Arch. Rational Mech. Anal.
Digital Object Identifier (DOI) https://doi.org/10.1007/s00205-018-1228-2

Time-Domain Analysis of an Acoustic–Elastic Interaction Problem

Gang Bao, Yixian Gao & Peijun Li

Communicated by F. Lin

Abstract

Consider the scattering of a time-domain acoustic plane wave by a bounded elastic obstacle which is immersed in a homogeneous air or fluid. This paper concerns the mathematical analysis of such a time-domain acoustic–elastic interaction problem. An exact transparent boundary condition (TBC) is developed to reduce the scattering problem from an open domain into an initial-boundary value problem in a bounded domain. The well-posedness and stability are established for the reduced problem. A priori estimates with explicit time dependence are achieved for the pressure of the acoustic wave field and the displacement of the elastic wave field. Our proof is based on the method of energy, the Lax–Milgram lemma, and the inversion theorem of the Laplace transform. In addition, a time-domain absorbing perfectly matched layer (PML) method is introduced to replace the nonlocal TBC by a Dirichlet boundary condition. A first order symmetric hyperbolic system is derived for the truncated PML problem. The well-posedness and stability are proved. The time-domain PML results are expected to be useful in the computational air/fluid–solid interaction problems.

1. Introduction

Let a time-domain acoustic plane wave be an incident on a bounded elastic obstacle which is immersed in an open domain occupied by a homogeneous, compressible, and inviscid air or fluid. The elastic obstacle is assumed to be made of a homogeneous and isotropic medium. When the incident wave hits on the surface of the obstacle, the scattered acoustic wave will be generated in the open air/fluid and propagate away from the obstacle. Meanwhile, an elastic wave is induced inside the obstacle due to the excitation of the incident wave on the surface. This scattering phenomenon leads to an air/fluid–solid interaction problem. The surface divides the whole space into the interior and exterior of the obstacle in which the elastic wave

Published online: 27 February 2018

and the acoustic wave reside, respectively. The governing acoustic and elastic wave equations are coupled on the surface through two continuity conditions. The first kinematic interface condition is imposed to ensure that the normal velocity of the air/fluid on one side of the boundary matches the accelerated velocity of the solid on another side. The second one is the dynamic condition which results from the balance of forces on two sides of the interface. The dynamic interaction between a structure and surrounding compressible and inviscid air/fluid medium is encountered in many areas of engineering and industrial design and identification, such as detection of submerged objects, vibration analysis for aircrafts and automobiles, and ultrasound vibro-acoustography [22,23,37,39].

This paper concerns the mathematical analysis of such a time-domain acoustic–elastic interaction problem. The goal of this work is threefold:

(1) Prove the well-posdeness and stability of the problem;
(2) Obtain a prior estimates of the solution with explicit time dependence;
(3) Establish the well-posedness and stability of the perfectly matched layer formulation of the problem.

This problem can be categorized into the class of obstacle scattering problems, which are of great interest to physicists, engineers, and applied mathematicians due to their significant applications in diverse scientific areas [14,15,46], such as radar and sonar, geophysical exploration, medical imaging, nondestructive testing, and near-field and nano- optics. The time-domain scattering problems have recently attracted considerable attention due to their capability of capturing wide-band signals and modeling more general material and nonlinearity [4,32,34,41,48]. Compared with the time-harmonic scattering problems, the time-domain problems are less studied due to the additional challenge of the temporal dependence. The analysis can be found in [7,47] for the time-domain acoustic and electromagnetic obstacle scattering problems. We refer to [35] and [25] for the analysis of the time-dependent electromagnetic scattering from an open cavity and a periodic structure, respectively.

The acoustic–elastic interaction problems have received much attention in both the mathematical and engineering communities [16,19,28–30,36]. Many approaches have been attempted to solve numerically the time-domain problems such as coupling of boundary element and finite element with different time quadratures [20,24,43]. Some numerical studies have been done for the inverse problems arising from the fluid–solid interaction such as reconstruction of surfaces of periodic structures or obstacles [31,49]. However, the rigorous mathematical study is open at present.

The perfectly matched layer (PML) method was introduced by Berenger in 1994 [2]. It has been widely used to simulate wave propagation in unbounded media since then. Spurious wave reflections from the boundary of the computational domain are avoided by adding a fictitious layer in which the waves rapidly decay regardless of the frequency and incident angle. This feature makes PML an effective approach for modeling a variety of wave phenomena [3,11,13,45]. After the application to electromagnetic waves, various PML formulations have been introduced for acoustic wave propagation in fluids [40] and elastic wave propagation in solids [1,10,33].

The studies of the PML method for two heterogeneous fluids and two or more solid media are reported in [33,42,50]. We refer to [9] for the convergence analysis of the PML method for a two-layered background medium, but fewer PML formulations are reported to have effectively truncated the air/fluid–solid domains near their interface. The two media need to be modeled by different equations with appropriate interface coupling boundary conditions. Comparing with the PML method for the time-harmonic scattering problems, the rigorous mathematical analysis is very rare for the time-domain PML method due to challenge of the dependence of the absorbing medium on all frequencies. For the time-domain PML method for acoustic waves, the planar PML method in one space direction is considered in [27] for the wave equation. In [17,18], the PML system with a point source is analyzed based on the Cagniard-de Hoop method. In [6], the convergence of the time-domain PML method with circular absorbing layer is proved by using the exponential decay estimate of the modified Bessel functions. In [8], a rectangle PML method is presented and the stability and convergence are obtained. To our best knowledge, the time-domain PML analysis is lacking for the air/fluid–solid interaction problem.

In this work, we intend to answer the mathematical questions on well-posedness and stability of the time-domain acoustic–elastic interaction problem in an open domain. The problem is reformulated equivalently into an initial-boundary value problem in a bounded domain by adopting an exact transparent boundary condition (TBC). Using the Laplace transform and energy method, we show that the reduced variational problem has a unique weak solution in the frequency domain. We also obtain the stability estimate for the solution in the time-domain. We achieve a priori estimates with explicit time dependence for the pressure of the acoustic wave and the displacement of the elastic wave by considering directly the time-domain variational problem and taking special test functions. In addition, we introduce the PML method in the rectangle domain and give the PML formulation for the fluid–solid heterogeneous media. By designing special PML medium property, we derive a first order symmetric hyperbolic system for the truncated PML problem. We show that the system has a unique strong solution and obtain the stability of the solution by using an energy function method. The time-domain PML results are expected to be useful for the computational air/fluid–solid interaction problems.

The paper is organized as follows. In Section 2, the mathematical model is introduced for the acoustic–elastic interaction problem. The time-domain TBC is developed to equivalently reduce the scattering problem into an initial-boundary value problem in a bounded domain. Section 3 is devoted to the analysis of the reduced problem, where the well-posdeness and stability are established in both the frequency domain and time-domain. Moreover, a priori energy estimates are obtained with explicit dependence on the time. In Section 4, the time-domain PML formulation is introduced for the scattering problem. A first order symmetric hyperbolic system is deduced for the truncated PML problem. The well-posedness and stability are established. The paper is concluded with some remarks and future work in Section 5.

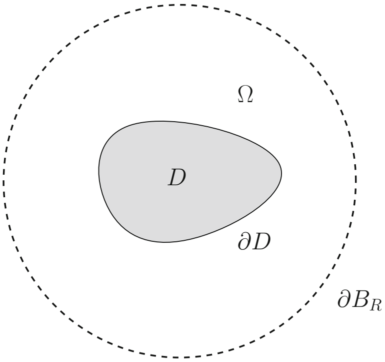

Fig. 1. Geometry of the acoustic–elastic interaction problem

2. Problem Formulation

In this section, we introduce model equations for the acoustic and elastic waves, present an interface problem for the acoustic–elastic interaction, and give some properties of the modified Bessel function. In addition, an exact time-domain transparent boundary condition is introduced to reformulate the scattering problem into an initial-boundary value problem in a bounded domain.

2.1. Problem Geometry

We consider an acoustic plane wave incident on a bounded elastic solid immersed in a homogeneous air/fluid in two dimensions. The problem geometry is shown in Fig. 1. Due to the wave interaction, an elastic wave is induced inside the solid, while the scattered acoustic wave is generated in the air/fluid. This process leads to an air/fluid–solid interaction problem. The solid's surface divides the whole space into the interior domain and the exterior domain where the elastic wave and the acoustic wave occupies, respectively. Let the solid $D \in \mathbb{R}^2$ be a bounded domain with Lipschitz boundary ∂D. We assume that D is occupied by an isotropic linearly elastic medium which is characterized by a constant mass density $\rho_2 > 0$ and two Lamé constants μ, λ satisfying $\mu > 0, \lambda + \mu > 0$. The exterior domain $D^c = \mathbb{R}^2 \backslash \bar{D}$, which is assumed to be connected and filled with a homogeneous, compressible, and inviscid air/fluid with a constant density $\rho_1 > 0$. Denote by $B_R = \left\{ \boldsymbol{r} = (x, y)^\top \in \mathbb{R}^2 : |\boldsymbol{r}| < R \right\}$ the circle with the boundary ∂B_R, where $R > 0$ is sufficiently large such that $\bar{D} \subset B_R$. Let $\Omega = B_R \backslash \bar{D}$ be the bounded region between ∂D and ∂B_R. Denote by $\boldsymbol{n}_D$ the unit normal vector on ∂D directed from D into D^c.

2.2. Acoustic Wave Equation

The acoustic wave field in the air/fluid is governed by the conservation and the dynamics equations in the time-domain

$$\nabla p(\boldsymbol{r}, t) = -\rho_1 \partial_t \boldsymbol{v}(\boldsymbol{r}, t), \quad c^2 \rho_1 \nabla \cdot \boldsymbol{v}(\boldsymbol{r}, t) = -\partial_t p(\boldsymbol{r}, t), \quad \boldsymbol{r} \in D^c, \ t > 0, \tag{2.1}$$

where p is the pressure, $\boldsymbol{v}$ is the velocity, $\rho_1 > 0$ and $c > 0$ are the density and wave speed in the air/fluid, respectively. Eliminating the velocity $\boldsymbol{v}$ from (2.1), we obtain the wave equation for the pressure p:

$$\Delta p(\boldsymbol{r}, t) - \frac{1}{c^2}\partial_t^2 p(\boldsymbol{r}, t) = 0, \quad \boldsymbol{r} \in D^c,\ t > 0. \tag{2.2}$$

Let

$$p^{\text{inc}}(\boldsymbol{r}, t) = f(\boldsymbol{d} \cdot \boldsymbol{r} - ct), \quad \boldsymbol{r} \in D^c, \quad t > 0$$

be a plane incident wave, where f is assumed to be a $C^k (k \geqq 3)$ smooth function and $\boldsymbol{d}$ is the unit propagation direction vector. It is clear to note that the incident wave satisfies acoustic wave equation (2.2). The scattered field $p^{\text{sc}} = p - p^{\text{inc}}$ is required to satisfy the outgoing radiation condition

$$\partial_r p^{\text{sc}} + \frac{1}{c}\partial_t p^{\text{sc}} = o(|\boldsymbol{r}|^{-1/2}), \quad r = |\boldsymbol{r}| \to \infty,\ t > 0. \tag{2.3}$$

The system is assumed to be quiescent at the beginning and the homogeneous initial conditions are prescribed:

$$p(\boldsymbol{r}, t)\big|_{t=0} = 0, \quad \partial_t p(\boldsymbol{r}, t)\big|_{t=0} = 0, \quad \boldsymbol{r} \in D^c.$$

2.3. Elastic Wave Equation

The elastic wave in a homogeneous isotropic solid satisfies the linear elasticity equation

$$\nabla \cdot \boldsymbol{\sigma}(\boldsymbol{u}(\boldsymbol{r}, t)) - \rho_2 \partial_t^2 \boldsymbol{u}(\boldsymbol{r}, t) = 0, \quad \boldsymbol{r} \in D,\ t > 0, \tag{2.4}$$

where $\boldsymbol{u} = (u_1, u_2)^\top$ is the displacement vector, $\rho_2 > 0$ is the density, and the symmetric stress tensor $\boldsymbol{\sigma}(\boldsymbol{u})$ is given by the generalized Hook law

$$\boldsymbol{\sigma}(\boldsymbol{u}) = 2\mu\boldsymbol{\epsilon}(\boldsymbol{u}) + \lambda \text{tr}\,(\boldsymbol{\epsilon}(\boldsymbol{u}))\, I, \quad \boldsymbol{\epsilon}(\boldsymbol{u}) = \frac{1}{2}\left(\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^\top\right). \tag{2.5}$$

Here λ, μ are the Lamé parameters satisfying $\mu > 0, \lambda + \mu > 0$, $I \in \mathbb{R}^{2\times 2}$ is the identity matrix, $\boldsymbol{\epsilon}(\boldsymbol{u})$ is the strain tensor, and $\nabla \boldsymbol{u}$ is the displacement gradient tensor

$$\nabla \boldsymbol{u} = \begin{bmatrix} \partial_x u_1 & \partial_y u_1 \\ \partial_x u_2 & \partial_y u_2 \end{bmatrix}.$$

Substituting (2.5) into (2.4), we obtain the time-domain Navier equation for the displacement $\boldsymbol{u}$:

$$\mu \Delta \boldsymbol{u}(\boldsymbol{r}, t) + (\lambda + \mu)\nabla\nabla \cdot \boldsymbol{u}(\boldsymbol{r}, t) - \rho_2 \partial_t^2 \boldsymbol{u}(\boldsymbol{r}, t) = 0, \quad \boldsymbol{r} \in D,\ t > 0. \tag{2.6}$$

Since the whole system is assumed to be quiescent, the displacement vector is constrained by the initial conditions

$$\boldsymbol{u}(\boldsymbol{r}, t)\big|_{t=0} = 0, \quad \partial_t \boldsymbol{u}(\boldsymbol{r}, t)\big|_{t=0} = 0, \quad \boldsymbol{r} \in D.$$

Next we introduce the Helmholtz decomposition in order to derive a first order system in the PML formulation in Section 4. Let $\boldsymbol{u} = (u_1, u_2)^\top$ and u be a vector and scalar function, respectively. Introduce a scalar curl operator and a vector curl operator

$$\mathrm{curl}\boldsymbol{u} = \partial_x u_2 - \partial_y u_1, \quad \mathbf{curl} u = (\partial_y u, -\partial_x u)^\top.$$

For any solution of the Navier equation (2.6), the Helmholtz decomposition reads

$$\boldsymbol{u} = \nabla\varphi_1 + \mathbf{curl}\varphi_2, \tag{2.7}$$

where φ_1 and φ_2 are two scalar potential functions. Substituting (2.7) into (2.6) yields

$$\nabla\left(\rho_2\partial_t^2\varphi_1 - (\lambda + 2\mu)\Delta\varphi_1\right) + \mathbf{curl}\left(\rho_2\partial_t^2\varphi_2 - \mu\Delta\varphi_2\right) = 0,$$

which is fulfilled if φ_1 and φ_2 satisfy the wave equations

$$\Delta\varphi_1 - \frac{1}{c_1^2}\partial_t^2\varphi_1 = 0, \quad \Delta\varphi_2 - \frac{1}{c_2^2}\partial_t^2\varphi_2 = 0, \tag{2.8}$$

where

$$c_1 = \left(\frac{\lambda + 2\mu}{\rho_2}\right)^{1/2}, \quad c_2 = \left(\frac{\mu}{\rho_2}\right)^{1/2}.$$

2.4. Interface Conditions

To couple the acoustic wave equation and the elastic wave equation, the kinematic interface condition is imposed to ensure the continuity of the normal component of the velocity on ∂D:

$$\boldsymbol{n}_D \cdot \boldsymbol{v}(\boldsymbol{r}, t) = \boldsymbol{n}_D \cdot \partial_t \boldsymbol{u}(\boldsymbol{r}, t), \quad \boldsymbol{r} \in \partial D,\ t > 0. \tag{2.9}$$

Noting $-\rho_1\partial_t\boldsymbol{v}(\boldsymbol{r}, t) = \nabla p(\boldsymbol{r}, t)$, we have from (2.9) that

$$\partial_{\boldsymbol{n}_D} p(\boldsymbol{r}, t) = \boldsymbol{n}_D \cdot \nabla p(\boldsymbol{r}, t) = -\rho_1\boldsymbol{n}_D \cdot \partial_t^2\boldsymbol{u}(\boldsymbol{r}, t), \quad \boldsymbol{r} \in \partial D,\ t > 0.$$

In addition, the following dynamic interface condition is also required:

$$-p(\boldsymbol{r}, t)\boldsymbol{n}_D = \mu\partial_{\boldsymbol{n}_D}\boldsymbol{u}(\boldsymbol{r}, t) + (\lambda + \mu)(\nabla \cdot \boldsymbol{u}(\boldsymbol{r}, t))\boldsymbol{n}_D, \quad \boldsymbol{r} \in \partial D,\ t > 0.$$

2.5. Laplace Transform and Some Functional Spaces

We first introduce some properties of the Laplace transform. For any $s = s_1 + \mathrm{i}s_2$ with $s_1 > 0$ and $\mathrm{i} = \sqrt{-1}$, define by $\breve{\boldsymbol{u}}(s)$ be the Laplace transform of the vector field $\boldsymbol{u}(t)$, that is,

$$\breve{\boldsymbol{u}}(s) = \mathscr{L}(\boldsymbol{u})(s) = \int_0^\infty e^{-st}\boldsymbol{u}(t)\mathrm{d}t.$$

Using the integration by parts yields

$$\int_0^t \boldsymbol{u}(\tau)\mathrm{d}\tau = \mathscr{L}^{-1}(s^{-1}\breve{\boldsymbol{u}}(s)),$$

where $\mathscr{L}^{-1}$ is the inverse Laplace transform. It can be verified form the formula of the inverse Laplace transform that

$$\boldsymbol{u}(t) = \mathscr{F}^{-1}\left(e^{s_1 t}\mathscr{L}(\boldsymbol{u})(s_1 + \mathrm{i}s_2)\right),$$

where $\mathscr{F}^{-1}$ denotes the inverse Fourier transform with respect to s_2. Recall the Plancherel or Parseval identity for the Laplace transform (cf. [12, (2.46)])

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \breve{\boldsymbol{u}}(s)\breve{\boldsymbol{v}}(s)\mathrm{d}s_2 = \int_0^\infty e^{-2s_1 t}\boldsymbol{u}(t)\boldsymbol{v}(t)\mathrm{d}t, \quad \forall\, s_1 > \zeta_0, \tag{2.10}$$

where $\breve{\boldsymbol{u}} = \mathscr{L}(\boldsymbol{u})$, $\breve{\boldsymbol{v}} = \mathscr{L}(\boldsymbol{v})$ and ζ_0 is abscissa of convergence for the Laplace transform of $\boldsymbol{u}$ and $\boldsymbol{v}$.

Hereafter, the expression $a \lesssim b$ stands for $a \leqq Cb$, where C is a positive constant and its specific value is not required but should be always clear from the context.

The following lemma (cf. [44, Theorem 43.1]) is an analogue of Paley–Wiener–Schwarz theorem for Fourier transform of the distributions with compact support in the case of Laplace transform:

Lemma 2.1. *Let $\breve{\boldsymbol{h}}(s)$ denote a holomorphic function in the half-plane $s_1 > \zeta_0$, valued in the Banach space $\mathbb{E}$. The two following conditions are equivalent:*

(1) there is a distribution $\breve{\boldsymbol{h}} \in \mathcal{D}'_+(\mathbb{E})$ whose Laplace transform is equal to $\breve{\boldsymbol{h}}(s)$;
(2) there is a real ζ_1 with $\zeta_0 \leqq \zeta_1 < \infty$ and an integer $m \geqq 0$ such that for all complex numbers s with $\operatorname{Re} s = s_1 > \zeta_1$, it holds that $\|\breve{\boldsymbol{h}}(s)\|_{\mathbb{E}} \lesssim (1 + |s|)^m$,

where $\mathcal{D}'_+(\mathbb{E})$ is the space of distributions on the real line which vanish identically in the open negative half line.

Let $\Omega \subset \mathbb{R}^2$ be a bounded Lipschitz domain with boundary $\partial\Omega$. Denote by

$$H^\nu(\Omega) = \left\{D^\alpha u \in L^2(\Omega) \text{ for all } |\alpha| \leqq \nu\right\}$$

the standard Sobolev space of square integrable functions with the order of derivatives up to ν. Denotes by $H^\nu(\partial\Omega)$ the trace functional space, where $\nu \in \mathbb{R}$. It is clear to note that the dual space of $H^{1/2}(\partial\Omega)$ is $H^{-1/2}(\partial\Omega)$ under the $L^2(\partial\Omega)$ inner produce

$$\langle u, v\rangle_{\partial\Omega} = \int_{\partial\Omega} u\bar{v}\mathrm{d}\gamma.$$

Here the bar denotes the complex conjugate.

Let $H^1(\Omega)^2$ and $H^{1/2}(\partial\Omega)^2$ be the Cartesian product spaces equipped with the corresponding 2-norms of $H^1(\Omega)$ and $H^{1/2}(\partial\Omega)$, respectively. For any $\boldsymbol{u}(\boldsymbol{r}) = (u_1(\boldsymbol{r}), u_2(\boldsymbol{r}))^\top \in H^1(\Omega)^2$, define

$$\|\nabla\boldsymbol{u}\|_{L^2(\Omega)^{2\times 2}} := \left(\sum_{j=1}^{2}\int_{\Omega} |\nabla u_j|^2\mathrm{d}\boldsymbol{r}\right)^{1/2}.$$

A simple calculation gives

$$\|\nabla\boldsymbol{u}\|^2_{L^2(\Omega)^{2\times 2}} + \|\nabla\cdot\boldsymbol{u}\|^2_{L^2(\Omega)} \lesssim \|\boldsymbol{u}\|^2_{H^1(\Omega)^2}. \tag{2.11}$$

2.6. The Modified Bessel Function

To describe the TBC operator for the acoustic wave equation, we introduce a modified Bessel function. For $n \in \mathbb{Z}$, the modified Bessel function $K_n(z)$, $z \in \mathbb{C}$ is the solution of the ordinary differential equation

$$z^2\frac{\mathrm{d}^2 f}{\mathrm{d}z^2} + z\frac{\mathrm{d}f}{\mathrm{d}z} - (z^2 + n^2)f = 0.$$

This satisfies the following asymptotic behavior:

$$K_n(z) \sim \sqrt{\frac{\pi}{2z}}e^{-z} \quad \text{as } |z| \to \infty.$$

The following lemma is proved in [6, Lemma 2.10]:

Lemma 2.2. *Let $R > 0$, $n \in \mathbb{Z}$, $s = s_1 + \mathrm{i}s_2$ with $s_1 > 0$. It then holds that*

$$-\mathrm{Re}\left(\frac{K_n'(sR)}{K_n(sR)}\right) \geqq 0.$$

2.7. Transparent Boundary Condition

In this subsection, we introduce an exact time-domain TBC to formulate the acoustic–elastic wave interaction problem into the following coupled initial-boundary value problem:

$$\begin{cases} \Delta p - \frac{1}{c^2}\partial_t^2 p = 0 & \text{in } \Omega,\ t > 0,\\ \mu\Delta\boldsymbol{u} + (\lambda+\mu)\nabla\nabla\cdot\boldsymbol{u} - \rho_2\partial_t^2\boldsymbol{u} = 0 & \text{in } D,\ t > 0,\\ p\big|_{t=0} = \partial_t p\big|_{t=0} = 0 & \text{in } \Omega,\\ \boldsymbol{u}\big|_{t=0} = \partial_t\boldsymbol{u}\big|_{t=0} = 0 & \text{in } D,\\ \partial_{\boldsymbol{n}_D} p = -\rho_1(\boldsymbol{n}_D\cdot\partial_t^2\boldsymbol{u}), \quad -p\boldsymbol{n}_D = \mu\partial_{\boldsymbol{n}_D}\boldsymbol{u} + (\lambda+\mu)(\nabla\cdot\boldsymbol{u})\boldsymbol{n}_D & \text{on } \partial D,\ t > 0,\\ \partial_r p = \mathscr{T}p + \rho, & \text{on } \partial B_R,\ t > 0, \end{cases} \tag{2.12}$$

where $\mathscr{T}$ is the time-domain TBC operator on ∂B_R, $\rho = \partial_r p^{\rm inc} - \mathscr{T} p^{\rm inc}$. In what follows, we derive the formulation of the operator $\mathscr{T}$ and present some of its properties.

