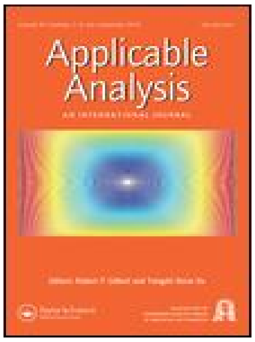

Applicable Analysis
An International Journal

ISSN: 0003-6811 (Print) 1563-504X (Online) Journal homepage: http://www.tandfonline.com/loi/gapa20

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To cite this article: Peijun Li, Yuliang Wang & Yue Zhao (2016) Convergence analysis in near-field imaging for elastic waves, Applicable Analysis, 95:11, 2339-2360, DOI: 10.1080/00036811.2015.1089238

To link to this article: http://dx.doi.org/10.1080/00036811.2015.1089238

Published online: 22 Sep 2015.

APPLICABLE ANALYSIS, 2016
VOL. 95, NO. 11, 2339–2360
http://dx.doi.org/10.1080/00036811.2015.1089238

Convergence analysis in near-field imaging for elastic waves

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ABSTRACT
A significant method has recently been developed for solving the inverse elastic surface scattering problem which arises from near-field imaging applications. The method utilizes the transformed field expansion along with the Fourier series expansion to deduce an analytic solution for the direct problem. Implemented via the fast Fourier transform, an explicit reconstruction formula is obtained to solve the linearized inverse problem. Numerical examples show that the method is efficient and effective to reconstruct scattering surfaces with subwavelength resolution. This paper is devoted to the mathematical analysis of the proposed method. The well-posedness is established for the solution of the direct problem. The convergence of the power series solution is examined. A local uniqueness result is proved for the inverse problem where a single incident field with a fixed frequency is needed. The error estimate is derived for the reconstruction formula. It provides a deep insight on the trade-off among resolution, accuracy, and stability of the solution for the inverse problem.

ARTICLE HISTORY
Received 23 February 2015
Accepted 28 August 2015

COMMUNICATED BY
J. Zou

KEYWORDS
Inverse elastic scattering; near-field imaging; convergence analysis; error estimate

AMS Subject Classification
65N21

1. Introduction

The elastic wave scattering problems have received considerable attention in both the engineering and mathematical communities for their important applications in diverse scientific areas, such as geophysics, seismology, and nondestructive testing.(1,2,3,4,5) There are two types of scattering problems. Given an incident field, the direct problems are to determine the wave field for such a known scatterer as a scattering surface, an impenetrable obstacle, or an inhomogeneous medium. The inverse problem is to reconstruct the properties of the scatterer, such as its geometry or material, from the measured wave field. This paper is concerned with the mathematical analysis of the direct and inverse elastic wave scattering by a periodic rough surface in two dimensions. In the context of the surface scattering, the direct problem is to determine the displacement for the given surface; the inverse problem is to reconstruct the shape of the surface from the measured displacement.

Periodic structures, also known as diffraction gratings in optics and electromagnetics, are of great importance in engineering and industrial applications. The direct problems have been investigated in (6,7,8,9,10) and in the references cited therein. The direct problems for general infinite rough surfaces have been studied mathematically in (11,12,13,14). The inverse problem has also been studied theoretically for its uniqueness (15) and solved numerically using the factorization method (16) and an optimization approach.(9)

Recently, we proposed a novel method for solving the inverse surface scattering problem in (17). The problem arises from near-field imaging applications. The periodic surface was considered to be a small and smooth perturbation of a planar surface. The half-space above the surface was

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filled with a linear, isotropic, and homogeneous elastic medium, while the half-space below the surface was elastically rigid. Given a time-harmonic plane incident wave, the inverse problem was to reconstruct the shape of the surface from the measurement of the displacement at a horizontal line above the surface. Compared with the wavelength of the incident wave, the distance was small between the measurement line and the surface. Thus, the near-field data could be obtained to achieve super resolution for the reconstruction. Due to the coupling of the displacement vector field in the governing equation, we used the Helmholtz decomposition and formulated the direct scattering as a boundary value problem with a transparent boundary condition imposed on the measurement line. Utilizing the transformed field expansion and the Fourier series expansion, we reduced the boundary value problem into a successive sequence of one-dimensional two-point boundary value problems which were solved in a closed form of a power series solution. By keeping only the leading and linear terms in the power series, we linearized the inverse problem and obtained an explicit reconstruction formula relating the Fourier coefficients of the data to the solution. It is shown numerically that the method is effective and efficient at reconstructing the surface with super resolution. The work extends the interesting findings to the area of elastic waves from solving a class of inverse surface scattering problems for acoustic and electromagnetic waves.(18,19,20,21,22,23,24) We refer to (25, 26,27,28,29) for related inverse surface scattering problems in near-field imaging applications. It may be found in (30,31,32,33,34) for the transformed field expansion method and the boundary perturbation technique to solve some direct surface scattering problems.

Motivated by the significant numerical results, we intend to carry out the theoretical analysis of the solutions for the direct and inverse problems in (17). Using the variational approach and the Lax–Milgram lemma, we establish the well-posedness and a regularity estimate for the solution of the direct problem. We prove the convergence of the power series by estimating the solutions for the successive sequence of boundary value problems. Applying the Helmholtz decomposition and a unique continuation argument, we present a local uniqueness result for the inverse problem by showing a Poincaré-type inequality. Finally, we show an error estimate which comes from three aspects: the measurement noise, the linearization, and the spectral cut-off regularization. The estimate confirms the numerical observations and provides a deep insight into the trade-off among resolution, accuracy, and stability of the solution for the inverse problem. A related convergence analysis can be found in (22) for near-field imaging of periodic surfaces with acoustic waves. The analysis in this work is more involved since the Navier equation for elastic waves has a richer structure than the Helmholtz equation for acoustic waves.

The rest of the paper is organized as follows. Section 2 is devoted to the well-posedness, regularity estimates, and convergence of the solution for the direct problem. The inverse problem is studied in Section 3 for its uniqueness. The reconstruction formula is briefly discussed and the error estimate is derived. The paper is concluded in Section 4 with remarks and directions for future research.

2. Direct scattering

In this section, we present a mathematical model for elastic scattering by periodic rigid surfaces and analyze the solution for the corresponding boundary value problem.

2.1. Problem formulation

Consider a periodic scattering surface in two dimensions, as seen in Figure 1. The space above the surface is filled with a linear, isotropic, and homogeneous elastic medium, which is characterized by the Lamé constants λ and μ satisfying $\mu > 0$ and $\lambda + \mu > 0$. Let the surface in one period be described by

$$S = \left\{(x, y) \in \mathbb{R}^2 : y = f(x),\ 0 < x < \Lambda\right\},$$

where $f \geq 0$ is a periodic function with period Λ. We assume that f takes the form

$$f(x) = \varepsilon g(x), \quad g \in C^k(\mathbb{R}),$$

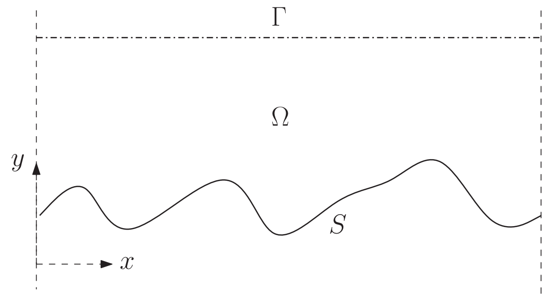

Figure 1. Schematic of the problem geometry for the elastic surface scattering.

where $k \geq 2$ is an integer, $\varepsilon > 0$ is a small constant termed the deformation parameter, and g is a normalized periodic function with period Λ. Denote an indicator of the surface smoothness:

$$K = \max_{0 \leq m \leq k} \sup_{x \in \mathbb{R}} \left| \frac{\mathrm{d}^m}{\mathrm{d}x^m} g(x) \right|, \tag{2.1}$$

which plays an important role in the convergence analysis and the error estimate. Let

$$\Gamma = \{(x, y) \in \mathbb{R}^2 : y = h,\ 0 < x < \Lambda\}$$

be a planar surface, where $h > \max_{x \in (0,\Lambda)} f(x)$ is the measurement distance. Denote by

$$\Omega = \{(x, y) \in \mathbb{R}^2 : f(x) < y < h,\ 0 < x < \Lambda\}$$

the bounded domain between S and Γ.

Let the scattering surface be illuminated from above by a time-harmonic incident wave

$$\boldsymbol{u}^{\mathrm{inc}} = [\sin\theta,\ -\cos\theta] e^{\mathrm{i}\kappa_{\mathrm{p}}(x\sin\theta - y\cos\theta)},$$

where $\theta \in (-\pi/2, \pi/2)$ is the incident angle, $\kappa_{\mathrm{p}} = \omega/(\lambda + 2\mu)^{1/2}$ is the pressure wavenumber, and ω is the angular frequency. For simplicity, we assume a normal incidence, i.e. the incident angle $\theta = 0$. The incident field reduces to

$$\boldsymbol{u}^{\mathrm{inc}} = [0,\ -1] e^{-\mathrm{i}\kappa_{\mathrm{p}} y}.$$

The elastic wave field satisfies the Navier equation:

$$\mu\Delta\boldsymbol{u} + (\lambda + \mu)\nabla\nabla \cdot \boldsymbol{u} + \omega^2 \boldsymbol{u} = 0 \quad \text{in } \Omega, \tag{2.2}$$

where $\boldsymbol{u} = [u_1,\ u_2]$ is the total displacement vector field. Since the substrate below S is elastically rigid, we have the Dirichlet boundary condition

$$\boldsymbol{u} = 0 \quad \text{on } S. \tag{2.3}$$

Due to the periodicity of scattering surface and the normal incidence of the plane incident field, the solution of (2.2) is periodic with period of Λ, i.e. $\boldsymbol{u}(x + \Lambda, y) = \boldsymbol{u}(x, y)$, and admits the Fourier series expansion:

$$\boldsymbol{u}(x, y) = \sum_{n \in \mathbb{Z}} \boldsymbol{u}^{(n)}(y) e^{\mathrm{i}\alpha_n x}, \quad \boldsymbol{u}^{(n)}(y) = \frac{1}{\Lambda} \int_0^\Lambda \boldsymbol{u}(x, y) e^{-\mathrm{i}\alpha_n x} \mathrm{d}x,$$

where $\alpha_n = 2\pi n/\Lambda$.

The total field $\boldsymbol{u}$ can be split into its pressure and shear parts via the Helmholtz decomposition:

$$\boldsymbol{u} = \nabla\phi + \mathbf{curl}\,\psi, \tag{2.4}$$

where ϕ and ψ are the scalar potentials, and the vector curl is $\mathbf{curl}\,\psi = [\partial_y\psi, -\partial_x\psi]$. Substituting (2.4) into (2.2), we may obtain the Helmholtz equations:

$$\left(\Delta + \kappa_{\rm p}^2\right)\phi = 0, \quad \left(\Delta + \kappa_{\rm s}^2\right)\psi = 0 \quad \text{in } \Omega,$$

where $\kappa_{\rm s} = \omega/\mu^{1/2}$ is the shear wavenumber.

