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Stability on the inverse random source scattering problem for the one-dimensional Helmholtz equation ☆

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ABSTRACT

Consider the one-dimensional stochastic Helmholtz equation where the source is assumed to be driven by the white noise. This paper concerns the stability analysis of the inverse random source problem which is to reconstruct the statistical properties of the source such as the mean and variance. Our results show that increasing stability can be obtained for the inverse problem by using suitable boundary data with multi-frequencies.

1. Introduction and problem statement

Consider the one-dimensional stochastic Helmholtz equation

$$u''(x,\kappa) + \kappa^2 u(x,\kappa) = f(x) + \sigma(x)\dot{W}_x, \tag{1.1}$$

where $\kappa > 0$ is the wavenumber, f and σ are deterministic functions which have compact supports contained in the interval [0, 1], W_x is the spatial Brownian motion and $\dot{W}_x$ is the white noise. In this model, f, σ, and σ^2 can be viewed as the mean, the standard deviation, and the variance of the random source, respectively. The radiated random wave field u is required to satisfy the outgoing wave conditions:

$$u'(0,\kappa) + \mathrm{i}\kappa u(0,\kappa) = 0, \quad u'(1,\kappa) - \mathrm{i}\kappa u(1,\kappa) = 0. \tag{1.2}$$

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Given f and σ, the direct source scattering problem is to determine the radiated wave field u. It is shown in [1] that (1.1)–(1.2) has a unique pathwise solution which is

$$u(x,\kappa)=\int_0^1 \frac{e^{\mathrm{i}\kappa|x-y|}}{2\mathrm{i}\kappa}f(y)\mathrm{d}y+\int_0^1 \frac{e^{\mathrm{i}\kappa|x-y|}}{2\mathrm{i}\kappa}\sigma(y)\mathrm{d}W_y. \tag{1.3}$$

Here the second integral at the right hand side of (1.3) is understood in the sense of Itô. This paper concerns the inverse source scattering problem, which is to determine f and $g=\sigma^2$ from boundary measurement of the radiated wave field u. Specifically, we propose the following two inverse problems:

(1) If f and g are complex function, the inverse problem is to determine f and g simultaneously by two-sided observation data $u(0,\kappa)$ and $u(1,\kappa), \kappa\in(0,K)$, where $K>1$ is a constant.
(2) If f is a real function, the inverse problem is to determine f by one-sided observation data $u(0,\kappa), \kappa\in(0,1)\cup\cup_{j=1}^{N} j\pi$, where $N\in\mathbb{N}$.

The inverse source problem has significant applications in medical and biomedical imaging [11]. Although the deterministic inverse source problem has been well studied [2,4], little is known for the stochastic case [7]. We refer to [1,14] for numerical solution of the one-dimensional inverse random source scattering problem. A related inverse random source problem can be found in [5]. However, there are no stability results available for the inverse random source scattering problem at present.

In this paper, we study stability of the above two inverse problems. As is known, the inverse source problem does not have a unique solution at a single frequency even for its deterministic counterpart [8,10]. Our goal is to establish increasing stability of the inverse problems with multi-frequencies. We refer to [3,6] for increasing stability of the deterministic inverse source problem. In [6], the authors discussed stability of the inverse source problem for the three-dimensional Helmholtz equation by using the Huygens principle. In [3], the authors studied the stability of the two- and three-dimensional Helmholtz equations via Green's functions. Related results can be found in [12,13] on increasing stability of determining potentials and in the continuation for the Helmholtz equation.

2. Main results

Let the triple $(\Omega,\mathcal{F},P)$ be a complete probability space on which the one-dimensional Brownian motion $\{W_x\}_{x\in[0,1]}$ is defined. If X is a random variable, $\mathbf{E}(X)$ and $\mathbf{V}(X)=\mathbf{E}(X-\mathbf{E}(X))^2$ are the expectation and variance of X, respectively. We remark that $\mathbf{V}(X)$ is not an ordinary variance if X is a complex-valued random variable. For convenience, we still call $\mathbf{V}(X)$ the variance of random variable X even if it is complex-valued. We refer to [9] for more details on notation of stochastic differential equations.

Define a complex-valued functional space:

$$\mathcal{C}_M=\{f\in H^n(0,1):\|f\|_{H^n(0,1)}\le M,\ \mathrm{supp} f\subset(0,1),\ f:(0,1)\to\mathbb{C}\}$$

and a real-valued functional space:

$$\mathcal{R}_M=\{f\in H^n(0,1):\|f\|_{H^n(0,1)}\le M,\ \mathrm{supp} f\subset(0,1),\ f:(0,1)\to\mathbb{R}\},$$

where $n\in\mathbb{N}$ and $M>1$ is a constant. Given two random functions u_1 and u_2, we define the function of expectation discrepancy:

$$v(x,\kappa)=\mathbf{E}u_1(x,\kappa)-\mathbf{E}u_2(x,\kappa)$$

and the function of the variance discrepancy:

$$w(x,\kappa) = \mathbf{V}u_1(x,\kappa) - \mathbf{V}u_2(x,\kappa).$$

Now we show the main stability result of the first inverse problem.

Theorem 2.1. *Let $f_j, g_j \in \mathcal{C}_M, j = 1, 2$, and let u_j be the solution (1.3) corresponding to f_j, g_j. Then there exist two positive constants C_1, C_2 independent of n, K, M, κ such that*

$$\|f_1 - f_2\|^2_{L^2(0,1)} \le C_1 \left(\epsilon_1^2 + \frac{M^2}{\left(\frac{K^{\frac{2}{3}}|\ln \epsilon_1|^{\frac{1}{4}}}{(6n-3)^3}\right)^{2n-1}} \right), \tag{2.1}$$

$$\|g_1 - g_2\|^2_{L^2(0,1)} \le C_2 \left(\epsilon_2^2 + \frac{M^2}{\left(\frac{K^{\frac{2}{3}}|\ln \epsilon_2|^{\frac{1}{4}}}{(6n-3)^3}\right)^{2n-1}} \right), \tag{2.2}$$

where $K > 1$ and

$$\epsilon_1 = \left(4 \int_0^K \kappa^2 \left(|v(0,\kappa)|^2 + |v(1,\kappa)|^2 \right) \mathrm{d}\kappa \right)^{\frac{1}{2}}, \tag{2.3}$$

$$\epsilon_2 = \left(16 \int_0^K \kappa^4 \left(|w(0,\kappa)|^2 + |w(1,\kappa)|^2 \right) \mathrm{d}\kappa \right)^{\frac{1}{2}}. \tag{2.4}$$

Remark 2.2. There are two parts in the stability estimates (2.1) and (2.2): the first part is the data discrepancy and the second part comes from the high frequency tails of the functions. It is clear to see that the stability increases as K increases, i.e., the problem is more stable as more frequencies data are used. We can also see that when $n < \left[\frac{K^{\frac{2}{9}}|\ln \epsilon_j|^{\frac{1}{12}}+3}{6} \right]$ the stability increases as n increases, i.e., the problem is more stable as the functions have suitably higher regularity.

