol{e}_\rho\\ &= \mathrm{i}\kappa_1\mathrm{e}^{\mathrm{i}\kappa_1 R\cos(\theta-\varphi)}((\lambda+\mu)+\mu\cos^2(\theta-\varphi),\ -\mu\cos(\theta-\varphi)\sin(\theta-\varphi))^\top.\end{aligned} \tag{3.23}$$

Substituting (3.20)–(3.22) into (3.23), we may also verify that $\|\mathscr{B}\boldsymbol{u}^{\mathrm{inc}}\|_{-1/2,\partial B} < \infty$. Hence $\|\boldsymbol{g}\|_{-1/2,\partial B} < \infty$. It follows from lemma 3.6 that

$$|g(\boldsymbol{v})| = |\langle\boldsymbol{g}, \boldsymbol{v}\rangle_{\partial B}| \leqslant \|\boldsymbol{g}\|_{-1/2,\partial B}\|\boldsymbol{v}\|_{1/2,\partial B} \lesssim \|\boldsymbol{v}\|_{1,\Omega}, \quad \forall\ \boldsymbol{v}\in H^1_{\partial D}(\Omega)^2,$$

which completes the proof. □

Theorem 3.10. *The variational problems admits a unique weak solution* $\boldsymbol{u}\in H^1_{\partial D}(\Omega)^2$.

Proof. Using the Cauchy–Schwarz inequality, lemma 3.4, and lemma 3.6, we have

$$\begin{aligned}|b(\boldsymbol{u},\boldsymbol{v})| &\leqslant \mu\|\nabla\boldsymbol{u}\|_{0,\Omega}\|\nabla\boldsymbol{v}\|_{0,\Omega} + (\lambda+\mu)\|\nabla\cdot\boldsymbol{u}\|_{0,\Omega}\|\nabla\cdot\boldsymbol{v}\|_{0,\Omega} + \omega^2\|\boldsymbol{u}\|_{0,\Omega}\|\boldsymbol{v}\|_{0,\Omega}\\ &\quad + \|\mathscr{T}\boldsymbol{u}\|_{-1/2,\partial B}\|\boldsymbol{v}\|_{1/2,\partial B}\\ &\lesssim \|\boldsymbol{u}\|_{1,\Omega}\|\boldsymbol{v}\|_{1,\Omega},\end{aligned}$$

which shows that the sesquilinear form $b(\cdot,\cdot)$ is bounded.

It follows from lemma 3.3 that there exists an $N_0\in\mathbb{N}$ such that $\hat{M}_n$ is positive definite for $|n| > N_0$. The sesquilinear form b can be written as

$$\begin{aligned}b(\boldsymbol{u},\boldsymbol{v}) = \mu\int_\Omega(\nabla\boldsymbol{u} : \nabla\bar{\boldsymbol{v}})\mathrm{d}\boldsymbol{x} + (\lambda+\mu)\int_\Omega(\nabla\cdot\boldsymbol{u})(\nabla\cdot\bar{\boldsymbol{v}})\,\mathrm{d}\boldsymbol{x} - \sum_{|n|>N_0}\langle M_n\boldsymbol{u}^{(n)}, \boldsymbol{v}^{(n)}\rangle\\ - \omega^2\int_\Omega\boldsymbol{u}\cdot\bar{\boldsymbol{v}}\,\mathrm{d}\boldsymbol{x} - \sum_{|n|\leqslant N_0}\langle M_n\boldsymbol{u}^{(n)}, \boldsymbol{v}^{(n)}\rangle.\end{aligned}$$

Taking the real part of b, and using lemma 3.3, 3.8, 3.7, we obtain

$$\begin{aligned}\mathrm{Re}\,b(\boldsymbol{u},\boldsymbol{u}) &= \mu\|\nabla\boldsymbol{u}\|^2_{0,\Omega} + (\lambda+\mu)\|\nabla\cdot\boldsymbol{u}\|^2_{0,\Omega} + \sum_{|n|>N_0}\langle\hat{M}_n\boldsymbol{u}^{(n)}, \boldsymbol{u}^{(n)}\rangle\\ &\quad - \omega^2\|\boldsymbol{u}\|_{0,\Omega} + \sum_{|n|\leqslant N_0}\langle\hat{M}_n\boldsymbol{u}^{(n)}, \boldsymbol{u}^{(n)}\rangle\\ &\geqslant C_1\|\boldsymbol{u}\|_{1,\Omega} - \omega^2\|\boldsymbol{u}\|_{0,\Omega} - C_2\|\boldsymbol{u}\|_{0,\partial B}\\ &\geqslant C_1\|\boldsymbol{u}\|_{1,\Omega} - \omega^2\|\boldsymbol{u}\|_{0,\Omega} - C_2\varepsilon\|\boldsymbol{u}\|_{1,\Omega} - C(\varepsilon)\|\boldsymbol{u}\|_{0,\Omega}\\ &= (C_1 - C_2\varepsilon)\|\boldsymbol{u}\|_{1,\Omega} - C_3\|\boldsymbol{u}\|_{0,\Omega}.\end{aligned}$$

Letting $\varepsilon > 0$ to be sufficiently small, we have $C_1 - C_2\varepsilon > 0$ and thus Gårding's inequality. Since the injection of $H^1_{\partial D}(\Omega)^2$ into $L^2(\Omega)^2$ is compact, the proof is completed by using the Fredholm alternative (see [28, theorem 5.4.5]) and the uniqueness result in theorem 3.5. □

4. Inverse scattering

In this section, we study the domain derivative and propose a continuation method for the inverse scattering problem.

4.1. Domain derivative

Given $h > 0$, introduce a domain Ω_h bounded by ∂D_h and ∂B, where

$$\partial D_h = \{\boldsymbol{x} + h\boldsymbol{p}(\boldsymbol{x}) : \boldsymbol{x} \in \partial D\},$$

where the obstacle's surface ∂D is assumed to be in C^2 and the function $\boldsymbol{p} \in C^2(\partial D)$. It is clear to note that $\partial D_h \in C^2$ is a perturbation of ∂D for sufficiently small h.

Consider the variational problem in the perturbed domain Ω_h: Find $\boldsymbol{u}_h \in H^1_{\partial D_h}(\Omega_h)^2$ such that

$$b^h(\boldsymbol{u}_h, \boldsymbol{v}_h) = \langle \boldsymbol{g}, \boldsymbol{v}_h \rangle_{\partial B}, \quad \forall \boldsymbol{v}_h \in H^1_{\partial D_h}(\Omega_h)^2, \tag{4.1}$$

where the sesquilinear form $b^h : H^1_{\partial D_h}(\Omega_h)^2 \times H^1_{\partial D_h}(\Omega_h)^2 \to \mathbb{C}$ is defined by

$$\begin{aligned} b^h(\boldsymbol{u}_h, \boldsymbol{v}_h) = \mu \int_{\Omega_h} \nabla \boldsymbol{u}_h : \nabla \bar{\boldsymbol{v}}_h \, \mathrm{d}\boldsymbol{y} + (\lambda + \mu) \int_{\Omega_h} (\nabla \cdot \boldsymbol{u}_h)(\nabla \cdot \bar{\boldsymbol{v}}_h) \mathrm{d}\boldsymbol{y} \\ - \omega^2 \int_{\Omega_h} \boldsymbol{u}_h \cdot \bar{\boldsymbol{v}}_h \, \mathrm{d}\boldsymbol{y} - \langle \mathscr{T} \boldsymbol{u}_h, \boldsymbol{v}_h \rangle_{\partial B}. \end{aligned} \tag{4.2}$$

We may repeat the proofs in section 3 to show that the variational problem (4.1) has a unique weak solution $\boldsymbol{u}_h \in H^1_{\partial D_h}(\Omega_h)^2$ for any h.