Denote

$$\beta_{\rm p}^{(n)} = \begin{cases} \left(\kappa_{\rm p}^2 - \alpha_n^2\right)^{1/2}, & |\alpha_n| < \kappa_{\rm p}, \\ {\rm i}\left(\alpha_n^2 - \kappa_{\rm p}^2\right)^{1/2}, & |\alpha_n| > \kappa_{\rm p}, \end{cases} \tag{2.5}$$

and

$$\beta_{\rm s}^{(n)} = \begin{cases} \left(\kappa_{\rm s}^2 - \alpha_n^2\right)^{1/2}, & |\alpha_n| < \kappa_{\rm s}, \\ {\rm i}\left(\alpha_n^2 - \kappa_{\rm s}^2\right)^{1/2}, & |\alpha_n| > \kappa_{\rm s}. \end{cases} \tag{2.6}$$

We exclude the resonance by assuming that $\beta_{\rm p}^{(n)} \neq 0$ and $\beta_{\rm s}^{(n)} \neq 0$ for $n \in \mathbb{Z}$, i.e. $|\alpha_n| \neq \kappa_{\rm p}$ and $|\alpha_n| \neq \kappa_{\rm s}$ for $n \in \mathbb{Z}$.

Using the Rayleigh expansions for ϕ and ψ, we deduce the transparent boundary conditions for the scalar potentials ϕ and ψ on Γ:

$$\partial_y\phi = \sum_{n\in\mathbb{Z}} {\rm i}\beta_{\rm p}^{(n)}\phi^{(n)}e^{{\rm i}\alpha_n x} + \rho, \quad \partial_y\psi = \sum_{n\in\mathbb{Z}} {\rm i}\beta_{\rm s}^{(n)}\psi^{(n)}e^{{\rm i}\alpha_n x}, \tag{2.7}$$

where $\rho = -2e^{-{\rm i}\kappa_{\rm p}h}$. It is shown in (17) that a transparent boundary condition may be obtained for the displacement $\boldsymbol{u}$ by using the Helmholtz decomposition (2.4) and the boundary condition (2.7):

$$\mu\partial_y\boldsymbol{u} + (\lambda + \mu)[0, 1]\nabla\cdot\boldsymbol{u} = \mathscr{T}\boldsymbol{u} + \boldsymbol{\rho}, \tag{2.8}$$

where

$$\mathscr{T}\boldsymbol{u} = \sum_{n\in\mathbb{Z}} M_n\boldsymbol{u}^{(n)}e^{{\rm i}\alpha_n x}, \quad \boldsymbol{\rho} = [0, 1]2{\rm i}\kappa_{\rm p}(\lambda + 2\mu)e^{-{\rm i}\kappa_{\rm p}h}, \tag{2.9}$$

and

$$M_n = {\rm i}\begin{bmatrix} \dfrac{\omega^2\beta_{\rm p}^{(n)}}{\alpha_n^2 + \beta_{\rm p}^{(n)}\beta_{\rm s}^{(n)}} & \mu\alpha_n - \dfrac{\omega^2\alpha_n}{\alpha_n^2 + \beta_{\rm p}^{(n)}\beta_{\rm s}^{(n)}} \\ \dfrac{\omega^2\alpha_n}{\alpha_n^2 + \beta_{\rm p}^{(n)}\beta_{\rm s}^{(n)}} - \mu\alpha_n & \dfrac{\omega^2\beta_{\rm s}^{(n)}}{\alpha_n^2 + \beta_{\rm p}^{(n)}\beta_{\rm s}^{(n)}} \end{bmatrix}: \tag{2.10}$$

Given the incident field $\boldsymbol{u}^{\rm inc}$, the direct problem is to determine the total field $\boldsymbol{u}$ from the known surface function f; the inverse problem is to reconstruct the surface function f from the measured displacement of $\boldsymbol{u}$ on Γ.

We impose the following two hypotheses:

$$(\mathrm{H1})\ h < c_1; \quad (\mathrm{H2})\ K\varepsilon h^{-1} < c_2,$$

where c_1, c_2 are sufficiently small positive constants independent of K, ε, h. The hypothesis (H1) ensures the well-posedness of the solution for the direct problem and is a reasonable assumption in near-field imaging applications. The hypothesis (H2) guarantees the convergence of the power series solution for the direct problem, which makes it possible to derive the error estimate for the inverse problem. It is clear to see that the hypothesis (H2) is satisfied if the surface deformation parameter ε is sufficiently small. Throughout the paper, the parameters $K, \varepsilon, \omega, \lambda, \mu$ are fixed constants. If not otherwise specified, the expression $a \lesssim b$ means $a \leq Cb$ for a constant C independent of n, K, ε, h.

2.2. Variational problem

Let us introduce some functional spaces for the variational formulation. Define a periodic Sobolev space

$$H^1_{S,\mathrm{p}}(\Omega) = \left\{u \in H^1(\Omega) : u = 0 \text{ on } S,\ u(0, y) = u(\Lambda, y)\right\},$$

which is equipped with the usual H^1-norm:

$$\|u\|_{1,\Omega} = \left[\sum_{|s|\leq 1} \int_\Omega \left|D^s u(x, y)\right|^2 \, \mathrm{d}x\mathrm{d}y\right]^{1/2}.$$

Let $H^1_{S,\mathrm{p}}(\Omega)^2 = H^1_{S,\mathrm{p}}(\Omega) \times H^1_{S,\mathrm{p}}(\Omega)$ be a Cartesian product space which is equipped with the norm:

$$\|\boldsymbol{u}\|_{1,\Omega} = \left(\|u_1\|^2_{1,\Omega} + \|u_2\|^2_{1,\Omega}\right)^{1/2}.$$

Denote by $H^{-1}_{S,\mathrm{p}}(\Omega)^2$ the dual space of $H^1_{S,\mathrm{p}}(\Omega)^2$, which consists of all bounded linear functionals on $H^1_{S,\mathrm{p}}(\Omega)^2$ and is equipped with the norm:

$$\|\boldsymbol{u}\|_{-1,\Omega} = \sup_{\boldsymbol{v}\in H^1_{S,\mathrm{p}}(\Omega)^2} \frac{|(\boldsymbol{u}, \boldsymbol{v})_\Omega|}{\|\boldsymbol{v}\|_{1,\Omega}},$$

where $(\cdot,\cdot)_\Omega$ is the inner product in Ω. Let $H^s(\Gamma), s \in \mathbb{R}$ be the trace Sobolev space of periodic functions on Γ with the norm

$$\|u\|_{s,\Gamma} = \left[\Lambda \sum_{n\in\mathbb{Z}} \left(1 + \alpha_n^2\right)^s \left|u^{(n)}(h)\right|^2\right]^{1/2}.$$

Define the product space $H^s(\Gamma)^2 = H^s(\Gamma) \times H^s(\Gamma)$ with the norm

$$\|\boldsymbol{u}\|_{s,\Gamma} = \left(\|u_1\|^2_{s,\Gamma} + \|u_2\|^2_{s,\Gamma}\right)^{1/2}.$$

It is easy to verify that $H^{-s}(\Gamma)^2$ is the dual space of $H^s(\Gamma)^2$ for any s with respect to the inner product

$$\langle \boldsymbol{u}, \boldsymbol{v}\rangle_\Gamma = \int_\Gamma \boldsymbol{u} \cdot \bar{\boldsymbol{v}} \, \mathrm{d}x.$$

Multiply (2.2) by the complex conjugate of a test function $\boldsymbol{v} \in H^1_{S,\mathrm{p}}(\Omega)^2$, applying integration by parts, and using the transparent boundary condition (2.9), we arrive at a variational problem: to find

$\boldsymbol{u} \in H^1_{S,\mathrm{p}}(\Omega)^2$ such that

$$a_\Omega(\boldsymbol{u}, \boldsymbol{v}) = b(\boldsymbol{v}) \quad \text{for all } \boldsymbol{v} \in H^1_{S,\mathrm{p}}(\Omega)^2, \tag{2.11}$$

where the sesquilinear form

$$a_\Omega(\boldsymbol{u}, \boldsymbol{v}) = \int_\Omega a_0(\boldsymbol{u}, \boldsymbol{v})\, \mathrm{d}x\mathrm{d}y - \omega^2 (\boldsymbol{u}, \boldsymbol{v})_\Omega - \langle \mathscr{T}\boldsymbol{u}, \boldsymbol{v}\rangle_\Gamma \tag{2.12}$$

with

$$\begin{aligned} a_0(\boldsymbol{u}, \boldsymbol{v}) = &(\lambda + 2\mu)(\partial_x u_1 \partial_x \bar{v}_1 + \partial_y u_2 \partial_y \bar{v}_2) + \mu(\partial_y u_1 \partial_y \bar{v}_1 + \partial_x u_2 \partial_x \bar{v}_2) \\ &+ (\lambda + \mu)(\partial_x u_1 \partial_y \bar{v}_2 + \partial_y u_2 \partial_x \bar{v}_1), \end{aligned}$$

and the linear functional

$$b(\boldsymbol{v}) = \langle \boldsymbol{\rho}, \boldsymbol{v}\rangle_\Gamma .$$

Lemma 2.1: It holds the estimate

$$\|\boldsymbol{u}\|_{0,\Omega} \le h\, \|\nabla \boldsymbol{u}\|_{0,\Omega} \quad \text{for any } \boldsymbol{u} \in H^1_{S,\mathrm{p}}(\Omega)^2.$$

Proof: Let

$$D = \{(x, y) \in \mathbb{R}^2 : 0 < x < \Lambda,\ 0 < y < h\}$$

be a rectangular domain containing Ω. For any $\boldsymbol{u} \in H^1_{S,\mathrm{p}}(\Omega)^2$, consider its zero extension to D:

$$\tilde{\boldsymbol{u}}(x, y) = \begin{cases} \boldsymbol{u}(x, y), & (x, y) \in \Omega, \\ 0, & (x, y) \in D \backslash \bar{\Omega}. \end{cases} \tag{2.13}$$

It follows from the Cauchy–Schwarz inequality that

$$|\tilde{\boldsymbol{u}}|^2 = \left| \int_0^y \partial_y \tilde{\boldsymbol{u}}\, \mathrm{d}y \right|^2 \le h \int_0^h \left|\partial_y \tilde{\boldsymbol{u}}\right|^2\, \mathrm{d}y.$$

Hence

$$\|\tilde{\boldsymbol{u}}\|^2_{0,D} = \int_0^h \int_0^\Lambda |\tilde{\boldsymbol{u}}|^2\, \mathrm{d}x\mathrm{d}y \le h \int_0^h \int_0^\Lambda \int_0^h \left|\partial_y \tilde{\boldsymbol{u}}\right|^2\, \mathrm{d}y\mathrm{d}x\mathrm{d}y \le h^2 \|\nabla \tilde{\boldsymbol{u}}\|^2_{0,D},$$

which completes the proof by noting that

$$\|\boldsymbol{u}\|_{0,\Omega} = \|\tilde{\boldsymbol{u}}\|_{0,D} \quad \text{and} \quad \|\nabla \boldsymbol{u}\|_{0,\Omega} = \|\nabla \tilde{\boldsymbol{u}}\|_{0,D}.$$

□

The following two trace regularity results are useful in our analysis.