Here is the main stability result of the second inverse problem.

Theorem 2.3. *Let $f_j \in \mathcal{R}_M, j = 1, 2$, and let u_j be the solution of (1.3) corresponding to f_j. Let*

$$\epsilon_3 = \left(\sum_{j=1}^{N} (2j\pi)^2 |\mathrm{Re} v(0, j\pi)|^2 \right)^{\frac{1}{2}}, \quad \epsilon_4 = \sup_{\kappa \in (0,1)} 2\kappa |\mathrm{Re} v(0,\kappa)| < 1.$$

Then there exists a positive constant C_3 independent of n, N, M, κ such that

$$\|f_1 - f_2\|^2_{L^2(0,1)} \le C_3 \left(\epsilon_3^2 + \frac{M^2}{\left(\frac{N^{\frac{5}{8}}|\ln \epsilon_4|^{\frac{1}{9}}}{(6n-3)^3}\right)^{2n-1}} \right).$$

Remark 2.4. Theorem 2.3 shows that only one-sided boundary observation data are needed for the wavenumbers in the set $(0,1)\cup\cup_{j=1}^{N} j\pi$ if one wants to determine the mean of the random source. The stability increases as N or $n < \left[\frac{N^{\frac{5}{24}}|\ln\epsilon_4|^{\frac{1}{27}}+3}{6}\right]$ increases.

The remainder of the paper is organized as follows. We prove Theorem 2.1 and Theorem 2.3 in section 3 and section 4, respectively.

3. Proof of Theorem 2.1

First we present several useful lemmas.

Lemma 3.1. *Let $f_j, g_j \in L^2(0,1), \mathrm{supp} f_j, \mathrm{supp} g_j \subset (0,1), j = 1, 2$. We have*

$$\|f_1 - f_2\|^2_{L^2(0,1)} = \frac{2}{\pi}\int_0^\infty \kappa^2\Big(|v(0,\kappa)|^2 + |v(1,\kappa)|^2\Big)\mathrm{d}\kappa,$$

$$\|g_1 - g_2\|^2_{L^2(0,1)} = \frac{16}{\pi}\int_0^\infty \kappa^4\Big(|w(0,\kappa)|^2 + |w(1,\kappa)|^2\Big)\mathrm{d}\kappa.$$

Proof. Letting $\xi \in \mathbb{R}$ with $|\xi| = \kappa$, we multiply $e^{-\mathrm{i}\xi x}$ on both sides of (1.1) and obtain

$$e^{-\mathrm{i}\xi x}u''(x,\kappa) + \kappa^2 e^{-\mathrm{i}\xi x}u(x,\kappa) = e^{-\mathrm{i}\xi x}f(x) + e^{-\mathrm{i}\xi x}\sigma(x)\dot{W}_x.$$

Since

$$(e^{-\mathrm{i}\xi x}u'(x,\kappa))' = e^{-\mathrm{i}\xi x}u''(x,\kappa) - \mathrm{i}\xi e^{-\mathrm{i}\xi x}u'(x,\kappa),$$

we have

$$(e^{-\mathrm{i}\xi x}u'(x,\kappa))' = e^{-\mathrm{i}\xi x}f(x) + e^{-\mathrm{i}\xi x}\sigma(x)\dot{W}_x - \kappa^2 e^{-\mathrm{i}\xi x}u(x,\kappa) - \mathrm{i}\xi e^{-\mathrm{i}\xi x}u'(x,\kappa). \tag{3.1}$$

Integrating (3.1) over $(0,1)$ with respect to x yields

$$\begin{aligned} e^{-\mathrm{i}\xi}u'(1,\kappa) - u'(0,\kappa) = &\int_0^1 e^{-\mathrm{i}\xi x}f(x)\mathrm{d}x + \int_0^1 e^{-\mathrm{i}\xi x}\sigma(x)\mathrm{d}W_x \\ &- \kappa^2\int_0^1 e^{-\mathrm{i}\xi x}u(x,\kappa)\mathrm{d}x - \mathrm{i}\xi\int_0^1 e^{-\mathrm{i}\xi x}u'(x,\kappa)\mathrm{d}x. \end{aligned} \tag{3.2}$$

It follows from the integration by parts that

$$-\mathrm{i}\xi\int_0^1 e^{-\mathrm{i}\xi x}u'(x,\kappa)\mathrm{d}x = -\mathrm{i}\xi e^{-\mathrm{i}\xi}u(1,\kappa) + \mathrm{i}\xi u(0,\kappa) + \kappa^2\int_0^1 e^{-\mathrm{i}\xi x}u(x,\kappa)\mathrm{d}x. \tag{3.3}$$

Substituting (3.3) into (3.2), we get

$$e^{-\mathrm{i}\xi}u'(1,\kappa)+\mathrm{i}\xi e^{-\mathrm{i}\xi}u(1,\kappa)-u'(0,\kappa)-\mathrm{i}\xi u(0,\kappa)=\int_0^1 e^{-\mathrm{i}\xi x}f(x)\mathrm{d}x+\int_0^1 e^{-\mathrm{i}\xi x}\sigma(x)\mathrm{d}W_x,$$

which gives after applying the outgoing wave conditions (1.2) that

$$\mathrm{i}(\kappa+\xi)e^{-\mathrm{i}\xi}u(1,\kappa)+\mathrm{i}(\kappa-\xi)u(0,\kappa)=\int_0^1 e^{-\mathrm{i}\xi x}f(x)\mathrm{d}x+\int_0^1 e^{-\mathrm{i}\xi x}\sigma(x)\mathrm{d}W_x. \tag{3.4}$$

Taking the expectation on both sides of (3.4), we obtain

$$\int_0^1 e^{-\mathrm{i}\xi x}f(x)\mathrm{d}x=\mathrm{i}(\kappa+\xi)e^{-\mathrm{i}\xi}\mathbf{E}u(1,\kappa)+\mathrm{i}(\kappa-\xi)\mathbf{E}u(0,\kappa),\quad |\xi|=\kappa\in(0,\infty). \tag{3.5}$$