Define a nonlinear scattering operator:

$$\mathscr{S} : \partial D_h \to \gamma \boldsymbol{u}_h,$$

where γ is the trace operator onto ∂B. The domain derivative of the operator $\mathscr{S}$ on the boundary ∂D along with the direction $\boldsymbol{p}$ is defined by

$$\mathscr{S}'(\partial D; \boldsymbol{p}) := \lim_{h \to 0} \frac{\mathscr{S}(\partial D_h) - \mathscr{S}(\partial D)}{h} = \lim_{h \to 0} \frac{\gamma \boldsymbol{u}_h - \gamma \boldsymbol{u}}{h}.$$

For a given $\boldsymbol{p} \in C^2(\partial D)$, we extend its domain to $\bar{\Omega}$ by requiring that $\boldsymbol{p} \in C^2(\Omega) \cap C(\bar{\Omega})$, $\boldsymbol{p} = 0$ on ∂B, and $\boldsymbol{y} = \xi^h(\boldsymbol{x}) = \boldsymbol{x} + h\boldsymbol{p}(\boldsymbol{x})$ maps Ω to Ω_h. It is clear to note that ξ^h is a diffeomorphism from Ω to Ω_h for sufficiently small h. Denote by $\eta^h(\boldsymbol{y}) : \Omega_h \to \Omega$ the inverse map of ξ^h.

Define $\breve{\boldsymbol{u}}(\boldsymbol{x}) = (\breve{u}_1, \breve{u}_2)^\top := (\boldsymbol{u}_h \circ \xi^h)(\boldsymbol{x})$. It follows from the change of variable $\boldsymbol{y} = \xi^h(\boldsymbol{x})$ that we have

$$\begin{aligned} \int_{\Omega_h} (\nabla \boldsymbol{u}_h : \nabla \bar{\boldsymbol{v}}_h) \mathrm{d}\boldsymbol{y} &= \sum_{j=1}^{2} \int_{\Omega} \nabla \breve{u}_j J_{\eta^h} J_{\eta^h}^\top \nabla \bar{\breve{v}}_j \det(J_{\xi^h}) \mathrm{d}\boldsymbol{x}, \\ \int_{\Omega_h} (\nabla \cdot \boldsymbol{u}_h)(\nabla \cdot \bar{\boldsymbol{v}}_h) \mathrm{d}\boldsymbol{y} &= \int_{\Omega} (\nabla \breve{\boldsymbol{u}} : J_{\eta^h}^\top)(\nabla \bar{\breve{\boldsymbol{v}}} : J_{\eta^h}^\top) \det(J_{\xi^h}) \mathrm{d}\boldsymbol{x}, \\ \int_{\Omega_h} \boldsymbol{u}_h \cdot \bar{\boldsymbol{v}}_h \, \mathrm{d}\boldsymbol{y} &= \int_{\Omega} \breve{\boldsymbol{u}} \cdot \bar{\breve{\boldsymbol{v}}} \det(J_{\xi^h}) \mathrm{d}\boldsymbol{x}, \end{aligned}$$

where $\breve{\boldsymbol{v}}(\boldsymbol{x}) = (\breve{v}_1, \breve{v}_2)^\top := (\boldsymbol{v}_h \circ \xi^h)(\boldsymbol{x})$, J_{η^h} and J_{ξ^h} are the Jacobian matrices of the transforms η^h and ξ^h, respectively.

For an arbitrary test function $\boldsymbol{v}_h$ in the domain Ω_h, it is easy to note that $\breve{\boldsymbol{v}}$ is a test function in the domain Ω according to the transform. Therefore, the sesquilinear form b^h in (4.2) becomes

$$b^h(\breve{\boldsymbol{u}}, \boldsymbol{v}) = \sum_{j=1}^{2}\mu\int_{\Omega} \nabla\breve{u}_j J_{\eta^h}J_{\eta^h}^{\top}\nabla\bar{v}_j \det(J_{\xi^h})\mathrm{d}\boldsymbol{x} + (\lambda+\mu)\int_{\Omega}(\nabla\breve{\boldsymbol{u}} : J_{\eta^h}^{\top}) \times (\nabla\bar{\boldsymbol{v}} : J_{\eta^h}^{\top})\det(J_{\xi^h})\mathrm{d}\boldsymbol{x} - \omega^2\int_{\Omega}\breve{\boldsymbol{u}}\cdot\bar{\boldsymbol{v}}\det(J_{\xi^h})\mathrm{d}\boldsymbol{x} - \langle\mathscr{T}\breve{\boldsymbol{u}}, \boldsymbol{v}\rangle_{\partial B},$$

which gives an equivalent variational formulation to (4.1):

$$b^h(\breve{\boldsymbol{u}}, \boldsymbol{v}) = \langle\boldsymbol{g}, \boldsymbol{v}\rangle_{\partial B}, \quad \forall\,\boldsymbol{v}\in H^1_{\partial D}(\Omega)^2. \tag{4.3}$$

A simple calculation yields

$$b(\breve{\boldsymbol{u}} - \boldsymbol{u}, \boldsymbol{v}) = b(\breve{\boldsymbol{u}}, \boldsymbol{v}) - \langle\boldsymbol{g}, \boldsymbol{v}\rangle_{\partial B} = b(\breve{\boldsymbol{u}}, \boldsymbol{v}) - b^h(\breve{\boldsymbol{u}}, \boldsymbol{v}) = b_1 + b_2 + b_3,$$

where

$$b_1 = \sum_{j=1}^{2}\mu\int_{\Omega}\nabla\breve{u}_j(I - J_{\eta^h}J_{\eta^h}^{\top}\det(J_{\xi^h}))\nabla\bar{v}_j\,\mathrm{d}\boldsymbol{x}, \tag{4.4}$$

$$b_2 = (\lambda+\mu)\int_{\Omega}(\nabla\cdot\breve{\boldsymbol{u}})(\nabla\cdot\bar{\boldsymbol{v}}) - (\nabla\breve{\boldsymbol{u}} : J_{\eta^h}^{\top})(\nabla\bar{\boldsymbol{v}} : J_{\eta^h}^{\top})\det(J_{\xi^h})\mathrm{d}\boldsymbol{x}, \tag{4.5}$$

$$b_3 = \omega^2\int_{\Omega}\breve{\boldsymbol{u}}\cdot\bar{\boldsymbol{v}}(\det(J_{\xi^h}) - 1)\mathrm{d}\boldsymbol{x}. \tag{4.6}$$