Lemma 2.2: It holds the estimate

$$\|\boldsymbol{u}\|_{1/2,\Gamma} \le \|\boldsymbol{u}\|_{1,\Omega} \quad \text{for any } \boldsymbol{u} \in H^1_{S,\mathrm{p}}(\Omega)^2.$$

Proof: For any $\boldsymbol{u} \in H^1_{S,\mathrm{p}}(\Omega)^2$, let $\tilde{\boldsymbol{u}}$ be its zero extension to the domain D defined in (2.13). By the basic arithmetic–geometric mean inequality, we have

$$\begin{aligned}|\tilde{\boldsymbol{u}}^{(n)}(h)|^2 &= \int_0^h \frac{\mathrm{d}}{\mathrm{d}y}|\tilde{\boldsymbol{u}}^{(n)}(y)|^2\,\mathrm{d}y\\ &\le \int_0^h 2\left|\tilde{\boldsymbol{u}}^{(n)}(y)\right|\left|\frac{\mathrm{d}}{\mathrm{d}y}\tilde{\boldsymbol{u}}^{(n)}(y)\right|\,\mathrm{d}y\\ &\le \left(1+\alpha_n^2\right)^{1/2}\int_0^h\left|\tilde{\boldsymbol{u}}^{(n)}(y)\right|^2\,\mathrm{d}y + \left(1+\alpha_n^2\right)^{-1/2}\int_0^h\left|\frac{\mathrm{d}}{\mathrm{d}y}\tilde{\boldsymbol{u}}^{(n)}(y)\right|^2\,\mathrm{d}y,\end{aligned}$$

which gives

$$\left(1+\alpha_n^2\right)^{1/2}\left|\tilde{\boldsymbol{u}}^{(n)}(h)\right|^2 \le \left(1+\alpha_n^2\right)\int_0^h\left|\tilde{\boldsymbol{u}}^{(n)}(y)\right|^2\,\mathrm{d}y + \int_0^h\left|\frac{\mathrm{d}}{\mathrm{d}y}\tilde{\boldsymbol{u}}^{(n)}(y)\right|^2\,\mathrm{d}y.$$

Using the Fourier series expansion of $\tilde{\boldsymbol{u}}$, we can verify that

$$\|\tilde{\boldsymbol{u}}\|^2_{1,D} = \Lambda\sum_{n\in\mathbb{Z}}\left[(1+\alpha_n^2)\int_0^h\left|\tilde{\boldsymbol{u}}^{(n)}(y)\right|^2\,\mathrm{d}y + \int_0^h\left|\frac{\mathrm{d}}{\mathrm{d}y}\tilde{\boldsymbol{u}}^{(n)}(y)\right|^2\,\mathrm{d}y\right].$$

Hence

$$\|\tilde{\boldsymbol{u}}\|^2_{1/2,\Gamma} = \Lambda\sum_{n\in\mathbb{Z}}\left(1+\alpha_n^2\right)^{1/2}|\tilde{\boldsymbol{u}}^{(n)}(h)|^2 \le \|\tilde{\boldsymbol{u}}\|^2_{1,D},$$

which completes the proof by noting that

$$\|\boldsymbol{u}\|_{1/2,\Gamma} = \|\tilde{\boldsymbol{u}}\|_{1/2,\Gamma}, \quad \|\boldsymbol{u}\|_{1,\Omega} = \|\tilde{\boldsymbol{u}}\|_{1,D}.$$

□

Lemma 2.3: It holds the estimate

$$\|\boldsymbol{u}\|^2_{-1/2,\Gamma} \le \eta^{-1}\|\boldsymbol{u}\|^2_{0,\Omega} + \eta\left\|\partial_y\boldsymbol{u}\right\|^2_{0,\Omega} \quad \text{for any } \boldsymbol{u} \in H^1_{S,\mathrm{p}}(\Omega)^2,$$

where $\eta > 0$ is a constant.

Proof: For any $\boldsymbol{u} \in H^1_{S,\mathrm{p}}(\Omega)^2$, let $\tilde{\boldsymbol{u}}$ be its zero extension to the domain D defined in (2.13). It follows from the basic arithmetic–geometric mean inequality that

$$\begin{aligned}|\tilde{\boldsymbol{u}}^{(n)}(h)|^2 &= \int_0^h \frac{\mathrm{d}}{\mathrm{d}y}|\tilde{\boldsymbol{u}}^{(n)}(y)|^2\,\mathrm{d}y \le \int_0^h 2|\tilde{\boldsymbol{u}}^{(n)}(y)|\left|\frac{\mathrm{d}}{\mathrm{d}y}\tilde{\boldsymbol{u}}^{(n)}(y)\right|\,\mathrm{d}y\\ &\le \eta^{-1}\left(1+\alpha_n^2\right)^{1/2}\int_0^h|\tilde{\boldsymbol{u}}^{(n)}(y)|^2\,\mathrm{d}y + \eta\left(1+\alpha_n^2\right)^{-1/2}\int_0^h\left|\frac{\mathrm{d}}{\mathrm{d}y}\tilde{\boldsymbol{u}}^{(n)}(y)\right|^2\,\mathrm{d}y,\end{aligned}$$

which gives

$$\left(1+\alpha_n^2\right)^{-1/2}|\tilde{\boldsymbol{u}}^{(n)}(h)|^2 \le \eta^{-1}\int_0^h|\tilde{\boldsymbol{u}}^{(n)}(y)|^2\mathrm{d}y + \eta(1+\alpha_n^2)^{-1}\int_0^h\left|\frac{\mathrm{d}}{\mathrm{d}y}\tilde{\boldsymbol{u}}^{(n)}(y)\right|^2\,\mathrm{d}y.$$

Using the Fourier series expansion of $\tilde{\boldsymbol{u}}$, we can verify that

$$\|\tilde{\boldsymbol{u}}\|^2_{0,\Omega} = \Lambda\sum_{n\in\mathbb{Z}}\int_0^h\left|\tilde{\boldsymbol{u}}^{(n)}(y)\right|^2\,\mathrm{d}y$$

and

$$\|\partial_y \tilde{\boldsymbol{u}}\|_{0,\Omega}^2 = \Lambda \sum_{n\in\mathbb{Z}} \int_0^h \left| \frac{\mathrm{d}}{\mathrm{d}y} \tilde{\boldsymbol{u}}^{(n)}(y) \right|^2 \mathrm{d}y.$$

Hence, we have

$$\begin{aligned}
\|\tilde{\boldsymbol{u}}|_{-1/2,\Gamma}^2 &= \Lambda \sum_{n\in\mathbb{Z}} \left(1+\alpha_n^2\right)^{-1/2} |\tilde{\boldsymbol{u}}^{(n)}(h)|^2 \\
&\leq \Lambda \sum_{n\in\mathbb{Z}} \eta^{-1} \int_0^h |\tilde{\boldsymbol{u}}^{(n)}(y)|^2 \,\mathrm{d}y + \eta \left(1+\alpha_n^2\right)^{-1} \int_0^h \left| \frac{\mathrm{d}}{\mathrm{d}y} \tilde{\boldsymbol{u}}^{(n)}(y) \right|^2 \mathrm{d}y \\
&\leq \Lambda \sum_{n\in\mathbb{Z}} \eta^{-1} \int_0^h |\tilde{\boldsymbol{u}}^{(n)}(y)|^2 \mathrm{d}y + \eta \int_0^h \left| \frac{\mathrm{d}}{\mathrm{d}y} \tilde{\boldsymbol{u}}^{(n)}(y) \right|^2 \mathrm{d}y \\
&= \eta^{-1} \|\tilde{\boldsymbol{u}}\|_{0,\Omega}^2 + \eta \left\|\partial_y \tilde{\boldsymbol{u}}\right\|_{0,\Omega}^2,
\end{aligned}$$

which completes the proof by noting that

$$\|\boldsymbol{u}\|_{-1/2,\Gamma} = \|\tilde{\boldsymbol{u}}\|_{-1/2,\Gamma}, \quad \|\boldsymbol{u}\|_{0,\Omega} = \|\tilde{\boldsymbol{u}}\|_{0,D}, \quad \left\|\partial_y \boldsymbol{u}\right\|_{0,\Omega} = \left\|\partial_y \tilde{\boldsymbol{u}}\right\|_{0,D}.$$

Lemma 2.4: The boundary operator $\mathscr{T} : H^{1/2}(\Gamma)^2 \to H^{-1/2}(\Gamma)^2$ is continuous, i.e.

$$\|\mathscr{T}\boldsymbol{u}\|_{-1/2,\Gamma} \lesssim \|\boldsymbol{u}\|_{1/2,\Gamma} \quad \text{for any } \boldsymbol{u} \in H^{1/2}(\Gamma)^2.$$

Proof: Using the definitions of $\beta_{\mathrm{p}}^{(n)}$ and $\beta_{\mathrm{s}}^{(n)}$, we have

$$\beta_{\mathrm{p}}^{(n)} = \mathrm{i}\left(\alpha_n^2 - \kappa_{\mathrm{p}}^2\right)^{1/2} \sim \mathrm{i}|n|, \quad \beta_{\mathrm{s}}^{(n)} = \mathrm{i}\left(\alpha_n^2 - \kappa_{\mathrm{s}}^2\right)^{1/2} \sim \mathrm{i}|n|,$$

and

$$\alpha_n^2 + \beta_{\mathrm{p}}^{(n)}\beta_{\mathrm{s}}^{(n)} = \alpha_n^2 \left[1 - \left(1 - \frac{\kappa_{\mathrm{p}}^2}{\alpha_n^2}\right)^{1/2} \left(1 - \frac{\kappa_{\mathrm{s}}^2}{\alpha_n^2}\right)^{1/2} \right] \sim \frac{1}{2}\left(\kappa_{\mathrm{p}}^2 + \kappa_{\mathrm{s}}^2\right) \quad \text{as } |n| \to \infty.$$

It follows from (2.10) that

$$\|M_n\|_2 \sim |n| \quad \text{as } |n| \to \infty.$$

Hence, there exist an interger N and two positive constants C_1 and C_2 such that $\|M_n\|_2^2 \leq C_1$ for $|n| < N$ and $\|M_n\|_2^2 \leq C_2|n|^2$ for $|n| \geq N$. Here $\|\cdot\|_2$ denotes the matrix norm induced by the 2-norm for vectors. A simple calculation yields

$$\begin{aligned}
\|\mathscr{T}\boldsymbol{u}\|_{-1/2,\Gamma}^2 &= \sum_{n\in\mathbb{Z}} \left(1+\alpha_n^2\right)^{-1/2} \left|M_n \boldsymbol{u}^{(n)}(h)\right|^2 \leq \sum_{n\in\mathbb{Z}} \left(1+\alpha_n^2\right)^{-1/2} \|M_n\|_2^2 \left|\boldsymbol{u}^{(n)}(h)\right|^2 \\
&\leq C_1 \sum_{|n|<N} \left(1+\alpha_n^2\right)^{1/2} \left|\boldsymbol{u}^{(n)}(h)\right|^2 + C_2 \sum_{|n|\geq N} \left(1+n^2\right)\left(1+\alpha_n^2\right)^{-1/2} \left|\boldsymbol{u}^{(n)}(h)\right|^2 \\
&\lesssim \sum_{n\in\mathbb{Z}} \left(1+\alpha_n^2\right)^{1/2} \left|\boldsymbol{u}^{(n)}(h)\right|^2 = \|\boldsymbol{u}\|_{1/2,\Gamma},
\end{aligned}$$

which completes the proof. □

Let

$$\hat{M}_n = -\frac{1}{2}\left(M_n + M_n^*\right)$$
$$= \mu \begin{bmatrix} \mathrm{Im}\left(\dfrac{\kappa_s^2\beta_p^{(n)}}{\alpha_n^2+\beta_p^{(n)}\beta_s^{(n)}}\right) & -\mathrm{i}\alpha_n\mathrm{Re}\left(1-\dfrac{\kappa_s^2}{\alpha_n^2+\beta_p^{(n)}\beta_s^{(n)}}\right) \\ \mathrm{i}\alpha_n\mathrm{Re}\left(1-\dfrac{\kappa_s^2}{\alpha_n^2+\beta_p^{(n)}\beta_s^{(n)}}\right) & \mathrm{Im}\left(\dfrac{\kappa_s^2\beta_s^{(n)}}{\alpha_n^2+\beta_p^{(n)}\beta_s^{(n)}}\right) \end{bmatrix}.$$

It is crucial to study the properties of the matrix $\hat{M}_n$ in order the show the well-posedness of the solution for the direct problem.