Since f_j is assumed to have a compact support in $[0,1]$, we have from (3.5) that

$$\hat{f}_j(\xi)=\int_{-\infty}^{\infty} e^{-\mathrm{i}\xi x}f_j(x)\mathrm{d}x=\mathrm{i}(\kappa+\xi)e^{-\mathrm{i}\xi}\mathbf{E}u_j(1,\kappa)+\mathrm{i}(\kappa-\xi)\mathbf{E}u_j(0,\kappa),\quad |\xi|=\kappa\in(0,\infty),$$

which gives

$$\begin{aligned}\hat{f}_1(\xi)-\hat{f}_2(\xi)=\int_{-\infty}^{\infty} e^{-\mathrm{i}\xi x}(f_1(x)-f_2(x))\mathrm{d}x&=\mathrm{i}(\kappa+\xi)e^{-\mathrm{i}\xi}(\mathbf{E}u_1(1,\kappa)-\mathbf{E}u_2(1,\kappa))\\&\quad+\mathrm{i}(\kappa-\xi)(\mathbf{E}u_1(0,\kappa)-\mathbf{E}u_2(0,\kappa)),\quad |\xi|=\kappa\in(0,\infty).\end{aligned}$$

Hence we have

$$\hat{f}_1(-\kappa)-\hat{f}_2(-\kappa)=2\mathrm{i}\kappa(\mathbf{E}u_1(0,\kappa)-\mathbf{E}u_2(0,\kappa))=2\mathrm{i}\kappa v(0,\kappa)$$

and

$$\hat{f}_1(\kappa)-\hat{f}_2(\kappa)=2\mathrm{i}e^{-\mathrm{i}\kappa}\kappa(\mathbf{E}u_1(1,\kappa)-\mathbf{E}u_2(1,\kappa))=2\mathrm{i}\kappa e^{-\mathrm{i}\kappa}v(1,\kappa).$$

It follows from the Plancherel theorem that

$$\begin{aligned}\|f_1-f_2\|^2_{L^2(0,1)}&=\frac{1}{2\pi}\int_{-\infty}^{\infty}|\hat{f}_1(\xi)-\hat{f}_2(\xi)|^2\mathrm{d}\xi\\&=\frac{1}{2\pi}\int_{-\infty}^{0}|\hat{f}_1(\xi)-\hat{f}_2(\xi)|^2\mathrm{d}\xi+\frac{1}{2\pi}\int_0^{\infty}|\hat{f}_1(\xi)-\hat{f}_2(\xi)|^2\mathrm{d}\xi\\&=\frac{1}{2\pi}\int_0^{\infty}|\hat{f}_1(-\kappa)-\hat{f}_2(-\kappa)|^2\mathrm{d}\kappa+\frac{1}{2\pi}\int_0^{\infty}|\hat{f}_1(\kappa)-\hat{f}_2(\kappa)|^2\mathrm{d}\kappa\\&=\frac{2}{\pi}\int_0^{\infty}\kappa^2|v(0,\kappa)|^2\mathrm{d}\kappa+\frac{2}{\pi}\int_0^{\infty}\kappa^2|v(1,\kappa)|^2\mathrm{d}\kappa.\end{aligned}$$

Noting

$$\mathbf{E}\Big(\int_0^1 e^{-\mathrm{i}\xi x}\sigma(x)\mathrm{d}W_x\Big)^2 = \int_0^1 e^{-2\mathrm{i}\xi x}\sigma^2(x)\mathrm{d}x = \int_0^1 e^{-2\mathrm{i}\xi x}g(x)\mathrm{d}x,$$

we have from (3.4) that

$$\int_0^1 e^{-2\mathrm{i}\xi x}g(x)\mathrm{d}x = \mathbf{V}\big(\mathrm{i}(\kappa+\xi)e^{-\mathrm{i}\xi}u(1,\kappa) + \mathrm{i}(\kappa-\xi)u(0,\kappa)\big), \quad |\xi| = \kappa \in (0,\infty). \tag{3.6}$$

Since g_j has a compact support in $(0,1)$, we get from (3.6) that

$$\hat{g}_j(2\xi) = \int_{-\infty}^{\infty} e^{-2\mathrm{i}\xi x}g_j(x)\mathrm{d}x = \mathbf{V}\big(\mathrm{i}(\kappa+\xi)e^{-\mathrm{i}\xi}u_j(1,\kappa) + \mathrm{i}(\kappa-\xi)u_j(0,\kappa)\big), \quad |\xi| = \kappa \in (0,\infty),$$

which gives

$$\begin{aligned}\hat{g}_1(2\xi) - \hat{g}_2(2\xi) &= \int_{-\infty}^{\infty} e^{-2\mathrm{i}\xi x}(g_1(x) - g_2(x))\mathrm{d}x \\ &= \mathbf{V}\big(\mathrm{i}(\kappa+\xi)e^{-\mathrm{i}\xi}u_1(1,\kappa) + \mathrm{i}(\kappa-\xi)u_1(0,\kappa)\big) - \mathbf{V}\big(\mathrm{i}(\kappa+\xi)e^{-\mathrm{i}\xi}u_2(1,\kappa) + \mathrm{i}(\kappa-\xi)u_2(0,\kappa)\big).\end{aligned}$$

Hence we have

$$\hat{g}_1(-2\kappa) - \hat{g}_2(-2\kappa) = (2\mathrm{i}\kappa)^2\big(\mathbf{V}u_1(0,\kappa) - \mathbf{V}u_2(0,\kappa)\big) = (2\mathrm{i}\kappa)^2 w(0,\kappa)$$

and

$$\hat{g}_1(2\kappa) - \hat{g}_2(2\kappa) = (2\mathrm{i}\kappa)^2 e^{-2\mathrm{i}\kappa}\big(\mathbf{V}u_1(1,\kappa) - \mathbf{V}u_2(1,\kappa)\big) = (2\mathrm{i}\kappa)^2 e^{-2\mathrm{i}\kappa} w(1,\kappa).$$