Here I is the identity matrix. Following the definitions of the Jacobian matrices, we may easily verify that

$$\begin{aligned} \det(J_{\xi^h}) &= 1 + h\nabla\cdot\boldsymbol{p} + O(h^2),\\ J_{\eta^h} &= J_{\xi^h}^{-1}\circ\eta^h = I - hJ_{\boldsymbol{p}} + O(h^2),\\ J_{\eta^h}J_{\eta^h}^{\top}\det(J_{\xi^h}) &= I - h(J_{\boldsymbol{p}} + J_{\boldsymbol{p}}^{\top}) + h(\nabla\cdot\boldsymbol{p})I + O(h^2), \end{aligned}$$

where the matrix $J_{\boldsymbol{p}} = \nabla\boldsymbol{p}$. Substituting the above estimates into (4.4)–(4.6), we obtain

$$\begin{aligned} b_1 &= \sum_{j=1}^{2}\mu\int_{\Omega}\nabla\breve{u}_j(h(J_{\boldsymbol{p}} + J_{\boldsymbol{p}}^{\top}) - h(\nabla\cdot\boldsymbol{p})I + O(h^2))\nabla\bar{v}_j\,\mathrm{d}\boldsymbol{x},\\ b_2 &= (\lambda+\mu)\int_{\Omega}h(\nabla\cdot\breve{\boldsymbol{u}})(\nabla\bar{\boldsymbol{v}} : J_{\boldsymbol{p}}^{\top}) + h(\nabla\cdot\bar{\boldsymbol{v}})(\nabla\breve{\boldsymbol{u}} : J_{\boldsymbol{p}}^{\top})\\ &\quad - h(\nabla\cdot\boldsymbol{p})(\nabla\cdot\breve{\boldsymbol{u}})(\nabla\cdot\bar{\boldsymbol{v}}) + O(h^2)\mathrm{d}\boldsymbol{x},\\ b_3 &= \omega^2\int_{\Omega}\breve{\boldsymbol{u}}\cdot\bar{\boldsymbol{v}}(h\nabla\cdot\boldsymbol{p} + O(h^2))\mathrm{d}\boldsymbol{x}. \end{aligned}$$

Hence we have

$$b\left(\frac{\breve{\boldsymbol{u}} - \boldsymbol{u}}{h}, \boldsymbol{v}\right) = g_1(\boldsymbol{p})(\breve{\boldsymbol{u}}, \boldsymbol{v}) + g_2(\boldsymbol{p})(\breve{\boldsymbol{u}}, \boldsymbol{v}) + g_3(\boldsymbol{p})(\breve{\boldsymbol{u}}, \boldsymbol{v}) + O(h), \tag{4.7}$$

where

$$g_1 = \sum_{j=1}^{2} \mu \int_{\Omega} \nabla \breve{u}_j ((J_{\boldsymbol{p}} + J_{\boldsymbol{p}}^{\top}) - (\nabla \cdot \boldsymbol{p}) I) \nabla \bar{v}_j \,\mathrm{d}\boldsymbol{x},$$

$$g_2 = (\lambda + \mu) \int_{\Omega} (\nabla \cdot \breve{\boldsymbol{u}})(\nabla \bar{\boldsymbol{v}} : J_{\boldsymbol{p}}^{\top}) + (\nabla \cdot \bar{\boldsymbol{v}})(\nabla \breve{\boldsymbol{u}} : J_{\boldsymbol{p}}^{\top}) - (\nabla \cdot \boldsymbol{p})(\nabla \cdot \breve{\boldsymbol{u}})(\nabla \cdot \bar{\boldsymbol{v}}) \mathrm{d}\boldsymbol{x},$$

$$g_3 = \omega^2 \int_{\Omega} (\nabla \cdot \boldsymbol{p}) \breve{\boldsymbol{u}} \cdot \bar{\boldsymbol{v}} \,\mathrm{d}\boldsymbol{x}.$$

Theorem 4.1. *Let $\boldsymbol{u}$ be the solution of the variational problem (3.18). Given $\boldsymbol{p} \in C^2(\partial D)$, the domain derivative of the scattering operator $\mathscr{S}$ is $\mathscr{S}'(\partial D; \boldsymbol{p}) = \gamma \boldsymbol{u}'$, where $\boldsymbol{u}'$ is the unique weak solution of the boundary value problem:*

$$\begin{cases} \mu \Delta \boldsymbol{u}' + (\lambda + \mu) \nabla \nabla \cdot \boldsymbol{u}' + \omega^2 \boldsymbol{u}' = 0 & \text{in } \Omega, \\ \boldsymbol{u}' = -(\boldsymbol{p} \cdot \boldsymbol{\nu}) \partial_{\boldsymbol{\nu}} \boldsymbol{u} & \text{on } \partial D, \\ \mathscr{B} \boldsymbol{u}' = \mathscr{T} \boldsymbol{u}' & \text{on } \partial B. \end{cases} \tag{4.8}$$

Proof. Given $\boldsymbol{p} \in C^2(\partial D)$, we extend it to $\bar{\Omega}$ as before. It follows from the well-posedness of the variational problem (3.18) that $\breve{\boldsymbol{u}} \to \boldsymbol{u}$ in $H^1_{\partial D}(\Omega)^2$ as $h \to 0$. Taking the limit $h \to 0$ in (4.7) gives

$$b\left(\lim_{h \to 0} \frac{\breve{\boldsymbol{u}} - \boldsymbol{u}}{h}, \boldsymbol{v} \right) = g_1(\boldsymbol{p})(\boldsymbol{u}, \boldsymbol{v}) + g_2(\boldsymbol{p})(\boldsymbol{u}, \boldsymbol{v}) + g_3(\boldsymbol{p})(\boldsymbol{u}, \boldsymbol{v}), \tag{4.9}$$

which shows that $(\breve{\boldsymbol{u}} - \boldsymbol{u})/h$ is convergent in $H^1_{\partial D}(\Omega)^2$ as $h \to 0$. Denote by $\dot{\boldsymbol{u}}$ this limit and rewrite (4.9) as

$$b(\dot{\boldsymbol{u}}, \boldsymbol{v}) = g_1(\boldsymbol{p})(\boldsymbol{u}, \boldsymbol{v}) + g_2(\boldsymbol{p})(\boldsymbol{u}, \boldsymbol{v}) + g_3(\boldsymbol{p})(\boldsymbol{u}, \boldsymbol{v}). \tag{4.10}$$

First we compute $g_1(\boldsymbol{p})(\boldsymbol{u}, \boldsymbol{v})$. Noting $\boldsymbol{p} = 0$ on ∂B and using the identity

$$\begin{aligned} \nabla u((J_{\boldsymbol{p}} + J_{\boldsymbol{p}}^{\top}) - (\nabla \cdot \boldsymbol{p}) I) \nabla \bar{v} = \nabla \cdot [(\boldsymbol{p} \cdot \nabla u) \nabla \bar{v} + (\boldsymbol{p} \cdot \nabla \bar{v}) \nabla u - (\nabla u \cdot \nabla \bar{v}) \boldsymbol{p}] \\ - (\boldsymbol{p} \cdot \nabla u) \Delta \bar{v} - (\boldsymbol{p} \cdot \nabla \bar{v}) \Delta u, \end{aligned}$$