Lemma 2.5: It holds:

(1) If $|\alpha_n| > \kappa_s$, then $\hat{M}_n$ is a positive definite matrix.
(2) If $|\alpha_n| < \kappa_s$, then $\|\hat{M}_n\|_2 \le c(\mu,\lambda)\omega$, where $c(\lambda,\mu)$ is a constant depending only on μ and λ.

Proof: For any $n \in \mathbb{Z}$, we denote $\gamma_p^{(n)} = \left|\kappa_p^2 - \alpha_n^2\right|^{1/2}$ and $\gamma_s^{(n)} = \left|\kappa_s^2 - \alpha_n^2\right|^{1/2}$.

(1) If $|\alpha_n| > \kappa_s$, then $\beta_p^{(n)} = \mathrm{i}\gamma_p^{(n)}$ and $\beta_s^{(n)} = \mathrm{i}\gamma_s^{(n)}$. It follows that

$$\hat{M}_n = \frac{\mu}{\zeta_n}\begin{bmatrix} \kappa_s^2\gamma_p^{(n)} & -\mathrm{i}\alpha_n\left(\zeta_n - \kappa_s^2\right) \\ \mathrm{i}\alpha_n\left(\zeta_n-\kappa_s^2\right) & \kappa_s^2\gamma_s^{(n)} \end{bmatrix},$$

where $\zeta_n = \alpha_n^2 - \gamma_p^{(n)}\gamma_s^{(n)}$. It is easy to verify $0 < \zeta_n < \kappa_s^2$. Since $\hat{M}_n$ is a Hermitian matrix and $\kappa_s^2\gamma_p^{(n)} > 0$, it suffices to show $\det(\hat{M}_n) > 0$ according to Sylvester's rule. Direct calculation yields

$$\begin{aligned} \det(\hat{M}_n) &= \kappa_s^4\gamma_p^{(n)}\gamma_s^{(n)} - \alpha_n^2\left(\zeta_n-\kappa_s^2\right)^2 \\ &= \kappa_s^4\gamma_p^{(n)}\gamma_s^{(n)} - \alpha_n^2\left(\zeta_n^2 - 2\zeta_n\kappa_s^2 + \kappa_s^4\right) \\ &= \kappa_s^4\left(\gamma_p^{(n)}\gamma_s^{(n)} - \alpha_n^2\right) - \zeta_n\alpha_n^2\left(\zeta_n - 2\kappa_s^2\right) \\ &= \zeta_n\left(2\alpha_n^2\kappa_s^2 - \alpha_n^2\zeta_n - \kappa_s^4\right) > \zeta_n\left(\alpha_n^2\kappa_s^2 - \kappa_s^4\right) > 0. \end{aligned}$$

(2) If $|\alpha_n| < \kappa_p < \kappa_s$, then $\beta_p^{(n)} = \gamma_p^{(n)}$ and $\beta_s^{(n)} = \gamma_s^{(n)}$. It follows that

$$\hat{M}_n = \frac{\mathrm{i}\alpha_n\mu}{\zeta_n}\begin{bmatrix} 0 & \kappa_s^2 - \zeta_n \\ \zeta_n - \kappa_s^2 & 0 \end{bmatrix},$$

where $\zeta_n = \alpha_n^2 + \gamma_p^{(n)}\gamma_s^{(n)}$. It can be verified that $\kappa_p^2 < \zeta_n \le \kappa_p\kappa_s$. Hence, we have

$$\|\hat{M}_n\|_2 = \frac{\mu|\alpha_n|}{\zeta_n}\left|\kappa_s^2 - \zeta_n\right| < \frac{\mu}{\kappa_p}\left(\kappa_s^2 - \kappa_p^2\right) = \omega\left(\lambda+\mu\right)\left(\lambda+2\mu\right)^{-1/2}. \tag{2.14}$$

If $\kappa_p < |\alpha_n| < \kappa_s$, then $\beta_p^{(n)} = \mathrm{i}\gamma_p^{(n)}$ and $\beta_s^{(n)} = \gamma_s^{(n)}$. It follows that

$$\hat{M}_n = \frac{\mu}{\zeta_n}\begin{bmatrix} \alpha_n^2\kappa_s^2\gamma_p^{(n)} & \mathrm{i}\alpha_n\kappa_p^2\left(\gamma_s^{(n)}\right)^2 \\ -\mathrm{i}\alpha_n\kappa_p^2\left(\gamma_s^{(n)}\right)^2 & -\kappa_s^2\gamma_p^{(n)}\left(\gamma_s^{(n)}\right)^2 \end{bmatrix},$$

where

$$\zeta_n = \left|\alpha_n + \mathrm{i}\gamma_{\rm p}\gamma_{\rm s}\right|^2 = \alpha_n^2\left(\kappa_{\rm p}^2 + \kappa_{\rm s}^2\right) - \kappa_{\rm p}^2\kappa_{\rm s}^2.$$

It is easy to verify that $\kappa_{\rm p}^4 < \zeta_n < \kappa_{\rm s}^4$ and

$$\begin{aligned}\alpha_n^2\kappa_{\rm s}^2\gamma_{\rm p}^{(n)} &< \kappa_{\rm s}^4\left(\kappa_{\rm s}^2 - \kappa_{\rm p}^2\right)^{1/2},\\ |\alpha_n|\,\kappa_{\rm p}^2\left(\gamma_{\rm s}^{(n)}\right)^2 &< \kappa_{\rm s}\kappa_{\rm p}^2\left(\kappa_{\rm s}^2 - \kappa_{\rm p}^2\right) < \kappa_{\rm s}^4\left(\kappa_{\rm s}^2 - \kappa_{\rm p}^2\right)^{1/2},\\ \kappa_{\rm s}^2\gamma_{\rm p}^{(n)}\left(\gamma_{\rm s}^{(n)}\right)^2 &< \kappa_{\rm s}^2\left(\kappa_{\rm s}^2 - \kappa_{\rm p}^2\right)\left(\kappa_{\rm s}^2 - \kappa_{\rm p}^2\right)^{1/2} < \kappa_{\rm s}^4\left(\kappa_{\rm s}^2 - \kappa_{\rm p}^2\right)^{1/2}.\end{aligned}$$

It follows that

$$\|\hat{M}_n\|_2 \le 2\|\hat{M}_n\|_{\max} < \mu\kappa_{\rm p}^{-4}\kappa_{\rm s}^4\left(\kappa_{\rm s}^2 - \kappa_{\rm p}^2\right)^{1/2} = \omega\left(\frac{\lambda + 2\mu}{\mu}\right)^{3/2}(\lambda+\mu)^{1/2}, \qquad (2.15)$$

where $\|\cdot\|_{\max}$ denotes the matrix max norm. The proof is completed combining (2.14) and (2.15). □

Theorem 2.6: The variational problem (2.11) has a unique solution $\boldsymbol{u} \in H^1_{S,\rm p}(\Omega)^2$, which satisfies

$$\|\boldsymbol{u}\|_{1,\Omega} \lesssim \|\boldsymbol{\rho}\|_{-1/2,\Gamma}.$$

Proof: It follows from the Cauchy–Schwarz inequality, Lemmas 2.2, and 2.4 that

$$\begin{aligned}|a_\Omega(\boldsymbol{u},\boldsymbol{v})| &\lesssim \|\boldsymbol{u}\|_{1,\Omega}\|\boldsymbol{v}\|_{1,\Omega} + \|\mathscr{T}\boldsymbol{u}\|_{-1/2,\Gamma}\|\boldsymbol{v}\|_{1/2,\Gamma}\\ &\lesssim \|\boldsymbol{u}\|_{1,\Omega}\|\boldsymbol{v}\|_{1,\Omega} + \|\boldsymbol{u}\|_{1/2,\Gamma}\|\boldsymbol{v}\|_{1/2,\Gamma}\\ &\lesssim \|\boldsymbol{u}\|_{1,\Omega}\|\boldsymbol{v}\|_{1,\Omega},\end{aligned}$$

which shows that the sesquilinear form is bounded.

Following (2.9) yields

$$-\mathrm{Re}\,\langle\mathscr{T}\boldsymbol{u},\boldsymbol{u}\rangle_\Gamma = \Lambda\sum_{n\in\mathbb{Z}}\left(\hat{M}_n\boldsymbol{u}^{(n)}(h)\right)\cdot\bar{\boldsymbol{u}}^{(n)}(h).$$

We have from Lemma 2.5 that

$$\sum_{|\alpha_n|>\kappa_{\rm s}}\left(\hat{M}_n\boldsymbol{u}^{(n)}(h)\right)\cdot\bar{\boldsymbol{u}}^{(n)}(h) \ge 0$$

and

$$\begin{aligned}\left|\sum_{|\alpha_n|<\kappa_{\rm s}}\left(\hat{M}_n\boldsymbol{u}^{(n)}(h)\right)\cdot\bar{\boldsymbol{u}}^{(n)}(h)\right| &\le c(\mu,\lambda)\omega\sum_{|\alpha_n|<\kappa_{\rm s}}\left|\boldsymbol{u}^{(n)}(h)\right|^2\\ &\lesssim c(\mu,\lambda)\omega\sum_{|\alpha_n|<\kappa_{\rm s}}\left(1+\alpha_n^2\right)^{-1/2}\left|\boldsymbol{u}^{(n)}(h)\right|^2 \le c(\mu,\lambda)\omega\|\boldsymbol{u}\|^2_{-1/2,\Gamma}.\end{aligned}$$