Using the Plancherel theorem again yields

$$\begin{aligned}\|g_1 - g_2\|^2_{L^2(0,1)} &= \frac{1}{2\pi}\int_{-\infty}^{\infty}|\hat{g}_1(\xi) - \hat{g}_2(\xi)|^2\mathrm{d}\xi = \frac{1}{\pi}\int_{-\infty}^{\infty}|\hat{g}_1(2\xi) - \hat{g}_2(2\xi)|^2\mathrm{d}\xi \\ &= \frac{1}{\pi}\int_0^{\infty}|\hat{g}_1(-2\kappa) - \hat{g}_2(-2\kappa)|^2\mathrm{d}\kappa + \frac{1}{\pi}\int_0^{\infty}|\hat{g}_1(2\kappa) - \hat{g}_2(2\kappa)|^2\mathrm{d}\kappa \\ &= \frac{16}{\pi}\int_0^{\infty}\kappa^4|w(0,\kappa)|^2\mathrm{d}\kappa + \frac{16}{\pi}\int_0^{\infty}\kappa^2|w(1,\kappa)|^4\mathrm{d}\kappa,\end{aligned}$$

which completes the proof. $\square$

Lemma 3.2. *Let* $f_j, g_j \in L^2(0,1), j = 1, 2$. *We have*

$$4\kappa^2|v(0,\kappa)|^2 = \Big|\int_0^1 e^{\mathrm{i}\kappa x}(f_1(x) - f_2(x))\mathrm{d}x\Big|^2,$$

$$4\kappa^2|v(1,\kappa)|^2 = \Big|\int_0^1 e^{-\mathrm{i}\kappa x}(f_1(x) - f_2(x))\mathrm{d}x\Big|^2,$$

$$16\kappa^4|w(0,\kappa)|^2 = \Big|\int_0^1 e^{2\mathrm{i}\kappa x}(g_1(x) - g_2(x))\mathrm{d}x\Big|^2,$$

$$16\kappa^4|w(1,\kappa)|^2 = \Big|\int_0^1 e^{-2\mathrm{i}\kappa x}(g_1(x) - g_2(x))\mathrm{d}x\Big|^2.$$

Proof. It follows from (1.3) that the solution of (1.1)–(1.2) is

$$2\mathrm{i}\kappa u_j(x,\kappa) = \int_0^1 e^{\mathrm{i}\kappa|x-y|} f_j(y)\mathrm{d}y + \int_0^1 e^{\mathrm{i}\kappa|x-y|}\sigma_j(y)\mathrm{d}W_y,$$

which gives

$$2\mathrm{i}\kappa u_j(0,\kappa) = \int_0^1 e^{\mathrm{i}\kappa x} f_j(x)\mathrm{d}x + \int_0^1 e^{\mathrm{i}\kappa x}\sigma_j(x)\mathrm{d}W_x, \tag{3.7}$$

$$2\mathrm{i}\kappa u_j(1,\kappa) = \int_0^1 e^{\mathrm{i}\kappa(1-x)} f_j(x)\mathrm{d}x + \int_0^1 e^{\mathrm{i}\kappa(1-x)}\sigma_j(x)\mathrm{d}W_x. \tag{3.8}$$

Taking expectation of (3.7) and (3.8), we may obtain

$$2\mathrm{i}\kappa v(0,\kappa) = \int_0^1 e^{\mathrm{i}\kappa x}(f_1(x) - f_2(x))\mathrm{d}x,$$

$$2\mathrm{i}\kappa v(1,\kappa) = \int_0^1 e^{\mathrm{i}\kappa(1-x)}(f_1(x) - f_2(x))\mathrm{d}x.$$

Taking the variance of (3.7) and (3.8) yields

$$-4\kappa^2 w(0,\kappa) = \int_0^1 e^{2\mathrm{i}\kappa x}(g_1(x) - g_2(x))\mathrm{d}x,$$

$$-4\kappa^2 w(1,\kappa) = \int_0^1 e^{2\mathrm{i}\kappa(1-x)}(g_1(x) - g_2(x))\mathrm{d}x,$$

which completes the proof by taking square of the amplitudes on both sides of the above four equations. □

Let

$$I_1(s) = \int_0^s \left(\int_0^1 e^{\mathrm{i}\kappa x}(f_1(x) - f_2(x))\mathrm{d}x \right) \left(\int_0^1 e^{-\mathrm{i}\kappa x}\overline{(f_1(x) - f_2(x))}\mathrm{d}x \right) \mathrm{d}\kappa$$
$$+ \int_0^s \left(\int_0^1 e^{-\mathrm{i}\kappa x}(f_1(x) - f_2(x))\mathrm{d}x \right) \left(\int_0^1 e^{\mathrm{i}\kappa x}\overline{(f_1(x) - f_2(x))}\mathrm{d}x \right) \mathrm{d}\kappa, \tag{3.9}$$

$$I_2(s) = \int_0^s \left(\int_0^1 e^{2\mathrm{i}\kappa x}(g_1(x) - g_2(x))\mathrm{d}x \right) \left(\int_0^1 e^{-2\mathrm{i}\kappa x}\overline{(g_1(x) - g_2(x))}\mathrm{d}x \right) \mathrm{d}\kappa$$
$$+ \int_0^s \left(\int_0^1 e^{-2\mathrm{i}\kappa x}(g_1(x) - g_2(x))\mathrm{d}x \right) \left(\int_0^1 e^{2\mathrm{i}\kappa x}\overline{(g_1(x) - g_2(x))}\mathrm{d}x \right) \mathrm{d}\kappa. \tag{3.10}$$

The integrands in (3.9) and (3.10) are entire analytic function of κ. The integrals with respect to κ can be taken over any path joining points 0 and s in the complex plane. Thus $I_1(s)$ and $I_2(s)$ are entire analytic functions of $s = s_1 + \mathrm{i}s_2, s_1, s_2 \in \mathbb{R}$.

Lemma 3.3. *Let $f_j, g_j \in L^2(0,1), j = 1, 2$. We have for any $s = s_1 + \mathrm{i}s_2, s_1, s_2 \in \mathbb{R}$ that*

$$|I_1(s)| \le 2|s|e^{2|s_2|} \int_0^1 |f_1(x) - f_2(x)|^2\mathrm{d}x,$$

$$|I_2(s)| \le 2|s|e^{4|s_2|} \int_0^1 |g_1(x) - g_2(x)|^2\mathrm{d}x.$$

Proof. Let $\kappa = st, t \in (0,1)$. A simple calculation yields

$$I_1(s) = s \int_0^1 \left(\int_0^1 e^{\mathrm{i}stx}(f_1(x) - f_2(x))\mathrm{d}x \right) \left(\int_0^1 e^{-\mathrm{i}stx}\overline{(f_1(x) - f_2(x))}\mathrm{d}x \right) \mathrm{d}t$$
$$+ s \int_0^1 \left(\int_0^1 e^{-\mathrm{i}stx}(f_1(x) - f_2(x))\mathrm{d}x \right) \left(\int_0^1 e^{\mathrm{i}stx}\overline{(f_1(x) - f_2(x))}\mathrm{d}x \right) \mathrm{d}t.$$