we obtain from the divergence theorem that

$$\begin{aligned} g_1(\boldsymbol{p})(\boldsymbol{u}, \boldsymbol{v}) &= -\sum_{j=1}^{2} \mu \int_{\Omega} (\boldsymbol{p} \cdot \nabla u_j) \Delta \bar{v}_j + (\boldsymbol{p} \cdot \nabla \bar{v}_j) \Delta u_j \,\mathrm{d}\boldsymbol{x} \\ &\quad - \sum_{j=1}^{2} \mu \int_{\partial D} (\boldsymbol{p} \cdot \nabla u_j)(\boldsymbol{\nu} \cdot \nabla \bar{v}_j) + (\boldsymbol{p} \cdot \nabla \bar{v}_j)(\boldsymbol{\nu} \cdot \nabla u_j) - (\boldsymbol{p} \cdot \boldsymbol{\nu})(\nabla u_j \cdot \nabla \bar{v}_j) \mathrm{d}s \\ &= -\mu \int_{\Omega} (\boldsymbol{p} \cdot \nabla \boldsymbol{u}) \cdot \Delta \bar{\boldsymbol{v}} + (\boldsymbol{p} \cdot \nabla \bar{\boldsymbol{v}}) \cdot \Delta \boldsymbol{u} \,\mathrm{d}\boldsymbol{x} \\ &\quad - \mu \int_{\partial D} (\boldsymbol{p} \cdot \nabla \boldsymbol{u}) \cdot (\boldsymbol{\nu} \cdot \nabla \bar{\boldsymbol{v}}) + (\boldsymbol{p} \cdot \nabla \bar{\boldsymbol{v}}) \cdot (\boldsymbol{\nu} \cdot \nabla \boldsymbol{u}) - (\boldsymbol{p} \cdot \boldsymbol{\nu})(\nabla \boldsymbol{u} : \nabla \bar{\boldsymbol{v}}) \mathrm{d}s. \end{aligned}$$

Since $\mu\Delta \boldsymbol{u} + (\lambda + \mu)\nabla\nabla \cdot \boldsymbol{u} + \omega^2 \boldsymbol{u} = 0$ in Ω, we have from the integration by parts that

$$\begin{aligned}\mu \int_\Omega (\boldsymbol{p} \cdot \nabla\bar{\boldsymbol{v}}) \cdot \Delta \boldsymbol{u} \, \mathrm{d}\boldsymbol{x} &= -(\lambda + \mu) \int_\Omega (\boldsymbol{p} \cdot \nabla\bar{\boldsymbol{v}}) \cdot (\nabla\nabla \cdot \boldsymbol{u}) \mathrm{d}\boldsymbol{x} - \omega^2 \int_\Omega (\boldsymbol{p} \cdot \nabla\bar{\boldsymbol{v}}) \cdot \boldsymbol{u} \, \mathrm{d}\boldsymbol{x} \\ &= (\lambda + \mu) \int_\Omega (\nabla \cdot \boldsymbol{u}) \nabla \cdot (\boldsymbol{p} \cdot \nabla\bar{\boldsymbol{v}}) \mathrm{d}\boldsymbol{x} \\ &\quad + (\lambda + \mu) \int_{\partial D} (\nabla \cdot \boldsymbol{u})(\boldsymbol{\nu} \cdot (\boldsymbol{p} \cdot \nabla\bar{\boldsymbol{v}})) \mathrm{d}s \\ &\quad - \omega^2 \int_\Omega (\boldsymbol{p} \cdot \nabla\bar{\boldsymbol{v}}) \cdot \boldsymbol{u} \, \mathrm{d}\boldsymbol{x}.\end{aligned}$$

Using the integration by parts again yields

$$\mu \int_\Omega (\boldsymbol{p} \cdot \nabla \boldsymbol{u}) \cdot \Delta\bar{\boldsymbol{v}} \, \mathrm{d}\boldsymbol{x} = -\mu \int_\Omega \nabla(\boldsymbol{p} \cdot \nabla \boldsymbol{u}) : \nabla\bar{\boldsymbol{v}} \, \mathrm{d}\boldsymbol{x} - \mu \int_{\partial D} (\boldsymbol{p} \cdot \nabla \boldsymbol{u}) \cdot (\boldsymbol{\nu} \cdot \nabla\bar{\boldsymbol{v}}) \mathrm{d}s.$$

Recall that $\boldsymbol{v} = 0$ on ∂D. We have $\partial_{\boldsymbol{\tau}} \boldsymbol{v} = 0$, which implies that

$$\nu_2 \partial_{x_1} v_1 = \nu_1 \partial_{x_2} v_1, \quad \nu_2 \partial_{x_1} v_2 = \nu_1 \partial_{x_2} v_2. \tag{4.11}$$

Using (4.11), we can verify that

$$\int_{\partial D} (\boldsymbol{p} \cdot \nabla\bar{\boldsymbol{v}}) \cdot (\boldsymbol{\nu} \cdot \nabla \boldsymbol{u}) - (\boldsymbol{p} \cdot \boldsymbol{\nu})(\nabla \boldsymbol{u} : \nabla\bar{\boldsymbol{v}}) \mathrm{d}s = 0.$$

Noting $\boldsymbol{v} = 0$ on ∂D and

$$(\nabla \cdot \boldsymbol{p})(\boldsymbol{u} \cdot \bar{\boldsymbol{v}}) + (\boldsymbol{p} \cdot \nabla\bar{\boldsymbol{v}}) \cdot \boldsymbol{u} = \nabla \cdot ((\boldsymbol{u} \cdot \bar{\boldsymbol{v}})\boldsymbol{p}) - (\boldsymbol{p} \cdot \nabla \boldsymbol{u}) \cdot \bar{\boldsymbol{v}},$$

we obtain by the divergence theorem that

$$\int_\Omega (\nabla \cdot \boldsymbol{p})(\boldsymbol{u} \cdot \bar{\boldsymbol{v}}) + (\boldsymbol{p} \cdot \nabla\bar{\boldsymbol{v}}) \cdot \boldsymbol{u} \, \mathrm{d}\boldsymbol{x} = -\int_\Omega (\boldsymbol{p} \cdot \nabla \boldsymbol{u}) \cdot \bar{\boldsymbol{v}} \, \mathrm{d}\boldsymbol{x}.$$

Combining the above identities, we conclude that

$$\begin{aligned} g_1(\boldsymbol{p})(\boldsymbol{u}, \boldsymbol{v}) + g_3(\boldsymbol{p})(\boldsymbol{u}, \boldsymbol{v}) &= \mu \int_\Omega \nabla(\boldsymbol{p} \cdot \nabla \boldsymbol{u}) : \nabla\bar{\boldsymbol{v}} \, \mathrm{d}\boldsymbol{x} \\ &\quad - (\lambda + \mu) \int_\Omega (\nabla \cdot \boldsymbol{u}) \nabla \cdot (\boldsymbol{p} \cdot \nabla\bar{\boldsymbol{v}}) \mathrm{d}\boldsymbol{x} \\ &\quad - \omega^2 \int_\Omega (\boldsymbol{p} \cdot \nabla \boldsymbol{u}) \cdot \bar{\boldsymbol{v}} \, \mathrm{d}\boldsymbol{x} + (\lambda + \mu) \int_{\partial D} (\nabla \cdot \boldsymbol{u})(\boldsymbol{\nu} \cdot (\boldsymbol{p} \cdot \nabla\bar{\boldsymbol{v}})) \mathrm{d}s. \end{aligned} \tag{4.12}$$