Combining the above estimates, we get

$$-\mathrm{Re}\,\langle \mathscr{T}\boldsymbol{u}, \boldsymbol{u}\rangle_\Gamma \geq -c(\lambda,\mu)\omega\|\boldsymbol{u}\|_{-1/2,\Gamma},$$

which gives together with Lemmas 2.1, and 2.3 that

$$\begin{aligned}
\mathrm{Re}[a_\Omega(\boldsymbol{u},\boldsymbol{u})] &= \mu\|\nabla\boldsymbol{u}\|_{0,\Omega}^2 + (\lambda+\mu)\|\mathrm{div}\,\boldsymbol{u}\|_{0,\Omega}^2 - \omega^2\|\boldsymbol{u}\|_{0,\Omega}^2 - \mathrm{Re}\,\langle\mathscr{T}\boldsymbol{u},\boldsymbol{u}\rangle_\Gamma \\
&\geq \mu\|\nabla\boldsymbol{u}\|_{0,\Omega}^2 - \omega^2\|\boldsymbol{u}\|_{0,\Omega}^2 - c(\mu,\lambda)\omega\|\boldsymbol{u}\|_{-1/2,\Gamma}^2 \\
&\geq \mu\|\nabla\boldsymbol{u}\|_{0,\Omega}^2 - \omega^2\|\boldsymbol{u}\|_{0,\Omega}^2 - c(\mu,\lambda)\omega\eta^{-1}\|\boldsymbol{u}\|_{0,\Omega}^2 - c(\lambda,\mu)\omega\eta\|\partial_y\boldsymbol{u}\|_{0,\Omega}^2 \\
&\geq (\mu - c(\mu,\lambda)\omega\eta)\|\nabla\boldsymbol{u}\|_{0,\Omega}^2 - \left(\omega^2 + c(\mu,\lambda)\eta^{-1}\right)\|\boldsymbol{u}\|_{0,\Omega}^2 \\
&\geq \left[(\mu - c(\mu,\lambda)\omega\eta) - \left(\omega^2 + c(\mu,\lambda)\omega\eta^{-1}\right)h^2\right]\|\nabla\boldsymbol{u}\|_{0,\Omega}^2 \\
&\geq \left[(\mu - c(\mu,\lambda)\omega\eta) - \left(\omega^2 + c(\mu,\lambda)\omega\eta^{-1}\right)h^2\right]\left(1+h^2\right)^{-1}\|\boldsymbol{u}\|_{1,\Omega}^2,
\end{aligned}$$

where $\eta > 0$ is an arbitrary constant. Taking $\eta = \mu/(2c(\mu,\lambda)\omega)$ yields

$$\mathrm{Re}[a_\Omega(\boldsymbol{u},\boldsymbol{u})] \geq \left[\frac{\mu}{2} - \left(1 + \frac{2c(\mu,\lambda)^2}{\mu}\right)\omega^2 h^2\right]\left(1+h^2\right)^{-1}\|\boldsymbol{u}\|_{1,\Omega}^2.$$

Hence, the sesquilinear form $a_\Omega(\cdot,\cdot)$ is coercive if the hypothesis (H1) is satisfied. The proof is completed by using the Lax–Milgram lemma. □

The well-posedness of the direct problem has been proved under more general conditions in (6) for smooth surfaces and in (8) for Lipschitz surfaces. We provide a simpler proof due to the special feature of the problem: the scattering surface is a small perturbation of a planar surface and thus the measurement distance can be chosen to be small. Moreover, the proof gives an energy estimate that is necessary to the subsequent analysis of the convergence for the direct problem and the error estimate for the inverse problem.

2.3. Transformed field expansion

Consider the change of variables

$$\tilde{x} = x, \quad \tilde{y} = h\left(\frac{y-f}{h-f}\right),$$

which maps the domain Ω to the rectangle $D = (0,\Lambda)\times(0,h)$.

Let $\tilde{\boldsymbol{u}}(\tilde{x},\tilde{y}) = \boldsymbol{u}(x,y)$. Under the change of variables, the Navier Equation (2.2), upon dropping the tildes, reduces to the following equations in D:

$$\begin{cases} \left(c_{1,1}\partial_{xx} + c_{1,2}\partial_{yy} + c_{1,3}\partial_{xy} + c_{1,4}\partial_y + c_{1,5}\right) u_1 \\ + \left(c_{1,6}\partial_{xy} + c_{1,7}\partial_y + c_{1,8}\partial_{yy}\right) u_2 = 0, \\ \left(c_{2,1}\partial_{xx} + c_{2,2}\partial_{yy} + c_{2,3}\partial_{xy} + c_{2,4}\partial_y + c_{2,5}\right) u_2 \\ + \left(c_{2,6}\partial_{xy} + c_{2,7}\partial_y + c_{2,8}\partial_{yy}\right) u_1 = 0, \end{cases} \tag{2.16}$$

where

$$\begin{aligned}
c_{1,1} &= (\lambda + 2\mu)(h - f)^2, & c_{1,2} &= (\lambda + 2\mu)[f'(h - y)]^2 + \mu h^2, \\
c_{1,3} &= (\lambda + 2\mu)[-2f'(h - y)(h - f)], & c_{1,4} &= -(\lambda + 2\mu)(h - y)[f''(h - f) + 2(f')^2], \\
c_{1,5} &= \omega^2(h - f)^2, & c_{1,6} &= (\lambda + \mu)h(h - f), \\
c_{1,7} &= (\lambda + \mu)f'h, & c_{1,8} &= -(\lambda + \mu)f'h(h - y), \\
c_{2,1} &= \mu(h - f)^2, & c_{2,2} &= \mu[f'(h - y)]^2 + (\lambda + 2\mu)h^2, \\
c_{2,3} &= \mu[-2f'(h - y)(h - f)], & c_{2,4} &= -\mu(h - y)[f''(h - f) + 2(f')^2], \\
c_{2,5} &= \omega^2(h - f)^2, & c_{2,6} &= (\lambda + \mu)h(h - f), \\
c_{2,7} &= (\lambda + \mu)f'h, & c_{2,8} &= -(\lambda + \mu)f'h(h - y).
\end{aligned}$$

The Dirichlet boundary condition (2.3) can be written as

$$\boldsymbol{u} = 0 \quad \text{on } y = 0. \tag{2.17}$$

The transparent boundary condition (2.8) becomes

$$\begin{aligned} &\mu\partial_y \boldsymbol{u} + (\lambda + \mu)[0,\ 1]\nabla\cdot\boldsymbol{u} \\ &\quad = (1 - h^{-1}f)\mathscr{T}\boldsymbol{u} + (\lambda + \mu)h^{-1}f\,[0,\ \ \partial_x u_1] \quad \text{on } y = h. \end{aligned} \tag{2.18}$$

Recalling the small perturbation assumption $f = \varepsilon g$, we consider a formal expansion of $\boldsymbol{u}$ as a power series of ε:

$$\boldsymbol{u}(x, y; \varepsilon) = \sum_{m=0}^{\infty} \boldsymbol{u}_m(x, y)\varepsilon^m. \tag{2.19}$$

Substituting (2.19) into (2.16)–(2.18) and $f = \varepsilon g$ into the coefficients $c_{i,j}$, we obtain a successive sequence of boundary value problems:

$$\begin{cases} \mu\Delta\boldsymbol{u}_m + (\lambda + \mu)\nabla\nabla\cdot\boldsymbol{u}_m + \omega^2\boldsymbol{u}_m = \boldsymbol{r}_m & \text{in } D, \\ \boldsymbol{u}_m = \boldsymbol{0} & \text{on } y = 0, \\ \mu\partial_y\boldsymbol{u}_m + (\lambda + \mu)[0,\ 1]\nabla\cdot\boldsymbol{u}_m = \mathscr{T}\boldsymbol{u}_m + \boldsymbol{\rho}_m & \text{on } y = h, \end{cases} \tag{2.20}$$

where

$$\begin{cases} \boldsymbol{\rho}_0 = \boldsymbol{\rho} = 2\mathrm{i}\kappa_{\mathrm{p}}(\lambda + 2\mu)[0, 1]e^{-\mathrm{i}\kappa_{\mathrm{p}}h}, \\ \boldsymbol{\rho}_1 = -gh^{-1}(\mathscr{T}\boldsymbol{u}_0 + \boldsymbol{\rho}) + (\lambda + \mu)gh^{-1}\,[0,\ \ \partial_x u_{1,0}], \\ \boldsymbol{\rho}_m = -gh^{-1}\mathscr{T}\boldsymbol{u}_{m-1} + (\lambda + \mu)gh^{-1}[0,\ \ \partial_x u_{1,m-1}], \quad m \geq 2, \end{cases} \tag{2.21}$$

and $\boldsymbol{r}_m = [r_{1,m},\ r_{2,m}]$ with

$$\begin{aligned} r_{1,m} = {}& h^{-1}\left[2(\lambda + 2\mu)g\partial_{xx} + 2(\lambda + 2\mu)g'(h - y)\partial_{xy} \right. \\ & \left. + (\lambda + 2\mu)g''(h - y)\partial_y + 2g\omega^2\right] u_{1,m-1} \\ & + h^{-1}\left[(\lambda + \mu)g\partial_{xy} - (\lambda + \mu)g'\partial_y + (\lambda + \mu)g'(h - y)\partial_{yy}\right] u_{2,m-1} \\ & + h^{-2}\left\{-(\lambda + 2\mu)g^2\partial_{xx} - (\lambda + 2\mu)\left(g'\right)^2(h - y)^2\partial_{yy} - 2(\lambda + 2\mu)gg'(h - y)\partial_{xy}\right. \end{aligned}$$

$$+\ (\lambda+2\mu)\left[2\left(g'\right)^2-gg''\right](h-y)\partial_y-\omega^2g^2\Big\}\,u_{1,m-2}, \tag{2.22}$$

$$\begin{aligned} r_{2,m} = {} & h^{-1}\left[2\mu g\partial_{xx}+2\mu g'(h-y)\partial_{xy}+\mu g''(h-y)\partial_y+2g\omega^2\right]u_{2,m-1} \\ & + h^{-1}\left[(\lambda+\mu)g\partial_{xy}-(\lambda+\mu)g'\partial_y+(\lambda+\mu)g'(h-y)\partial_{yy}\right]u_{1,m-1} \\ & + h^{-2}\Big\{-\mu g^2\partial_{xx}-\mu\left(g'\right)^2(h-y)^2\partial_{yy}-2\mu gg'(h-y)\partial_{xy} \\ & + \mu(2\left(g'\right)^2-gg'')(h-y)\partial_y-\omega^2g^2\Big\}\,u_{2,m-2}. \end{aligned} \tag{2.23}$$

It is understood that $\boldsymbol{u}_m = 0$ if $m < 0$ in the above recurrence relations. Note that the boundary value problem for $\boldsymbol{u}_m$ depends only on the solutions of $\boldsymbol{u}_{m-1}$ and $\boldsymbol{u}_{m-2}$. The recursive boundary value problem (2.20) is mainly used for the mathematical analysis of the direct and inverse problems. The derivation of the reconstruction formula is based on the transformed field expansion of the scalar potential functions ϕ and ψ.

2.4. Convergence of power series expansion

In this section, we shall prove the well-posedness of the boundary value problem (2.20) and show the convergence of the power series solution (2.19).

Define a periodic Sobolev space:

$$H^1_{0,\mathrm{p}}(D) = \{u \in H^1(D) : u(x,0) = 0,\ u(0,y) = u(\Lambda,y)\}.$$

Let $H^1_{0,\mathrm{p}}(D)^2 = H^1_{0,\mathrm{p}}(D) \times H^1_{0,\mathrm{p}}(D)$. Denote by $H^{-1}_{0,\mathrm{p}}(D)^2$ the dual space of $H^1_{0,\mathrm{p}}(D)^2$. Similarly, we can deduce the variational problem for (2.20): to find $\boldsymbol{u}_m \in H^1_{0,\mathrm{p}}(D)^2$ such that

$$a_D(\boldsymbol{u}_m,\boldsymbol{v}) = b_m(\boldsymbol{v}) \quad \text{for all } \boldsymbol{v} \in H^1_{0,\mathrm{p}}(D)^2, \tag{2.24}$$

where the sesquilinear form a_D is defined by (2.12) with Ω replaced by D and the linear functional

$$b_m(\boldsymbol{v}) = \langle \boldsymbol{\rho}_m,\boldsymbol{v}\rangle_\Gamma - (\boldsymbol{r}_m,\boldsymbol{v})_D.$$

Following the same proof of Theorem 2.6, we may show the well-posedness of the solution for the variational problem (2.24).