Noting that $|e^{\pm \mathrm{i}stx}| \le e^{|s_2|}$ for all $x \in (0,1)$, we have

$$|I_1(s)| \le 2|s| \int_0^1 \Big(\int_0^1 e^{2|s_2|}|f_1(x) - f_2(x)|^2\mathrm{d}x \Big)\mathrm{d}t \le 2|s|e^{2|s_2|} \int_0^1 |f_1(x) - f_2(x)|^2\mathrm{d}x.$$

Similarly, we can show that

$$|I_2(s)| \le 2|s|e^{4|s_2|} \int_0^1 |g_1(x) - g_2(x)|^2\mathrm{d}x,$$

which completes the proof. □

Lemma 3.4. *Let $f_j, g_j \in H^n(0,1), \mathrm{supp} f_j, \mathrm{supp} g_j \subset (0,1), j = 1, 2$. We have for any $s > 0$ that*

$$4\int_s^\infty \kappa^2\big(|v(0,\kappa)|^2 + |v(1,\kappa)|^2\big)\mathrm{d}\kappa \le 2s^{-(2n-1)}\|f_1 - f_2\|^2_{H^n(0,1)},$$

$$16\int_s^\infty \kappa^4\big(|w(0,\kappa)|^2 + |w(1,\kappa)|^2\big)\mathrm{d}\kappa \le 2s^{-(2n-1)}\|g_1 - g_2\|^2_{H^n(0,1)}.$$

Proof. It follows from Lemma 3.2 that we have

$$\begin{aligned}
&4\int_s^\infty \kappa^2|v(0,\kappa)|^2\mathrm{d}\kappa + 4\int_s^\infty \kappa^2|v(1,\kappa)|^2\mathrm{d}\kappa \\
&= \int_s^\infty\Big|\int_0^1 e^{\mathrm{i}\kappa x}(f_1(x) - f_2(x))\mathrm{d}x\Big|^2\mathrm{d}\kappa + \int_s^\infty\Big|\int_0^1 e^{-\mathrm{i}\kappa x}(f_1(x) - f_2(x))\mathrm{d}x\Big|^2\mathrm{d}\kappa
\end{aligned}$$

Using integration by parts and noting $\mathrm{supp} f_j \subset (0,1)$, we obtain

$$\int_0^1 e^{\pm\mathrm{i}\kappa x}(f_1(x) - f_2(x))\mathrm{d}x = \frac{1}{(\pm\mathrm{i}\kappa)^n}\int_0^1 e^{\pm\mathrm{i}\kappa x}(f_1^{(n)}(x) - f_2^{(n)}(x))\mathrm{d}x,$$

which gives

$$\Big|\int_0^1 e^{\pm\mathrm{i}\kappa x}(f_1(x) - f_2(x))\mathrm{d}x\Big|^2 \le \kappa^{-2n}\|f_1^{(n)} - f_2^{(n)}\|^2_{H^n(0,1)}$$

Hence we get

$$\begin{aligned}
\int_s^\infty\Big|\int_0^1 e^{\pm\mathrm{i}\kappa x}(f_1(x) - f_2(x))\mathrm{d}x\Big|^2\mathrm{d}\kappa &\le \|f_1^{(n)} - f_2^{(n)}\|^2_{H^n(0,1)}\int_s^\infty \kappa^{-2n}\mathrm{d}\kappa \\
&= \frac{s^{-(2n-1)}}{(2n-1)}\|f_1^{(n)} - f_2^{(n)}\|^2_{H^n(0,1)}
\end{aligned}$$

Again, we have from Lemma 3.2 that

$$\begin{aligned}
&16\int_s^\infty \kappa^4|w(0,\kappa)|^2\mathrm{d}\kappa + 16\int_s^\infty \kappa^4|w(1,\kappa)|^2\mathrm{d}\kappa \\
&= \int_s^\infty\Big|\int_0^1 e^{2\mathrm{i}\kappa x}(g_1(x) - g_2(x))\mathrm{d}x\Big|^2\mathrm{d}\kappa + \int_s^\infty\Big|\int_0^1 e^{-2\mathrm{i}\kappa x}(g_1(x) - g_2(x))\mathrm{d}x\Big|^2\mathrm{d}\kappa.
\end{aligned}$$

Similarly, we have

$$\int_0^1 e^{\pm 2\mathrm{i}\kappa x}(g_1(x) - g_2(x))\mathrm{d}x = \frac{1}{(\pm 2\mathrm{i}\kappa)^n}\int_0^1 e^{\pm 2\mathrm{i}\kappa x}(g_1^{(n)}(x) - g_2^{(n)}(x))\mathrm{d}x,$$

which gives

$$\Big|\int_0^1 e^{\pm 2\mathrm{i}\kappa x}(g_1(x)-g_2(x))\mathrm{d}x\Big|^2 \le (2\kappa)^{-2n}\|g_1^{(n)}-g_2^{(n)}\|^2_{H^n(0,1)}.$$

Therefore, we get

$$\int_s^\infty\Big|\int_0^1 e^{\pm 2\mathrm{i}\kappa x}(g_1(x)-g_2(x))\mathrm{d}x\Big|^2\mathrm{d}\kappa \le \|g_1^{(n)}-g_2^{(n)}\|^2_{H^n(0,1)}\int_s^\infty(2\kappa)^{-2n}\mathrm{d}\kappa$$
$$= \frac{s^{-(2n-1)}}{(2n-1)4^n}\|g_1^{(n)}-g_2^{(n)}\|^2_{H^n(0,1)},$$

which completes the proof. □

The following lemma is proved in [6].

Lemma 3.5. *Denote $S=\{z=x+\mathrm{i}y\in\mathbb{C}:-\frac{\pi}{4}<\arg z<\frac{\pi}{4}\}$. Let $J(z)$ be analytic in S and continuous in $\bar{S}$ satisfying*

$$\begin{cases}|J(z)|\le\epsilon, & z\in(0,\ L],\\ |J(z)|\le V, & z\in S,\\ |J(0)|=0.\end{cases}$$

Then there exits a function $\mu(z)$ satisfying

$$\begin{cases}\mu(z)\ge\frac{1}{2}, & z\in(L,\ 2^{\frac{1}{4}}L),\\ \mu(z)\ge\frac{1}{\pi}((\frac{z}{L})^4-1)^{-\frac{1}{2}}, & z\in(2^{\frac{1}{4}}L,\ \infty)\end{cases}$$

such that

$$|J(z)|\le V\epsilon^{\mu(z)},\quad \forall\, z\in(L,\ \infty).$$

Lemma 3.6. *Let $f_j, g_j\in\mathcal{C}_M$. Then there exists a function $\mu(z)$ satisfying*

$$\begin{cases}\mu(s)\ge\frac{1}{2}, & s\in(K,\ 2^{\frac{1}{4}}K),\\ \mu(s)\ge\frac{1}{\pi}((\frac{s}{K})^4-1)^{-\frac{1}{2}}, & s\in(2^{\frac{1}{4}}K,\ \infty),\end{cases}\tag{3.11}$$

such that

$$|I_1(s)|\le CM^2e^{3s}\epsilon_1^{2\mu(s)},\quad |I_2(s)|\le CM^2e^{5s}\epsilon_2^{2\mu(s)},\quad \forall s\in(K,\ \infty).$$