Let $\breve{p}(\boldsymbol{r}, s) = \mathscr{L}(p)(\boldsymbol{r}, t)$ and $\breve{\boldsymbol{u}}(\boldsymbol{r}, s) = \mathscr{L}(\boldsymbol{u})(\boldsymbol{r}, t)$ be the Laplace transform of $p(\boldsymbol{r}, t)$ and $\boldsymbol{u}(\boldsymbol{r}, t)$ with respect to t, respectively. Recall that

$$\mathscr{L}(\partial_t p) = s\breve{p}(\cdot, s) - p(\cdot, 0), \quad \mathscr{L}(\partial_t^2 p) = s^2\breve{p}(\cdot, s) - sp(\cdot, 0) - \partial_t p(\cdot, 0),$$
$$\mathscr{L}(\partial_t \boldsymbol{u}) = s\breve{\boldsymbol{u}}(\cdot, s) - \boldsymbol{u}(\cdot, 0), \quad \mathscr{L}(\partial_t^2 \boldsymbol{u}) = s^2\breve{\boldsymbol{u}}(\cdot, s) - s\boldsymbol{u}(\cdot, 0) - \partial_t \boldsymbol{u}(\cdot, 0).$$

Taking the Laplace transform of (2.12) and using the initial conditions, we obtain the time-harmonic acoustic–elastic wave interaction problem in s-domain

$$\begin{cases} \Delta\breve{p} - \frac{s^2}{c^2}\breve{p} = 0 & \text{in } \Omega, \\ \mu\Delta\breve{\boldsymbol{u}} + (\lambda+\mu)\nabla\nabla\cdot\breve{\boldsymbol{u}} - \rho_2 s^2\breve{\boldsymbol{u}} = 0 & \text{in } D, \\ \partial_{\boldsymbol{n}_D}\breve{p} = -\rho_1 s^2 \boldsymbol{n}_D\cdot\breve{\boldsymbol{u}}, \quad -\breve{p}\boldsymbol{n}_D = \mu\partial_{\boldsymbol{n}_D}\breve{\boldsymbol{u}} + (\lambda+\mu)(\nabla\cdot\breve{\boldsymbol{u}})\boldsymbol{n}_D & \text{on } \partial D, \\ \partial_r\breve{p} = \mathscr{B}\breve{p} + \breve{\rho} & \text{on } \partial B_R, \end{cases}$$

where $s = s_1 + {\rm i}s_2$ with $s_1 > 0$, $\breve{\rho} = \mathscr{L}(\rho)$, and $\mathscr{B}$ is the TBC operator on ∂B_R in the s-domain satisfying $\mathscr{T} = \mathscr{L}^{-1} \circ \mathscr{B} \circ \mathscr{L}$.

In order to deduce the TBC operator, we consider the Helmholtz equation with a complex wavenumber:

$$\Delta\breve{p} - \frac{s^2}{c^2}\breve{p} = 0 \quad \text{in } \mathbb{R}^2\backslash B_R. \tag{2.13}$$

Since $\breve{p}^{\rm inc} = \mathscr{L}(p^{\rm inc})$ satisfies (2.13), the scattered field $\breve{p}^{\rm sc} = \mathscr{L}(p^{\rm sc})$ also satisfies

$$\Delta\breve{p}^{\rm sc} - \frac{s^2}{c^2}\breve{p}^{\rm sc} = 0 \quad \text{in } \mathbb{R}^2\backslash B_R, \tag{2.14}$$

together with the radiation condition (see (2.3))

$$\partial_r\breve{p}^{\rm sc} + \frac{s}{c}\breve{p}^{\rm sc} = o(|\boldsymbol{r}|^{-1/2}), \quad r = |\boldsymbol{r}| \to \infty.$$

In the domain $\mathbb{R}^2\backslash B_R$, it follows from (2.14) and above radiation condition that

$$\breve{p}^{\rm sc}(r,\theta) = \sum_{n\in\mathbb{Z}} \frac{K_n(\frac{s}{c}r)}{K_n(\frac{s}{c}R)}\breve{p}_n^{\rm sc}(R)e^{{\rm i}n\theta}, \quad \breve{p}_n^{\rm sc}(R) = \frac{1}{2\pi}\int_0^{2\pi} p^{\rm sc}(R,\theta)e^{-{\rm i}n\theta}{\rm d}\theta.$$

A simple calculation yields

$$\partial_r\breve{p}^{\rm sc}(r,\theta)\big|_{\partial B_R} = \sum_{n\in\mathbb{Z}} \frac{s}{c}\frac{K_n'(\frac{s}{c}R)}{K_n(\frac{s}{c}R)}\breve{p}_n^{\rm sc}(R)e^{{\rm i}n\theta}. \tag{2.15}$$

For any function u defined in $\mathbb{R}^2\backslash B_R$ with the series expansion

$$u(r,\theta) = \sum_{n\in\mathbb{Z}} u_n(r)e^{{\rm i}n\theta}, \quad u_n(r) = \frac{1}{2\pi}\int_0^{2\pi} u(r,\theta)e^{-{\rm i}n\theta}{\rm d}\theta,$$

we define the Dirichlet-to-Neumann (DtN) operator $\mathscr{B}: H^{1/2}(\partial B_R) \to H^{-1/2}(\partial B_R)$:

$$\mathscr{B}u(R,\theta) = \partial_r u(R,\theta) = \sum_{n\in\mathbb{Z}} \frac{s}{c}\frac{K_n'(\frac{s}{c}R)}{K_n(\frac{s}{c}R)} u_n(R)e^{\mathrm{i}n\theta}. \tag{2.16}$$

It follows from (2.15) that the total field $\breve{p}$ satisfies

$$\partial_r \breve{p} = \mathscr{B}\breve{p} + \breve{\rho} \quad \text{on } \partial B_R, \tag{2.17}$$

where $\breve{\rho} = \partial_r \breve{p}^{\text{inc}} - \mathscr{B}\breve{p}^{\text{inc}}$. Taking the inverse Laplace of (2.17) yields the TBC in the time-domain

$$\partial_r p(\boldsymbol{r},t) = \mathscr{T} p(\boldsymbol{r},t) + \rho(\boldsymbol{r},t) \quad \text{on } \partial B_R.$$

Lemma 2.3. *It holds that*

$$-\mathrm{Re}\langle s^{-1}\mathscr{B}w,\ w\rangle_{\partial B_R} \geqq 0, \quad \forall w \in H^{1/2}(\partial B_R).$$

Proof. Given $w \in H^{1/2}(\partial B_R)$, we have

$$w(R,\theta) = \sum_{n\in\mathbb{Z}} w_n(R)e^{\mathrm{i}n\theta}.$$

It follows from (2.16) and Lemma 2.2 that

$$-\mathrm{Re}\langle s^{-1}\mathscr{B}w,\ w\rangle_{\partial B_R} = \frac{2\pi R}{c}\sum_{n\in\mathbb{Z}} -\mathrm{Re}\left(\frac{K_n'(\frac{s}{c}R)}{K_n(\frac{s}{c}R)}\right)|w_n(R)|^2 \geqq 0,$$

which completes the proof. □

Lemma 2.4. *For any* $w(\boldsymbol{r},t) \in L^2\left(0,T;\ H^{1/2}(\partial B_R)\right)$ *with initial value* $w(\cdot,0) = 0$*, it holds that*

$$-\mathrm{Re}\int_0^T \langle \mathscr{T}w, \partial_t w\rangle_{\partial B_R}\mathrm{d}t \geqq 0.$$

Proof. Let $\tilde{w}(\boldsymbol{r},t)$ be the extension of $w(\boldsymbol{r},t)$ with respect to t in $\mathbb{R}$ such that $\tilde{w}(\boldsymbol{r},t) = 0$ outside the interval $[0,T]$. Using the Parseval identity (2.10) and Lemma 2.3, we get

$$\begin{aligned}
-\mathrm{Re}\int_0^T e^{-2s_1 t}\langle \mathscr{T}w, \partial_t w\rangle_{\partial B_R}\mathrm{d}t &= -\mathrm{Re}\int_0^T e^{-2s_1 t}\int_{\partial B_R}(\mathscr{T}w)\partial_t\bar{w}\mathrm{d}\gamma\mathrm{d}t \\
&= -\mathrm{Re}\int_{\partial B_R}\int_0^\infty e^{-2s_1 t}(\mathscr{T}\tilde{w})\partial_t\bar{\tilde{w}}\mathrm{d}t\mathrm{d}\gamma \\
&= -\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{Re}\langle \mathscr{B}\breve{\tilde{w}}, s\breve{\tilde{w}}\rangle_{\partial B_R}\mathrm{d}s_2 \\
&= -\frac{1}{2\pi}\int_{-\infty}^{\infty}|s|^2\mathrm{Re}\langle s^{-1}\mathscr{B}\breve{\tilde{w}}, \breve{\tilde{w}}\rangle_{\partial B_R}\mathrm{d}s_2 \geqq 0,
\end{aligned}$$

which completes the proof after taking $s_1 \to 0$. □

The following trace theorem is useful in our subsequent stability estimates and the proof can be found in [21]:

Lemma 2.5. (trace theorem) *Let $\Omega \subset \mathbb{R}^2$ be a bounded Lipschitz domain with boundary $\partial\Omega$. The trace operator $\mathscr{R} : H^\nu(\Omega) \to H^{\nu-1/2}(\partial\Omega)$ is bounded for $1/2 < \nu < 3/2$.*

3. The Reduced Problem

In this section, we present the well-posedness, prove the stability, and present a priori estimates for the reduced problem.

3.1. Well-posedness in the s-domain

Consider the reduced boundary value problem

$$\begin{cases} \Delta \breve{p} - \dfrac{s^2}{c^2}\breve{p} = 0 & \text{in } \Omega, \quad (3.1a)\\ \mu\Delta\breve{\boldsymbol{u}} + (\lambda+\mu)\nabla\nabla\cdot\breve{\boldsymbol{u}} - \rho_2 s^2\breve{\boldsymbol{u}} = 0. & \text{in } D, \quad (3.1b)\\ \partial_{\boldsymbol{n}_D}\breve{p} = -\rho_1 s^2\boldsymbol{n}_D\cdot\breve{\boldsymbol{u}}, \quad -\breve{p}\boldsymbol{n}_D = \mu\partial_{\boldsymbol{n}_D}\breve{\boldsymbol{u}} + (\lambda+\mu)(\nabla\cdot\breve{\boldsymbol{u}})\boldsymbol{n}_D & \text{on } \partial D, \quad (3.1c)\\ \partial_r\breve{p} = \mathscr{B}\breve{p} + \breve{\rho}, & \text{on } \partial B_R. \quad (3.1d) \end{cases}$$

We introduce a variational formulation for the problem (3.1) and give a proof of its well-posedness in the space $H^1(\Omega) \times H^1(D)^2$.

Multiplying (3.1a) and (3.1b) by the complex conjugate of a test function $q \in H^1(\Omega)$ and a test functional $\boldsymbol{v} \in H^1(D)^2$, respectively, using the integration by parts and boundary conditions, which include the TBC (3.1d), kinematic interface condition and dynamic interface condition (3.1c), we arrive at the variational problem of finding $(\breve{p}, \breve{\boldsymbol{u}}) \in H^1(\Omega) \times H^1(D)^2$ such that

$$\int_\Omega \left(\frac{1}{s}\nabla\breve{p}\cdot\nabla\bar{q} + \frac{s}{c^2}\breve{p}\,\bar{q}\right)\mathrm{d}\boldsymbol{r} - \langle s^{-1}\mathscr{B}\breve{p}, q\rangle_{\partial B_R} - \rho_1 s\int_{\partial D}(\boldsymbol{n}_D\cdot\breve{\boldsymbol{u}})\,\bar{q}\mathrm{d}\gamma$$
$$= \frac{1}{s}\langle\breve{\rho}, q\rangle_{\partial B_R} \tag{3.2}$$

and

$$\int_D \left(\frac{1}{s}\left(\mu(\nabla\breve{\boldsymbol{u}} : \nabla\bar{\boldsymbol{v}}) + (\lambda+\mu)(\nabla\cdot\breve{\boldsymbol{u}})(\nabla\cdot\bar{\boldsymbol{v}})\right) + \rho_2 s\breve{\boldsymbol{u}}\cdot\bar{\boldsymbol{v}}\right)\mathrm{d}\boldsymbol{r}$$
$$+\frac{1}{s}\int_{\partial D}(\breve{p}\boldsymbol{n}_D)\cdot\bar{\boldsymbol{v}}\mathrm{d}\gamma = 0, \tag{3.3}$$

where $A : B = \operatorname{tr}\left(AB^\top\right)$ is the Frobenius inner product of square matrices A and B.

We multiply (3.3) by $\rho_1|s|^2$ and add it to (3.2) to obtain an equivalent variational problem: To find $(\breve{p}, \breve{\boldsymbol{u}}) \in H^1(\Omega) \times H^1(D)^2$ such that

$$a\left(\breve{p}, \breve{\boldsymbol{u}}; q, \boldsymbol{v}\right) = \frac{1}{s}\langle \breve{\rho},\ q\rangle_{\partial B_R}, \quad \forall (q, \boldsymbol{v}) \in H^1(\Omega) \times H^1(D)^2, \tag{3.4}$$

where the sesquilinear form

$$\begin{aligned} a\left(\breve{p}, \breve{\boldsymbol{u}}; q, \boldsymbol{v}\right) = & \int_\Omega \left(\frac{1}{s}\nabla \breve{p} \cdot \nabla \bar{q} + \frac{s}{c^2}\breve{p}\,\bar{q}\right) \mathrm{d}\boldsymbol{r} \\ & + \int_D \left(\rho_1 \bar{s}\left(\mu(\nabla \breve{\boldsymbol{u}} : \nabla \bar{\boldsymbol{v}}) + (\lambda+\mu)(\nabla \cdot \breve{\boldsymbol{u}})(\nabla \cdot \bar{\boldsymbol{v}})\right)\right. \\ & \left. + \rho_1 \rho_2 s |s|^2 \breve{\boldsymbol{u}} \cdot \bar{\boldsymbol{v}}\right) \mathrm{d}\boldsymbol{r} \\ & - \langle s^{-1}\mathscr{B}\breve{p}, q\rangle_{\partial B_R} + \rho_1 \int_{\partial D} \left(\bar{s}(\breve{p}\boldsymbol{n}_D) \cdot \bar{\boldsymbol{v}} - s\bar{q}(\boldsymbol{n}_D \cdot \breve{\boldsymbol{u}})\right) \mathrm{d}\gamma. \end{aligned} \tag{3.5}$$

Theorem 3.1. *The variational problem* (3.4) *has a unique weak solution* $(\breve{p}, \breve{\boldsymbol{u}}) \in H^1(\Omega) \times H^1(D)^2$, *which satisfies*

$$\|\nabla \breve{p}\|_{L^2(\Omega)^2} + \|s\breve{p}\|_{L^2(\Omega)} \lesssim \frac{1+|s|}{s_1}\|\breve{p}^{\mathrm{inc}}\|_{H^{1/2}(\partial B_R)}, \tag{3.6}$$

$$\|\nabla \breve{\boldsymbol{u}}\|_{L^2(D)^{2\times 2}} + \|\nabla \cdot \breve{\boldsymbol{u}}\|_{L^2(D)} + \|s\breve{\boldsymbol{u}}\|_{L^2(D)^2} \lesssim \frac{1+|s|}{s_1|s|}\|\breve{p}^{\mathrm{inc}}\|_{H^{1/2}(\partial B_R)}. \tag{3.7}$$

Proof. Using the Cauchy–Schwarz inequality and Lemma 2.5, we have

$$\begin{aligned} |a\left(\breve{p}, \breve{\boldsymbol{u}}; q, \boldsymbol{v}\right)| \leqq & \frac{1}{|s|}\|\nabla \breve{p}\|_{L^2(\Omega)^2}\|\nabla q\|_{L^2(\Omega)^2} + \frac{|s|}{c^2}\|\breve{p}\|_{L^2(\Omega)}\|q\|_{L^2(\Omega)} \\ & + \rho_1|s|\left(\mu\|\nabla \breve{\boldsymbol{u}}\|_{L^2(D)^{2\times 2}}\|\nabla \boldsymbol{v}\|_{L^2(D)^{2\times 2}}\right. \\ & \left. + (\lambda+\mu)\|\nabla \cdot \breve{\boldsymbol{u}}\|_{L^2(D)}\|\nabla \cdot \boldsymbol{v}\|_{L^2(D)}\right) \\ & + \rho_1\rho_2|s|^3\|\breve{\boldsymbol{u}}\|_{L^2(D)^2}\|\boldsymbol{v}\|_{L^2(D)^2} \\ & + \frac{1}{|s|}\|\mathscr{B}\breve{p}\|_{H^{-1/2}(\partial B_R)}\|q\|_{H^{1/2}(\partial B_R)} \\ & + \rho_1|s|\left(\|p\boldsymbol{n}_D\|_{H^{-1/2}(\partial D)^2}\|\boldsymbol{v}\|_{H^{1/2}(\partial D)^2}\right. \\ & \left. + \|q\|_{H^{1/2}(\partial D)}\|\boldsymbol{n}_D \cdot \breve{\boldsymbol{u}}\|_{H^{-1/2}(\partial D)}\right) \\ \lesssim & \|\breve{p}\|_{H^1(\Omega)}\|q\|_{H^1(\Omega)} + \|\breve{\boldsymbol{u}}\|_{H^1(D)^2}\|\boldsymbol{v}\|_{H^1(D)^2} \\ & + \|\breve{p}\|_{H^{1/2}(\partial D)}\|q\|_{H^{1/2}(\partial D)} \\ & + \|\breve{p}\|_{H^{1/2}(\partial D)}\|\boldsymbol{v}\|_{H^{1/2}(\partial D)^2} + \|q\|_{H^{1/2}(\partial D)}\|\breve{\boldsymbol{u}}\|_{H^{1/2}(\partial D)^2} \\ \lesssim & \|\breve{p}\|_{H^1(\Omega)}\|q\|_{H^1(\Omega)} + \|\breve{\boldsymbol{u}}\|_{H^1(D)^2}\|\boldsymbol{v}\|_{H^1(D)^2} \\ & + \|\breve{p}\|_{H^1(\Omega)}\|q\|_{H^1(\Omega)} \\ & + \|\breve{p}\|_{H^1(\Omega)}\|\boldsymbol{v}\|_{H^1(D)^2} + \|q\|_{H^1(\Omega)}\|\breve{\boldsymbol{u}}\|_{H^1(D)^2}, \end{aligned}$$

which shows that the sesquilinear form is bounded.

Letting $(q, \boldsymbol{v}) = (\breve{p}, \breve{\boldsymbol{u}})$ in (3.5) yields

$$a(\breve{p}, \breve{\boldsymbol{u}};\ \breve{p}, \breve{\boldsymbol{u}}) = \int_\Omega \left(\frac{1}{s}|\nabla \breve{p}|^2 + \frac{s}{c^2}|\breve{p}|^2\right) \mathrm{d}\boldsymbol{r} \\ + \int_D \left(\rho_1 \bar{s} \left(\mu(\nabla \breve{\boldsymbol{u}} : \nabla \bar{\breve{\boldsymbol{u}}}) + (\lambda + \mu)|\nabla \cdot \breve{\boldsymbol{u}}|^2\right)\right. \\ \left. + \rho_1 \rho_2 s |s|^2 |\breve{\boldsymbol{u}}|^2\right) \mathrm{d}\boldsymbol{r} - \langle s^{-1} \mathscr{B} \breve{p}, \breve{p} \rangle_{\partial B_R} + \rho_1 \int_{\partial D} \left(\bar{s}(\breve{p} \boldsymbol{n}_D) \cdot \bar{\breve{\boldsymbol{u}}}\right. \\ \left. - s \bar{\breve{p}} (\boldsymbol{n}_D \cdot \breve{\boldsymbol{u}})\right) \mathrm{d}\gamma. \tag{3.8}$$

Taking the real part of (3.8) and using Lemma 2.3, we obtain

$$\mathrm{Re}\,(a(\breve{p}, \breve{\boldsymbol{u}};\ \breve{p}, \breve{\boldsymbol{u}})) = \int_\Omega \left(\frac{s_1}{|s|^2}|\nabla \breve{p}|^2 + \frac{s_1}{c^2}|\breve{p}|^2\right) \mathrm{d}\boldsymbol{r} \\ + \rho_1 s_1 \left(\mu \|\nabla \breve{\boldsymbol{u}}\|^2_{L^2(D)^{2\times 2}} + (\lambda + \mu)\|\nabla \cdot \breve{\boldsymbol{u}}\|^2_{L^2(D)}\right) \\ + \rho_1 \rho_2 s_1 |s|^2 |\breve{\boldsymbol{u}}|^2_{L^2(D)^2} - \mathrm{Re}\langle s^{-1} \mathscr{B} \breve{p}, \breve{p} \rangle_{\partial B_R} \\ \gtrsim \frac{s_1}{|s|^2} \left(\|\nabla \breve{p}\|^2_{L^2(\Omega)^2} + \|s \breve{p}\|^2_{L^2(\Omega)}\right) \\ + s_1 \left(\|\nabla \breve{\boldsymbol{u}}\|^2_{L^2(D)^{2\times 2}} + \|\nabla \cdot \breve{\boldsymbol{u}}\|^2_{L^2(D)} + \|s \breve{\boldsymbol{u}}\|^2_{L^2(D)^2}\right). \tag{3.9}$$

It follows from the Lax–Milgram lemma that the variational problem (3.4) has a unique solution $(\breve{p}, \breve{\boldsymbol{u}}) \in H^1(\Omega) \times H^1(D)^2$.