Theorem 2.7: The variational problem (2.24) has a unique solution $\boldsymbol{u}_m \in H^1_{0,\mathrm{p}}(D)^2$, which satisfies

$$\|\boldsymbol{u}_m\|_{1,D} \lesssim \|\boldsymbol{\rho}_m\|_{-1/2,\Gamma} + \|\boldsymbol{r}_m\|_{-1,D}.$$

In order to show the convergence of the power series, it is necessary to estimate $\|\boldsymbol{\rho}_m\|_{-1/2,\Gamma}$ and $\|\boldsymbol{r}_m\|_{-1,D}$. The proof of the following lemma can be found in (17).

Lemma 2.8: It holds the estimate

$$\|g\boldsymbol{v}\|_{1/2,\Gamma} \lesssim K\|\boldsymbol{v}\|_{1/2,\Gamma} \quad \text{for any } \boldsymbol{v} \in H^{1/2}(\Gamma)^2.$$

Lemma 2.9: It holds the estimate

$$\|\boldsymbol{\rho}_m\|_{-1/2,\Gamma} \lesssim Kh^{-1}\|\boldsymbol{u}_{m-1}\|_{1,D}, \quad m \geq 2.$$

Proof: It follows from (2.2), Lemmas 2.4 and 2.8 that we have

$$\begin{aligned}
\left|\langle \boldsymbol{\rho}_m, \boldsymbol{v}\rangle_\Gamma\right| &\le h^{-1}\left|\langle g\mathscr{T}\boldsymbol{u}_{m-1}, \boldsymbol{v}\rangle_\Gamma\right| + h^{-1}(\lambda+\mu)\left|\langle g[0,\ \partial_x u_{1,m-1}], \boldsymbol{v}\rangle_\Gamma\right| \\
&= h^{-1}\left|\langle \mathscr{T}\boldsymbol{u}_{m-1}, g\boldsymbol{v}\rangle_\Gamma\right| + h^{-1}(\lambda+\mu)\left|\langle [0,\ \partial_x u_{1,m-1}], g\boldsymbol{v}\rangle_\Gamma\right| \\
&\lesssim h^{-1}\left(\|\mathscr{T}\boldsymbol{u}_{m-1}\|_{-1/2,\Gamma}\|g\boldsymbol{v}\|_{1/2,\Gamma} + \left\|[0,\ \partial_x u_{1,m-1}]\right\|_{-1/2,\Gamma}\|g\boldsymbol{v}\|_{1/2,\Gamma}\right) \\
&\lesssim Kh^{-1}\left(\|\boldsymbol{u}_{m-1}\|_{1/2,\Gamma}\|\boldsymbol{v}\|_{1/2,\Gamma} + \|\boldsymbol{u}_{m-1}\|_{1/2,\Gamma}\|\boldsymbol{v}\|_{1/2,\Gamma}\right) \\
&\lesssim Kh^{-1}\|\boldsymbol{u}_{m-1}\|_{1,D}\|\boldsymbol{v}\|_{1/2,\Gamma} \quad \text{for any } \boldsymbol{v}\in H^{1/2}(\Gamma)^2,
\end{aligned}$$

which completes the proof. □

Lemma 2.10: It holds the estimate

$$\|\boldsymbol{r}_m\|_{-1,D} \lesssim \left(Kh^{-1}\right)\|\boldsymbol{u}_{m-1}\|_{1,D} + \left(Kh^{-1}\right)^2\|\boldsymbol{u}_{m-2}\|_{1,D}.$$

Proof: It is easy to verify that

$$\left\|\frac{\mathrm{d}^j g}{\mathrm{d}x^j}v\right\|_{0,D} \le K\|v\|_{0,D}, \quad \left\|\frac{\mathrm{d}^j g}{\mathrm{d}x^j}v\right\|_{1,D} \le K\|v\|_{1,D}, \quad j=0,1,2,$$

for any $v\in H^1(D)$ and $g\in C^k(\mathbb{R}), k\ge 2$. It also holds that

$$\|(h-y)v\|_{0,D} \lesssim \|v\|_{0,D}, \quad \|(h-y)v\|_{1,D} \lesssim \|v\|_{1,D},$$

for any $v\in H^1(D)$.

Combining the above estimates, Lemmas 2.8, and 2.2, we have from the integration by parts for any $u\in H^1_{0,\mathrm{p}}(D), v\in H^1_{0,\mathrm{p}}(D)$ that

$$\begin{aligned}
\left|\left(g\partial_{xx}u, v\right)_D\right| &= \left|\left(\partial_{xx}u, gv\right)_D\right| = \left|\left(\partial_x u, \partial_x\left(gv\right)\right)_D\right| \le K\|u\|_{1,D}\|v\|_{1,D}, \\
\left|\left(g'(h-y)\partial_{xy}u, v\right)_D\right| &= \left|\left(\partial_{xy}u, g'(h-y)v\right)_D\right| = \left|\left(\partial_y u, (h-y)\partial_x(g'v)\right)_D\right| \\
&\lesssim K\|u\|_{1,D}\|v\|_{1,D}, \\
\left|\left(g''(h-y)\partial_y u, v\right)_D\right| &\le \|g''\partial_y u\|_{0,D}\|v\|_{0,D} \lesssim K\|u\|_{1,D}\|v\|_{1,D}, \\
\left|\left(gu, v\right)_D\right| &\le K\|u\|_{1,D}\|v\|_{1,D}, \\
\left|\left(g\partial_{xy}u, v\right)_D\right| &= \left|\left(\partial_y u, \partial_x(gv)\right)_D\right| \le K\|u\|_{1,D}\|v\|_{1,D}, \\
\left|\left(g'\partial_y u, v\right)_D\right| &\le K\|u\|_{1,D}\|v\|_{1,D}, \\
\left|\left(g'(h-y)\partial_{yy}u, v\right)_D\right| &\le \left|\left(\partial_y u, g'\partial_y\left[(h-y)v\right]\right)_D\right| + \left|\langle \partial_y u, g'(h-y)v\rangle_\Gamma\right| \\
&= \left|\left(\partial_y u, g'\partial_y\left[(h-y)v\right]\right)_D\right| \le K\|u\|_{1,D}\|v\|_{1,D}.
\end{aligned}$$

Using the integration by parts, we have for any $u\in H^1_{0,\mathrm{p}}(D), v\in H^1_{0,\mathrm{p}}(D)$ that

$$\begin{aligned}
\left|\left(g^2\partial_{xx}u, v\right)_D\right| &\lesssim K^2\|u\|_{1,D}\|v\|_{1,D}, \\
\left|\left(\left(g'\right)^2(h-y)^2\partial_{xx}u, v\right)_D\right| &\lesssim K^2\|u\|_{1,D}\|v\|_{1,D}, \\
\left|\left(gg'(h-y)\partial_{xy}u, v\right)_D\right| &\lesssim K^2\|u\|_{1,D}\|v\|_{1,D}, \\
\left|\left(\left[2\left(g'\right)^2 - gg''\right](h-y)\partial_y u, v\right)_D\right| &\lesssim K^2\|u\|_{1,D}\|v\|_{1,D}, \\
\left|\left(g^2 u, v\right)_D\right| &\lesssim K^2\|u\|_{1,D}\|v\|_{1,D}.
\end{aligned}$$

Combining the above estimates yields

$$\left|\left(r_{1,m}, v\right)_D\right| \lesssim \left[\left(Kh^{-1}\right)\|\boldsymbol{u}_{m-1}\|_{1,D} + \left(Kh^{-1}\right)^2\|\boldsymbol{u}_{m-2}\|_{1,D}\right]\|v\|_{1,D}$$

for any $v \in H^1_{0,\mathrm{p}}(D)$.

Similarly, we can show that

$$\left|\left(r_{2,m}, v\right)_D\right| \lesssim \left[\left(Kh^{-1}\right)\|\boldsymbol{u}_{m-1}\|_{1,D} + \left(Kh^{-1}\right)^2\|\boldsymbol{u}_{m-2}\|_{1,D}\right]\|v\|_{1,D}$$

for any $v \in H^1_{0,\mathrm{p}}(D)$. Hence, we obtain

$$\left|\left(\boldsymbol{r}_m, \boldsymbol{v}\right)_D\right| \lesssim \left[\left(Kh^{-1}\right)\|\boldsymbol{u}_{m-1}\|_{1,D} + \left(Kh^{-1}\right)^2\|\boldsymbol{u}_{m-2}\|_{1,D}\right]\|\boldsymbol{v}\|_{1,D}$$

for any $\boldsymbol{v} \in H^1_{0,\mathrm{p}}(D)^2$, which completes the proof. □

Theorem 2.11: Let $\boldsymbol{u}_m$ be the solution of the variational problem (2.20). It satisfies

$$\|\boldsymbol{u}_m\|_{1,D} \leq \left(cKh^{-1}\right)^m, \quad m \geq 0,$$

where $c > 0$ is a constant independent of K, m, ε, h.

Proof: Combining Theorem 2.7, Lemmas 2.9, and 2.10 yields

$$\|\boldsymbol{u}_m\|_{1,D} \lesssim \left[\left(Kh^{-1}\right)\|\boldsymbol{u}_{m-1}\|_{1,D} + \left(Kh^{-1}\right)^2\|\boldsymbol{u}_{m-2}\|_{1,D}\right].$$

Hence there exists a constant $\tilde{c}$ which is independent of K, m, ε, h such that

$$\|\boldsymbol{u}_m\|_{1,D} \leq \tilde{c}\left[\left(Mh^{-1}\right)\|\boldsymbol{u}_{m-1}\|_{1,D} + \left(Mh^{-1}\right)^2\|\boldsymbol{u}_{m-2}\|_{1,D}\right].$$

Consider the following recurrence inequality:

$$x_m \leq \tilde{c}\left(ax_{m-1} + a^2x_{m-2}\right),$$

where $\tilde{c}, a, x_0, x_1$ are nonnegative. The goal is to show that there exists a constant $c > 0$ such that

$$x_m \leq (ca)^m, \quad m \geq 0.$$

It suffices to prove that

$$\tilde{c}\left[a(ca)^{m-1} + a^2(ca)^{m-2}\right] \leq (ca)^m,$$

which leads to an inequality:

$$\tilde{c}(1 + c) \leq c^2.$$

The proof is completed by taking $c \geq \frac{1}{2}\left[\tilde{c} + \left(\tilde{c}^2 + 4\tilde{c}\right)^{1/2}\right]$. □

The following convergence result follows immediately.

Theorem 2.12: The power series solution (2.19) converges strongly.

Proof: It follows from Theorem 2.11 that we have

$$\left\|\boldsymbol{u}_m\varepsilon^m\right\|_{1,D} \leq \left(cK\varepsilon h^{-1}\right)^m.$$

It follows from the dominated convergence theorem that the power series solution (2.19) converges if the hypothesis (H2) is satisfied. □

3. Inverse Scattering

In this section, we give a local uniqueness result and establish an error estimate for the inverse problem.

3.1. Uniqueness

Let $G \subset \mathbb{R}^2$ be a bounded domain with Lipschitz boundary ∂G. Define the depth of G with respect to the y variable:

$$\mathrm{dep}(G) = \sup\{|y_1 - y_2| : (x_1, y_1), (x_2, y_2) \in G\}.$$

Lemma 3.1: The boundary value problem

$$\begin{cases} \mu\Delta \boldsymbol{u} + (\lambda+\mu)\nabla\nabla\cdot\boldsymbol{u} + \omega^2\boldsymbol{u} = 0 & \text{in } G, \\ \boldsymbol{u} = 0 & \text{on } \partial G \end{cases}$$

has only the trivial solution if $\omega\,\mathrm{dep}(G) < \mu^{1/2}$.