Proof. We only show the proof of the estimate for $I_1(s)$ since the proof is the same for $I_2(s)$. It follows from Lemma 3.3 that

$$|I_1(s)e^{-3s}|\le CM^2,\quad \forall s\in S.$$

Recalling (2.3), (3.9), and Lemma 3.2, we have

$$|I_1(s)e^{-3s}| \le \epsilon_1^2, \quad s \in [0,\ K].$$

A direct application of Lemma 3.5 shows that there exists a function $\mu(s)$ satisfying (3.11) such that

$$|I_1(s)e^{-3s}| \le CM^2\epsilon_1^{2\mu}, \quad \forall s \in (K,\ \infty),$$

which completes the proof. □

Now we show the proof of Theorem 2.1.

Proof. It suffices to show the estimate (2.1) since the proof is similar for the estimate (2.2). We can assume that $\epsilon_1 < e^{-1}$, otherwise the estimate is obvious. Let

$$s = \begin{cases} \frac{1}{(3\pi)^{\frac{1}{3}}} K^{\frac{2}{3}} |\ln \epsilon_1|^{\frac{1}{4}}, & 2^{\frac{1}{4}}(3\pi)^{\frac{1}{3}}K^{\frac{1}{3}} < |\ln \epsilon_1|^{\frac{1}{4}}, \\ K, & |\ln \epsilon_1| \le 2^{\frac{1}{4}}(3\pi)^{\frac{1}{3}}K^{\frac{1}{3}}. \end{cases}$$

If $2^{\frac{1}{4}}(3\pi)^{\frac{1}{3}}K^{\frac{1}{3}} < |\ln \epsilon_1|^{\frac{1}{4}}$, then we have

$$\begin{aligned} |I_1(s)| &\le CM^2 e^{3s} e^{-\frac{2|\ln \epsilon_1|}{\pi}\left(\left(\frac{s}{K}\right)^4 - 1\right)^{-\frac{1}{2}}} \le CM^2 e^{\frac{3}{(3\pi)^{\frac{1}{3}}} K^{\frac{2}{3}} |\ln \epsilon_1|^{\frac{1}{4}} - \frac{2|\ln \epsilon_1|}{\pi}\left(\frac{K}{s}\right)^2} \\ &= CM^2 e^{-2\left(\frac{9}{\pi}\right)^{\frac{1}{3}} K^{\frac{2}{3}} |\ln \epsilon_1|^{\frac{1}{2}} \left(1 - \frac{1}{2}|\ln \epsilon_1|^{-\frac{1}{4}}\right)}. \end{aligned}$$

Noting $\frac{1}{2}|\ln \epsilon_1|^{-\frac{1}{4}} < \frac{1}{2}$, $\left(\frac{9}{\pi}\right)^{\frac{1}{3}} > 1$ we have

$$|I_1(s)| \le CM^2 e^{-K^{\frac{2}{3}} |\ln \epsilon_1|^{\frac{1}{2}}}.$$

Using the elementary inequality

$$e^{-x} \le \frac{(6n-3)!}{x^{3(2n-1)}}, \quad x > 0,$$

we get

$$|I_1(s)| \le \frac{CM^2}{\left(\frac{K^2 |\ln \epsilon_1|^{\frac{3}{2}}}{(6n-3)^3}\right)^{2n-1}}.$$

If $|\ln \epsilon_1| \le 2^{\frac{1}{4}}(3\pi)^{\frac{1}{3}}K^{\frac{1}{3}}$, then $s = K$. We have from (2.3), (3.9), and Lemma 3.2 that

$$|I_1(s)| \le \epsilon_1^2.$$

Hence we obtain from Lemma 3.4 that

$$\begin{aligned} &4\int_0^\infty \kappa^2 \left(|v(0,\kappa)|^2 + |v(1,\kappa)|^2\right) \mathrm{d}\kappa \\ &\le \epsilon_1^2 + \frac{CM^2}{\left(\frac{K^2 |\ln \epsilon_1|^{\frac{3}{2}}}{(6n-3)^3}\right)^{2n-1}} + \frac{\|f_1 - f_2\|^2_{H^n(0,1)}}{\left(2^{-\frac{1}{4}}(3\pi)^{-\frac{1}{3}} K^{\frac{2}{3}} |\ln \epsilon_1|^{\frac{1}{4}}\right)^{2n-1}}. \end{aligned}$$

By Lemma 3.1, we have

$$\|f_1 - f_2\|^2_{L^2(0,1)} \le C\left(\epsilon_1^2 + \frac{M^2}{\left(\frac{K^2|\ln\epsilon_1|^{\frac{3}{2}}}{(6n-3)^3}\right)^{2n-1}} + \frac{M^2}{\left(\frac{K^{\frac{2}{3}}|\ln\epsilon_1|^{\frac{1}{4}}}{(6n-3)^3}\right)^{2n-1}}\right).$$

Since $K^{\frac{2}{3}}|\ln\epsilon_1|^{\frac{1}{4}} \le K^2|\ln\epsilon_1|^{\frac{3}{2}}$ when $K > 1$ and $|\ln\epsilon_1| > 1$, we obtain the stability estimate. □

4. Proof of Theorem 2.3

Lemma 4.1. *Let $f_j \in L^2(0,1), j = 1, 2$ be real functions. We have for all $\kappa \in (0, \infty)$ that*

$$2\kappa\mathbf{E}\mathrm{Re}v(0,\kappa) = \int_0^1 \sin(\kappa x)(f_1(x) - f_2(x))\mathrm{d}x.$$

Proof. It can be easily obtained from (1.3) that

$$2\mathrm{i}\kappa u_j(0,\kappa) = \int_0^1 e^{\mathrm{i}\kappa x} f_j(x)\mathrm{d}x + \int_0^1 e^{\mathrm{i}\kappa x}\sigma_j(x)\mathrm{d}W_x.$$

Taking the expectation of the above equation gives

$$2\mathrm{i}\kappa\mathbf{E}u_j(0,\kappa) = \int_0^1 e^{\mathrm{i}\kappa x} f_j(x)\mathrm{d}x.$$