Moreover, we have from (3.4) and the definition of $\breve{\rho}$ that

$$|a(\breve{p}, \breve{\boldsymbol{u}};\ \breve{p}, \breve{\boldsymbol{u}})| \lesssim \frac{1}{|s|^2} \|\breve{\rho}\|_{H^{-1/2}(\partial B_R)} \|s \breve{p}\|_{H^{1/2}(\partial B_R)} \\ \lesssim \frac{1}{|s|^2} \|\breve{p}^{\mathrm{inc}}\|_{H^{1/2}(\partial B_R)} \|s \breve{p}\|_{H^1(\Omega)} \\ \lesssim \frac{1 + |s|}{|s|^2} \|\breve{p}^{\mathrm{inc}}\|_{H^{1/2}(\partial B_R)} \left(\|\nabla \breve{p}\|^2_{L^2(\Omega)^2} + \|s \breve{p}\|^2_{L^2(\Omega)}\right)^{1/2}. \tag{3.10}$$

Combing (3.9) and (3.10) leads to

$$\frac{s_1}{|s|^2} \left(\|\nabla \breve{p}\|^2_{L^2(\Omega)^2} + \|s \breve{p}\|^2_{L^2(\Omega)}\right) \lesssim |a(\breve{p}, \breve{\boldsymbol{u}};\ \breve{p}, \breve{\boldsymbol{u}})| \\ \lesssim \frac{1 + |s|}{|s|^2} \|\breve{p}^{\mathrm{inc}}\|_{H^{1/2}(\partial B_R)} \\ \left(\|\nabla \breve{p}\|^2_{L^2(\Omega)^2} + \|s \breve{p}\|^2_{L^2(\Omega)}\right)^{1/2},$$

which completes the proof of (3.6) after applying the Cauchy–Schwarz inequality.

Similarly, we get from (3.9) and (3.10) that

$$
\begin{aligned}
&|s|^2 \left(\|\nabla \check{\boldsymbol{u}}\|^2_{L^2(D)^{2\times 2}} + \|\nabla \cdot \check{\boldsymbol{u}}\|^2_{L^2(D)} + \|s\check{\boldsymbol{u}}\|^2_{L^2(D)^2} \right) \\
&\quad \lesssim \left(\|\nabla \check{p}\|^2_{L^2(\Omega)^2} + \|s\check{p}\|^2_{L^2(\Omega)} \right) \\
&\qquad + |s|^2 \left(\|\nabla \check{\boldsymbol{u}}\|^2_{L^2(D)^{2\times 2}} + \|\nabla \cdot \check{\boldsymbol{u}}\|^2_{L^2(D)} + \|s\check{\boldsymbol{u}}\|^2_{L^2(D)^2} \right) \\
&\quad \lesssim \frac{|s|^2}{s_1} |a(\check{p}, \check{\boldsymbol{u}}; \check{p}, \check{\boldsymbol{u}})| \lesssim \frac{1+|s|}{s_1} \|\check{p}^{\mathrm{inc}}\|_{H^{1/2}(\partial B_R)} \left(\|\nabla \check{p}\|^2_{L^2(\Omega)^2} + \|s\check{p}\|^2_{L^2(\Omega)} \right)^{1/2} \\
&\quad \lesssim \frac{1+|s|}{s_1} \|\check{p}^{\mathrm{inc}}\|_{H^{1/2}(\partial B_R)} \Big(\|\nabla \check{p}\|^2_{L^2(\Omega)^2} + \|s\check{p}\|^2_{L^2(\Omega)} \\
&\qquad + |s|^2 \left(\|\nabla \check{\boldsymbol{u}}\|^2_{L^2(D)^{2\times 2}} + \|\nabla \cdot \check{\boldsymbol{u}}\|^2_{L^2(D)} + \|s\check{\boldsymbol{u}}\|^2_{L^2(D)^2} \right) \Big)^{1/2}.
\end{aligned}
$$

Applying the Cauchy–Schwarz inequality again, we obtain

$$
\|\nabla \check{\boldsymbol{u}}\|_{L^2(D)^{2\times 2}} + \|\nabla \cdot \check{\boldsymbol{u}}\|_{L^2(D)} + \|s\check{\boldsymbol{u}}\|_{L^2(D)^2} \lesssim \frac{1+|s|}{s_1|s|} \|\check{p}^{\mathrm{inc}}\|_{H^{1/2}(\partial B_R)},
$$

which completes the estimate of (3.7). □

3.2. Well-Posedness in the Time-Domain

We now consider the reduced problem in the time-domain

$$
\begin{cases}
\Delta p - \dfrac{1}{c^2} \partial_t^2 p = 0 & \text{in } \Omega,\ t > 0, & (3.11\text{a}) \\
\mu \Delta \boldsymbol{u} + (\lambda + \mu) \nabla \nabla \cdot \boldsymbol{u} - \rho_2 \partial_t^2 \boldsymbol{u} = 0 & \text{in } D,\ t > 0, & (3.11\text{b}) \\
p\big|_{t=0} = \partial_t p\big|_{t=0} = 0 & \text{in } \Omega, & (3.11\text{c}) \\
\boldsymbol{u}\big|_{t=0} = \partial_t \boldsymbol{u}\big|_{t=0} = 0 & \text{in } D, & (3.11\text{d}) \\
\partial_{\boldsymbol{n}_D} p = -\rho_1 \boldsymbol{n}_D \cdot \partial_t^2 \boldsymbol{u} & \text{on } \partial D,\ t > 0, & (3.11\text{e}) \\
-p \boldsymbol{n}_D = \mu \partial_{\boldsymbol{n}_D} \boldsymbol{u} + (\lambda + \mu)(\nabla \cdot \boldsymbol{u}) \boldsymbol{n}_D & \text{on } \partial D,\ t > 0, & (3.11\text{f}) \\
\partial_r p = \mathscr{T} p + \rho & \text{on } \partial B_R,\ t > 0. & (3.11\text{g})
\end{cases}
$$

To show the well-posedness of the reduced problem (3.11), we make the following assumption for the incident field:

$$
p^{\mathrm{inc}}(\boldsymbol{r}, t) \text{ is a } C^k (k \geqq 3) \text{ smooth function with respect to } t \text{ for any } \boldsymbol{r} \in \mathbb{R}^2 \backslash \bar{D}, \tag{3.12}
$$

which is satisfied since we assume that f is a $C^k (k \geqq 3)$ smooth function.

Theorem 3.2. *The initial-boundary value problem* (3.11) *has a unique solution* $(p(\boldsymbol{r}, t), \boldsymbol{u}(\boldsymbol{r}, t))$*, which satisfies*

$$p(\boldsymbol{r},t) \in L^2\left(0,T;\ H^1(\Omega)\right) \cap H^1\left(0,T;\ L^2(\Omega)\right),$$
$$\boldsymbol{u}(\boldsymbol{r},t) \in L^2\left(0,T;\ H^1(D)^2\right) \cap H^1\left(0,T;\ L^2(D)^2\right),$$

and the stability estimates

$$\begin{aligned}\max_{t\in[0,T]} &\left(\|\partial_t p\|_{L^2(\Omega)} + \|\nabla \partial_t p\|_{L^2(\Omega)^2}\right)\\ &\lesssim \|\rho\|_{L^1(0,T;H^{-1/2}(\partial B_R))} + \max_{t\in[0,T]} \|\partial_t \rho\|_{H^{-1/2}(\partial B_R)}\\ &\quad + \|\partial_t^2 \rho\|_{L^1(0,T;H^{-1/2}(\partial B_R))},\end{aligned} \tag{3.13}$$

$$\begin{aligned}\max_{t\in[0,T]} &\left(\|\partial_t \boldsymbol{u}\|_{L^2(D)^2} + \|\nabla\cdot \boldsymbol{u}\|_{L^2(D)} + \|\nabla \boldsymbol{u}\|_{L^2(D)^{2\times 2}}\right)\\ &\lesssim \|\rho\|_{L^1(0,T;H^{-1/2}(\partial B_R))} + \max_{t\in[0,T]} \|\partial_t \rho\|_{H^{-1/2}(\partial B_R)}\\ &\quad + \|\partial_t^2 \rho\|_{L^1(0,T;H^{-1/2}(\partial B_R))}.\end{aligned} \tag{3.14}$$

Proof. First for the pressure p, we have

$$\begin{aligned}\int_0^T &\left(\|\nabla p\|^2_{L^2(\Omega)^2} + \|\partial_t p\|^2_{L^2(\Omega)}\right) \mathrm{d}t\\ &\leqq \int_0^T e^{-2s_1(t-T)} \left(\|\nabla p\|^2_{L^2(\Omega)^2} + \|\partial_t p\|^2_{L^2(\Omega)}\right) \mathrm{d}t\\ &= e^{2s_1 T} \int_0^T e^{-2s_1 t} \left(\|\nabla p\|^2_{L^2(\Omega)^2} + \|\partial_t p\|^2_{L^2(\Omega)}\right) \mathrm{d}t\\ &\lesssim \int_0^\infty e^{-2s_1 t} \left(\|\nabla p\|^2_{L^2(\Omega)^2} + \|\partial_t p\|^2_{L^2(\Omega)}\right) \mathrm{d}t.\end{aligned}$$

Similarly, we have for the elastic displacement $\boldsymbol{u}$ that

$$\begin{aligned}\int_0^T &\left(\|\partial_t \boldsymbol{u}\|^2_{L^2(D)^2} + \|\nabla \boldsymbol{u}\|^2_{L^2(D)^{2\times 2}} + \|\nabla\cdot \boldsymbol{u}\|^2_{L^2(D)}\right) \mathrm{d}t\\ &\lesssim \int_0^\infty e^{-2s_1 t}\Big(\|\partial_t \boldsymbol{u}\|^2_{L^2(D)^2}\\ &\quad + \|\nabla \boldsymbol{u}\|^2_{L^2(D)^{2\times 2}} + \|\nabla\cdot \boldsymbol{u}\|^2_{L^2(D)}\Big)\mathrm{d}t.\end{aligned}$$

Hence it suffices to estimate the integrals

$$\int_0^\infty e^{-2s_1 t} \left(\|\nabla p\|^2_{L^2(\Omega)^2} + \|\partial_t p\|^2_{L^2(\Omega)}\right) \mathrm{d}t$$

and

$$\int_0^\infty e^{-2s_1 t} \left(\|\partial_t \boldsymbol{u}\|^2_{L^2(D)^2} + \|\nabla \boldsymbol{u}\|^2_{L^2(D)^{2\times 2}} + \|\nabla\cdot \boldsymbol{u}\|^2_{L^2(D)}\right) \mathrm{d}t.$$

Taking the Laplace transform of (3.11), we obtain the reduced acoustic–elastic interaction problem in the s-domain (3.1). It follows from Theorem 3.1 that $\breve{p}$

and $\breve{\boldsymbol{u}}$ satisfy the stability estimate (3.6) and (3.7), respectively. It follows from [44, Lemma 44.1] that $\breve{p}$ and $\breve{\boldsymbol{u}}$ are holomorphic functions of s on the half plane $s_1 > \zeta_0 > 0$, where ζ_0 is any positive constant. Hence we have from Lemma 2.1 that the inverse Laplace transform of $\breve{p}$ and $\breve{\boldsymbol{u}}$ exist and are supported in $[0, \infty)$, that is, the initial-boundary value problem (3.11) has a unique solution $(p, \boldsymbol{u})$.

It follows from the estimate (3.6)–(3.7) and the trace theorem in Lemma 2.5 that

$$\begin{aligned}\|\nabla \breve{p}\|^2_{L^2(\Omega)^2} + \|s\breve{p}\|^2_{L^2(\Omega)} &\lesssim \frac{1+|s|^2}{s_1^2}\|\breve{p}^{\rm inc}\|^2_{H^{1/2}(\partial B_R)} \\ &\lesssim \frac{1+|s|^2}{s_1^2}\|\breve{p}^{\rm inc}\|^2_{H^1(\Omega)}, \end{aligned} \tag{3.15}$$

$$\begin{aligned}\|\nabla \breve{\boldsymbol{u}}\|^2_{L^2(D)^{2\times 2}} + \|\nabla\cdot\breve{\boldsymbol{u}}\|^2_{L^2(D)} + \|s\breve{\boldsymbol{u}}\|^2_{L^2(D)} &\lesssim \frac{1+|s|^2}{s_1^2|s|^2}\|\breve{p}^{\rm inc}\|^2_{H^{1/2}(\partial B_R)} \\ &\lesssim \frac{1+|s|^2}{s_1^4}\|\breve{p}^{\rm inc}\|^2_{H^1(\Omega)}. \end{aligned} \tag{3.16}$$

Using the Parseval identity (2.10) and the stability estimate (3.15), we have

$$\begin{aligned}&\int_0^\infty e^{-2s_1 t}\left(\|\nabla p\|^2_{L^2(\Omega)^2} + \|\partial_t p\|^2_{L^2(\Omega)}\right)\mathrm{d}t \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\|\nabla \breve{p}\|^2_{L^2(\Omega)^2} + \|s\breve{p}\|^2_{L^2(\Omega)}\right)\mathrm{d}s_2 \\ &\lesssim s_1^{-2}\int_{-\infty}^{\infty}\left(\|\breve{p}^{\rm inc}\|^2_{H^1(\Omega)} + \||s|\breve{p}^{\rm inc}\|^2_{H^1(\Omega)}\right)\mathrm{d}s_2 \\ &= s_1^{-2}\int_{-\infty}^{\infty}\left(\|\mathscr{L}p^{\rm inc}\|^2_{H^1(\Omega)} + \|\mathscr{L}(\partial_t p^{\rm inc})\|^2_{H^1(\Omega)}\right)\mathrm{d}s_2 \\ &\lesssim s_1^{-2}\int_0^\infty e^{-2s_1 t}\left(\|p^{\rm inc}\|^2_{H^1(\Omega)} + \|\partial_t p^{\rm inc}\|^2_{H^1(\Omega)}\right)\mathrm{d}t,\end{aligned}$$

which shows that

$$p(\boldsymbol{r}, t) \in L^2\left(0, T; H^1(\Omega)\right) \cap H^1\left(0, T; L^2(\Omega)\right).$$

Since $\breve{\boldsymbol{u}} = \mathscr{L}(\boldsymbol{u}) = \mathscr{F}(e^{-s_1 t}\boldsymbol{u})$, where $\mathscr{F}$ is the Fourier transform with respect to s_2, we have from the Parseval identity (2.10) and the stability estimate (3.16) that

$$\begin{aligned}&\int_0^\infty e^{-2s_1 t}\left(\|\partial_t \boldsymbol{u}\|^2_{L^2(D)^2} + \|\nabla\boldsymbol{u}\|^2_{L^2(D)^{2\times 2}} + \|\nabla\cdot\boldsymbol{u}\|^2_{L^2(D)}\right)\mathrm{d}t \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\|s\breve{\boldsymbol{u}}\|^2_{L^2(D)^2} + \|\nabla\breve{\boldsymbol{u}}\|^2_{L^2(D)^{2\times 2}} + \|\nabla\cdot\breve{\boldsymbol{u}}\|^2_{L^2(D)}\right)\mathrm{d}s_2 \\ &\lesssim s_1^{-4}\int_{-\infty}^{\infty}\left(\|\breve{p}^{\rm inc}\|^2_{H^1(\Omega)} + \||s|\breve{p}^{\rm inc}\|^2_{H^1(\Omega)}\right)\mathrm{d}s_2\end{aligned}$$

$$= s_1^{-4} \int_{-\infty}^{\infty} \left(\|\mathscr{L} p^{\rm inc}\|^2_{H^1(\Omega)} + \|\mathscr{L}(\partial_t p^{\rm inc})\|^2_{H^1(\Omega)} \right) {\rm d}s_2$$
$$\lesssim s_1^{-4} \int_0^{\infty} e^{-2s_1 t} \left(\|p^{\rm inc}\|^2_{H^1(\Omega)} + \|\partial_t p^{\rm inc}\|^2_{H^1(\Omega)} \right) {\rm d}t.$$

It follows from (2.11) that

$$\boldsymbol{u}(\boldsymbol{r}, t) \in L^2 \left(0, T; H^1(D)^2\right) \cap H^1 \left(0, T; L^2(D)^2\right).$$

Next we prove the stability. Taking the partial derivative of (3.11b), (3.11d), and (3.11f) with respect to t yields

$$\begin{cases} \mu\Delta(\partial_t \boldsymbol{u}) + (\lambda+\mu)\nabla\nabla\cdot(\partial_t \boldsymbol{u}) - \rho_2 \partial_t^2(\partial_t \boldsymbol{u}) = 0 & \text{in } D,\ t > 0, \\ \partial_t \boldsymbol{u}\big|_{t=0} = 0, \quad \partial_t^2 \boldsymbol{u}\big|_{t=0} = \rho_2^{-1} \left(\mu\Delta\boldsymbol{u} + (\lambda+\mu)\nabla\nabla\cdot\boldsymbol{u}\right)\big|_{t=0} = 0 & \text{in } D, \\ -\partial_t p \boldsymbol{n}_D = \mu \partial_{\boldsymbol{n}_D}(\partial_t \boldsymbol{u}) + (\lambda+\mu)(\nabla\cdot\partial_t \boldsymbol{u})\boldsymbol{n}_D & \text{on } \partial D,\ t > 0. \end{cases} \tag{3.17}$$

For any $0 < t < T$, consider the energy function

$$E(t) = E_1(t) + E_2(t),$$

where

$$E_1(t) = \|\frac{1}{c}\partial_t p\|^2_{L^2(\Omega)} + \|\nabla p\|^2_{L^2(\Omega)^2},$$

and

$$E_2(t) = \|(\rho_1\rho_2)^{1/2}\partial_t^2 \boldsymbol{u}\|^2_{L^2(D)^2} + \|\rho_1^{1/2}(\lambda+\mu)^{1/2}\nabla\cdot(\partial_t \boldsymbol{u})\|^2_{L^2(D)}$$
$$+ \|(\rho_1\mu)^{1/2}\nabla(\partial_t \boldsymbol{u})\|^2_{L^2(D)^{2\times 2}}.$$

It is easy to note that

$$E(t) - E(0) = \int_0^t E'(\tau){\rm d}\tau = \int_0^t \left(E_1'(\tau) + E_2'(\tau)\right){\rm d}\tau. \tag{3.18}$$

It follows from the integration by parts and (3.11a), (3.11c), (3.11e), (3.11g) that

$$\int_0^t E_1'(\tau){\rm d}\tau = 2{\rm Re}\int_0^t \int_\Omega \left(\frac{1}{c^2}\partial_\tau^2 p\ \partial_\tau \bar{p} + \nabla(\partial_\tau p)\cdot\nabla\bar{p}\right){\rm d}\boldsymbol{r}{\rm d}\tau$$
$$= 2{\rm Re}\int_0^t \int_\Omega \left(\Delta p\ \partial_\tau \bar{p} + \nabla(\partial_\tau p)\cdot\nabla\bar{p}\right){\rm d}\boldsymbol{r}{\rm d}\tau$$
$$= \int_0^t \int_\Omega 2{\rm Re}\left(-\nabla p\cdot\nabla(\partial_t \bar{p}) + \nabla(\partial_t p)\cdot\nabla\bar{p}\right){\rm d}\boldsymbol{r}{\rm d}\tau$$
$$+ 2{\rm Re}\int_0^t \int_{\partial B_R} \partial_r p\ \partial_\tau \bar{p}{\rm d}\gamma{\rm d}\tau - 2{\rm Re}\int_0^t \int_{\partial D} \partial_{\boldsymbol{n}_D} p\ \partial_\tau \bar{p}{\rm d}\gamma{\rm d}\tau$$
$$= 2{\rm Re}\int_0^t \int_{\partial B_R} (\mathscr{T} p)\ \partial_t \bar{p}{\rm d}\gamma{\rm d}\tau + 2{\rm Re}\int_0^t \int_{\partial B_R} \rho\ \partial_\tau \bar{p}{\rm d}\gamma{\rm d}\tau$$

$$
\begin{aligned}
&+2\mathrm{Re}\int_0^t\int_{\partial D}\rho_1(\boldsymbol{n}_D\cdot\partial_\tau^2\boldsymbol{u})\,\partial_\tau\bar{p}\mathrm{d}\gamma\mathrm{d}\tau\\
&=2\mathrm{Re}\int_0^t\langle\mathscr{T}p,\ \partial_\tau p\rangle_{\partial B_R}\mathrm{d}\tau+2\mathrm{Re}\int_0^t\langle\rho,\ \partial_\tau p\rangle_{\partial B_R}\mathrm{d}\tau\\
&+2\mathrm{Re}\int_0^t\rho_1\langle\boldsymbol{n}_D\cdot\partial_\tau^2\boldsymbol{u},\ \partial_\tau p\rangle_{\partial D}\mathrm{d}\tau. \qquad (3.19)
\end{aligned}
$$

Similarly, we have from (3.17) and the integration by parts that

$$
\begin{aligned}
\int_0^t E_2'(\tau)\mathrm{d}\tau &= 2\mathrm{Re}\int_0^t\int_D\rho_1\rho_2\partial_\tau^2(\partial_\tau\boldsymbol{u})\cdot\partial_\tau^2\bar{\boldsymbol{u}}\mathrm{d}\boldsymbol{r}\mathrm{d}\tau\\
&+2\mathrm{Re}\int_0^\tau\int_D\rho_1(\lambda+\mu)(\nabla\cdot\partial_\tau^2\boldsymbol{u})(\nabla\cdot\partial_\tau\bar{\boldsymbol{u}})+\rho_1\mu(\nabla\partial_\tau^2\boldsymbol{u}):(\nabla\partial_\tau\bar{\boldsymbol{u}})\mathrm{d}\boldsymbol{r}\mathrm{d}\tau\\
&=2\mathrm{Re}\int_0^t\int_D\rho_1\left(\mu\Delta\partial_\tau\boldsymbol{u}+(\lambda+\mu)\nabla\nabla\cdot\partial_\tau u\right)\cdot\partial_\tau^2\bar{\boldsymbol{u}}\mathrm{d}\boldsymbol{r}\mathrm{d}\tau\\
&+2\mathrm{Re}\int_0^\tau\int_D\rho_1(\lambda+\mu)(\nabla\cdot\partial_\tau^2\boldsymbol{u})(\nabla\cdot\partial_\tau\bar{\boldsymbol{u}})+\rho_1\mu(\nabla\partial_\tau^2\boldsymbol{u}):(\nabla\partial_\tau\bar{\boldsymbol{u}})\mathrm{d}\boldsymbol{r}\mathrm{d}\tau\\
&=2\mathrm{Re}\int_0^t\int_D-\Big(\rho_1\mu(\nabla\partial_\tau\boldsymbol{u}):(\nabla\partial_\tau^2\bar{\boldsymbol{u}})+\rho_1(\lambda+\mu)(\nabla\cdot\partial_\tau\boldsymbol{u})(\nabla\cdot\partial_\tau^2\bar{\boldsymbol{u}})\Big)\mathrm{d}\boldsymbol{r}\mathrm{d}\tau\\
&+2\mathrm{Re}\int_0^\tau\int_D\rho_1(\lambda+\mu)(\nabla\cdot\partial_\tau^2\boldsymbol{u})(\nabla\cdot\partial_\tau\bar{\boldsymbol{u}})+\rho_1\mu(\nabla\partial_\tau^2\boldsymbol{u}):(\nabla\partial_\tau\bar{\boldsymbol{u}})\mathrm{d}\boldsymbol{r}\mathrm{d}\tau\\
&+2\mathrm{Re}\int_0^t\rho_1\langle\mu\partial_{\boldsymbol{n}_D}(\partial_\tau\boldsymbol{u})+(\lambda+\mu)(\nabla\cdot\partial_\tau\boldsymbol{u})\boldsymbol{n}_D,\ \partial_\tau^2\boldsymbol{u}\rangle_{\partial D}\mathrm{d}\tau\\
&=-2\mathrm{Re}\int_0^t\rho_1\langle\partial_\tau p,\boldsymbol{n}_D\cdot\partial_\tau^2\boldsymbol{u}\rangle_{\partial D}\mathrm{d}\tau. \qquad (3.20)
\end{aligned}
$$

Since $E(0)=0$, combining (3.18)–(3.20) and using Lemma 2.4 and Lemma 2.5, we obtain

$$
\begin{aligned}
E(t) &= 2\mathrm{Re}\int_0^t\langle\mathscr{T}p,\ \partial_\tau p\rangle_{\partial B_R}\mathrm{d}\tau+2\mathrm{Re}\int_0^t\langle\rho,\ \partial_\tau p\rangle_{\partial B_R}\mathrm{d}\tau\\
&\leqq 2\mathrm{Re}\int_0^t\langle\rho,\ \partial_\tau p\rangle_{\partial B_R}\mathrm{d}\tau\leqq 2\int_0^t\|\rho\|_{H^{-1/2}(\partial B_R)}\|\partial_\tau p\|_{H^{1/2}(\partial B_R)}\mathrm{d}\tau\\
&\lesssim 2\int_0^t\|\rho\|_{H^{-1/2}(\partial B_R)}\|\partial_\tau p\|_{H^1(\Omega)}\mathrm{d}\tau\\
&\lesssim 2\max_{t\in[0,T]}\|\partial_\tau p\|_{H^1(\Omega)}\|\rho\|_{L^1(0,T;H^{-1/2}(\partial B_R))}. \qquad (3.21)
\end{aligned}
$$

Using Young's inequality, we obtain

$$
\begin{aligned}
\left(\|\partial_t p\|^2_{L^2(\Omega)}+\|\nabla p\|^2_{L^2(\Omega)^2}\right)\lesssim E(t)&\lesssim\epsilon\max_{t\in[0,T]}\left(\|\partial_t p\|^2_{L^2(\Omega)}+\|\nabla\partial_t p\|^2_{L^2(\Omega)^2}\right)\\
&+\frac{1}{\epsilon}\|\rho\|^2_{L^1(0,T;H^{-1/2}(\partial B_R))}. \qquad (3.22)
\end{aligned}
$$

Since the right-hand side of (3.22) contains the term $\nabla\partial_t p$, which can not be controlled by the left-hand side of (3.22), hence we need to consider a new reduced system.