Next we compute $g_2(\boldsymbol{p})(\boldsymbol{u}, \boldsymbol{v})$. It is easy to verify that

$$\begin{aligned} \int_\Omega (\nabla \cdot \boldsymbol{u})(\nabla\bar{\boldsymbol{v}} : J_{\boldsymbol{p}}^\top) + (\nabla \cdot \bar{\boldsymbol{v}})(\nabla \boldsymbol{u} : J_{\boldsymbol{p}}^\top) \mathrm{d}\boldsymbol{x} &= \int_\Omega (\nabla \cdot \boldsymbol{u}) \nabla \cdot (\boldsymbol{p} \cdot \nabla\bar{\boldsymbol{v}}) \mathrm{d}\boldsymbol{x} \\ &\quad - \int_\Omega (\nabla \cdot \boldsymbol{u})(\boldsymbol{p} \cdot (\nabla \cdot (\nabla\bar{\boldsymbol{v}})^\top)) \mathrm{d}\boldsymbol{x} + \int_\Omega (\nabla \cdot \bar{\boldsymbol{v}}) \nabla \cdot (\boldsymbol{p} \cdot \nabla \boldsymbol{u}) \mathrm{d}\boldsymbol{x} \\ &\quad - \int_\Omega (\nabla \cdot \bar{\boldsymbol{v}})(\boldsymbol{p} \cdot (\nabla \cdot (\nabla \boldsymbol{u})^\top)) \mathrm{d}\boldsymbol{x}. \end{aligned}$$

Using the integration by parts, we obtain

$$\begin{aligned}\int_\Omega (\nabla\cdot\boldsymbol{p})(\nabla\cdot\boldsymbol{u})(\nabla\cdot\bar{\boldsymbol{v}})\mathrm{d}\boldsymbol{x} &= -\int_\Omega \boldsymbol{p}\cdot\nabla((\nabla\cdot\boldsymbol{u})(\nabla\cdot\bar{\boldsymbol{v}}))\mathrm{d}\boldsymbol{x}\\ &\quad -\int_{\partial D}(\nabla\cdot\boldsymbol{u})(\nabla\cdot\bar{\boldsymbol{v}})(\boldsymbol{\nu}\cdot\boldsymbol{p})\mathrm{d}s\\ &= -\int_\Omega(\nabla\cdot\bar{\boldsymbol{v}})(\boldsymbol{p}\cdot(\nabla\cdot(\nabla\boldsymbol{u})^\top))\mathrm{d}\boldsymbol{x}\\ &\quad -\int_\Omega(\nabla\cdot\boldsymbol{u})(\boldsymbol{p}\cdot(\nabla\cdot(\nabla\bar{\boldsymbol{v}})^\top))\mathrm{d}\boldsymbol{x}\\ &\quad -\int_{\partial D}(\nabla\cdot\boldsymbol{u})(\nabla\cdot\bar{\boldsymbol{v}})(\boldsymbol{\nu}\cdot\boldsymbol{p})\mathrm{d}s.\end{aligned}$$

Using (4.11) again, we get

$$\int_{\partial D}(\nabla\cdot\boldsymbol{u})(\nabla\cdot\bar{\boldsymbol{v}})(\boldsymbol{\nu}\cdot\boldsymbol{p})\mathrm{d}s = \int_{\partial D}(\nabla\cdot\boldsymbol{u})(\boldsymbol{\nu}\cdot(\boldsymbol{p}\cdot\nabla\bar{\boldsymbol{v}}))\mathrm{d}s.$$

Combining the above identities gives

$$\begin{aligned} g_2(\boldsymbol{p})(\boldsymbol{u},\boldsymbol{v}) &= (\lambda+\mu)\int_\Omega(\nabla\cdot\boldsymbol{u})\nabla\cdot(\boldsymbol{p}\cdot\nabla\bar{\boldsymbol{v}})\mathrm{d}\boldsymbol{x} + (\lambda+\mu)\int_\Omega\nabla\cdot(\boldsymbol{p}\cdot\nabla\boldsymbol{u})(\nabla\cdot\bar{\boldsymbol{v}})\mathrm{d}\boldsymbol{x}\\ &\quad -(\lambda+\mu)\int_{\partial D}(\nabla\cdot\boldsymbol{u})(\boldsymbol{\nu}\cdot(\boldsymbol{p}\cdot\nabla\bar{\boldsymbol{v}}))\mathrm{d}s. \end{aligned}\tag{4.13}$$

Noting (4.10), and adding (4.12) and (4.13), we obtain

$$\begin{aligned} b(\dot{\boldsymbol{u}},\boldsymbol{v}) &= \mu\int_\Omega\nabla(\boldsymbol{p}\cdot\nabla\boldsymbol{u}):\nabla\bar{\boldsymbol{v}}\,\mathrm{d}\boldsymbol{x} + (\lambda+\mu)\\ &\quad\int_\Omega\nabla\cdot(\boldsymbol{p}\cdot\nabla\boldsymbol{u})(\nabla\cdot\bar{\boldsymbol{v}})\mathrm{d}\boldsymbol{x} - \omega^2\int_\Omega(\boldsymbol{p}\cdot\nabla\boldsymbol{u})\cdot\bar{\boldsymbol{v}}\,\mathrm{d}\boldsymbol{x}. \end{aligned}$$

Define $\boldsymbol{u}' = \dot{\boldsymbol{u}} - \boldsymbol{p}\cdot\nabla\boldsymbol{u}$. It is clear to note that $\boldsymbol{p}\cdot\nabla\boldsymbol{u} = 0$ on ∂B since $\boldsymbol{p} = 0$ on ∂B. Hence, we have

$$b(\boldsymbol{u}',\boldsymbol{v}) = 0, \quad \forall\ \boldsymbol{v}\in H^1_{\partial D}(\Omega)^2, \tag{4.14}$$

which shows that $\boldsymbol{u}'$ is the weak solution of the boundary value problem (4.8). To verify that boundary condition of $\boldsymbol{u}'$ on ∂B, we recall the definition of $\boldsymbol{u}'$ and have

$$\boldsymbol{u}' = \lim_{h\to 0}\frac{\breve{\boldsymbol{u}}-\boldsymbol{u}}{h} - \boldsymbol{p}\cdot\nabla\boldsymbol{u} = -\boldsymbol{p}\cdot\nabla\boldsymbol{u} \quad \text{on } \partial D,$$

since $\breve{\boldsymbol{u}} = \boldsymbol{u} = 0$ on ∂D. Noting $\boldsymbol{u} = 0$ and thus $\partial_\tau\boldsymbol{u} = 0$ on ∂D, we have

$$\boldsymbol{p}\cdot\nabla\boldsymbol{u} = (\boldsymbol{p}\cdot\boldsymbol{\nu})\partial_\nu\boldsymbol{u} + (\boldsymbol{p}\cdot\boldsymbol{\tau})\partial_\tau\boldsymbol{u} = (\boldsymbol{p}\cdot\boldsymbol{\nu})\partial_\nu\boldsymbol{u}, \tag{4.15}$$

which completes the proof by combining (4.14) and (4.15). □

4.2. Reconstruction method

As a closed curve, the obstacle's surface has a parametric equation

$$\partial D = \{\boldsymbol{r}\in\mathbb{R}^2 : \boldsymbol{r}(t) = (r_1(t), r_2(t))^\top,\ t\in[0, 2\pi]\},$$

where r_1, r_2 are twice continuously differentiable and 2π-periodic functions and they can be represented as Fourier series

$$r_j(t) = r_j^{(0)} + \sum_{n=1}^{\infty}(r_j^{(2n-1)}\cos(nt) + r_j^{(2n)}\sin(nt)).$$

To reconstruct the surface, it suffices to determine the Fourier coefficients $r_j^{(n)}$. In practice, a cut-off approximation will be taken

$$r_{j,N}(t) = r_j^{(0)} + \sum_{n=1}^{N}(r_j^{(2n-1)}\cos(nt) + r_j^{(2n)}\sin(nt)).$$

For large N, $r_{j,N}$ differ from r_j in high frequency modes which represent small details of the obstacle's surface.