Noting that $f_j, j = 1, 2$ are real functions, we have

$$2\kappa\mathbf{E}\mathrm{Re}u_j(0,\kappa) = \int_0^1 \sin(\kappa x) f_j(x)\mathrm{d}x,$$

which completes the proof. □

Lemma 4.2. *Let $\|f_1 - f_2\|_{L^2(0,1)} \le M$. There exists a function $\mu(\kappa)$ satisfying*

$$\begin{cases} \mu(\kappa) \ge \frac{1}{2}, & \kappa \in (1,\ 2^{\frac{1}{4}}), \\ \mu(\kappa) \ge \frac{1}{\pi}(\kappa^4 - 1)^{-\frac{1}{2}}, & \kappa \in (2^{\frac{1}{4}},\ \infty), \end{cases} \tag{4.1}$$

such that

$$\Big|\int_0^1 \sin(\kappa x)(f_1(x) - f_2(x))\mathrm{d}x\Big|^2 \le CM^2 e^{4\kappa}\epsilon_4^{2\mu(\kappa)}, \quad \forall\kappa \in (1,\ \infty).$$

Proof. Let $\kappa = \kappa_1 + \mathrm{i}\kappa_2, \kappa_1, \kappa_2 \in \mathbb{R}$. It is easy to show that

$$\Big|\int_0^1 \sin(\kappa x)(f_1(x) - f_2(x))\mathrm{d}x\Big|^2 \le e^{|\kappa_2|}\|f_1 - f_2\|_{L^2(0,1)}.$$

Noting $|\kappa_2| \le \kappa_1$ for $\kappa \in S$, we have

$$\begin{aligned}\big|e^{-2\kappa}\big|\Big|\int_0^1 \sin(\kappa x)(f_1(x) - f_2(x))\mathrm{d}x\Big|^2 &\le \big|e^{-2\kappa}\big|e^{|\kappa_2|}\|f_1 - f_2\|_{L^2(0,1)} \\ &\le e^{-\kappa_1}\|f_1 - f_2\|_{L^2(0,1)} \le M.\end{aligned}$$

It follows from Lemma 4.1 that

$$\big|e^{-2\kappa}\big|\left|\int_0^1 \sin(\kappa x)(f_1(x) - f_2(x))\mathrm{d}x\right| \le \epsilon_4, \quad \kappa \in (0, 1].$$

We conclude from Lemma 3.5 that there exists a function μ satisfying (4.1) such that

$$\big|e^{-2\kappa}\big|\Big|\int_0^1 \sin(\kappa x)(f_1(x) - f_2(x))\mathrm{d}x \le CM\epsilon_4^{\mu(\kappa)}, \quad \kappa \in (1,\ \infty),$$

which completes the proof. □

Lemma 4.3. *Let $f_j \in H^n(0,1), \mathrm{supp} f_j \subset (0,1), j = 1, 2$. It holds that*

$$\sum_{j=T}^{\infty}(2j\pi)^2|\mathbf{E}\mathrm{Re}v(0, j\pi)|^2 \le \frac{1}{T^{2n-1}}\|f_1 - f_2\|^2_{H^n(0,1)}.$$

Proof. It follows from Lemma 4.1 that

$$\sum_{j=T}^{\infty} 4(j\pi)^2|\mathbf{E}\mathrm{Re}v(0, j\pi)|^2 = \sum_{j=T}^{\infty}\Big|\int_0^1 \sin(j\pi x)(f_1(x) - f_2(x))\mathrm{d}x\Big|^2.$$

Noting that f_j has a compact support in $(0, 1)$, we have from the integration by parts that

$$\begin{aligned}\Big|\int_0^1 \sin(j\pi x)(f_1(x) - f_2(x))\mathrm{d}x\Big|^2 &= \Big|\frac{1}{(j\pi)^n}\int_0^1 \sin(j\pi x + n\pi/2)(f_1^{(n)}(x) - f_2^{(n)}(x))\mathrm{d}x\Big|^2 \\ &\le \frac{1}{(j\pi)^{2n}}\|f_1 - f_2\|^2_{H^n(0,1)}.\end{aligned}$$

Combining the above estimates yields

$$\sum_{j=T}^{\infty} 4(j\pi)^2|\mathbf{E}\mathrm{Re}v(0,j\pi)|^2 \le \left(\sum_{j=T}^{\infty}\frac{1}{(j\pi)^{2n}}\right)\|f_1-f_2\|^2_{H^n(0,1)}$$
$$\le \frac{1}{\pi^{2n}}\left(\int_0^{\infty}\frac{1}{(T+x)^{2n}}\mathrm{d}x\right)\|f_1-f_2\|^2_{H^n(0,1)}$$
$$= \frac{1}{(2n-1)\pi^{2n}}\frac{1}{T^{2n-1}}\|f_1-f_2\|^2_{H^n(0,1)},$$

which completes the proof. □

Now we show the proof of Theorem 2.3.

Proof. We can assume that $\epsilon_3 < 1$, otherwise the estimate is obvious. Applying Lemma 4.1 and the Parseval identity, we have

$$\int_0^1 |f_1(x)-f_2(x)|^2\mathrm{d}x = \sum_{j=1}^{\infty} 4(j\pi)^2|\mathbf{E}\mathrm{Re}v(0,j\pi)|^2$$
$$= \sum_{j=1}^{T} 4(j\pi)^2|\mathbf{E}\mathrm{Re}v(0,j\pi)|^2 + \sum_{j=T+1}^{\infty} 4(j\pi)^2|\mathbf{E}\mathrm{Re}v(0,j\pi)|^2$$

Let

$$T = \begin{cases} [N^{\frac{3}{4}}|\ln\epsilon_4|^{\frac{1}{9}}], & N^{\frac{3}{8}} < \frac{1}{2^{\frac{5}{6}}\pi^{\frac{2}{3}}}|\ln\epsilon_4|^{\frac{1}{9}}, \\ N, & N^{\frac{3}{8}} \ge \frac{1}{2^{\frac{5}{6}}\pi^{\frac{2}{3}}}|\ln\epsilon_4|^{\frac{1}{9}}. \end{cases}$$

Using Lemma 4.2 leads to

$$\Big|\int_0^1 \sin(\kappa x)(f_1(x)-f_2(x))\mathrm{d}x\Big|^2 \le CM^2 e^{4\kappa}\epsilon_4^{2\mu(\kappa)} \le CM^2 e^{4\kappa} e^{-2\mu(\kappa)|\ln\epsilon_4|}$$
$$\le CM^2 e^{4\kappa} e^{-\frac{2}{\pi}(\kappa^4-1)^{-\frac{1}{2}}|\ln\epsilon_4|} \le CM^2 e^{4\kappa-\frac{2}{\pi}\kappa^{-2}|\ln\epsilon_4|}$$
$$\le CM^2 e^{-\frac{2}{\pi}\kappa^{-2}|\ln\epsilon_4|(1-2\pi\kappa^3|\ln\epsilon_4|^{-1})}, \quad \forall\kappa\in(2^{\frac{1}{4}},\ \infty).$$