Taking the first partial derivative of (3.11a), (3.11c), (3.11e) and (3.11g), and the second partial derivative of (3.11b), (3.11d) and (3.11f) with respect to t, respectively, we get

$$\begin{cases} \Delta\partial_t p - \frac{1}{c^2}\partial_t^2(\partial_t p) = 0 & \text{in } \Omega,\ t>0,\\ \mu\Delta\partial_t^2\boldsymbol{u} + (\lambda+\mu)\nabla\nabla\cdot\partial_t^2\boldsymbol{u} - \rho_2\partial_t^2(\partial_t^2\boldsymbol{u}) = 0 & \text{in } D,\ t>0,\\ \partial_t p\big|_{t=0} = 0, \quad \partial_t^2 p\big|_{t=0} = c^2\Delta p\big|_{t=0} = 0 & \text{in } \Omega,\\ \partial_t^2\boldsymbol{u}\big|_{t=0} = 0, \quad \partial_t(\partial_t^2\boldsymbol{u}) = \rho_2^{-1}\left(\mu\Delta(\partial_t\boldsymbol{u}) + (\lambda+\mu)\nabla\nabla\cdot(\partial_t\boldsymbol{u})\right)\big|_{t=0} = 0 & \text{in } D,\\ \partial_{\boldsymbol{n}_D}(\partial_t p) = -\rho_1\boldsymbol{n}_D\cdot\partial_t^2(\partial_t\boldsymbol{u}) & \text{on } \partial D,\ t>0,\\ -\partial_t^2 p\boldsymbol{n}_D = \mu\partial_{\boldsymbol{n}_D}(\partial_t^2\boldsymbol{u}) + (\lambda+\mu)(\nabla\cdot\partial_t^2\boldsymbol{u})\boldsymbol{n}_D & \text{on } \partial D,\ t>0,\\ \partial_r(\partial_t p) = \mathscr{T}(\partial_t p) + \partial_t\rho & \text{on } \partial B_R,\ t>0. \end{cases}$$

We consider the energy function

$$F(t) = F_1(t) + F_2(t),$$

where

$$F_1(t) = \|\frac{1}{c}\partial_t^2 p\|^2_{L^2(\Omega)} + \|\nabla\partial_t p\|^2_{L^2(\Omega)^2},$$

and

$$F_2(t) = \|(\rho_1\rho_2)^{1/2}\partial_t^2(\partial_t\boldsymbol{u})\|^2_{L^2(D)^2} + \|\rho_1^{1/2}(\lambda+\mu)^{1/2}\nabla\cdot(\partial_t^2\boldsymbol{u})\|^2_{L^2(D)} + \|(\rho_1\mu)^{1/2}\nabla(\partial_t^2\boldsymbol{u})\|^2_{L^2(D)^{2\times 2}}.$$

It is clear to note that $F(0) = 0$.

Similarly, as with Lemma 2.4, we may show that

$$\mathrm{Re}\int_0^t \langle\mathscr{T}\partial_\tau p,\ \partial_\tau^2 p\rangle_{\partial B_R}\mathrm{d}\tau \leqq 0. \tag{3.23}$$

In fact, let $\tilde{p}$ be the extension of p with respect to t in $\mathbb{R}$ such that $\tilde{p} = 0$ outside the interval $[0, t]$. Using the Parseval identity (2.10) and Lemma 2.3, we get

$$\begin{aligned} \mathrm{Re}\int_0^t e^{-2s_1\tau}\langle\mathscr{T}\partial_\tau p, \partial_\tau^2 p\rangle_{\partial B_R}\mathrm{d}\tau &= \mathrm{Re}\int_0^t e^{-2s_1\tau}\int_{\partial B_R}(\mathscr{T}\partial_\tau p)\ \partial_\tau^2\bar{p}\mathrm{d}\gamma\mathrm{d}\tau \\ = \mathrm{Re}\int_{\partial B_R}\int_0^\infty e^{-2s_1\tau}(\mathscr{T}\partial_\tau\tilde{p})\ \partial_\tau^2\tilde{p}\ \mathrm{d}\gamma\mathrm{d}\tau &= \frac{1}{2\pi}\int_{-\infty}^\infty \mathrm{Re}\langle s\mathscr{B}\breve{\tilde{p}},\ s^2\breve{\tilde{p}}\rangle_{\partial B_R}\mathrm{d}s_2 \\ = \frac{1}{2\pi}\int_{-\infty}^\infty |s|^4\mathrm{Re}\langle s^{-1}\mathscr{B}\breve{\tilde{p}},\ \breve{\tilde{p}}\rangle_{\partial B_R}\mathrm{d}s_2 \leqq 0, \end{aligned}$$

which implies (3.23), after taking $s_1 \to 0$.

Following the same steps to prove the inequality (3.21), we obtain from (3.23) and integration by parts that

$$\begin{aligned}F(t) &\leqq 2\mathrm{Re}\int_0^t \langle \partial_\tau \rho, \partial_\tau^2 p\rangle_{\partial B_R} \mathrm{d}\tau + 2\mathrm{Re}\int_0^t \langle \mathscr{T}\partial_\tau p,\ \partial_\tau^2 p\rangle_{\partial B_R} \mathrm{d}\tau \\ &\leqq 2\mathrm{Re}\int_{\partial B_R}\int_0^t \partial_\tau \rho\ \partial_\tau^2 \bar{p} \mathrm{d}\tau \mathrm{d}\gamma \\ &= 2\mathrm{Re}\int_{\partial B_R} \partial_\tau \rho(\cdot,\tau)\partial_\tau \bar{p}(\cdot,\tau)\Big|_0^t - 2\mathrm{Re}\int_0^t \langle \partial_\tau^2 \rho, \partial_\tau p\rangle_{\partial B_R} \mathrm{d}\tau \\ &\lesssim 2 \max_{t\in[0,T]} \|\partial_t p\|_{H^1(\Omega)}\left(\max_{t\in[0,T]} \|\partial_t \rho\|_{H^{-1/2}(\partial B_R)} + \|\partial_t^2 \rho\|_{L^1(0,T;H^{-1/2}(\partial B_R))}\right).\end{aligned} \tag{3.24}$$

Combining (3.22) and (3.24), using Young's inequality, we have

$$\begin{aligned}\left(\|\partial_t p\|^2_{L^2(\Omega)} + \|\nabla\partial_t p\|^2_{L^2(\Omega)^2}\right) &\lesssim E(t) + F(t) \\ &\lesssim 2\epsilon \max_{t\in[0,T]}\left(\|\partial_t p\|^2_{L^2(\Omega)} + \|\nabla\partial_t p\|^2_{L^2(\Omega)^2}\right) \\ &\quad + \frac{1}{\epsilon}\|\rho\|^2_{L^1(0,T;H^{-1/2}(\partial B_R))} \\ &\quad + \frac{1}{\epsilon}\max_{t\in[0,T]}\|\partial_t\rho\|^2_{H^{-1/2}(\partial B_R)} \\ &\quad + \frac{1}{\epsilon}\|\partial_t^2\rho\|^2_{L^1(0,T;H^{-1/2}(\partial B_R))}.\end{aligned} \tag{3.25}$$

Hence we choose $\epsilon > 0$ small enough such that $2\epsilon < 1/2$, for example, $\epsilon = 1/8$. It follows from (3.25) and the Cauchy–Schwarz inequality that

$$\begin{aligned}\max_{t\in[0,T]}\left(\|\partial_t p\|_{L^2(\Omega)} + \|\nabla\partial_t p\|_{L^2(\Omega)^2}\right) \lesssim &\Big(\|\rho\|_{L^1(0,T;H^{-1/2}(\partial B_R))} \\ &+ \max_{t\in[0,T]}\|\partial_t\rho\|_{H^{-1/2}(\partial B_R)} \\ &+ \|\partial_t^2\rho\|_{L^1(0,T;H^{-1/2}(\partial B_R))}\Big),\end{aligned}$$

which completes the estimate (3.13).

For the elastic displacement $\boldsymbol{u}$, we can also obtain that

$$\begin{aligned}&\left(\|\partial_t^2 \boldsymbol{u}\|^2_{L^2(D)^2} + \|\nabla\cdot(\partial_t \boldsymbol{u})\|^2_{L^2(D)} + \|\nabla(\partial_t \boldsymbol{u})\|^2_{L^2(D)^{2\times 2}}\right) \\ &\quad + \left(\|\partial_t p\|^2_{L^2(\Omega)} + \|\nabla\partial_t p\|^2_{L^2(\Omega)^2}\right) \\ &\lesssim E(t) + F(t) \lesssim 2\epsilon \max_{t\in[0,T]}\left(\|\partial_t p\|^2_{L^2(\Omega)} + \|\nabla\partial_t p\|^2_{L^2(\Omega)^2}\right)\end{aligned}$$

$$+\frac{1}{\epsilon}\|\rho\|^2_{L^1(0,T;H^{-1/2}(\partial B_R))}$$
$$+\frac{1}{\epsilon}\max_{t\in[0,T]}\|\partial_t\rho\|^2_{H^{-1/2}(\partial B_R)}+\frac{1}{\epsilon}\|\partial_t^2\rho\|^2_{L^1(0,T;H^{-1/2}(\partial B_R))}.$$

Using Young's inequality again, we obtain

$$\max_{t\in[0,T]}\left(\|\partial_t^2\boldsymbol{u}\|^2_{L^2(D)^2}+\|\nabla\cdot(\partial_t\boldsymbol{u})\|^2_{L^2(D)}+\|\nabla(\partial_t\boldsymbol{u})\|^2_{L^2(D)^{2\times 2}}\right)$$
$$\lesssim\|\rho\|^2_{L^1(0,T;H^{-1/2}(\partial B_R))}+\max_{t\in[0,T]}\|\partial_t\rho\|^2_{H^{-1/2}(\partial B_R)}$$
$$+\|\partial_t^2\rho\|^2_{L^1(0,T;H^{-1/2}(\partial B_R))}. \tag{3.26}$$

For any $0<t\leqq T$, using the Young inequality leads to

$$\|\partial_t\boldsymbol{u}\|^2_{L^2(D)^2}=\int_0^t\partial_\tau\|\partial_\tau\boldsymbol{u}(\cdot,\tau)\|^2_{L^2(D)^2}\mathrm{d}\tau\leqq\epsilon T\|\partial_t\boldsymbol{u}\|^2_{L^2(D)^2}+\frac{T}{\epsilon}\|\partial_t^2\boldsymbol{u}\|^2_{L^2(D)^2}.$$

Here we choose ϵ small enough such that $\epsilon T<1$, for example, $\epsilon=\frac{1}{2T}$. Hence, we have

$$\|\partial_t\boldsymbol{u}\|^2_{L^2(D)^2}\leqq 2T^2\|\partial_t^2\boldsymbol{u}\|^2_{L^2(D)^2}\lesssim\|\partial_t^2\boldsymbol{u}\|^2_{L^2(D)^2}. \tag{3.27}$$

Similarly, we can obtain

$$\|\nabla\cdot\boldsymbol{u}\|^2_{L^2(D)}\lesssim T^2\|\nabla\cdot(\partial_t\boldsymbol{u})\|^2_{L^2(D)},\quad\|\nabla\boldsymbol{u}\|^2_{L^2(D)^{2\times 2}}\lesssim T^2\|\nabla(\partial_t\boldsymbol{u})\|^2_{L^2(D)^{2\times 2}}. \tag{3.28}$$

Combining (3.26)–(3.28), we have

$$\max_{t\in[0,T]}\left(\|\partial_t\boldsymbol{u}\|^2_{L^2(D)^2}+\|\nabla\cdot\boldsymbol{u}\|^2_{L^2(D)}+\|\nabla\boldsymbol{u}\|^2_{L^2(D)^{2\times 2}}\right)$$
$$\lesssim\|\rho\|^2_{L^1(0,T;H^{-1/2}(\partial B_R))}+\max_{t\in[0,T]}\|\partial_t\rho\|^2_{H^{-1/2}(\partial B_R)}$$
$$+\|\partial_t^2\rho\|^2_{L^1(0,T;H^{-1/2}(\partial B_R))},$$

which shows the estimate (3.14), after applying the Cauchy–Schwarz inequality. □

3.3. A Priori Estimates

In this section, we derive a priori stability estimates for the pressure p and the displacement $\boldsymbol{u}$ with a minimum regularity requirement for the data and an explicit dependence on the time.

We consider the elastic wave equation of $\partial_t \boldsymbol{u}$ in order to match the interface conditions when deducing the stability estimates. Taking the partial derivative of (3.11b), (3.11d), and (3.11f) with respect to t, we obtain a new reduced problem:

$$\begin{cases} \Delta p - \frac{1}{c^2}\partial_t^2 p = 0 & \text{in } \Omega,\ t > 0 \\ \partial_r p = \mathscr{T} p + \rho & \text{on } \partial B_R,\ t > 0, \\ \partial_{\boldsymbol{n}_D} p = -\rho_1 \boldsymbol{n}_D \cdot \partial_t^2 \boldsymbol{u}, & \text{on } \partial D,\ t > 0, \\ p\big|_{t=0} = \partial_t p\big|_{t=0} = 0, & \text{in } \Omega, \\ \mu\Delta(\partial_t \boldsymbol{u}) + (\lambda+\mu)\nabla\nabla\cdot(\partial_t \boldsymbol{u}) - \rho_2 \partial_t^2(\partial_t \boldsymbol{u}) = 0 & \text{in } D,\ t > 0, \\ \partial_t \boldsymbol{u}\big|_{t=0} = 0, & \text{in } D, \\ \partial_t^2 \boldsymbol{u}\big|_{t=0} = \rho_2^{-1}\left(\mu\Delta \boldsymbol{u} + (\lambda+\mu)\nabla\nabla\cdot \boldsymbol{u}\right)\big|_{t=0} = 0 & \text{in } D, \\ -\partial_t p \boldsymbol{n}_D = \mu \partial_{\boldsymbol{n}_D}(\partial_t \boldsymbol{u}) + (\lambda+\mu)(\nabla\cdot\partial_t \boldsymbol{u})\boldsymbol{n}_D & \text{on } \partial D,\ t > 0. \end{cases} \tag{3.29}$$

The variational problem of (3.29) is to find $(p, \boldsymbol{u}) \in H^1(\Omega) \times \in H^1(D)^2$ for all $t > 0$ such that

$$\begin{aligned} \int_\Omega \frac{1}{c^2}\partial_t^2 p\, \bar{q} \mathrm{d}\boldsymbol{r} = &- \int_\Omega \nabla p \cdot \nabla \bar{q} \mathrm{d}\boldsymbol{r} \\ &+ \int_{\partial B_R} \partial_r p\, \bar{q} \mathrm{d}\gamma - \int_{\partial D} \partial_{\boldsymbol{n}_D} p\, \bar{q} \mathrm{d}\gamma \\ = &- \int_\Omega \nabla p \cdot \nabla \bar{q} \mathrm{d}\boldsymbol{r} + \int_{\partial B_R} (\mathscr{T} p + \rho)\, \bar{q} \mathrm{d}\gamma \\ &+ \int_{\partial D} \rho_1 (\boldsymbol{n}_D \cdot \partial_t^2 \boldsymbol{u})\, \bar{q} \mathrm{d}\gamma, \quad \forall q \in H^1(\Omega), \end{aligned} \tag{3.30}$$

$$\begin{aligned} \int_D \rho_2 \partial_t^2(\partial_t \boldsymbol{u}) \cdot \bar{\boldsymbol{v}} \mathrm{d}\boldsymbol{r} = &- \int_D \left(\mu \nabla(\partial_t \boldsymbol{u}) : \nabla \bar{\boldsymbol{v}} + (\lambda+\mu)(\nabla\cdot(\partial_t \boldsymbol{u}))(\nabla\cdot\bar{\boldsymbol{v}})\right) \mathrm{d}\boldsymbol{r} \\ &+ \int_{\partial D} \left(\mu \partial_{\boldsymbol{n}_D}(\partial_t \boldsymbol{u}) + (\lambda+\mu)(\nabla\cdot\partial_t \boldsymbol{u})\boldsymbol{n}_D\right) \cdot \bar{\boldsymbol{v}} \mathrm{d}\gamma \\ = &- \int_D \left(\mu \nabla(\partial_t \boldsymbol{u}) : \nabla \bar{\boldsymbol{v}} + (\lambda+\mu)(\nabla\cdot(\partial_t \boldsymbol{u}))(\nabla\cdot\bar{\boldsymbol{v}})\right) \mathrm{d}\boldsymbol{r} \\ &- \int_{\partial D} (\partial_t p)(\boldsymbol{n}_D \cdot \bar{\boldsymbol{v}}) \mathrm{d}\gamma, \quad \forall \boldsymbol{v} \in H^1(D)^2. \end{aligned} \tag{3.31}$$

To show the stability of its solution, we follow the argument in [44] but with a careful study of the TBC. The two lemmas that follow are useful for the subsequent analysis.

Lemma 3.3. *Given $\theta \geqq 0$ and $p(\cdot, t) \in H^1(\Omega)$ with $p(\cdot, 0) = 0$, it holds that*

$$\mathrm{Re} \int_{\partial B_R} \int_0^\theta \left(\int_0^t \mathscr{T} p(\cdot, \tau) \mathrm{d}\tau \right) \bar{p}(\cdot, t) \mathrm{d}t \mathrm{d}\gamma \leqq 0.$$

Proof. Let $\tilde{p}$ be the extension of p with respect to t in $\mathbb{R}$ such that $\tilde{p} = 0$ outsider the interval $[0, \theta]$. We obtain from the Parseval identity (2.10) and Lemma 2.3 that

$$\mathrm{Re} \int_{\partial B_R} \int_0^\theta e^{-2s_1 t} \left(\int_0^t \mathscr{T} p(\cdot, \tau) \mathrm{d}\tau \right) \bar{p}(\cdot, t) \mathrm{d}t \mathrm{d}\gamma$$

$$
\begin{aligned}
&= \mathrm{Re}\int_{\partial B_R}\int_0^\infty e^{-2s_1 t}\left(\int_0^t \mathscr{T}\tilde{p}(\cdot,\tau)\mathrm{d}\tau\right)\bar{\tilde{p}}(\cdot,t)\mathrm{d}t\mathrm{d}\gamma\\
&= \mathrm{Re}\int_{\partial B_R}\int_0^\infty e^{-2s_1 t}\left(\int_0^t \mathscr{L}^{-1}\circ\mathscr{B}\circ\mathscr{L}\tilde{p}(\cdot,\tau)\mathrm{d}\tau\right)\bar{\tilde{p}}(\cdot,t)\mathrm{d}t\mathrm{d}\gamma\\
&= \mathrm{Re}\int_{\partial B_R}\int_0^\infty e^{-2s_1 t}\left(\mathscr{L}^{-1}\circ(s^{-1}\mathscr{B})\circ\mathscr{L}\tilde{p}(\cdot,t)\ \bar{\tilde{p}}(\cdot,t)\right)\mathrm{d}t\mathrm{d}\gamma\\
&= \frac{1}{2\pi}\int_{-\infty}^\infty \mathrm{Re}\int_{\partial B_R} s^{-1}\mathscr{B}\breve{\tilde{p}}(\cdot,s)\ \bar{\breve{\tilde{p}}}(\cdot,s)\mathrm{d}\gamma\mathrm{d}s_2\\
&= \frac{1}{2\pi}\int_{-\infty}^\infty \mathrm{Re}\langle s^{-1}\mathscr{B}\breve{\tilde{p}},\ \breve{\tilde{p}}\rangle_{\partial B_R}\mathrm{d}s_2 \leqq 0,
\end{aligned}
$$

where we use the fact that

$$
\int_0^t p(\cdot,\tau)\mathrm{d}\tau = \mathscr{L}^{-1}\left(s^{-1}\breve{p}(\cdot,s)\right).
$$

The proof is completed after taking the limit $s_1 \to 0$. □

Lemma 3.4. *Given $\theta > 0$ and $p(\cdot,t)\in H^1(\Omega)$ with $p(\cdot,0)=0$, it holds that*

$$
\mathrm{Re}\int_{\partial B_R}\int_0^\theta\left(\int_0^t \mathscr{T}\partial_\tau p(\cdot,\tau)\mathrm{d}\tau\right)\partial_t\bar{p}(\cdot,t)\mathrm{d}t\mathrm{d}\gamma \leqq 0.
$$