Denote by D_N the obstacle with boundary ∂D_N, which has a parametric form

$$\partial D_N = \{\boldsymbol{r}_N(t) \in \mathbb{R}^2 : \boldsymbol{r}_N(t) = (r_{1,N}(t), r_{2,N}(t))^\top, t \in [0, 2\pi]\}.$$

Let $\Omega_N = B\backslash\bar{D}_N$. Denote a vector of Fourier coefficients $\mathbf{C} = (c_1, c_2, \ldots, c_{4N+2})^\top \in \mathbb{R}^{4N+2}$, where $c_{2n+1} = r_1^{(n)}$, $c_{2n+2} = r_2^{(n)}$, $n = 0, 1, \ldots, 2N$, and a vector of scattering data $U = (\boldsymbol{u}(\boldsymbol{x}_1), \boldsymbol{u}(\boldsymbol{x}_2), \ldots, \boldsymbol{u}(\boldsymbol{x}_M))^\top \in \mathbb{C}^{2M}$, where $\boldsymbol{x}_m \in \partial B$, $m = 1, \ldots, M$. The inverse problem can be formulated to solve an approximate nonlinear equation

$$\mathscr{F}_N(\mathbf{C}) = \mathbf{U},$$

where the operator $\mathscr{F}_N$ maps a vector in $\mathbb{R}^{4N+2}$ into another vector in $\mathbb{C}^{2M}$.

Theorem 4.2. *Let $\boldsymbol{u}_N$ be the solution of the variational problem (3.18) with the obstacle D_N. The operator $\mathscr{F}_N$ is differentiable and its derivatives are given by*

$$\frac{\partial \mathscr{F}_{N,m}(\mathbf{C})}{\partial c_n} = \boldsymbol{u}_n'(\boldsymbol{x}_m), \quad n = 1, \ldots, 4N+2, m = 1, \ldots, M,$$

where $\boldsymbol{u}_n'$ is the unique weak solution of the boundary value problem

$$\begin{cases} \mu\Delta\boldsymbol{u}_n' + (\lambda+\mu)\nabla\nabla\cdot\boldsymbol{u}_n' + \omega^2\boldsymbol{u}_n' = 0 & \text{in } \Omega_N, \\ \boldsymbol{u}_n'(\boldsymbol{r}_N(t)) = q_n(t)(\partial_\nu \boldsymbol{u}_N(\boldsymbol{r}_N(t))) & \text{for } t \in [0, 2\pi], \\ \mathscr{B}\boldsymbol{u}_n' = \mathscr{T}\boldsymbol{u}_n' & \text{on } \partial B. \end{cases} \tag{4.16}$$

Here

$$q_1(t) = \frac{r_{2,N}'(t)}{\sqrt{(r_{1,N}'(t))^2 + (r_{2,N}'(t))^2}}, \quad q_2(t) = -\frac{r_{1,N}'(t)}{\sqrt{(r_{1,N}'(t))^2 + (r_{2,N}'(t))^2}}$$

and

$$q_n(t) = \begin{cases} q_1(t)\cos(jt), & n = 4j-1, \\ q_2(t)\cos(jt), & n = 4j, \\ q_1(t)\sin(jt), & n = 4j+1, \\ q_2(t)\sin(jt), & n = 4j+2, \end{cases}$$

for $j = 1, \ldots, N$.

Proof. Let $\boldsymbol{\nu} = (\nu_1, \nu_2)^\top$ be the unit outward norm on ∂D_N. Explicitly, we have

$$\nu_1 = -\frac{r_{2,N}'(t)}{\sqrt{(r_{1,N}'(t))^2 + (r_{2,N}'(t))^2}}, \quad \nu_2 = \frac{r_{1,N}'(t)}{\sqrt{(r_{1,N}'(t))^2 + (r_{2,N}'(t))^2}}.$$

Fix $n \in \{1, \dots, 4N + 2\}$ and $m \in \{1, \dots, M\}$, and let $\{\mathbf{e}_1, \dots, \mathbf{e}_{4N+2}\}$ be the set of natural basis vectors in $\mathbb{R}^{4N+2}$. By definition, we have

$$\frac{\partial \mathscr{F}_{N,m}(\mathbf{C})}{\partial c_n} = \lim_{h\to 0} \frac{\mathscr{F}_{N,m}(\mathbf{C} + h\mathbf{e}_n) - \mathscr{F}_{N,m}(C)}{h}.$$

A direct application of theorem 4.1 shows that the above limit exists and is the unique weak solution of the boundary value problem (4.16). □

Consider an objective function

$$f(\mathbf{C}) = \frac{1}{2}\|\mathscr{F}_N(\mathbf{C}) - U\|^2 = \frac{1}{2}\sum_{m=1}^{M} |\mathscr{F}_{N,m}(\mathbf{C}) - \boldsymbol{u}(\boldsymbol{x}_m)|^2.$$

The inverse problem can be formulated as the minimization problem:

$$\min_{\mathbf{C}} f(\mathbf{C}), \quad \mathbf{C} \in \mathbb{R}^{4N+2}.$$

To apply the descent method, it is necessary to compute the gradient of the objective function. Using theorem 4.2, we have from a simple calculation that

$$\nabla f(\mathbf{C}) = \left(\frac{\partial f(\mathbf{C})}{\partial c_1}, \dots, \frac{f(\mathbf{C})}{\partial c_{4N+2}}\right)^\top,$$

where

$$\frac{\partial f(\mathbf{C})}{\partial c_n} = \mathrm{Re} \sum_{m=1}^{M} \boldsymbol{u}_n'(\boldsymbol{x}_m) \cdot (\bar{\mathscr{F}}_{N,m}(C) - \bar{\boldsymbol{u}}(\boldsymbol{x}_m)).$$

We assume that the scattering data $\mathbf{U}$ is available over a range of frequencies $\omega \in [\omega_{\min}, \omega_{\max}]$, which may be divided into $\omega_{\min} = \omega_0 < \omega_1 < \cdots < \omega_K = \omega_{\max}$. Correspondingly, the compressional wavenumber may be divided into $\kappa_{1,\min} = \kappa_{1,0} < \kappa_{1,1} < \cdots < \kappa_{1,K} = \kappa_{1,\max}$ and the shear wavenumber may be divided into $\kappa_{2,\min} = \kappa_{2,0} < \kappa_{2,1} < \cdots < \kappa_{2,K} = \kappa_{2,\max}$. Let $k_i = [\kappa_{1,i}]$ or $k_i = [\kappa_{2,i}]$, $i = 0, 1, \dots, K$ be the greatest integer less than or equal to $\kappa_{1,i}$ or $\kappa_{2,i}$. We now propose an algorithm to reconstruct the Fourier coefficients c_n, $n = 1, \dots, 4N + 2$.