Hence we have

$$\Big|\int_0^1 \sin(\kappa x)(f_1(x)-f_2(x))\mathrm{d}x\Big|^2 \le CM^2 e^{-\frac{2}{\pi^3}T^{-2}|\ln\epsilon_4|(1-2\pi^4T^3|\ln\epsilon_4|^{-1})}, \quad \forall\ \kappa\in(2^{\frac{1}{4}},\ T\pi]. \tag{4.2}$$

If $N^{\frac{3}{8}} < \frac{1}{2^{\frac{5}{6}}\pi^{\frac{2}{3}}}|\ln\epsilon_4|^{\frac{1}{9}}$, then $2\pi^4T^3|\ln\epsilon_4|^{-1} < \frac{1}{2}$ and

$$e^{-\frac{2}{\pi^3}\frac{|\ln\epsilon_4|}{T^2}} \le e^{-\frac{2}{\pi^3}\frac{|\ln\epsilon_4|}{N^{\frac{3}{2}}|\ln\epsilon_4|^{\frac{2}{9}}}} \le e^{-\frac{2}{\pi^3}\frac{|\ln\epsilon_4|^{\frac{7}{9}}}{N^{\frac{3}{2}}}} \le e^{-\frac{2}{\pi^3}\frac{2^5\pi^4|\ln\epsilon_4|^{\frac{1}{9}}N^{\frac{9}{4}}}{N^{\frac{3}{2}}}} = e^{-64\pi|\ln\epsilon_4|^{\frac{1}{9}}N^{\frac{3}{4}}}. \tag{4.3}$$

Combining (4.2) and (4.3), we obtain

$$\Big|\int_0^1 \sin(\kappa x)(f_1(x)-f_2(x))\mathrm{d}x\Big|^2 \le CM^2 e^{-\frac{2}{\pi^3}T^{-2}|\ln\epsilon_4|(1-2\pi^4T^3|\ln\epsilon_4|^{-1})}$$
$$\le CM^2 e^{-\frac{1}{\pi^3}T^{-2}|\ln\epsilon_4|} \le CM^2 e^{-32\pi|\ln\epsilon_4|^{\frac{1}{9}}N^{\frac{3}{4}}}, \quad \forall\kappa\in(2^{\frac{1}{4}},\ T\pi].$$

It is easy to note that

$$e^{-x}\le\frac{(6n-3)!}{x^{3(2n-1)}}\quad\text{for } x>0.$$

We have

$$\Big|\int_0^1 \sin(j\pi x)(f_1(x)-f_2(x))\mathrm{d}x\Big|^2 \le CM^2\frac{1}{\left(\frac{|\ln\epsilon_4|^{\frac{1}{3}}N^{\frac{9}{4}}}{(6n-3)^3}\right)^{2n-1}},\quad j=1,\dots,T.$$

Consequently, we obtain

$$\sum_{j=1}^{T}\Big|\int_0^1 \sin(j\pi x)(f_1(x)-f_2(x))\mathrm{d}x\Big|^2 \le CM^2\frac{T}{\left(\frac{|\ln\epsilon_4|^{\frac{1}{3}}N^{\frac{9}{4}}}{(6n-3)^3}\right)^{2n-1}}$$
$$\le CM^2\frac{N^{\frac{3}{4}}|\ln\epsilon_4|^{\frac{1}{9}}}{\left(\frac{|\ln\epsilon_4|^{\frac{1}{3}}N^{\frac{9}{4}}}{(6n-3)^3}\right)^{2n-1}} \le CM^2\frac{1}{\left(\frac{|\ln\epsilon_4|^{\frac{2}{9}}N^{\frac{3}{2}}}{(6n-3)^3}\right)^{2n-1}}$$
$$\le CM^2\frac{1}{\left(\frac{|\ln\epsilon_4|^{\frac{1}{9}}N^{\frac{3}{2}}}{(6n-3)^3}\right)^{2n-1}}.$$

Here we have noted that $|\ln\epsilon_4|>1$ when $N^{\frac{3}{8}}<\frac{1}{2^{\frac{5}{6}}\pi^{\frac{2}{3}}}|\ln\epsilon_4|^{\frac{1}{9}}$. If $N^{\frac{3}{8}}<\frac{1}{2^{\frac{5}{6}}\pi^{\frac{2}{3}}}|\ln\epsilon_4|^{\frac{1}{9}}$, we also have

$$\frac{1}{\left([|\ln\epsilon_4|^{\frac{1}{9}}N^{\frac{3}{4}}]+1\right)^{2n-1}}\le\frac{1}{\left(|\ln\epsilon_4|^{\frac{1}{9}}N^{\frac{3}{4}}\right)^{2n-1}}.$$

If $N^{\frac{3}{8}}\ge\frac{1}{2^{\frac{5}{6}}\pi^{\frac{2}{3}}}|\ln\epsilon_4|^{\frac{1}{9}}$, then $T=N$. It follows from Lemma 4.1 that

$$\sum_{j=1}^{T}\Big|\int_0^1 \sin(j\pi x)(f_1(x)-f_2(x))\mathrm{d}x\Big|^2=\epsilon_3^2.$$

Combining the above estimates, we obtain

$$\Big|\int_0^1 \sin(\kappa x)(f_1(x)-f_2(x))\mathrm{d}x\Big|^2 \le C\epsilon_3^2+CM^2\frac{1}{\left(\frac{|\ln\epsilon_4|^{\frac{1}{9}}N^{\frac{3}{2}}}{(6n-3)^3}\right)^{2n-1}}$$
$$+CM^2\frac{1}{\left(|\ln\epsilon_4|^{\frac{1}{9}}N^{\frac{3}{4}}\right)^{2n-1}}+CM^2\frac{(2^{\frac{5}{6}}\pi^{\frac{2}{3}})^{2n-1}}{\left(|\ln\epsilon_4|^{\frac{1}{9}}N^{\frac{5}{8}}\right)^{2n-1}}.$$

Noting that $N^{\frac{5}{8}} \leq N^{\frac{3}{4}} \leq N^{\frac{3}{2}}$ and $2^{\frac{5}{6}}\pi^{\frac{2}{3}} \leq (6n-3)^3$, $\forall n \in \mathbb{N}$. The proof is completed by combining the above estimates. □

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