Proof. Similarly to the above proof, we have

$$
\begin{aligned}
\mathrm{Re}&\int_{\partial B_R}\int_0^\theta e^{-2s_1 t}\left(\int_0^t \mathscr{T}\partial_\tau p(\cdot,\tau)\mathrm{d}\tau\right)\partial_t\bar{p}(\cdot,t)\mathrm{d}t\mathrm{d}\gamma\\
&= \mathrm{Re}\int_{\partial B_R}\int_0^\infty e^{-2s_1 t}\left(\int_0^t \mathscr{T}\partial_\tau\tilde{p}(\cdot,\tau)\mathrm{d}\tau\right)\partial_t\bar{\tilde{p}}(\cdot,t)\mathrm{d}t\mathrm{d}\gamma\\
&= \mathrm{Re}\int_{\partial B_R}\int_0^\infty e^{-2s_1 t}\left(\int_0^t \mathscr{L}^{-1}\circ\mathscr{B}\circ\mathscr{L}\partial_\tau\tilde{p}(\cdot,\tau)\mathrm{d}\tau\right)\partial_t\bar{\tilde{p}}(\cdot,t)\mathrm{d}t\mathrm{d}\gamma\\
&= \mathrm{Re}\int_{\partial B_R}\int_0^\infty e^{-2s_1 t}\left(\mathscr{L}^{-1}\circ(s^{-1}\mathscr{B})\circ\mathscr{L}\partial_t\tilde{p}(\cdot,t)\ \partial_t\bar{\tilde{p}}(\cdot,t)\right)\mathrm{d}t\mathrm{d}\gamma\\
&= \frac{1}{2\pi}\int_{-\infty}^\infty \mathrm{Re}\int_{\partial B_R} |s|^2 s^{-1}\mathscr{B}\breve{\tilde{p}}(\cdot,s)\ \bar{\breve{\tilde{p}}}(\cdot,s)\mathrm{d}\gamma\mathrm{d}s_2\\
&= \frac{1}{2\pi}\int_{-\infty}^\infty |s|^2\mathrm{Re}\langle s^{-1}\mathscr{B}\breve{\tilde{p}},\ \breve{\tilde{p}}\rangle_{\partial B_R}\mathrm{d}s_2 \leqq 0,
\end{aligned}
$$

which completes the proof after taking the limit $s_1 \to 0$. □

Theorem 3.5. *Let $(p,\boldsymbol{u})\in H^1(\Omega)\times H^1(D)^2$ be the solution of (3.30)–(3.31). Given $f\in C^k (k\geqq 3)$, it holds for any $T>0$ that*

$$
\begin{aligned}
&\|p\|_{L^\infty(0,T;L^2(\Omega))} + \|\nabla p\|_{L^\infty(0,T;L^2(\Omega)^2)}\\
&\lesssim T\|\rho\|_{L^1(0,T;H^{-1/2}(\partial B_R))} + \|\partial_t\rho\|_{L^1(0,T;H^{-1/2}(\partial B_R))},
\end{aligned}
\tag{3.32}
$$

$$
\begin{aligned}
&\|\partial_t \boldsymbol{u}\|_{L^\infty(0,T;\ L^2(D)^2)} + \|\nabla \boldsymbol{u}\|_{L^\infty(0,T;\ L^2(D)^{2\times 2})} + \|\nabla\cdot\boldsymbol{u}\|_{L^\infty(0,T;\ L^2(D))} \\
&\lesssim T^3\|\rho\|_{L^1(0,T;H^{-1/2}(\partial B_R))} + T^2\|\partial_t\rho\|_{L^1(0,T;H^{-1/2}(\partial B_R))}, \qquad (3.33)\\
&\|p\|_{L^2(0,T;L^2(\Omega))} + \|\nabla p\|_{L^2(0,T;L^2(\Omega)^2)} \\
&\lesssim T^{3/2}\|\rho\|_{L^1(0,T;H^{-1/2}(\partial B_R))} + T^{1/2}\|\partial_t\rho\|_{L^1(0,T;H^{-1/2}(\partial B_R))}, \qquad (3.34)\\
&\|\partial_t \boldsymbol{u}\|_{L^2(0,T;L^2(D)^2)} + \|\nabla \boldsymbol{u}\|_{L^2(0,T;L^2(D)^{2\times 2})} + \|\nabla\cdot\boldsymbol{u}\|_{L^2(0,T;L^2(D))} \\
&\lesssim T^{7/2}\|\rho\|_{L^1(0,T;H^{-1/2}(\partial B_R))} + T^{5/2}\|\partial_t\rho\|_{L^1(0,T;H^{-1/2}(\partial B_R))}. \qquad (3.35)
\end{aligned}
$$

Proof. Let $0 < \theta < T$ and define an auxiliary function

$$
\psi_1(\boldsymbol{r}, t) = \int_t^\theta p(\boldsymbol{r}, \tau)\mathrm{d}\tau, \quad \boldsymbol{r}\in\Omega, \quad 0 \leqq t \leqq \theta.
$$

It is clear to note that

$$
\psi_1(\boldsymbol{r}, \theta) = 0, \quad \partial_t\psi_1(\boldsymbol{r}, t) = -p(\boldsymbol{r}, t). \tag{3.36}
$$

For any $\phi(\boldsymbol{r}, t) \in L^2\left(0, \theta;\ L^2(\Omega)\right)$, we have

$$
\int_0^\theta \phi(\boldsymbol{r}, t)\bar{\psi}_1(\boldsymbol{r}, t)\mathrm{d}t = \int_0^\theta \left(\int_0^t \phi(\boldsymbol{r}, \tau)\mathrm{d}\tau\right)\bar{p}(\boldsymbol{r}, t)\mathrm{d}t. \tag{3.37}
$$

Indeed, using integration by parts and (3.36), we have

$$
\begin{aligned}
\int_0^\theta \phi(\boldsymbol{r}, t)\bar{\psi}_1(\boldsymbol{r}, t)\mathrm{d}t &= \int_0^\theta \left(\phi(\boldsymbol{r}, t)\int_t^\theta \bar{p}(\boldsymbol{r}, \tau)\mathrm{d}\tau\right)\mathrm{d}t \\
&= \int_0^\theta\int_t^\theta \bar{p}(\boldsymbol{r}, \tau)\mathrm{d}\tau\mathrm{d}\left(\int_0^t \phi(\boldsymbol{r}, \varsigma)\mathrm{d}\varsigma\right) \\
&= \int_t^\theta \bar{p}(\boldsymbol{r}, \tau)\mathrm{d}\tau\int_0^t \phi(\boldsymbol{r}, \varsigma)\mathrm{d}\varsigma\Big|_0^\theta + \int_0^\theta\left(\int_0^t \phi(\boldsymbol{r}, \varsigma)\mathrm{d}\varsigma\right)\bar{p}(\boldsymbol{r}, t)\mathrm{d}t \\
&= \int_0^\theta\left(\int_0^t \phi(\boldsymbol{r}, \tau)\mathrm{d}\tau\right)\bar{p}(\boldsymbol{r}, t)\mathrm{d}t.
\end{aligned}
$$

Next, we take the test function $q = \psi_1$ in (3.30) and get

$$
\begin{aligned}
\int_\Omega \frac{1}{c^2}\partial_t^2 p\ \bar{\psi}_1\mathrm{d}\boldsymbol{r} = &-\int_\Omega \nabla p\cdot\nabla\bar{\psi}_1\mathrm{d}\boldsymbol{r} + \int_{\partial B_R}(\mathscr{T}p + \rho)\ \bar{\psi}_1\mathrm{d}\gamma \\
&+ \int_{\partial D}\rho_1\boldsymbol{n}_D\cdot\partial_t^2\boldsymbol{u}\bar{\psi}_1\mathrm{d}\gamma.
\end{aligned} \tag{3.38}
$$

It follows from (3.36) and the initial condition (3.11d) that

$$\begin{aligned}\mathrm{Re}\int_0^\theta\int_\Omega \frac{1}{c^2}\partial_t^2 p\ \bar{\psi}_1 \mathrm{d}\boldsymbol{x}\mathrm{d}t &= \mathrm{Re}\int_\Omega\int_0^\theta \frac{1}{c^2}\left(\partial_t\left(\partial_t p\ \bar{\psi}_1\right) + \partial_t p\ \bar{p}\right)\mathrm{d}t\mathrm{d}\boldsymbol{r}\\ &= \mathrm{Re}\int_\Omega \frac{1}{c^2}\left(\partial_t p\ \bar{\psi}_1\big|_0^\theta + \frac{1}{2}|p|^2\big|_0^\theta\right)\mathrm{d}\boldsymbol{r}\\ &= \frac{1}{2}\|\frac{1}{c}p(\cdot,\theta)\|^2_{L^2(\Omega)}.\end{aligned}$$

It is easy to verify that

$$\begin{aligned}\mathrm{Re}\int_0^\theta\int_{\partial D}\rho_1(\boldsymbol{n}_D\cdot\partial_t^2\boldsymbol{u})\ \bar{\psi}_1\mathrm{d}\gamma\mathrm{d}t &= \mathrm{Re}\int_{\partial D}\int_0^\theta \rho_1\left(\partial_t((\boldsymbol{n}_D\cdot\partial_t\boldsymbol{u})\ \bar{\psi}_1)\right.\\ &\quad\left. + (\boldsymbol{n}_D\cdot\partial_t\boldsymbol{u})\ \bar{p}\right)\mathrm{d}t\mathrm{d}\gamma\\ &= \mathrm{Re}\int_{\partial D}\rho_1\left((\boldsymbol{n}_D\cdot\partial_t\boldsymbol{u})\ \bar{\psi}_1\big|_0^\theta\right)\mathrm{d}\gamma\\ &\quad + \mathrm{Re}\int_0^\theta\int_{\partial D}\rho_1(\boldsymbol{n}_D\cdot\partial_t\boldsymbol{u})\ \bar{p}\mathrm{d}\gamma\mathrm{d}t\\ &= \mathrm{Re}\int_0^\theta\int_{\partial D}\rho_1(\boldsymbol{n}_D\cdot\partial_t\boldsymbol{u})\ \bar{p}\mathrm{d}\gamma\mathrm{d}t.\end{aligned}$$

Integrating (3.38) from $t = 0$ to $t = \theta$ and taking the real parts yields

$$\begin{aligned}&\frac{1}{2}\|\frac{1}{c}p(\cdot,\theta)\|^2_{L^2(\Omega)} + \mathrm{Re}\int_0^\theta\int_\Omega \nabla p\cdot\nabla\bar{\psi}_1\mathrm{d}\boldsymbol{r}\mathrm{d}t\\ &\quad= \frac{1}{2}\|\frac{1}{c}p(\cdot,\theta)\|^2_{L^2(\Omega)} + \frac{1}{2}\int_\Omega\left|\int_0^\theta \nabla p(\cdot,t)\mathrm{d}t\right|^2\mathrm{d}\boldsymbol{r}\\ &\quad= \mathrm{Re}\int_0^\theta\langle\mathscr{T}p+\rho,\ \psi_1\rangle_{\partial B_R}\mathrm{d}t + \mathrm{Re}\int_0^\theta\int_{\partial D}\rho_1(\boldsymbol{n}_D\cdot\partial_t^2\boldsymbol{u})\ \bar{\psi}_1\mathrm{d}\gamma\mathrm{d}t\\ &\quad= \mathrm{Re}\int_0^\theta\langle\mathscr{T}p+\rho,\ \psi_1\rangle_{\partial B_R}\mathrm{d}t + \mathrm{Re}\int_0^\theta\int_{\partial D}\rho_1(\boldsymbol{n}_D\cdot\partial_t\boldsymbol{u})\ \bar{p}\mathrm{d}\gamma\mathrm{d}t. \qquad (3.39)\end{aligned}$$

We define another auxiliary function

$$\boldsymbol{\psi}_2(\boldsymbol{r},t) = \int_t^\theta \partial_\tau\boldsymbol{u}(\boldsymbol{r},\tau)\mathrm{d}\tau,\quad \boldsymbol{r}\in D,\quad 0\leqq t\leqq\theta<T.$$

Clearly, we have

$$\boldsymbol{\psi}_2(\boldsymbol{r},\theta) = 0,\quad \partial_t\boldsymbol{\psi}_2(\boldsymbol{r},t) = -\partial_t\boldsymbol{u}(\boldsymbol{r},t). \qquad (3.40)$$

Using a proof similar to that for (3.37), for any $\boldsymbol{\phi}(\boldsymbol{r},t)\in L^2\left(0,\theta;\ L^2(D)^2\right)$, we may show that

$$\int_0^\theta \boldsymbol{\phi}(\boldsymbol{r},t)\cdot\bar{\boldsymbol{\psi}}_2(\boldsymbol{r},t)\mathrm{d}t = \int_0^\theta\left(\int_0^t \boldsymbol{\phi}(\boldsymbol{r},\tau)\mathrm{d}\tau\right)\cdot\partial_t\bar{\boldsymbol{u}}(\boldsymbol{r},t)\mathrm{d}t.$$

Taking the test function $\boldsymbol{v} = \boldsymbol{\psi}_2$ in (3.31), we can get

$$\int_D \rho_2 \partial_t^2(\partial_t \boldsymbol{u}) \cdot \bar{\boldsymbol{\psi}}_2 \mathrm{d}\boldsymbol{x} = -\int_D \left(\mu \nabla(\partial_t \boldsymbol{u}) : \nabla \bar{\boldsymbol{\psi}}_2 + (\lambda + \mu)(\nabla \cdot (\partial_t \boldsymbol{u}))(\nabla \cdot \bar{\boldsymbol{\psi}}_2)\right) \mathrm{d}\boldsymbol{r} - \int_{\partial D} (\partial_t p)(\boldsymbol{n}_D \cdot \bar{\boldsymbol{\psi}}_2) \mathrm{d}\gamma. \tag{3.41}$$

It follows from (3.40) and the initial condition in (3.29) that

$$\begin{aligned}
\mathrm{Re} \int_0^\theta \int_D \rho_2 \partial_t^2(\partial_t \boldsymbol{u}) \cdot \bar{\boldsymbol{\psi}}_2 \mathrm{d}\boldsymbol{x} \mathrm{d}t &= \mathrm{Re} \int_D \int_0^\theta \rho_2 \left(\partial_t(\partial_t^2 \boldsymbol{u} \cdot \bar{\boldsymbol{\psi}}_2) + \partial_t^2 \boldsymbol{u} \cdot \partial_t \bar{\boldsymbol{u}}\right) \mathrm{d}t \mathrm{d}\boldsymbol{r} \\
&= \mathrm{Re} \int_D \rho_2 \left((\partial_t^2 \boldsymbol{u} \cdot \bar{\boldsymbol{\psi}}_2)\Big|_0^\theta + \frac{1}{2} |\partial_t \boldsymbol{u}|^2 \Big|_0^\theta\right) \mathrm{d}\boldsymbol{r} \\
&= \frac{\rho_2}{2} \|\partial_t \boldsymbol{u}(\cdot,\ \theta)\|^2_{L^2(D)^2}
\end{aligned}$$

and

$$\begin{aligned}
&\mathrm{Re} \int_0^\theta \int_{\partial D} (\partial_t p)(\boldsymbol{n}_D \cdot \bar{\boldsymbol{\psi}}_2) \mathrm{d}\gamma \mathrm{d}t \\
&= \mathrm{Re} \int_{\partial D} \int_0^\theta \left(\partial_t \left(p\ (\boldsymbol{n}_D \cdot \bar{\boldsymbol{\psi}}_2)\right) + p\ (\boldsymbol{n}_D \cdot \partial_t \bar{\boldsymbol{u}})\right) \mathrm{d}t \mathrm{d}\gamma \\
&= \mathrm{Re} \int_{\partial D} \left(p\ (\boldsymbol{n}_D \cdot \bar{\boldsymbol{\psi}}_2)\right) \Big|_0^\theta \mathrm{d}\gamma + \mathrm{Re} \int_0^\theta \int_{\partial D} p\ (\boldsymbol{n}_D \cdot \partial_t \bar{\boldsymbol{u}}) \mathrm{d}\gamma \mathrm{d}t \\
&= \mathrm{Re} \int_0^\theta \int_{\partial D} p\ (\boldsymbol{n}_D \cdot \partial_t \bar{\boldsymbol{u}}) \mathrm{d}\gamma \mathrm{d}t.
\end{aligned}$$

Integrating (3.41) from $t = 0$ to $t = \theta$ and taking the real parts yields

$$\begin{aligned}
&\frac{\rho_2}{2} \|\partial_t \boldsymbol{u}(\cdot, \theta)\|^2_{L^2(D)^3} + \mathrm{Re} \int_0^\theta \int_D \left(\mu \nabla(\partial_t \boldsymbol{u}(\cdot, t)) : \nabla \bar{\boldsymbol{\psi}}_2(\cdot, t)\right. \\
&\qquad \left. + (\lambda + \mu) \left(\nabla \cdot (\partial_t \boldsymbol{u}(\cdot, t))\right)(\nabla \cdot \bar{\boldsymbol{\psi}}_2(\cdot, t))\right) \mathrm{d}\boldsymbol{r} \mathrm{d}t \\
&= \frac{\rho_2}{2} \|\partial_t \boldsymbol{u}(\cdot, \theta)\|^2_{L^2(D)^2} + \frac{1}{2} \int_D \left(\mu \left| \int_0^\theta \nabla(\partial_t \boldsymbol{u}(\cdot, t)) \mathrm{d}t \right|^2_{L^2(D)^{2\times 2}} \right. \\
&\qquad \left. + (\lambda + \mu) \left| \int_0^\theta \nabla \cdot (\partial_t \boldsymbol{u}(\cdot, t)) \mathrm{d}t \right|^2 \right) \mathrm{d}\boldsymbol{r} \\
&= -\mathrm{Re} \int_0^\theta \int_{\partial D} p\ (\boldsymbol{n}_D \cdot \partial_t \bar{\boldsymbol{u}}) \mathrm{d}\gamma \mathrm{d}t,
\end{aligned} \tag{3.42}$$

where we have used the fact that

$$\left| \int_0^\theta \nabla(\partial_t \boldsymbol{u}(\cdot, t)) \mathrm{d}t \right|^2_{L^2(D)^{2\times 2}} := \int_0^\theta \nabla(\partial_t \boldsymbol{u}(\cdot, t)) \mathrm{d}t : \int_0^\theta \nabla(\partial_t \bar{\boldsymbol{u}}(\cdot, t)) \mathrm{d}t.$$

Multiplying (3.42) by ρ_1 and then adding it to (3.39) gives

$$\frac{1}{2}\|\frac{1}{c}p(\cdot,\theta)\|^2_{L^2(\Omega)}+\frac{1}{2}\int_\Omega\left|\int_0^\theta\nabla p(\cdot,t)\mathrm{d}t\right|^2\mathrm{d}\boldsymbol{r}+\frac{\rho_2\rho_1}{2}\|\partial_t\boldsymbol{u}(\cdot,\theta)\|^2_{L^2(D)^2}$$
$$+\frac{\rho_1}{2}\int_D\left(\mu\left|\int_0^\theta\nabla(\partial_t\boldsymbol{u}(\cdot,t))\mathrm{d}t\right|^2_{L^2(D)^{2\times 2}}\right.$$
$$\left.+(\lambda+\mu)\left|\int_0^\theta\nabla\cdot(\partial_t\boldsymbol{u}(\cdot,t))\mathrm{d}t\right|^2\right)\mathrm{d}\boldsymbol{r}$$
$$=\mathrm{Re}\int_0^\theta\langle\mathscr{T}p+\rho,\psi_1\rangle_{\partial B_R}\mathrm{d}t+\mathrm{Re}\int_0^\theta\int_{\partial D}\rho_1(\boldsymbol{n}_D\cdot\partial_t\boldsymbol{u})\,\bar{p}\mathrm{d}\gamma\mathrm{d}t$$
$$-\mathrm{Re}\int_0^\theta\int_{\partial D}\rho_1 p\,(\boldsymbol{n}_D\cdot\partial_t\bar{\boldsymbol{u}})\mathrm{d}\gamma\mathrm{d}t$$
$$=\mathrm{Re}\int_0^\theta\langle\mathscr{T}p,\psi_1\rangle_{\partial B_R}\mathrm{d}t+\mathrm{Re}\int_0^\theta\int_{\partial B_R}\rho\,\bar{\psi}_1\mathrm{d}\gamma\mathrm{d}t. \tag{3.43}$$

In what follows, we estimate the two terms on the right-hand side of (3.43) separately. Using (3.37) and Lemma 3.3, we obtain

$$\mathrm{Re}\int_0^\theta\langle\mathscr{T}p,\psi_1\rangle_{\partial B_R}\mathrm{d}t=\mathrm{Re}\int_0^\theta\int_{\partial B_R}\mathscr{T}p\bar{\psi}_1\mathrm{d}\gamma\mathrm{d}t$$
$$=\mathrm{Re}\int_{\partial B_R}\int_0^\theta\left(\int_0^t\mathscr{T}p(\cdot,\tau)\mathrm{d}\tau\right)\bar{p}(\cdot,t)\mathrm{d}t\mathrm{d}\gamma\leqq 0. \tag{3.44}$$

For $0\leqq t\leqq\theta\leqq T$, we have from (3.37) that

$$\mathrm{Re}\int_0^\theta\int_{\partial B_R}\rho\bar{\psi}_1\mathrm{d}\gamma\,\mathrm{d}t=\mathrm{Re}\int_0^\theta\left(\int_0^t\int_{\partial B_R}\rho(\cdot,\tau)\mathrm{d}\gamma\mathrm{d}\tau\right)\bar{p}(\cdot,t)\mathrm{d}t$$
$$\lesssim\int_0^\theta\int_0^t\|\rho(\cdot,\tau)\|_{H^{-1/2}(\partial B_R)}\|p(\cdot,t)\|_{H^{1/2}(\partial B_R)}\mathrm{d}\tau\mathrm{d}t$$
$$\lesssim\int_0^\theta\int_0^t\|\rho(\cdot,\tau)\|_{H^{-1/2}(\partial B_R)}\|p(\cdot,t)\|_{H^1(\Omega)}\mathrm{d}\tau\mathrm{d}t$$
$$\lesssim\left(\int_0^\theta\|\rho(\cdot,t)\|_{H^{-1/2}(\partial B_R)}\mathrm{d}t\right)\left(\int_0^\theta\|p(\cdot,t)\|_{H^1(\Omega)}\mathrm{d}t\right). \tag{3.45}$$

Substituting (3.44) and (3.45) into (3.43), we have for any $\theta\in[0,T]$ that

$$\frac{1}{2}\|\frac{1}{c}p(\cdot,\theta)\|^2_{L^2(\Omega)}+\frac{\rho_2\rho_1}{2}\|\partial_t\boldsymbol{u}(\cdot,\theta)\|^2_{L^2(D)^2}$$
$$\leqq\frac{1}{2}\|\frac{1}{c}p(\cdot,\theta)\|^2_{L^2(\Omega)}+\frac{1}{2}\int_\Omega\left|\int_0^\theta\nabla p(\cdot,t)\mathrm{d}t\right|^2\mathrm{d}\boldsymbol{r}$$

$$
\begin{aligned}
&+\frac{\rho_2\rho_1}{2}\|\partial_t \boldsymbol{u}(\cdot,\theta)\|^2_{L^2(D)^2}\\
&+\frac{\rho_1}{2}\int_D \left(\mu\left|\int_0^\theta \nabla(\partial_t \boldsymbol{u}(\cdot,t))\mathrm{d}t\right|^2_{L^2(D)^{2\times 2}}\right.\\
&\left.+(\lambda+\mu)\left|\int_0^\theta \nabla\cdot(\partial_t \boldsymbol{u}(\cdot,t))\mathrm{d}t\right|^2\right)\mathrm{d}\boldsymbol{r}\\
\lesssim&\left(\int_0^\theta \|\rho(\cdot,t)\|_{H^{-1/2}(\partial B_R)}\mathrm{d}t\right)\left(\int_0^\theta \|p(\cdot,t)\|_{H^1(\Omega)}\mathrm{d}t\right).
\end{aligned}
\tag{3.46}
$$