(1) Set an initial approximation $c_3 = c_6 = R_0 > 0$ and $c_n = 0$ otherwise, i.e., the initial approximation is a circle with radius R_0.
(2) Begin with the smallest frequency ω_0, and seek an approximation to the functions $r_{j,N}$ by Fourier series with Fourier modes not exceeding k_0:

$$r_{j,k_0} = r_j^{(0)} + \sum_{n=1}^{k_0} (r_j^{(2n-1)}\cos(nt) + r_j^{(2n)}\sin(nt)).$$

Denote $\mathbf{C}_{k_0} = (c_1, c_2, \dots, c_{2k_0+2})^\top$ and consider the iteration

$$\mathbf{C}_{k_0}^{(l+1)} = \mathbf{C}_{k_0}^{(l)} - \varepsilon\nabla f(C_{k_0}^{(l)}), \quad l = 1, \dots, L,$$

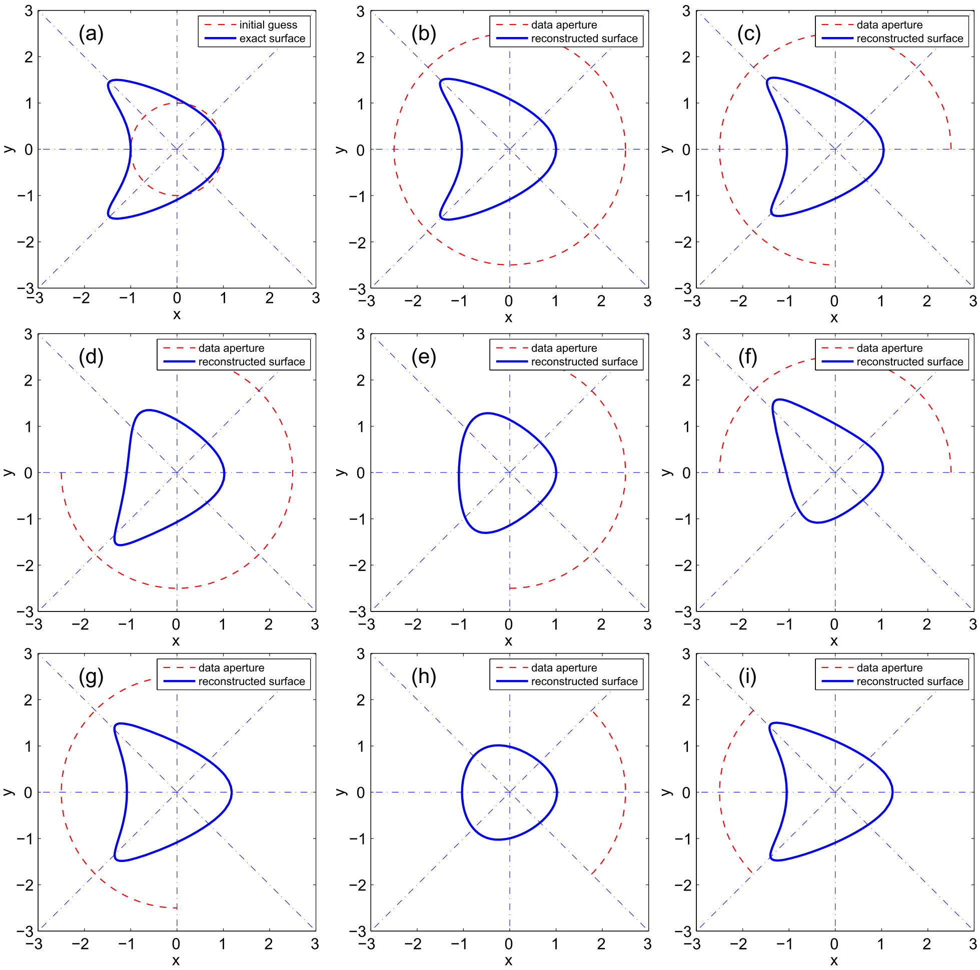

Figure 1. example 1: the kite-shaped obstacle. (a) exact surface and initial guess of the unit circle; (b) $\psi \in [0, 2\pi]$; (c) $\psi \in \left[0, \frac{3\pi}{2}\right]$; (d) $\psi \in \left[0, \frac{\pi}{2}\right] \cup [\pi, 2\pi]$; (e) $\psi \in \left[0, \frac{\pi}{2}\right] \cup \left[\frac{3\pi}{2}, 2\pi\right]$; (f) $\psi \in [0, \pi]$; (g) $\psi \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$; (h) $\psi \in \left[0, \frac{\pi}{4}\right] \cup \left[\frac{7\pi}{4}, 2\pi\right]$; (i) $\psi \in \left[\frac{3\pi}{4}, \frac{5\pi}{4}\right]$.

where $\varepsilon > 0$ and $L > 0$ are the step size and the total number of iterations for the descent method, respectively.

(3) Increase to the next higher frequency ω_1 of the available data. Repeat Step 2 with the previous approximation to $r_{j,N}$ as the starting point. More precisely, approximate $r_{j,N}$ by

$$r_{j,k_1} = \tilde{r}_j^{(0)} + \sum_{n=1}^{k_1} (\tilde{r}_j^{(2n-1)} \cos(nt) + \tilde{r}_j^{(2n)} \sin(nt))$$

and determine the coefficients $\tilde{c}_n$, $n = 1, 2, \ldots, 2k_1 + 2$ by using the descent method starting from the previous result:

$$\tilde{c}_n = \begin{cases} c_n & \text{for } 1 \leqslant n \leqslant 2k_0 + 2, \\ 0 & \text{for } 2k_0 + 2 < n \leqslant 2k_1 + 2, \end{cases}$$

where the coefficients c_n come from Step 2. The resulting solution in this step represents the Fourier coefficients of $r_{j,N}$ corresponding to the frequencies not exceeding k_1.

(4) Repeat Step 3 until a prescribed highest frequency ω_K is reached.

We need to choose the prescribed frequency larger than the highest Fourier mode of the surface in order to get a complete reconstruction.

5. Numerical experiments

In this section, we present two representative examples to show the results of the proposed method. The scattering data is obtained from solving the direct problem by using the finite element method with the perfectly matched layer technique, which is implemented via FreeFem++ [13]. The finite element solution is interpolated uniformly on ∂B. To test the stability, we add an amount of relative noise to the data

$$\boldsymbol{u}^\delta(\boldsymbol{x}_i) = \boldsymbol{u}(\boldsymbol{x}_i)(1 + \delta \text{ rand}), \quad i = 1, \ldots, M,$$

where rand are uniformly distributed random numbers in $[-1, 1]$. Since the measurement points $\boldsymbol{x}_i \in \partial B$, we have $\boldsymbol{x}_i = (R\cos\psi_i, R\sin\psi_i)^\top$, where $\psi_i \in [0, 2\pi]$ is the observation angle.

In the following two examples, we take the Lamé constants $\lambda = 2$, $\mu = 1$, which account for the compressional wavenumber $\kappa_1 = \omega/2$ and the shear wavenumber $\kappa_2 = \omega$. The radius of the ball B is $R = 2.5$ and the radius of the initial guess of the circle $R_0 = 1.0$. The noise level $\delta = 5\%$. The step size is $\varepsilon = 0.005/k_i$, $i = 0, 1, \ldots, K$. The total number of iterations and measurement points are $L = 10$ and $M = 64$, respectively.