Proof: Let $\boldsymbol{u} \neq 0$ be a solution to the boundary value problem. It is easy to verify that the solution $\boldsymbol{u}$ satisfies

$$\mu\|\nabla\boldsymbol{u}\|_{0,D}^2 + (\lambda+\mu)\|\mathrm{div}\,\boldsymbol{u}\|_{0,D}^2 = \omega^2\|\boldsymbol{u}\|_{0,D}^2.$$

Following the proof of Lemma 2.1, we have

$$\|\boldsymbol{u}\|_{0,G} \leq \mathrm{dep}(G)\|\nabla\boldsymbol{u}\|_{0,G}.$$

Combining the above estimates yields $\omega\mathrm{dep}(G) \geq \mu^{1/2}$, which contradicts with the assumption. □

Theorem 3.2: Let $f_j = \varepsilon g_j, g_j \in C^k(\mathbb{R}), j = 1, 2$ be periodic functions with period Λ. Define $\Omega_j = \{(x, y) \in \mathbb{R}^2 : 0 < x < \Lambda, f_j < y < h\}$. Let $\boldsymbol{u}_j$ be the unique weak solution of (2.11) in Ω_j. If ε is sufficiently small and $\boldsymbol{u}_1 = \boldsymbol{u}_2$ on Γ, then $f_1 = f_2$.

Proof: Define $S_j = \{(x, y) \in \mathbb{R}^2 : y = f_j(x), 0 < x < \Lambda\}$. Let $\Omega = \Omega_1 \cap \Omega_2$. Suppose $f_1 \neq f_2$, then $\Omega_1\backslash\Omega$ or $\Omega_2\backslash\Omega$ is a nonempty set. Without loss of generality, let $G = \Omega_1\backslash\Omega$ be a nonempty bounded set. Let $\partial G = C_1 \cup C_2$ with $C_j \subset S_j$. It is easy to see that

$$\mathrm{dep}(G) \leq \varepsilon \max\{\|g_1\|_\infty, \|g_2\|_\infty\},$$

which gives $\omega\mathrm{dep}(G) < \mu^{1/2}$ for sufficiently small ε.

Let $\boldsymbol{u} = \boldsymbol{u}_1 - \boldsymbol{u}_2$, and let $\phi_1, \psi_1, \phi_2, \psi_2$, and ϕ, ψ be the scalar potential function for $\boldsymbol{u}_1$, $\boldsymbol{u}_2$, and $\boldsymbol{u}$, respectively, where $\phi = \phi_1 - \phi_2$ and $\psi = \psi_1 - \psi_2$. It follows from the Helmholtz decomposition (2.4) and the transparent boundary conditions (2.7) that

$$\boldsymbol{u}^{(n)}(h) = \mathrm{i}\begin{bmatrix} \alpha_n & \beta_{\mathrm{s}}^{(n)} \\ \beta_{\mathrm{p}}^{(n)} & -\alpha_n \end{bmatrix}\begin{bmatrix} \phi^{(n)}(h) \\ \psi^{(n)}(h) \end{bmatrix}.$$

Since $\boldsymbol{u} = 0$ at Γ and $\alpha_n^2 + \beta_{\mathrm{p}}^{(n)}\beta_{\mathrm{s}}^{(n)} \neq 0$ for all $n \in \mathbb{Z}$, we have $\phi = \psi = 0$ at Γ. Using the transparent boundary conditions (2.7) again implies $\partial_y\phi = \partial_y\psi = 0$ on Γ. It follows from the Rayleigh expansion or the Holmgren uniqueness theorem that $\phi = \psi = 0$ above Γ. By unique continuation, we have $\phi = \psi = 0$ in $\bar{\Omega}$. It follows from the Helmholtz decomposition again that $\boldsymbol{u} = 0$ in $\bar{\Omega}$, and in particular $\boldsymbol{u} = 0$ on C_2. It follows from $\boldsymbol{u}_2 = 0$ on C_2 that we have $\boldsymbol{u}_1 = 0$ on C_2 and the following boundary value problem:

$$\begin{cases} \mu\Delta\boldsymbol{u}_1 + (\lambda+\mu)\nabla\nabla\cdot\boldsymbol{u}_1 + \omega^2\boldsymbol{u}_1 = 0 & \text{in } G, \\ \boldsymbol{u}_1 = 0 & \text{on } \partial G. \end{cases}$$

According to Lemma 3.1, the above boundary value problem only has a trivial solution $\boldsymbol{u}_1 = 0$ in G. An application of the Helmholtz decomposition and the unique continuation gives $\boldsymbol{u}_1 = 0$ in Ω_1. But this contradicts the nonhomogeneous transparent boundary condition (2.8). □

3.2. The reconstruction formula

Let us briefly present the explicit reconstruction formula for the inverse problem. We refer to (17) for the details. In Section 2.3, we studied the power series expansion of $\boldsymbol{u}$ and derived a successive sequence of boundary value problems for $\boldsymbol{u}_m$. However, it is inconvenient to use the expansion of $\boldsymbol{u}$ for the inverse problem due to the coupling in the equations. Instead, we study the scalar potential functions ϕ, ψ via the Helmholtz decomposition (2.4).

Consider the power series expansion:

$$\phi(x,y;\varepsilon) = \sum_{m=0}^{\infty}\phi_m(x,y)\varepsilon^m, \quad \psi(x,y;\varepsilon) = \sum_{m=0}^{\infty}\psi_m(x,y)\varepsilon^m.$$

Applying the transformed field expansion and the Fourier series expansion, we may obtain a successive sequence of coupled two-point boundary value problems:

$$\begin{cases} \partial_{yy}\phi_k^{(n)} + \left(\beta_{\rm p}^{(n)}\right)^2\phi_k^{(n)} = v_k^{(n)}, & 0 < y < h, \\ \partial_y\phi_k^{(n)} = p_k^{(n)} + {\rm i}\alpha_n\psi_k^{(n)}, & y = 0, \\ \partial_y\phi_k^{(n)} - {\rm i}\beta_{\rm p}^{(n)}\phi_k^{(n)} = r_k^{(n)}, & y = h, \end{cases}$$

and

$$\begin{cases} \partial_{yy}\psi_k^{(n)} + \left(\beta_{\rm s}^{(n)}\right)^2\psi_k^{(n)} = w_k^{(n)}, & 0 < y < h, \\ \partial_y\psi_k^{(n)} = q_k^{(n)} - {\rm i}\alpha_n\phi_k^{(n)}, & y = 0, \\ \partial_y\psi_k^{(n)} - {\rm i}\beta_{\rm s}^{(n)}\psi_k^{(n)} = s_k^{(n)}, & y = h, \end{cases}$$

where the inhomogeneous terms $v_k^{(n)}, p_k^{(n)}, r_k^{(n)}$ and $w_k^{(n)}, q_k^{(n)}, s_k^{(n)}$ can be computed from ϕ_{k-1}, ϕ_{k-2} and ψ_{k-1}, ψ_{k-2}, respectively.

Using variation of parameters, we can solve the above recursive system and obtain its solution analytically. In particular, the leading terms are given by

$$\phi_0(x,y) = 2({\rm i}\kappa_{\rm p})^{-1}\cos(\kappa_{\rm p}y), \quad \psi_0(x,y) = 0.$$

Using the Helmholtz decomposition, we may obtain the leading term

$$\boldsymbol{u}_0 = \left[u_{1,0},\ u_{2,0}\right] = \left[0,\ e^{{\rm i}\kappa_{\rm p}y} - e^{-{\rm i}\kappa_{\rm p}y}\right].$$

The Fourier coefficients of the linear terms are given by

$$\phi_1^{(n)}(h) = \left(\frac{-2\kappa_{\rm p}\beta_{\rm s}^{(n)}}{\alpha_n^2 + \beta_{\rm p}^{(n)}\beta_{\rm s}^{(n)}} \right) e^{{\rm i}\beta_{\rm p}^{(n)}h} g^{(n)}, \quad \psi_1^{(n)}(h) = \left(\frac{2\kappa_{\rm p}\alpha_n}{\alpha_n^2 + \beta_{\rm p}^{(n)}\beta_{\rm s}^{(n)}} \right) e^{{\rm i}\beta_{\rm s}^{(n)}h} g^{(n)},$$

where $g^{(n)}$ is the Fourier coefficients of the surface profile function g. The following key identity is also derived:

$$\alpha_n \left(\beta_{\rm s}^{(n)}\right)^{-1} u_{1,1}^{(n)}(h) + u_{2,1}^{(n)}(h) = -2{\rm i}\kappa_{\rm p} e^{{\rm i}\beta_{\rm p}^{(n)}h} g^{(n)},$$

where $\boldsymbol{u}_1 = [u_{1,1}, u_{2,1}]$ is the linear term in the power series expansion of $\boldsymbol{u}$. Substituting the above equations into the power series expansion for $\boldsymbol{u}$ yields the representation of the Fourier coefficients of the scattering surface function f:

$$\begin{aligned} f^{(n)} = & -(2{\rm i}\kappa_{\rm p})^{-1} \Big[\alpha_n \left(\beta_{\rm s}^{(n)}\right)^{-1} u_1^{(n)}(h) + u_2^{(n)}(h) - u_{2,0}^{(n)}(h) \\ & -\alpha_n(\beta_{\rm s}^{(n)})^{-1} e_1^{(n)}(h) - e_2^{(n)}(h) \Big] e^{-{\rm i}\beta_{\rm p}^{(n)}h}, \end{aligned} \tag{3.1}$$

where the remainder $\boldsymbol{e} = [e_1, e_2]$ has the expansion:

$$\boldsymbol{e}(x,y) = \sum_{m=2}^{\infty} \boldsymbol{u}_m(x,y)\varepsilon^m.$$

Dropping the remainder in (3.1) yields the reconstruction of the Fourier coefficients

$$f_\varepsilon^{(n)} = -(2{\rm i}\kappa_{\rm p})^{-1} \left[\alpha_n \left(\beta_{\rm s}^{(n)}\right)^{-1} u_1^{(n)}(h) + u_2^{(n)}(h) - u_{2,0}^{(n)}(h) \right] e^{-{\rm i}\beta_{\rm p}^{(n)}h}.$$

The above formula assumes a noise-free data. In practice, the data contains a certain amount of noise. Let $\boldsymbol{u}^\delta$ be the noisy data such that

$$\|\boldsymbol{u}^\delta - \boldsymbol{u}\|_{0,\Gamma} \le \delta, \tag{3.2}$$

where $\delta > 0$ represents the noise level. After incorporating the data noise, the Fourier coefficients take the form

$$f_{\varepsilon,\delta}^{(n)} = -(2{\rm i}\kappa_{\rm p})^{-1} \left[\alpha_n \left(\beta_{\rm s}^{(n)}\right)^{-1} u_1^{\delta,(n)}(h) + u_2^{\delta,(n)}(h) - u_{2,0}^{(n)}(h) \right] e^{-{\rm i}\beta_{\rm p}^{(n)}h}. \tag{3.3}$$

The reconstructed surface profile function is finally obtained by

$$f_{\varepsilon,\delta}(x) = \sum_{|\alpha_n| \le \kappa_{\rm c}} f_{\varepsilon,\delta}^{(n)} e^{{\rm i}\alpha_n x}, \tag{3.4}$$

where $\kappa_{\rm c} > 0$ is the cut-off wavenumber and plays the role of the regularization parameter.