Since the right-hand side of (3.46) contain the term

$$
\int_0^\theta \int_\Omega \left(|\nabla p(\cdot,t)|^2+|p(\cdot,t)|^2\mathrm{d}\boldsymbol{r}\right)^{1/2}\mathrm{d}t,
$$

we may want to control the term $\|\nabla p\|^2_{L^2(\Omega)^2}$ by the left-hand side after applying the Young inequality. Thus, we consider new test functions

$$
\psi_3(\boldsymbol{r},t)=\int_t^\theta \partial_\tau p(\boldsymbol{r},\tau)\mathrm{d}\tau,\quad \boldsymbol{r}\in\Omega,\ \ 0\leqq t\leqq\theta\leqq T
$$

and

$$
\boldsymbol{\psi}_4(\boldsymbol{r},t)=\int_t^\theta \partial_\tau^2 \boldsymbol{u}(\boldsymbol{r},\tau)\mathrm{d}\tau,\quad \boldsymbol{r}\in D,\ 0\leqq t\leqq\theta\leqq T
$$

for the system

$$
\begin{cases}
\Delta\partial_t p-\frac{1}{c^2}\partial_t^2(\partial_t p)=0 & \text{in }\Omega,\ \ t>0,\\
\mu\Delta\partial_t^2\boldsymbol{u}+(\lambda+\mu)\nabla\nabla\cdot\partial_t^2\boldsymbol{u}-\rho_2\partial_t^2(\partial_t^2\boldsymbol{u})=0 & \text{in }D,\ \ t>0,\\
\partial_t p\big|_{t=0}=0,\quad \partial_t^2 p\big|_{t=0}=c^2\Delta p\big|_{t=0}=0 & \text{in }\Omega,\\
\partial_t^2\boldsymbol{u}\big|_{t=0}=0, & \text{in }\ D,\\
\partial_t(\partial_t^2\boldsymbol{u})=\rho_2^{-1}\left(\mu\Delta(\partial_t\boldsymbol{u})+(\lambda+\mu)\nabla\nabla\cdot(\partial_t\boldsymbol{u})\right)\big|_{t=0}=0 & \text{in }\ D,\\
\partial_{\boldsymbol{n}_D}(\partial_t p)=-\rho_1\boldsymbol{n}_D\cdot\partial_t^2(\partial_t\boldsymbol{u}) & \text{on }\partial D,\ \ t>0,\\
-\partial_t^2 p\boldsymbol{n}_D=\mu\partial_{\boldsymbol{n}_D}(\partial_t^2\boldsymbol{u})+(\lambda+\mu)(\nabla\cdot\partial_t^2\boldsymbol{u})\boldsymbol{n}_D & \text{on }\partial D,\ \ t>0,\\
\partial_r(\partial_t p)=\mathscr{T}(\partial_t p)+\partial_t\rho & \text{on }\partial B_R,\ \ t>0.
\end{cases}
\tag{3.47}
$$

Following the same steps as those used when proving the inequality (3.39), we obtain for (3.47) that

$$\begin{aligned}
&\frac{1}{2}\|\frac{1}{c}\partial_t p(\cdot,\theta)\|^2_{L^2(\Omega)} + \mathrm{Re}\int_0^\theta\int_\Omega \partial_t(\nabla p)\cdot\nabla\bar{\psi}_3 \mathrm{d}\boldsymbol{r}\mathrm{d}t \\
&= \frac{1}{2}\|\frac{1}{c}\partial_t p(\cdot,\theta)\|^2_{L^2(\Omega)} + \frac{1}{2}\int_\Omega \left|\int_0^\theta \partial_t(\nabla p(\cdot,t))\mathrm{d}t\right|^2 \mathrm{d}\boldsymbol{r} \\
&= \frac{1}{2}\|\frac{1}{c}\partial_t p(\cdot,\theta)\|^2_{L^2(\Omega)} + \frac{1}{2}\|\nabla p(\cdot,\theta)\|^2_{L^2(\Omega)^2} \\
&= \mathrm{Re}\int_0^\theta \langle \mathscr{T}\partial_t p + \partial_t\rho, \psi_3\rangle_{\partial B_R}\mathrm{d}t + \mathrm{Re}\int_0^\theta\int_{\partial D} \rho_1(\boldsymbol{n}_D\cdot\partial_t^2\boldsymbol{u})\partial_t\bar{p}\mathrm{d}\gamma\mathrm{d}\tau.
\end{aligned} \tag{3.48}$$

Similarly, we have

$$\begin{aligned}
&\frac{\rho_2}{2}\|\partial_t^2\boldsymbol{u}(\cdot,\theta)\|^2_{L^2(D)^3} + \mathrm{Re}\int_0^\theta\int_D \Big(\mu\nabla(\partial_t^2\boldsymbol{u}(\cdot,t)) : \nabla\bar{\boldsymbol{\psi}}_4(\cdot,t) \\
&\quad + (\lambda+\mu)\Big(\nabla\cdot(\partial_t^2\boldsymbol{u}(\cdot,t)))(\nabla\cdot\bar{\boldsymbol{\psi}}_4(\cdot,t)\Big)\Big)\mathrm{d}\boldsymbol{r}\mathrm{d}t \\
&\quad = \frac{\rho_2}{2}\|\partial_t^2\boldsymbol{u}(\cdot,\theta)\|^2_{L^2(D)^2} + \frac{1}{2}\int_D\left(\mu\left|\int_0^\theta \nabla(\partial_t^2\boldsymbol{u}(\cdot,t))\mathrm{d}t\right|^2_{L^2(D)^{2\times 2}}\right. \\
&\quad \left. + (\lambda+\mu)\left|\int_0^\theta \nabla\cdot(\partial_t^2\boldsymbol{u}(\cdot,t))\mathrm{d}t\right|^2\right)\mathrm{d}\boldsymbol{r} \\
&= \frac{\rho_2}{2}\|\partial_t^2\boldsymbol{u}(\cdot,\theta)\|^2_{L^2(D)^2} + \frac{1}{2}\mu\|\nabla\partial_t\boldsymbol{u}(\cdot,\theta)\|^2_{L^2(D)^{2\times 2}} \\
&\quad + \frac{1}{2}(\lambda+\mu)\|\nabla\cdot\partial_t\boldsymbol{u}(\cdot,\theta)\|^2_{L^2(D)} \\
&= -\mathrm{Re}\int_0^\theta\int_{\partial D}\partial_t p\,(\boldsymbol{n}_D\cdot\partial_t^2\bar{\boldsymbol{u}})\mathrm{d}\gamma\mathrm{d}t.
\end{aligned} \tag{3.49}$$

Combining (3.48)–(3.49), we deduce

$$\begin{aligned}
&\frac{1}{2}\|\frac{1}{c}\partial_t p(\cdot,\theta)\|^2_{L^2(\Omega)} + \frac{1}{2}\|\nabla p(\cdot,\theta)\|^2_{L^2(\Omega)^2} + \frac{\rho_1\rho_2}{2}\|\partial_t^2\boldsymbol{u}(\cdot,\theta)\|^2_{L^2(D)^2} \\
&\quad + \frac{\rho_1}{2}\mu\|\nabla\partial_t\boldsymbol{u}(\cdot,\theta)\|^2_{L^2(D)^{2\times 2}} \\
&\quad + \frac{\rho_1}{2}(\lambda+\mu)\|\nabla\cdot\partial_t\boldsymbol{u}(\cdot,\theta)\|^2_{L^2(D)} = \mathrm{Re}\int_0^\theta\langle\mathscr{T}\partial_t p, \psi_3\rangle_{\partial B_R}\mathrm{d}t \\
&\quad + \mathrm{Re}\int_0^\theta\int_{\partial B_R}\partial_t\rho\;\bar{\psi}_3\mathrm{d}\gamma\mathrm{d}t.
\end{aligned} \tag{3.50}$$

For the first term on the right-hand side of (3.50), we derive from Lemma 3.4 that

$$\mathrm{Re}\int_0^\theta\langle\mathscr{T}\partial_t p, \psi_3\rangle_{\partial B_R}\mathrm{d}t = \mathrm{Re}\int_{\partial B_R}\int_0^\theta\left(\int_0^t \mathscr{T}\partial_\tau p(\cdot,\tau)\mathrm{d}\tau\right)\partial_t\bar{p}(\cdot,t)\mathrm{d}t \leqq 0. \tag{3.51}$$

For the second term on the right-hand side of (3.50), it follows from the integration by parts and the trace theorem that

$$\begin{aligned}
\operatorname{Re}\int_0^\theta\int_{\partial B_R}\partial_t\rho\bar{\psi}_3\mathrm{d}\gamma\mathrm{d}t &= \operatorname{Re}\int_0^\theta\int_0^t\left(\int_{\partial B_R}\partial_t\rho(\cdot,\tau)\mathrm{d}\gamma\right)\mathrm{d}\tau\partial_t p(\cdot,t)\mathrm{d}t\\
&= \operatorname{Re}\int_{\partial B_R}\left(\Big(\int_0^t\partial_\tau\rho(\cdot,\tau)\mathrm{d}\tau\Big)\bar{p}(\cdot,t)\right)\Big|_0^\theta\mathrm{d}\gamma\\
&\quad -\operatorname{Re}\int_0^\theta\int_{\partial B_R}\partial_t\rho(\cdot,t)\bar{p}(\cdot,t)\mathrm{d}\gamma\mathrm{d}t\\
&\lesssim \int_0^\theta\|\partial_t(\cdot,t)\|_{H^{-1/2}(\partial B_R)}\|p(\cdot,t)\|_{H^{1/2}(\partial B_R)}\mathrm{d}t\\
&\lesssim \int_0^\theta\|\partial_t\rho(\cdot,t)\|_{H^{-1/2}(\partial B_R)}\|p(\cdot,t)\|_{H^1(\Omega)}\mathrm{d}t. \qquad (3.52)
\end{aligned}$$

Combing the estimates (3.46), (3.50)–(3.52), we obtain

$$\begin{aligned}
\|p(\cdot,\theta)\|^2_{L^2(\Omega)}+\|\nabla p(\cdot,\theta)\|^2_{L^2(\Omega)} &\lesssim \left(\int_0^\theta\|\rho(\cdot,t)\|_{H^{-1/2}(\partial B_R)}\mathrm{d}t\right)\\
&\quad\left(\int_0^\theta\|p(\cdot,t)\|_{H^1(\Omega)}\mathrm{d}t\right)\\
&\quad+\int_0^\theta\|\partial_t\rho(\cdot,t)\|_{H^{-1/2}(\partial B_R)}\|p(\cdot,t)\|_{H^1(\Omega)}\mathrm{d}t.
\end{aligned}\tag{3.53}$$

Taking the L^∞-norm with respect to θ on both side of (3.53) yields

$$\begin{aligned}
&\|p\|^2_{L^\infty(0,T;L^2(\Omega))}+\|\nabla p\|^2_{L^\infty(0,T;L^2(\Omega)^2)}\\
&\quad\lesssim T\|\rho\|_{L^1(0,T;H^{-1/2}(\partial B_R))}\|p\|_{L^\infty(0,T;H^1(\Omega))}\\
&\qquad+\|\partial_t\rho\|_{L^1(0,T;H^{-1/2}(\partial B_R))}\|p\|_{L^\infty(0,T;H^1(\Omega))}.
\end{aligned}$$

Using the ϵ-inequality, we obtain

$$\begin{aligned}
\|p\|^2_{L^\infty(0,T;L^2(\Omega))}+\|\nabla p\|^2_{L^\infty(0,T;L^2(\Omega)^2)} &\lesssim T^2\|\rho\|^2_{L^1(0,T;H^{-1/2}(\partial B_R))}\\
&\quad+\|\partial_t\rho\|^2_{L^1(0,T;H^{-1/2}(\partial B_R))},
\end{aligned}$$

which implies the estimate (3.32) after applying the Cauchy–Schwarz inequality. For the elastic wave, it follows from (3.46) and (3.50)– (3.52) that

$$\begin{aligned}
&\|\partial_t^2\boldsymbol{u}(\cdot,\theta)\|^2_{L^2(D)^2}+\|\nabla\partial_t\boldsymbol{u}(\cdot,\theta)\|^2_{L^2(D)^{2\times2}}+\|\nabla\cdot\partial_t\boldsymbol{u}(\cdot,\theta)\|^2_{L^2(D)}\\
&\quad+\|p(\cdot,\theta)\|^2_{L^2(\Omega)}+\|\nabla p(\cdot,\theta)\|^2_{L^2(\Omega)}\\
&\lesssim\left(\int_0^\theta\|\rho(\cdot,t)\|_{H^{-1/2}(\partial B_R)}\mathrm{d}t\right)\left(\int_0^\theta\|p(\cdot,t)\|_{H^1(\Omega)}\mathrm{d}t\right)\\
&\quad+\int_0^\theta\|\partial_t\rho(\cdot,t)\|_{H^{-1/2}(\partial B_R)}\|p(\cdot,t)\|_{H^1(\Omega)}\mathrm{d}t. \qquad (3.54)
\end{aligned}$$

For any $0 < t < T$, since $\partial_t \boldsymbol{u}(\boldsymbol{r}, 0) = 0$, using the Young inequality yields

$$\|\partial_t \boldsymbol{u}(\cdot,\theta)\|^2_{L^2(D)^2} = \int_0^\theta \partial_\tau \|\partial_\tau \boldsymbol{u}(\cdot,\tau)\|^2_{L^2(D)^2} \mathrm{d}\tau \\ \leqq \frac{T}{\epsilon}\|\partial_t^2 \boldsymbol{u}(\cdot,\theta)\|^2_{L^2(D)^2} + \epsilon T \|\partial_t \boldsymbol{u}(\cdot,\theta)\|^2_{L^2(D)^2}. \tag{3.55}$$

Choosing $\epsilon = \frac{1}{2T}$, we have from (3.55) that

$$\|\partial_t \boldsymbol{u}(\cdot,\theta)\|^2_{L^2(D)^2} \lesssim T^2 \|\partial_t^2 \boldsymbol{u}(\cdot,\theta)\|^2_{L^2(D)^2}. \tag{3.56}$$

Similarly, we have

$$\|\nabla \boldsymbol{u}(\cdot,\theta)\|^2_{L^2(D)^{2\times 2}} \lesssim T^2 \|\nabla \partial_t \boldsymbol{u}(\cdot,\theta)\|^2_{L^2(D)^{2\times 2}}, \\ \|\nabla \cdot \boldsymbol{u}(\cdot,\theta)\|^2_{L^2(D)} \lesssim T^2 \|\nabla \cdot \partial_t \boldsymbol{u}(\cdot,\theta)\|^2_{L^2(D)}. \tag{3.57}$$

It follows from (3.54)–(3.57) that

$$\begin{aligned} &\|\partial_t \boldsymbol{u}(\cdot,\theta)\|^2_{L^2(D)^2} + \|\nabla \boldsymbol{u}(\cdot,\theta)\|^2_{L^2(D)^{2\times 2}} + \|\nabla \cdot \boldsymbol{u}(\cdot,\theta)\|^2_{L^2(D)} \\ &\quad + T^2 \|p(\cdot,\theta)\|^2_{L^2(\Omega)} + T^2 \|\nabla p(\cdot,\theta)\|^2_{L^2(\Omega)} \\ &\lesssim T^2 \Big(\|\partial_t^2 \boldsymbol{u}(\cdot,\theta)\|^2_{L^2(D)^2} + \|\nabla \partial_t \boldsymbol{u}(\cdot,\theta)\|^2_{L^2(D)^{2\times 2}} \\ &\quad + \|\nabla \cdot \partial_t \boldsymbol{u}(\cdot,\theta)\|^2_{L^2(D)} + \|p(\cdot,\theta)\|^2_{L^2(\Omega)} + \|\nabla p(\cdot,\theta)\|^2_{L^2(\Omega)} \Big) \\ &\lesssim T^2 \left(\int_0^\theta \|\rho(\cdot,t)\|_{H^{-1/2}(\partial B_R)} \mathrm{d}t \right) \left(\int_0^\theta \|p(\cdot,t)\|_{H^1(\Omega)} \mathrm{d}t \right) \\ &\quad + T^2 \int_0^\theta \|\partial_t \rho(\cdot,t)\|_{H^{-1/2}(\partial B_R)} \|p(\cdot,t)\|_{H^1(\Omega)} \mathrm{d}t. \end{aligned} \tag{3.58}$$

Taking the L^∞-norm with respect to θ on both sider of (3.58) yields

$$\begin{aligned} &\|\partial_t \boldsymbol{u}\|^2_{L^\infty(0,T;\, L^2(D)^2)} + \|\nabla \boldsymbol{u}\|^2_{L^\infty(0,T;\, L^2(D)^{2\times 2})} + \|\nabla \cdot \boldsymbol{u}\|^2_{L^\infty(0,T;\, L^2(D))} \\ &\quad + T^2 \left(\|p\|^2_{L^\infty(0,T;L^2(\Omega))} + \|\nabla p\|^2_{L^\infty(0,T;L^2(\Omega)^2)} \right) \\ &\lesssim T^3 \|\rho\|_{L^1(0,T;H^{-1/2}(\partial B_R))} \|p\|_{L^\infty(0,T;H^1(\Omega))} \\ &\quad + T^2 \|\partial_t \rho\|_{L^1(0,T;H^{-1/2}(\partial B_R))} \|p\|_{L^\infty(0,T;H^1(\Omega))}. \end{aligned}$$

Using Young's inequality again, we get

$$\begin{aligned} &\|\partial_t \boldsymbol{u}\|^2_{L^\infty(0,T;\, L^2(D)^2)} + \|\nabla \boldsymbol{u}\|^2_{L^\infty(0,T;\, L^2(D)^{2\times 2})} + \|\nabla \cdot \boldsymbol{u}\|^2_{L^\infty(0,T;\, L^2(D))} \\ &\quad + (T^2 - \epsilon) \left(\|p\|^2_{L^\infty(0,T;L^2(\Omega))} + \|\nabla p\|^2_{L^\infty(0,T;L^2(\Omega)^2)} \right) \\ &\lesssim \frac{1}{\epsilon} \left(T^6 \|\rho\|^2_{L^1(0,T;H^{-1/2}(\partial B_R))} + T^4 \|\partial_t \rho\|^2_{L^1(0,T;H^{-1/2}(\partial B_R))} \right). \end{aligned}$$

Letting ϵ to be a small enough positive constant gives

$$\begin{aligned}
&\|\partial_t \boldsymbol{u}\|^2_{L^\infty(0,T;\ L^2(D)^2)} + \|\nabla \boldsymbol{u}\|^2_{L^\infty(0,T;\ L^2(D)^{2\times 2})} + \|\nabla\cdot\boldsymbol{u}\|^2_{L^\infty(0,T;\ L^2(D))}\\
&\quad\lesssim T^6\|\rho\|^2_{L^1(0,T;H^{-1/2}(\partial B_R))} + T^4\|\partial_t\rho\|^2_{L^1(0,T;H^{-1/2}(\partial B_R))},
\end{aligned}$$

which shows the estimate (3.33) after applying the Cauchy–Schwarz inequality.

Integrating (3.53) with respect to θ from 0 to T and using the Cauchy–Schwarz inequality, we obtain

$$\begin{aligned}
&\|p\|^2_{L^2(0,T;L^2(\Omega))} + \|\nabla p\|^2_{L^2(0,T;L^2(\Omega)^2)}\\
&\quad\lesssim T^{3/2}\|\rho\|_{L^1(0,T;H^{-1/2}(\partial B_R))}\|p\|_{L^2(0,T;H^1(\Omega))}\\
&\qquad + T^{1/2}\|\partial_t\rho\|_{L^1(0,T;H^{-1/2}(\partial B_R))}\|p\|_{L^2(0,T;H^1(\Omega))}.
\end{aligned}$$

Using the Young equality yields

$$\begin{aligned}
&\|p\|^2_{L^2(0,T;L^2(\Omega))} + \|\nabla p\|^2_{L^2(0,T;L^2(\Omega)^2)} \lesssim T^3\|\rho\|_{L^1(0,T;H^{-1/2}(\partial B_R))}\\
&\qquad + T\|\partial_t\rho\|_{L^1(0,T;H^{-1/2}(\partial B_R))},
\end{aligned}$$

which gives the estimate (3.34) by using the Cauchy–Schwarz inequality.

For the elastic wave $\boldsymbol{u}$, integrating (3.58) with respect to θ and using the Cauchy–Schwarz inequality yields

$$\begin{aligned}
&\|\partial_t \boldsymbol{u}\|^2_{L^2(0,T;L^2(D)^2)} + \|\nabla \boldsymbol{u}\|^2_{L^2(0,T;L^2(D)^{2\times 2})} + \|\nabla\cdot\boldsymbol{u}\|^2_{L^2(0,T;L^2(D))}\\
&\quad + T^2\left(\|p\|^2_{L^2(0,T;L^2(\Omega))} + \|\nabla p\|^2_{L^2(0,T;L^2(\Omega))}\right)\\
&\lesssim T^{7/2}\|\rho\|_{L^1(0,T;H^{-1/2}(\partial B_R))}\|p\|_{L^2(0,T;H^1(\Omega))}\\
&\quad + T^{5/2}\|\partial_t\rho\|_{L^1(0,T;H^{-1/2}(\partial B_R))}\|p\|_{L^2(0,T;H^1(\Omega))}.
\end{aligned}$$

Using the Young inequality leads to

$$\begin{aligned}
&\|\partial_t \boldsymbol{u}\|^2_{L^2(0,T;L^2(D)^2)} + \|\nabla \boldsymbol{u}\|^2_{L^2(0,T;L^2(D)^{2\times 2})} + \|\nabla\cdot\boldsymbol{u}\|^2_{L^2(0,T;L^2(D))}\\
&\quad + (T^2-\epsilon)\left(\|p\|^2_{L^2(0,T;L^2(\Omega))} + \|\nabla p\|^2_{L^2(0,T;L^2(\Omega))}\right)\\
&\lesssim \frac{1}{\epsilon}\left(T^7\|\rho\|^2_{L^1(0,T;H^{-1/2}(\partial B_R))} + T^5\|\partial_t\rho\|^2_{L^1(0,T;H^{-1/2}(\partial B_R))}\right).
\end{aligned}$$

Similarly, we choose ϵ small enough and obtain

$$\begin{aligned}
&\|\partial_t \boldsymbol{u}\|^2_{L^2(0,T;L^2(D)^2)} + \|\nabla \boldsymbol{u}\|^2_{L^2(0,T;L^2(D)^{2\times 2})} + \|\nabla\cdot\boldsymbol{u}\|^2_{L^2(0,T;L^2(D))}\\
&\quad\lesssim T^7\|\rho\|_{L^1(0,T;H^{-1/2}(\partial B_R))} + T^5\|\partial_t\rho\|_{L^1(0,T;H^{-1/2}(\partial B_R))},
\end{aligned}$$

which implies the estimate (3.35) by using the Cauchy–Schwarz inequality again.
□

4. The Time-Domain PML Problem

In this section, we introduce the absorbing PML formulation for the acoustic–elastic interaction problem, and establish the well-posedness and stability of the PML system.