Example 1. Consider a commonly used benchmark test example, a kite-shaped obstacle, which has the parametric equation

$$r_1(t) = -0.65 + \cos(t) + 0.65\cos(2t), \quad r_2(t) = 1.5\sin(t), \quad t \in [0, 2\pi].$$

The obstacle is non-convex. It has become a criterion to judge the quality of a reconstruction method whether the concave part of the obstacle can be successfully recovered. Our approach is essentially a Fourier spectral method and aims to recover the Fourier coefficients. Since the surface only contains a couple of low Fourier modes, it is expected that our method works very well even by using few scattering data. We use only a single compressional plane wave with the incident angle $\varphi = 0$ to illuminate the obstacle. We take the scattering data at two frequencies $\omega_0 = 2$ and $\omega_1 = 4$, i.e., the compressional wavenumbers are $\kappa_{1,0} = 1$, $\kappa_{1,1} = 2$. We have $k_0 = 1$, $k_1 = 2$, $K = 2$. Figure 1(a) shows the exact surface and the initial guess of the unit circle. Figure 1(b) shows the reconstructed surface by using the full aperture data, i.e., the observation angle $\psi \in [0, 2\pi]$. The result is almost perfect. We also investigate how the data aperture influences the quality of the reconstruction. Figures 1(c)–(i) plot the reconstructed surfaces and the corresponding data apertures for the construction. It is clear to note that the results are as good as the one by using the full aperture data as long as the observation angles cover the concave part of the obstacle; the results deteriorate as the aperture gets smaller if the observation angles do not cover the concave part of the obstacle.

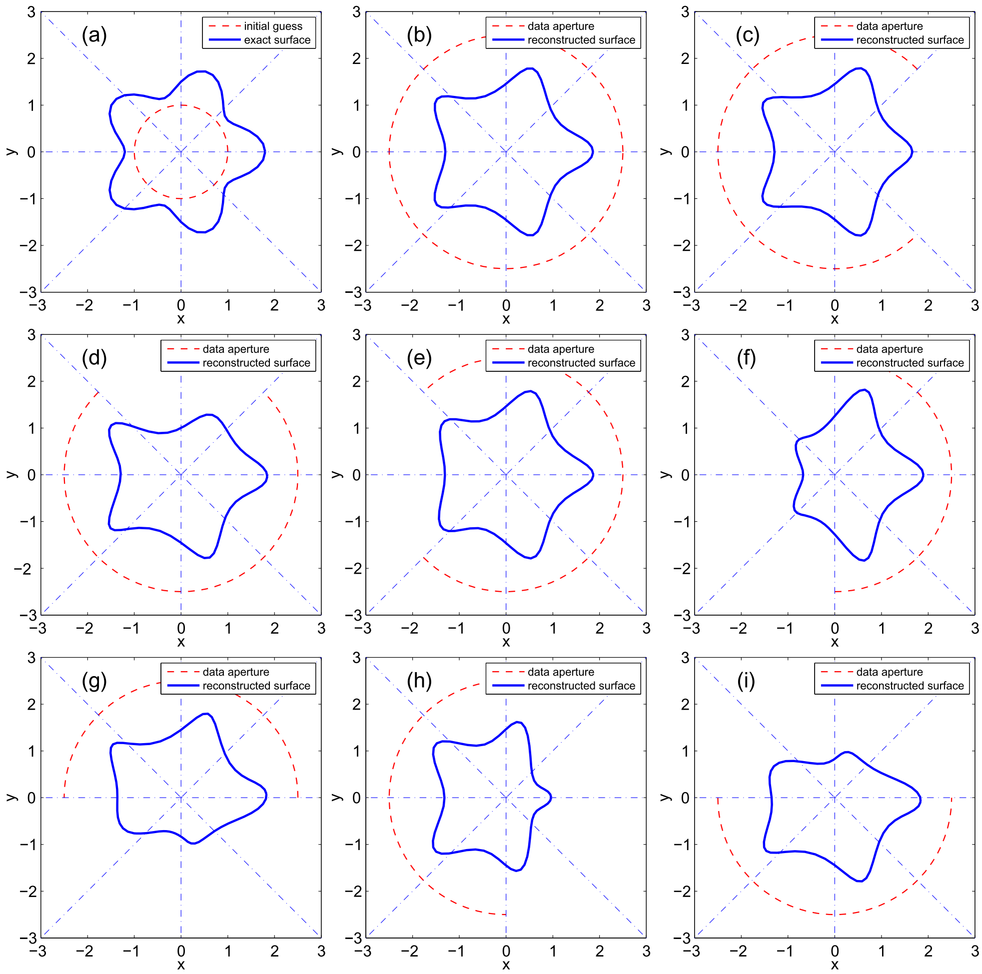

Figure 2. example 2: the star-shaped obstacle. (a) exact surface and initial guess of the unit circle; (b) $\psi \in [0, 2\pi]$; (c) $\psi \in \left[\frac{\pi}{4}, \frac{7\pi}{4}\right]$; (d) $\psi \in \left[0, \frac{\pi}{4}\right] \cup \left[\frac{3\pi}{4}, 2\pi\right]$; (e) $\psi \in \left[0, \frac{3\pi}{4}\right] \cup \left[\frac{5\pi}{4}, 2\pi\right]$; (f) $\psi \in \left[0, \frac{\pi}{2}\right] \cup \left[\frac{3\pi}{2}, 2\pi\right]$; (g) $\psi \in [0, \pi]$; (h) $\psi \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$; (i) $\psi \in [\pi, 2\pi]$.

Example 2. Consider a star-shaped obstacle, which has the parametric equation

$$r_1(t) = 1.5\cos(t) + 0.15\cos(4t) + 0.15\cos(6t),$$
$$r_2(t) = 1.5\sin(t) - 0.15\sin(4t) + 0.15\sin(6t),$$

where $t \in [0, 2\pi]$. Due to the oscillatory feature, this surface contains many more high Fourier modes and is more difficult than the first example. The scattering data with higher frequencies is required to completely recover the obstacle. In this example, we use a single shear plane wave with the incident angle $\varphi = 0$ to illuminate the obstacle. We take the scattering data at three frequencies $\omega_0 = 2$, $\omega_1 = 4$, $\omega_2 = 6$, i.e., the shear wavenumbers are $\kappa_{2,0} = 2$, $\kappa_{2,1} = 4$, $\kappa_{2,3} = 6$. We have $k_0 = 2$, $k_1 = 4$, $k_2 = 6$, $K = 3$. Figure 2(a) shows the exact surface and the initial guess of the unit circle. Figures 2(b)–(i) show the reconstructed surfaces by using different data apertures. We obtain that the part of the surface can be accurately reconstructed as long as it can be seen, i.e., the observation angles can cover that part.

6. Concluding remarks

In this paper, we study the direct and inverse obstacle scattering problems for elastic waves in two dimensions. We develop an exact transparent boundary condition and show that the direct problem has a unique weak solution. We examine the domain derivative of the total displacement with respect to the surface of the obstacle. We propose a continuation method in frequency for solving the inverse scattering problem. Numerical examples are presented to demonstrate the effectiveness of the proposed method. The results show that the method is stable and accurate to reconstruct surfaces with both full and limited aperture data.

We point out some current and future work. This paper considers rigid obstacles where the total displacement of the field vanishes on the surfaces. It is worthwhile to investigate the surfaces for different boundary conditions. Clearly, modifications are needed for the transparent boundary conditions, the proofs of the direct problem, and the boundary conditions for the domain derivatives. It is interesting to consider multiple obstacles where each obstacle's surface has a parametric equation. A challenging problem is to solve the three dimensional problem where a more efficient direct solver is needed for the large scale computation. We hope to be able to address these issues and report the progress elsewhere in the future.

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