3.3. Error estimate

In this section, we derive an error estimate for the explicit reconstruction formula (3.4). We need the following standard result about the decaying rate of the Fourier coefficients of a smooth function. The proof may be found in (22).

Lemma 3.3: If $g \in C^k(\mathbb{R})$ is periodic with period Λ, then

$$\left|g^{(n)}\right| \le \frac{K}{|\alpha_n|^k}.$$

Lemma 3.4: It holds the estimate

$$\left|\alpha_n \left(\beta_{\mathrm{s}}^{(n)}\right)^{-1}\right| \lesssim 1, \quad n \in \mathbb{Z}. \tag{3.5}$$

Proof: It follows from (2.6) that

$$\left|\alpha_n \left(\beta_{\mathrm{s}}^{(n)}\right)^{-1}\right|^2 = \frac{\alpha_n^2}{\left|\kappa_{\mathrm{s}}^2 - \alpha_n^2\right|}.$$

Consider the function

$$z(t) = \frac{t}{|\kappa_{\mathrm{s}}^2 - t|}, \quad t \ge 0.$$

It is easy to verify that $z(t)$ is increasing for $t < \kappa_{\mathrm{s}}^2$ and decreasing for $t > \kappa_{\mathrm{s}}^2$. Let n_- be the greatest integer such that $\alpha_n < \kappa_{\mathrm{s}}$ and n_+ be the smallest integer such that $\alpha_n > \kappa_{\mathrm{s}}$. We have

$$\left|\alpha_n \left(\beta_{\mathrm{s}}^{(n)}\right)^{-1}\right|^2 \le \max\left(\frac{\alpha_{n_-}^2}{\kappa_{\mathrm{s}}^2 - \alpha_{n_-}^2}, \frac{\alpha_{n_+}^2}{\alpha_{n_+}^2 - \kappa_{\mathrm{s}}^2}\right),$$

which completes the proof. □

Theorem 3.5: Let f be the exact surface function and $f_{\varepsilon,\delta}$ be the reconstructed surface function using (3.4). It holds the error estimate

$$\|f_{\varepsilon,\delta} - f\|_{0,\Gamma} \lesssim \left|e^{h\left(\kappa_{\mathrm{c}}^2 - \kappa_{\mathrm{p}}^2\right)^{1/2}}\right| \left[\delta + \left(K\varepsilon h^{-1}\right)^2\right] + K\varepsilon\kappa_{\mathrm{c}}^{-(2k-1)/2}. \tag{3.6}$$

Proof: It follows from (3.1), (3.3), and (3.4) that

$$\|f_{\varepsilon,\delta} - f\|_{0,\Gamma}^2 \lesssim E_1 + E_2 + E_3, \tag{3.7}$$

where

$$E_1 = \sum_{|\alpha_n| \le \kappa_{\mathrm{c}}} \left|e^{\mathrm{i}\beta_{\mathrm{p}}^{(n)} h}\right|^2 \left|\alpha_n \left(\beta_{\mathrm{s}}^{(n)}\right)^{-1} \left(u_1^{\delta,(n)} - u_1^{(n)}\right) + \left(u_2^{\delta,(n)} - u_2^{(n)}\right)\right|^2,$$

$$E_2 = \sum_{|\alpha_n| \le \kappa_{\mathrm{c}}} \left|e^{\mathrm{i}\beta_{\mathrm{p}}^{(n)} h}\right|^2 \left|\alpha_n \left(\beta_{\mathrm{s}}^{(n)}\right)^{-1} e_1^{(n)} + e_2^{(n)}\right|^2,$$

$$E_3 = \sum_{|\alpha_n| > \kappa_{\mathrm{c}}} \left|f^{(n)}\right|^2.$$

A simple calculation yields

$$\left|e^{\mathrm{i}\beta_{\mathrm{p}}^{(n)} h}\right| \le \left|e^{h\left(\kappa_{\mathrm{c}}^2 - \kappa_{\mathrm{p}}^2\right)^{1/2}}\right| \quad \text{for } |\alpha_n| \le \kappa_{\mathrm{c}},$$

which, combined with (3.5) implies

$$E_1^{1/2} \lesssim \left| e^{h\left(\kappa_c^2-\kappa_p^2\right)^{1/2}} \right| \left\| \boldsymbol{u}^\delta - \boldsymbol{u} \right\|_{0,\Gamma} \leq \left| e^{h\left(\kappa_c^2-\kappa_p^2\right)^{1/2}} \right| \delta. \tag{3.8}$$

By Theorem 2.11, we have

$$\begin{aligned} \|\boldsymbol{e}\|_{0,\Gamma} &= \left\| \sum_{m=2}^{\infty} \boldsymbol{u}_m \, \varepsilon^m \right\|_{0,\Gamma} \leq \sum_{m=2}^{\infty} \|\boldsymbol{u}_m\|_{0,\Gamma} \, \varepsilon^m \leq \sum_{m=2}^{\infty} \|\boldsymbol{u}_m\|_{1/2,\Gamma} \, \varepsilon^m \\ &\leq \sum_{m=2}^{\infty} \|\boldsymbol{u}_m\|_{1,D} \, \varepsilon^m \leq \sum_{m=2}^{\infty} \left(cK\varepsilon h^{-1}\right)^m \lesssim \left(cK\varepsilon h^{-1}\right)^2 \lesssim \left(K\varepsilon h^{-1}\right)^2, \end{aligned}$$

which gives

$$E_2^{1/2} \lesssim \left| e^{h\left(\kappa_c^2-\kappa_p^2\right)^{1/2}} \right| \left(K\varepsilon h^{-1}\right)^2. \tag{3.9}$$

Using Lemma 3.3 and the integral test for series, we deduce that

$$E_3 = \varepsilon^2 \sum_{|\alpha_n|>\kappa_c} \left| g^{(n)} \right|^2 \leq (K\varepsilon)^2 \sum_{|\alpha_n|>\kappa_c} |\alpha_n|^{-2k} \lesssim (K\varepsilon)^2 \kappa_c^{-2k+1}. \tag{3.10}$$

The proof is completed combining (3.7)–(3.10). □

The parameters in the error estimate (3.6) can be divided into two groups:

(1) the intrinsic parameters $K, \varepsilon, \delta, \kappa_p$, which are associated with the problem;
(2) the user-specified parameters h, κ_c, which can be chosen in practice.

The implicit constant in the estimate is independent on all the parameters except the pressure and shear wavenumbers κ_p and κ_s. The estimate shows that the error arises from three parts: the data noise, the linearization, and the regularization. Specifically, we have the following observations:

(1) all the three types of error decrease as functions of the intrinsic parameters K, ε, δ, which are consistent with the physical intuition;
(2) as the measurement distance h decreases, the error due to the data noise stays at a constant level if $\kappa_c \leq \kappa_p$ and decreases if $\kappa_c > \kappa_p$;
(3) if $\kappa_c \leq \kappa_p$, then the linearization error increases as h decreases;
(4) if $\kappa_c > \kappa_p$, then the linearization error decreases at the beginning but increases later if h becomes too small;
(5) as κ_c increases from 0 to κ_p, the error due to the data noise and linearization stays at a constant level and the regularization error decreases. Thus, it is stable to reconstruct the Fourier modes of the surface function within the diffraction limit;
(6) as κ_c increases from κ_p to ∞, the error due to the data noise and linearization increase at an exponential rate and the regularization error decreases. Thus it is increasingly more unstable to reconstruct the higher Fourier modes of the surface beyond the diffraction limit.

In view of the above observations, one should choose appropriate values for the tunable parameters h and κ_c in order to obtain a stable reconstructions with super resolution. For example, if we take

$h = \varepsilon^{1/2}$, then the estimate (3.6) reduces to

$$\|f_{\varepsilon,\delta} - f\|_{0,\Gamma} \lesssim \left| e^{\left(\kappa_c^2 - \kappa_p^2\right)^{1/2} \varepsilon^{1/2}} \right| (\delta + K^2\varepsilon) + K\varepsilon\kappa_c^{-(2k-1)/2}. \tag{3.11}$$

It is clear to note that the error estimate in (3.11) is completely characterized by the intrinsic parameters. Moreover, we have

$$\|f_{\varepsilon,\delta} - f\|_{0,\Gamma} \to 0 \quad \text{as } \varepsilon, \delta \to 0,$$

while the other parameters are held fixed.

To find an appropriate value for the cut-off wavenumber κ_c, we note from (3.11) that

$$\|f_{\varepsilon,\delta} - f\|_{0,\Gamma} \lesssim \left| e^{\kappa_c \varepsilon^{1/2}} \right| (\delta + K^2\varepsilon) + K\varepsilon\kappa_c^{-(2k-1)/2}. \tag{3.12}$$

We may take κ_c such that

$$e^{\kappa_c \varepsilon^{1/2}} = \min\left\{\varepsilon^{-p}, \delta^{-q}\right\},$$

where $0 < p, q < 1$ are user-specified constants. Then, we have

(1) If $\varepsilon^{-p} \le \delta^{-q}$, then $e^{\kappa_c \varepsilon^{1/2}} = \varepsilon^{-p}$ and $\kappa_c \sim |\ln \varepsilon|/\varepsilon^{1/2}$. Substituting into (3.12) and noting that $\delta \le \varepsilon^{p/q}$ yields a reduced error estimate:

$$\|f_{\varepsilon,\delta} - f\|_{0,\Gamma} \lesssim \varepsilon^{p(1-q)/q} + K^2\varepsilon^{1-p} + K\varepsilon^{(2k+3)/4}|\ln \varepsilon|^{-(2k-1)/2} \to 0 \quad \text{as } \varepsilon \to 0.$$

(2) If $\delta^{-q} \le \varepsilon^{-p}$, then $e^{\kappa_c \varepsilon^{1/2}} = \delta^{-q}$ and $\kappa_c \sim |\ln \delta|/\varepsilon^{1/2}$. Substituting it into (3.12) and noting that $\varepsilon \le \delta^{q/p}$ yields a reduced error estimate:

$$\|f_{\varepsilon,\delta} - f\|_{0,\Gamma} \lesssim \delta^{1-q} + K^2\delta^{q(1-p)/p} + K\delta^{(2k+3)q/4p}|\ln \delta|^{-(2k-1)} \to 0 \quad \text{as } \delta \to 0.$$

These error estimates are completely characterized by the intrinsic parameters ε or δ.

4. Conclusion

We studied mathematically a significant numerical method for solving the inverse elastic surface scattering problem. A simple proof was presented for the well-posedness of the solution for the direct problem using the Lax–Milgram lemma. The condition was established for the convergence of the power series solution. The inverse problem was shown to have a unique solution for a single illumination. An error estimate was derived, which clearly displays the dependence of the error on various parameters of the model problem and provides a deep insight into the trade-off among accuracy, resolution, and stability of the solution for the inverse problem.

As for future works, we would like to extend the analysis to the three-dimensional surface scattering problems. A challenging and interesting problem is to study the convergence of the inverse obstacle or cavity scattering problems for elastic waves. We hope to report the progress on these problems elsewhere in the near future.

Disclosure statement

No potential conflict of interest was reported by the authors.

Funding

The research was supported in part by the NSF [grant number DMS-1151308].

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