4.1. The PML Equations and Well-Posedness

It follows from the Helmholtz decomposition in (2.7) that the reduced problem (3.11) can be written into

$$\begin{cases} \Delta p - \frac{1}{c^2}\partial_t^2 p = 0 & \text{in } \Omega, \quad t > 0, \\ \Delta \varphi_1 - \frac{1}{c_1^2}\partial_t^2 \varphi_1 = 0, & \text{in } D, \quad t > 0, \\ \Delta \varphi_2 - \frac{1}{c_2^2}\partial_t^2 \varphi_1 = 0, & \text{in } D, \quad t > 0, \\ p\big|_{t=0} = \partial_t p\big|_{t=0} = 0 & \text{in } \Omega, \\ \varphi_j\big|_{t=0} = \partial_t \varphi_j\big|_{t=0} = 0, \quad j = 1,\ 2 & \text{in } D, \\ \partial_{\boldsymbol{n}_D} p = -\rho_1 \boldsymbol{n}_D \cdot \big(\nabla(\partial_t^2 \varphi_1) + \mathbf{curl}(\partial_t^2 \varphi_2)\big) & \text{on } \partial D, \quad t > 0, \\ -p\boldsymbol{n}_D = \mu \partial_{\boldsymbol{n}_D}(\nabla \varphi_1 + \mathbf{curl}\varphi_2) + \frac{(\lambda+\mu)}{c_1^2}\partial_t^2 \varphi_1 \boldsymbol{n}_D & \text{on } \partial D, \quad t > 0, \\ \partial_r p = \mathscr{T} p + \rho & \text{on } \partial B_R, \quad t > 0. \end{cases} \tag{4.1}$$

Let

$$c_1 \nabla \varphi_1 = -\partial_t \boldsymbol{\Phi}_1, \quad c_2 \nabla \varphi_2 = -\partial_t \boldsymbol{\Phi}_2,$$

The acoustic–elastic interaction problem (4.1) can be equivalently written into the first order system

$$\begin{cases} \partial_t p(\boldsymbol{r}, t) = -c^2 \rho_1 \nabla \cdot \boldsymbol{v}(\boldsymbol{r}, t), \quad \partial_t \boldsymbol{v}(\boldsymbol{r}, t) = -\rho_1^{-1} \nabla p(\boldsymbol{r}, t) & \boldsymbol{r} \in D^{\mathrm{c}}, \ t > 0, \\ \partial_t \varphi_1(\boldsymbol{r}, t) = -c_1 \nabla \cdot \boldsymbol{\Phi}_1(\boldsymbol{r}, t), \quad \partial_t \boldsymbol{\Phi}_1(\boldsymbol{r}, t) = -c_1 \nabla \varphi_1(\boldsymbol{r}, t) & \boldsymbol{r} \in D, \ t > 0, \\ \partial_t \varphi_2(\boldsymbol{r}, t) = -c_2 \nabla \cdot \boldsymbol{\Phi}_2(\boldsymbol{r}, t), \quad \partial_t \boldsymbol{\Phi}_2(\boldsymbol{r}, t) = -c_2 \nabla \varphi_2(\boldsymbol{r}, t) & \boldsymbol{r} \in D, \ t > 0, \\ p(\boldsymbol{r}, t)\big|_{t=0} = 0, \quad \boldsymbol{v}(\boldsymbol{r}, t)\big|_{t=0} = 0 & \boldsymbol{r} \in D^{\mathrm{c}}, \\ \varphi_j(\boldsymbol{r}, t)\big|_{t=0} = 0, \quad \boldsymbol{\Phi}_j(\boldsymbol{r}, t)\big|_{t=0} = 0, \quad j = 1, 2 & \boldsymbol{r} \in D, \\ \boldsymbol{n}_D \cdot \boldsymbol{v}(\boldsymbol{r}, t) = \boldsymbol{n}_D \cdot (\partial_t \nabla \varphi_1(\boldsymbol{r}, t) + \partial_t \mathbf{curl}\varphi_2(\boldsymbol{r}, t)) & \boldsymbol{r} \in \partial D, \ t > 0, \\ -p(\boldsymbol{r}, t)\boldsymbol{n}_D = \mu \partial_{\boldsymbol{n}_D}(\nabla \varphi_1 + \mathbf{curl}\varphi_2) + \frac{(\lambda+\mu)}{c_1^2}\partial_t^2 \varphi_1 \boldsymbol{n}_D, & \boldsymbol{r} \in \partial D, \ t > 0, \end{cases} \tag{4.2}$$

where p and $\boldsymbol{v}$ are the pressure and the velocity field of the acoustic wave, φ_1 and φ_2 are the compressional and shear scalar potential functions for the elastic wave field. We pick two positive numbers L_1, L_2 which are sufficiently large

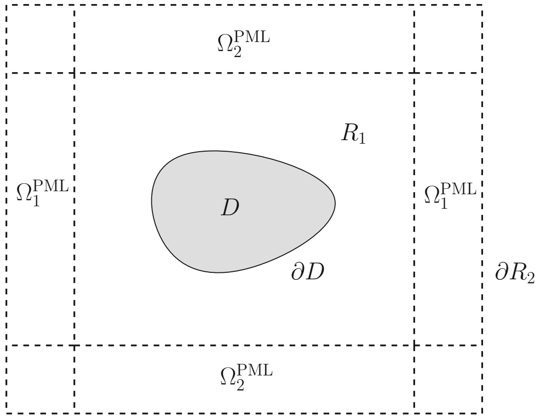

Fig. 2. Geometry of the PML problem

such that the elastic solid D is contained in the interior of the rectangle $R_1 = \{\boldsymbol{r} : |x| < L_1/2, |y| < L_2/2\}$. Let $R_2 = \{\boldsymbol{r} : |x| < L_1/2 + d_1, |y| < L_2/2 + d_2\}$ be the rectangle which contains R_1, where d_1, d_2 are the thickness of the PML region along x and y, respectively. The geometry of the PML problem is shown in Fig. 2.

Let $\alpha_1(x) = 1 + s^{-1}\sigma_1(x), \alpha_2(y) = 1 + s^{-1}\sigma_2(y)$ be the model medium property which satisfy

$$\sigma_1(x) = \begin{cases} \sigma_0 & \text{if } |x| > L_1/2, \\ 0 & \text{if } |x| \leqq L_1/2, \end{cases} \quad \sigma_2(y) = \begin{cases} \sigma_0 & \text{if } |y| > L_2/2, \\ 0 & \text{if } |y| \leqq L_2/2, \end{cases}$$

where σ_0 is a positive constant. Let $\hat{\boldsymbol{r}} = (\hat{x}, \hat{y})$ be the complex coordinates, where

$$\hat{x} = \int_0^x \alpha_1(\tau)\mathrm{d}\tau, \quad \hat{y} = \int_0^y \alpha_2(\tau)\mathrm{d}\tau.$$

Clearly, we have $\frac{\mathrm{d}\hat{x}}{\mathrm{d}x} = \alpha_1, \frac{\mathrm{d}\hat{y}}{\mathrm{d}y} = \alpha_2$.

Taking the Laplace transform of the first equation in (4.2) and using the initial conditions, we get

$$s\breve{p} = -c^2\rho_1\nabla\cdot\breve{\boldsymbol{v}}, \quad s\breve{\boldsymbol{v}} = -\rho_1^{-1}\nabla\breve{p} \quad \text{in } \mathbb{R}^2\backslash\bar{R}_1. \tag{4.3}$$

Let $\hat{\breve{p}}, \hat{\breve{\boldsymbol{v}}}$ be the PML extensions of the pressure field $\breve{p}$ and the velocity field $\breve{\boldsymbol{v}}$ with respect to the complex variable $\hat{\boldsymbol{r}}$ in $\mathbb{R}^2\backslash\bar{R}_1$, respectively. It follows from (4.3) and the chain rule that

$$\begin{cases} s\hat{\breve{p}} + (\sigma_1+\sigma_2)\hat{\breve{p}} + \frac{\sigma_1\sigma_2}{s}\hat{\breve{p}} = -c^2\rho_1\left(\partial_x(1+s^{-1}\sigma_2)\hat{\breve{v}}_1 + \partial_y(1+s^{-1}\sigma_1)\hat{\breve{v}}_2\right), \\ s\hat{\breve{v}}_1 + \sigma_1\hat{\breve{v}}_1 = -\rho_1^{-1}\partial_x\hat{\breve{p}}, \\ s\hat{\breve{v}}_2 + \sigma_2\hat{\breve{v}}_2 = -\rho_1^{-1}\partial_y\hat{\breve{p}}, \end{cases} \tag{4.4}$$

where $\hat{\hat{\boldsymbol{v}}} = (\hat{\hat{v}}_1, \hat{\hat{v}}_2)^\top$. To introduce the first order system for the time-domain PML problem, we define

$$\frac{1}{s}\hat{\hat{v}}_1 = \hat{\hat{v}}_1^*, \quad \frac{1}{s}\hat{\hat{v}}_2 = \hat{\hat{v}}_2^*, \quad \frac{\sigma_1\sigma_2\hat{\hat{p}}}{s} = \hat{\hat{p}}^*. \tag{4.5}$$

It follows from (4.5) that (4.4) becomes

$$\begin{cases} s\hat{\hat{p}} = -c^2\rho_1\left(\partial_x(\hat{\hat{v}}_1 + \sigma_2\hat{\hat{v}}_1^*) + \partial_y(\hat{\hat{v}}_2 + \sigma_1\hat{\hat{v}}_2^*)\right) - (\sigma_1+\sigma_2)\hat{\hat{p}} - \hat{\hat{p}}^*, \\ s\hat{\hat{v}}_1 = -\rho_1^{-1}\partial_x\hat{\hat{p}} - \sigma_1\hat{\hat{v}}_1, \\ s\hat{\hat{v}}_2 = -\rho_1^{-1}\partial_y\hat{\hat{p}} - \sigma_2\hat{\hat{v}}_2. \end{cases} \tag{4.6}$$

For $\boldsymbol{r} \in \mathbb{R}^2\backslash\bar{R}_1$, let

$$\hat{p} = \mathscr{L}^{-1}(\hat{\hat{p}}), \quad \hat{\boldsymbol{v}} = \mathscr{L}^{-1}(\hat{\hat{\boldsymbol{v}}}), \quad \hat{p}^* = \mathscr{L}^{-1}(\hat{\hat{\boldsymbol{p}}}^*), \quad \hat{\boldsymbol{v}}^* = \mathscr{L}^{-1}(\hat{\hat{\boldsymbol{v}}}^*) \tag{4.7}$$

with the initial conditions

$$\hat{p}\big|_{t=0}, \quad \hat{\boldsymbol{v}}\big|_{t=0} = 0, \quad \hat{p}^*\big|_{t=0} = 0, \quad \hat{\boldsymbol{v}}^*\big|_{t=0} = 0, \tag{4.8}$$

where $\hat{\hat{\boldsymbol{v}}}^* = \left(\hat{\hat{v}}_1^*, \hat{\hat{v}}_2^*\right)^\top$.

Taking the inverse Laplace transform in (4.5)–(4.6) and using the initial conditions (4.8), we have

$$\begin{cases} \partial_t\hat{p} = -c^2\rho_1\left(\partial_x(\hat{v}_1 + \sigma_2\hat{v}_1^*) + \partial_y(\hat{v}_2 + \sigma_1\hat{v}_2^*)\right) - (\sigma_1+\sigma_2)\hat{p} - \hat{p}^*, \\ \partial_t\hat{v}_1 = -\rho_1^{-1}\partial_x\hat{p} - \sigma_1\hat{v}_1, \quad \partial_t\hat{v}_2 = -\rho_1^{-1}\partial_y\hat{p} - \sigma_2\hat{v}_2, \\ \partial_t\hat{p}^* = \sigma_1\sigma_2\hat{p}, \quad \partial_t\hat{\boldsymbol{v}}^* = \hat{\boldsymbol{v}}. \end{cases} \tag{4.9}$$

Define a new variable $\hat{\boldsymbol{V}} = \left(\hat{V}_1, \hat{V}_2\right)^\top$ with

$$\hat{V}_1 = \hat{v}_1 + \sigma_2\hat{v}_1^*, \quad \hat{V}_2 = \hat{v}_2 + \sigma_1\hat{v}_2^*. \tag{4.10}$$

In $\mathbb{R}^2\backslash\bar{R}_1$, the PML system (4.9) can be written as

$$\begin{cases} \partial_t\hat{p} = -c^2\rho_1(\partial_x\hat{V}_1 + \partial_y\hat{V}_2) - (\sigma_1+\sigma_2)\hat{p} - \hat{p}^*, \\ \partial_t\hat{V}_1 = -\rho_1^{-1}\partial_x p + (\sigma_2-\sigma_1)(\hat{V}_1 - \sigma_2\hat{v}_1^*), \\ \partial_t\hat{V}_2 = -\rho_1^{-1}\partial_y p + (\sigma_1-\sigma_2)(\hat{V}_2 - \sigma_1\hat{v}_2^*), \\ \partial_t\hat{v}_1^* = \hat{V}_1 - \sigma_2\hat{v}_1^*, \quad \partial_t\hat{v}_2^* = \hat{V}_2 - \sigma_1\hat{v}_2^*, \\ \partial_t\hat{p}^* = \sigma_1\sigma_2\hat{p}. \end{cases} \tag{4.11}$$

The first order system (4.11) for $(\hat{p}, \hat{\boldsymbol{V}}, \hat{p}^*, \hat{\boldsymbol{v}}^*)$ gives the time-domain PML formulation for the acoustic wave field in $\mathbb{R}^2\backslash\bar{R}_1$. Since $\hat{p} = p, \hat{\boldsymbol{V}} = \boldsymbol{v}$ on ∂R_1, $(\hat{p}, \hat{\boldsymbol{V}})$ can be viewed as the extension of the solution $(p, \boldsymbol{v})$ of the problem (4.2) for the acoustic wave. Moreover, since $\sigma_1 = 0, \sigma_2 = 0$ inside the rectangle R_1, if we set $\hat{p} = p, \hat{p}^* = 0, \hat{\boldsymbol{V}} = \hat{\boldsymbol{v}}, \boldsymbol{v}^* = \int_0^t \hat{\boldsymbol{V}}(\cdot, \tau)\mathrm{d}\tau$ in $R_1\backslash\bar{D}$ for $t > 0$. Then in $D^{c} = \mathbb{R}^2 \setminus \bar{D}$, $(\hat{p}, \hat{\boldsymbol{V}}, \hat{p}^*, \hat{\boldsymbol{v}}^*)$ satisfies (4.11). Thus we deduce the time-domain PML system

$$
\begin{cases}
\partial_t \hat{p} + (\sigma_1 + \sigma_2)\hat{p} + \hat{p}^* + c^2 \rho_1 \nabla \cdot \hat{\boldsymbol{V}} = 0, \quad \partial_t \hat{p}^* = \sigma_1 \sigma_2 \hat{p} & \text{in } D^{c},\ t > 0,\\
\partial_t \hat{\boldsymbol{V}} = -\rho_1^{-1} \nabla \hat{p} + \Lambda_1 \hat{\boldsymbol{V}} - \Lambda_1 \Lambda_2 \hat{\boldsymbol{v}}^*, \quad \partial_t \hat{\boldsymbol{v}}^* = \hat{\boldsymbol{V}} - \Lambda_2 \hat{\boldsymbol{v}}^* & \text{in } D^{c},\ t > 0,\\
\partial_t \varphi_1 = -c_1 \nabla \cdot \boldsymbol{\Phi}_1, \quad \partial_t \boldsymbol{\Phi}_1 = -c_1 \nabla \varphi_1 & \text{in } D,\ t > 0,\\
\partial_t \varphi_2 = -c_2 \nabla \cdot \boldsymbol{\Phi}_2, \quad \partial_t \boldsymbol{\Phi}_2 = -c_2 \nabla \varphi_2 & \text{in } D,\ t > 0,\\
\hat{p}\big|_{t=0} = 0, \quad \hat{\boldsymbol{V}}\big|_{t=0} = 0, \quad \hat{p}^*\big|_{t=0} = 0, \quad \hat{\boldsymbol{v}}^*\big|_{t=0} = 0 & \text{in } D^{c},\\
\varphi_j\big|_{t=0} = 0, \quad \boldsymbol{\Phi}_j\big|_{t=0} = 0, \quad j = 1, 2 & \text{in } D,\\
\boldsymbol{n}_D \cdot \hat{\boldsymbol{V}} = \boldsymbol{n}_D \cdot (\partial_t \nabla \varphi_1 + \partial_t \mathbf{curl}\, \varphi_2) & \text{on } \partial D,\ t > 0,\\
-p \boldsymbol{n}_D = \mu \partial_{\boldsymbol{n}_D} (\nabla \varphi_1 + \mathbf{curl} \varphi_2) + \frac{(\lambda+\mu)}{c_1^2} \partial_t^2 \varphi_1 \boldsymbol{n}_D, & \text{on } \partial D,\ t > 0,
\end{cases}
\tag{4.12}
$$

where the matrices $\Lambda_1 = \operatorname{diag}((\sigma_2 - \sigma_1), (\sigma_1 - \sigma_2))$ and $\Lambda_2 = \operatorname{diag}(\sigma_2, \sigma_1)$.

The above discussion can be summarized in the following lemma:

Lemma 4.1. *Let* $(p, \boldsymbol{v}, \varphi_1, \boldsymbol{\Phi}_1, \varphi_2, \boldsymbol{\Phi}_2)$ *be the solution of the first order system of equations* (4.2) *which is extended to be* $(\hat{p}, \hat{\boldsymbol{V}}, \varphi_1, \boldsymbol{\Phi}_1, \varphi_2, \boldsymbol{\Phi}_2)$ *outside* R_1 *according to* (4.7) *and* (4.10). *Let* $(\hat{p}^*, \hat{\boldsymbol{v}}^*)$ *be defined in* (4.7) *for* $\boldsymbol{r} \in \mathbb{R}^2 \setminus \bar{R}_1$, *and* $(\hat{p}^*, \boldsymbol{v}^*) = (0, \int_0^t \hat{\boldsymbol{V}}(\cdot, \tau) d\tau)$ *for* $\boldsymbol{r} \in R_1 \setminus \bar{D}$. *Then* $(\hat{p}, \hat{\boldsymbol{V}}, \hat{p}^*, \hat{\boldsymbol{v}}^*, \varphi_1, \boldsymbol{\Phi}_1, \varphi_2, \boldsymbol{\Phi}_2)$ *satisfies the PML system* (4.12).

Due to the exponential decay of the scattered field $p^{\rm sc}$ in $\mathbb{R}^2 \setminus \bar{R}_1$, we define the following initial-boundary value problem for $(\hat{p}, \hat{\boldsymbol{V}}, \hat{p}^*, \hat{\boldsymbol{v}}^*, \varphi_1, \boldsymbol{\Phi}_1, \varphi_2, \boldsymbol{\Phi}_2)$, which is the truncated time-domain PML problem

$$
\begin{cases}
\partial_t \hat{p} + (\sigma_1 + \sigma_2)\hat{p} + \hat{p}^* + c^2 \rho_1 \nabla \cdot \hat{\boldsymbol{V}} = 0, \quad \partial_t \hat{p}^* = \sigma_1 \sigma_2 \hat{p} & \text{in } \Omega_2,\ t > 0,\\
\partial_t \hat{\boldsymbol{V}} = -\rho_1^{-1} \nabla \hat{p} + \Lambda_1 \hat{\boldsymbol{V}} - \Lambda_1 \Lambda_2 \hat{\boldsymbol{v}}^*, \quad \partial_t \hat{\boldsymbol{v}}^* = \hat{\boldsymbol{V}} - \Lambda_2 \hat{\boldsymbol{v}}^* & \text{in } \Omega_2,\ t > 0,\\
\partial_t \hat{\varphi}_1 = -c_1 \nabla \cdot \hat{\boldsymbol{\Phi}}_1, \quad \partial_t \hat{\boldsymbol{\Phi}}_1 = -c_1 \nabla \hat{\varphi}_1 & \text{in } D,\ t > 0,\\
\partial_t \hat{\varphi}_2 = -c_2 \nabla \cdot \hat{\boldsymbol{\Phi}}_2, \quad \partial_t \hat{\boldsymbol{\Phi}}_2 = -c_2 \nabla \varphi_2 & \text{in } D,\ t > 0,\\
\hat{p}\big|_{t=0} = 0, \quad \hat{\boldsymbol{V}}\big|_{t=0} = 0, \quad \hat{p}^*\big|_{t=0} = 0, \quad \hat{\boldsymbol{v}}^*\big|_{t=0} = 0 & \text{in } \Omega_2,\\
\varphi_j\big|_{t=0} = \boldsymbol{\Phi}_j\big|_{t=0} = 0, \quad j = 1, 2 & \text{in } D,\\
\boldsymbol{n}_D \cdot \hat{\boldsymbol{V}} = \boldsymbol{n}_D \cdot \left(\partial_t \nabla \hat{\varphi}_1 + \partial_t \mathbf{curl}\, \hat{\varphi}_2\right) & \text{on } \partial D,\ t > 0,\\
-\hat{p} \boldsymbol{n}_D = \mu \partial_{\boldsymbol{n}_D} (\nabla \hat{\varphi}_1 + \mathbf{curl} \hat{\varphi}_2) + \frac{(\lambda+\mu)}{c_1^2} \partial_t^2 \hat{\varphi}_1 \boldsymbol{n}_D & \text{on } \partial D,\ t > 0,\\
\hat{p} = p^{\rm inc} & \text{on } \partial R_2,\ t > 0,
\end{cases}
\tag{4.13}
$$

where $\Omega_2 = R_2 \setminus \bar{D}$.

Remark 4.2. It is clear to note that the solutions differ between the original PML problem (4.12) and the truncated PML problem (4.13). For simplicity of notation, we still take $(\hat{p}, \hat{\boldsymbol{V}}, \hat{p}^*, \hat{\boldsymbol{v}}^*, \varphi_1, \boldsymbol{\Phi}_1, \varphi_2, \boldsymbol{\Phi}_2)$ to be the solution of the truncated PML problem (4.13).

To show the well-posedness, we rewrite the PML system (4.13) into the vector form

$$\begin{cases} \partial_t \boldsymbol{p} = \boldsymbol{A}_1 \partial_x \boldsymbol{p} + \boldsymbol{B}_1 \partial_y \boldsymbol{p} + \boldsymbol{C}_1 \boldsymbol{p} & \text{in } \Omega_2,\ t > 0, \\ \partial_t \boldsymbol{q} = \boldsymbol{A}_2 \partial_x \boldsymbol{q} + \boldsymbol{B}_2 \partial_y \boldsymbol{q} + \boldsymbol{C}_2 \boldsymbol{q} & \text{in } D,\ t > 0, \end{cases} \tag{4.14}$$

where the vectors

$$\boldsymbol{p} = \left(\hat{p}, c\rho_1 \hat{V}_1, c\rho_1 \hat{V}_2, \hat{p}^*, c\rho_1 \hat{v}_1^*, c\rho_1 \hat{v}_2^*\right)^\top, \quad \boldsymbol{q} = (\varphi_1, \Phi_{11}, \Phi_{12}, \varphi_2, \Phi_{21}, \Phi_{22})^\top,$$

and $\boldsymbol{A}_j, \boldsymbol{B}_j, j = 1, 2$ are symmetric matrices given by

$$\boldsymbol{A}_1 = \begin{pmatrix} 0 & -c & 0 & 0 & 0 & 0 \\ -c & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \boldsymbol{A}_2 = \begin{pmatrix} 0 & -c_1 & 0 & 0 & 0 & 0 \\ -c_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -c_2 & 0 \\ 0 & 0 & 0 & -c_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\boldsymbol{B}_1 = \begin{pmatrix} 0 & 0 & -c & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -c & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \boldsymbol{B}_2 = \begin{pmatrix} 0 & 0 & -c_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -c_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -c_2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -c_2 & 0 & 0 \end{pmatrix},$$

and $\boldsymbol{C}_2 = 0$,

$$\boldsymbol{C}_1 = \begin{pmatrix} -(\sigma_1 + \sigma_2) & 0 & 0 & -1 & 0 & 0 \\ 0 & \sigma_2 - \sigma_1 & 0 & 0 & \sigma_2(\sigma_2 - \sigma_1) & 0 \\ 0 & 0 & \sigma_1 - \sigma_2 & 0 & 0 & \sigma_1(\sigma_1 - \sigma_2) \\ \sigma_1\sigma_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -\sigma_2 & 0 \\ 0 & 0 & 1 & 0 & 0 & -\sigma_1 \end{pmatrix}.$$

Notice that the truncated PML system (4.14) is a first order real symmetric hyperbolic system whose well-posedness follows from the standard theory (see for example [5] or [26]). Here we state the well-posedness of the PML problem (4.13) or (4.14) and omit the proof.

Theorem 4.3. *The truncated PML system* (4.13) *is has a unique strong solution.*

4.2. The Stability of the PML System

In this subsection, we rewrite the first order elastic wave equation into the second order Navier equation, and consider the stability of the initial-boundary value problem of the following PML system:

$$\begin{cases} \partial_t \hat{p} + (\sigma_1+\sigma_2)\hat{p} + \hat{p}^* + c^2\rho_1 \nabla\cdot\hat{\boldsymbol{V}} = 0, \quad \partial_t \hat{p}^* = \sigma_1\sigma_2\hat{p} & \text{in } \Omega_2\times(0,T), \\ \partial_t \hat{\boldsymbol{V}} = -\rho_1^{-1}\nabla\hat{p} + \Lambda_1\hat{\boldsymbol{V}} - \Lambda_1\Lambda_2\hat{\boldsymbol{v}}^*, \quad \partial_t\hat{\boldsymbol{v}}^* = \hat{\boldsymbol{V}} - \Lambda_2\hat{\boldsymbol{v}}^* & \text{in } \Omega_2\times(0,T), \\ \mu\Delta\boldsymbol{u} + (\lambda+\mu)\nabla\nabla\cdot\boldsymbol{u} - \rho_2\partial_t^2\boldsymbol{u} = 0 & \text{in } D\times(0,T), \\ \hat{p}\big|_{t=0} = 0, \quad \hat{\boldsymbol{V}}\big|_{t=0} = 0, \quad \hat{p}^*\big|_{t=0} = 0, \quad \hat{\boldsymbol{v}}^*\big|_{t=0} = 0, & \text{in } \Omega_2 \\ \boldsymbol{u}\big|_{t=0} = 0, \quad \partial_t\boldsymbol{u}\big|_{t=0} = 0 & \text{in } D, \\ \nabla\hat{p}\cdot\boldsymbol{n}_D = -\rho_1\boldsymbol{n}_D\cdot\partial_t^2\boldsymbol{u}, & \text{on } \partial D\times(0,T), \\ -\hat{p}\boldsymbol{n}_D = \mu\partial_{\boldsymbol{n}_D}\boldsymbol{u} + (\lambda+\mu)(\nabla\cdot\boldsymbol{u})\boldsymbol{n}_D, & \text{on } \partial D\times(0,T), \\ \hat{p} = p^{\mathrm{inc}} & \text{on } \partial R_2\times(0,T), \end{cases} \tag{4.15}$$

where $\hat{p}^* = 0$ in $R_1\backslash\bar{D}$. Denote $\Omega^{\mathrm{PML}} = R_2\backslash\bar{R}_1$ and

$$\Omega_1^{\mathrm{PML}} = \left\{\boldsymbol{r}\in\Omega^{\mathrm{PML}} : |x| > L_1/2,\ |y| < L_2/2\right\},$$
$$\Omega_2^{\mathrm{PML}} = \left\{\boldsymbol{r}\in\Omega^{\mathrm{PML}} : |x| < L_1/2,\ |y| > L_2/2\right\},$$
$$\Omega_{\mathrm{c}}^{\mathrm{PML}} = \left\{\boldsymbol{r}\in\Omega^{\mathrm{PML}} : |x| > L_1/2,\ |y| > L_2/2\right\}.$$

The following theorem is the main result of this section and shows the stability estimate of the PML system (4.15):

Theorem 4.4. *Let* $\left(\hat{p}, \hat{\boldsymbol{V}}, \hat{p}^*, \hat{\boldsymbol{v}}^*, \boldsymbol{u}\right)$ *be the solution of the PML problem* (4.15) *and* $p^{\mathrm{inc}} \in H^2(0,T;H^1(\Omega_2))$. *Then we have the following stability estimate:*

$$\max_{t\in[0,T]}\Big(\|\partial_t\hat{p}\|_{L_2(\Omega_2)} + \|\partial_t\boldsymbol{V}\|_{L^2(\Omega_2)^2} + \|\partial_t\hat{p}^*\|_{L_2(\Omega_2)} + \|\partial_t\boldsymbol{v}^*\|_{L^2(\Omega_2)^2}$$
$$+ \|\partial_t\boldsymbol{u}\|_{L^2(D)^2} + \|\nabla\cdot\boldsymbol{u}\|_{L^2(D)} + \|\nabla\boldsymbol{u}\|_{L^2(D)^{2\times 2}}\Big)$$
$$\lesssim (1+T)\max_{t\in[0,T]}\left(\|\partial_t^2 p^{\mathrm{inc}}\|_{H^{1/2}(\partial R_2)} + \|\partial_t p^{\mathrm{inc}}\|_{H^{1/2}(\partial R_2)} + \|p^{\mathrm{inc}}\|_{H^{1/2}(\partial R_2)}\right).$$

Before giving the proof of Theorem 4.4, we first consider the stability of an auxiliary system with zero boundary condition on ∂R_2. Consider the system

$$
\begin{cases}
\partial_t \phi + (\sigma_1+\sigma_2)\phi + \phi^* + c^2\rho_1 \nabla\cdot\boldsymbol{\Psi} = f_1, \quad \partial_t\phi^* = \sigma_1\sigma_2\phi & \text{in } \Omega_2\times(0,T), \quad (4.16a)\\
\partial_t\boldsymbol{\Psi} + \rho_1^{-1}\nabla\phi - \Lambda_1\boldsymbol{\Psi} + \Lambda_1\Lambda_2\boldsymbol{\psi}^* = \boldsymbol{f}_2, \quad \partial_t\boldsymbol{\psi}^* = \boldsymbol{\Psi} - \Lambda_2\boldsymbol{\psi}^* & \text{in } \Omega_2\times(0,T), \quad (4.16b)\\
\mu\Delta\boldsymbol{u} + (\lambda+\mu)\nabla\nabla\cdot\boldsymbol{u} - \rho_2\partial_t^2\boldsymbol{u} = 0 & \text{in } D\times(0,T), \quad (4.16c)\\
\phi\big|_{t=0} = 0, \quad \boldsymbol{\Psi}\big|_{t=0} = 0, \quad \phi^*\big|_{t=0} = 0, \quad \boldsymbol{\psi}^*\big|_{t=0} = 0 & \text{in } \Omega_2, \quad (4.16d)\\
\boldsymbol{u}\big|_{t=0} = 0, \quad \partial_t\boldsymbol{u}\big|_{t=0} = 0 & \text{in } D, \quad (4.16e)\\
\nabla\phi\cdot\boldsymbol{n}_D = -\rho_1\boldsymbol{n}_D\cdot\partial_t^2\boldsymbol{u} & \text{on } \partial D\times(0,T), \quad (4.16f)\\
-\phi\boldsymbol{n}_D = \mu\partial_{\boldsymbol{n}_D}\boldsymbol{u} + (\lambda+\mu)(\nabla\cdot\boldsymbol{u})\boldsymbol{n}_D & \text{on } \partial D\times(0,T), \quad (4.16g)\\
\phi = 0 & \text{on } \partial R_2\times(0,T), \quad (4.16h)
\end{cases}
$$

where $f_1 \in H^1(0,T;L^2(\Omega_2))$, $\boldsymbol{f}_2 \in H^1(0,T;L^2(\Omega_2)^2)$, $\boldsymbol{f}_2|_{\partial D} = 0$, and $\boldsymbol{\psi}^* = \int_0^t \boldsymbol{\Psi}\mathrm{d}\tau$, $\phi^* = 0$ in $R_1\backslash\bar{D}$.

Theorem 4.5. *Let $\left(\phi, \boldsymbol{\Psi}, \phi^*, \boldsymbol{\psi}^*, \boldsymbol{u}\right)$ be the solution of the auxiliary system (4.16). We have the following stability estimates:*

$$
\begin{aligned}
\max_{t\in[0,T]} &\Bigg(\|\partial_t^2\phi(\cdot,t) + \tilde{\sigma}_0\phi(\cdot,t)\|_{L^2(\Omega_2)} + \|\partial_t\boldsymbol{\Psi}(\cdot,t)\|_{L^2(\Omega_2)^2} \\
&+ \sum_j^2 \|\sigma_0\Psi_j(\cdot,t)\|_{L^2(\Omega_j^{\mathrm{PML}})}\Bigg) \\
&\lesssim \|f_1(\cdot,0)\|_{L^2(\Omega_2)} + \|\boldsymbol{f}_2(\cdot,0)\|_{L^2(\Omega_2)^2} \\
&+ \int_0^T \left(\|\partial_t f_1\|_{L^2(\Omega_2)} + \|\partial_t\boldsymbol{f}_2 + \tilde{\sigma}_0\boldsymbol{f}_2\|_{L^2(\Omega_2)^2}\right)\mathrm{d}t
\end{aligned}
\tag{4.17}
$$

and

$$
\begin{aligned}
\max_{t\in[0,T]} &\left(\|\partial_t\boldsymbol{u}\|_{L^2(D)^2} + \|\nabla\cdot\boldsymbol{u}\|_{L^2(D)} + \|\nabla\boldsymbol{u}\|_{L^2(D)^{2\times2}}\right) \\
&\lesssim \|f_1(\cdot,0)\|_{L^2(\Omega_2)} + \|\boldsymbol{f}_2(\cdot,0)\|_{L^2(\Omega_2)^2} \\
&+ \int_0^T \left(\|\partial_t f_1\|_{L^2(\Omega_2)} + \|\partial_t\boldsymbol{f}_2 + \tilde{\sigma}_0\boldsymbol{f}_2\|_{L^2(\Omega_2)^2}\right)\mathrm{d}t,
\end{aligned}
\tag{4.18}
$$

where $\tilde{\sigma}_0$ is defined by (4.21).

Proof. Taking the derivative of the first equation in (4.16a) with respect to t and using the second equation in (4.16a), we obtain

$$\partial_t^2\phi + (\sigma_1+\sigma_2)\partial_t\phi + \sigma_1\sigma_2\phi + c^2\rho_1\nabla\cdot(\partial_t\boldsymbol{\Psi}) = \partial_t f_1 \quad \text{in } \Omega_2\times(0,T). \tag{4.19}$$

Multiplying (4.19) by $\partial_t\phi + \tilde{\sigma}_0\phi$ and integrating over Ω_2 yields

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_t\phi+\tilde{\sigma}_0\phi\|^2_{L^2(\Omega_2)} &+ \int_{\Omega_2}((\sigma_1+\sigma_2-\tilde{\sigma}_0)\partial_t\phi+\sigma_1\sigma_2\phi)\,(\partial_t\phi+\tilde{\sigma}_0\phi)\mathrm{d}\boldsymbol{r}\\ &+\int_{\Omega_2}c^2\rho_1\nabla\cdot(\partial_t\boldsymbol{\Psi})(\partial_t\phi+\tilde{\sigma}_0\phi)\mathrm{d}\boldsymbol{r} = \int_{\Omega_2}\partial_t f_1(\partial_t\phi+\tilde{\sigma}_0\phi)\mathrm{d}\boldsymbol{r},\end{aligned} \tag{4.20}$$

where

$$\tilde{\sigma}_0 = \begin{cases}\sigma_0 & \text{in } R_2\backslash\bar{R}_1,\\ 0 & \text{in } R_1\backslash\bar{D}.\end{cases} \tag{4.21}$$

Since

$$\sigma_1+\sigma_2-\tilde{\sigma}_0 = \begin{cases}0 & \text{in } R_1\backslash\bar{D},\\ 0 & \text{in } \Omega_1^{\mathrm{PML}}\cup\Omega_2^{\mathrm{PML}},\\ \sigma_0 & \text{in } \Omega_{\mathrm{c}}^{\mathrm{PML}},\end{cases}$$

we have

$$\int_{\Omega_2}((\sigma_1+\sigma_2-\tilde{\sigma}_0)\partial_t\phi+\sigma_1\sigma_2\phi)\,(\partial_t\phi+\tilde{\sigma}_0\phi)\mathrm{d}\boldsymbol{r} = \sigma_0\int_{\Omega_{\mathrm{c}}^{\mathrm{PML}}}|\partial_t\phi+\tilde{\sigma}_0\phi|^2\mathrm{d}\boldsymbol{r} \geqq 0.$$

Integrating (4.20) from 0 to t, and using the above inequality and Green's first identity, we get

$$\begin{aligned}&\frac{1}{2}\|\partial_t\phi(\cdot,t)+\tilde{\sigma}_0\phi(\cdot,t)\|^2_{L^2(\Omega_2)} - \int_0^t\int_{\Omega_2}c^2\rho_1\partial_t\boldsymbol{\Psi}\cdot\nabla(\partial_t\phi+\tilde{\sigma}_0\phi)\mathrm{d}\boldsymbol{r}\mathrm{d}\tau\\ &\quad-\int_0^t\int_{\partial D}c^2\rho_1\partial_t\boldsymbol{\Psi}\cdot\boldsymbol{n}_D(\partial_t\phi+\tilde{\sigma}_0\phi)\mathrm{d}\gamma\mathrm{d}\tau\\ &=\frac{1}{2}\|\partial_t\phi(\cdot,t)+\tilde{\sigma}_0\phi(\cdot,t)\|^2_{L^2(\Omega_2)} - \int_0^t\int_{\Omega_2}c^2\rho_1\partial_t\boldsymbol{\Psi}\cdot\nabla(\partial_t\phi+\tilde{\sigma}_0\phi)\mathrm{d}\boldsymbol{r}\mathrm{d}\tau\\ &\quad-\int_0^t\int_{\partial D}c^2\rho_1(\partial_t\boldsymbol{\Psi}\cdot\boldsymbol{n}_D)\partial_t\phi\mathrm{d}\gamma\mathrm{d}\tau\\ &\leqq\frac{1}{2}\|\partial_t\phi(0,\cdot)\|^2_{L^2(\Omega_2)} + \int_0^t\int_{\Omega_2}\partial_t f_1(\partial_t\phi+\tilde{\sigma}_0\phi)\mathrm{d}\boldsymbol{r}\mathrm{d}\tau.\end{aligned} \tag{4.22}$$

Here we have used the fact that $\tilde{\sigma}_0 = 0$ on ∂D, $\phi(\cdot,0) = 0$ and $\phi\big|_{\partial R_2} = 0$. Since, $\Lambda_1 = \Lambda_2 = 0$ in $R_1\backslash\bar{D}$ and $\boldsymbol{f}_2|_{\partial D} = 0$, it follows from the first equation in (4.16b) and the boundary condition (4.16f) that

$$\partial_t\boldsymbol{\Psi}\cdot\boldsymbol{n}_D = -\rho_1^{-1}\nabla\phi\cdot\boldsymbol{n}_D = \boldsymbol{n}_D\cdot\partial_t^2\boldsymbol{u} \quad \text{on } \partial D.$$

Thus (4.22) can be written into

$$\frac{1}{2}\|\partial_t\phi(\cdot,t)+\tilde{\sigma}_0\phi(\cdot,t)\|^2_{L^2(\Omega_2)} - \int_0^t\int_{\Omega_2}c^2\rho_1\partial_t\boldsymbol{\Psi}\cdot\nabla(\partial_t\phi+\tilde{\sigma}_0\phi)\mathrm{d}\boldsymbol{r}\mathrm{d}\tau$$

$$
\begin{aligned}
&- \int_0^t \int_{\partial D} c^2 \rho_1 \partial_t \phi (\boldsymbol{n}_D \cdot \partial_t^2 \boldsymbol{u}) \mathrm{d}\gamma \mathrm{d}\tau \\
&\leqq \frac{1}{2} \|\partial_t \phi(0, \cdot)\|^2_{L^2(\Omega_2)} + \int_0^t \int_{\Omega_2} \partial_t f_1 (\partial_t \phi + \tilde{\sigma}_0 \phi) \mathrm{d}\boldsymbol{r} \mathrm{d}\tau.
\end{aligned} \tag{4.23}
$$

Next, taking the derivative of the first equation in (4.16b) with respect to t, multiplying the first equation (4.16b) by $\tilde{\sigma}_0$, and adding the two equations, we get

$$
\begin{aligned}
&\partial_t^2 \boldsymbol{\Psi} + \rho_1^{-1} \nabla (\partial_t \phi + \tilde{\sigma}_0 \phi) + (\tilde{\sigma}_0 I - \Lambda_1) \partial_t \boldsymbol{\Psi} + \Lambda_1 (\Lambda_2 - \tilde{\sigma}_0 I) \boldsymbol{\Psi} \\
&\quad + \Lambda_1 \Lambda_2 (\sigma_0 I - \Lambda_2) \boldsymbol{\psi}^* = \partial_t \boldsymbol{f}_2 \\
&\quad + \tilde{\sigma}_0 \boldsymbol{f}_2 \quad \text{in } \Omega_2 \times (0, T).
\end{aligned} \tag{4.24}
$$

Multiplying (4.24) by $\rho_1^2 c^2 \partial_t \boldsymbol{\Psi}$ and integration over Ω_2, we obtain

$$
\begin{aligned}
&\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \|c \rho_1 \partial_t \boldsymbol{\Psi}\|^2_{L^2(\Omega_2)^2} + \int_{\Omega_2} c^2 \rho_1 \nabla (\partial_t \phi + \tilde{\sigma}_0 \phi) \cdot \partial_t \boldsymbol{\Psi} \mathrm{d}\boldsymbol{r} \\
&\quad + \int_{\Omega_2} (\tilde{\sigma}_0 I - \Lambda_1) |c \rho_1 \partial_t \boldsymbol{\Psi}|^2 \mathrm{d}\boldsymbol{r} \\
&\quad + \int_{\Omega_2} c^2 \rho_1^2 \Lambda_1 (\Lambda_2 - \tilde{\sigma}_0 I) \boldsymbol{\Psi} \cdot \partial_t \boldsymbol{\Psi} \\
&\quad + \int_{\Omega_2} c^2 \rho_1^2 \Lambda_1 \Lambda_2 (\tilde{\sigma}_0 I - \Lambda_2) \boldsymbol{\psi}^* \cdot \partial_t \boldsymbol{\Psi} \mathrm{d}\boldsymbol{r} \\
&= \int_{\Omega_2} c^2 \rho_1^2 (\partial_t \boldsymbol{f}_2 + \tilde{\sigma}_0 \boldsymbol{f}_2) \cdot \partial_t \boldsymbol{\Psi} \mathrm{d}\boldsymbol{r}.
\end{aligned} \tag{4.25}
$$

Since the matrix $\tilde{\sigma}_0 I - \Lambda_1 = \mathrm{diag}(\tilde{\sigma}_0 + \sigma_1 - \sigma_2, \tilde{\sigma}_0 + \sigma_2 - \sigma_1)$ is symmetric positive definite in Ω_2, that is, $\tilde{\sigma}_0 I - \Lambda_1 \geqq 0$, the third term on the left hand side of (4.25) satisfies

$$
\int_{\Omega_2} (\tilde{\sigma}_0 I - \Lambda_1) |c \rho_1 \partial_t \boldsymbol{\Psi}|^2 \mathrm{d}\boldsymbol{r} \geqq 0. \tag{4.26}
$$

Note that $\Lambda_1 = 0$ in $R_1 \backslash D$ and $\Lambda_2 - \tilde{\sigma}_0 I = 0$ in $\Omega_{\mathrm{c}}^{\mathrm{PML}}$, the fourth term on the left hand side of (4.25) can be written

$$
\int_{\Omega_2} c^2 \rho_1^2 \Lambda_1 (\Lambda_2 - \tilde{\sigma}_0 I) \boldsymbol{\Psi} \cdot \partial_t \boldsymbol{\Psi} \mathrm{d}\boldsymbol{r} = \int_{\Omega_1^{\mathrm{PML}}} + \int_{\Omega_2^{\mathrm{PML}}} c^2 \rho_1^2 \Lambda_1 (\Lambda_2 - \tilde{\sigma}_0 I) \boldsymbol{\Psi} \cdot \partial_t \boldsymbol{\Psi} \mathrm{d}\boldsymbol{r}.
$$

In Ω_1^{PML}, by the definitions of σ_j, we have $\Lambda_1 = \mathrm{diag}(-\sigma_0, \sigma_0)$, $\Lambda_2 - \tilde{\sigma}_0 = \mathrm{diag}(-\sigma_0, 0)$ and

$$
\int_{\Omega_1^{\mathrm{PML}}} c^2 \rho_1^2 \Lambda_1 (\Lambda_2 - \tilde{\sigma}_0 I) \boldsymbol{\Psi} \cdot \partial_t \boldsymbol{\Psi} \mathrm{d}\boldsymbol{r} = \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \|c \rho_1 \sigma_0 \Psi_1\|^2_{L^2(\Omega_1^{\mathrm{PML}})}. \tag{4.27}
$$

In Ω_2^{PML}, it follows from $\Lambda_1 = \mathrm{diag}(\sigma_0, -\sigma_0)$ and $\Lambda_2 - \tilde{\sigma}_0 I = \mathrm{diag}(0, -\sigma_0)$ that

$$
\int_{\Omega_2^{\mathrm{PML}}} c^2 \rho_1^2 \Lambda_1 (\Lambda_2 - \tilde{\sigma}_0 I) \boldsymbol{\Psi} \cdot \partial_t \boldsymbol{\Psi} \mathrm{d}\boldsymbol{r} = \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \|c \rho_1 \sigma_0 \Psi_2\|^2_{L^2(\Omega_2^{\mathrm{PML}})}, \tag{4.28}
$$

where $\boldsymbol{\Psi} = (\Psi_1, \Psi_2)^\top$. For the fifth term on the left hand side of (4.25), it also holds that $\Lambda_1 = 0$ in $R_1 \backslash D$ and $\Lambda_2 - \tilde{\sigma}_0 I = 0$ in $\Omega_{\mathrm{c}}^{\mathrm{PML}}$. Furthermore, we have that

$$\Lambda_1\Lambda_2 = \mathrm{diag}(0, \sigma_0^2), \quad \tilde{\sigma}_0 - \Lambda_2 = \mathrm{diag}(\sigma_0, 0) \quad \text{in} \quad \Omega_1^{\mathrm{PML}},$$
$$\Lambda_1\Lambda_2 = \mathrm{diag}(\sigma_0^2, 0), \quad \tilde{\sigma}_0 - \Lambda_2 = \mathrm{diag}(0, \sigma_0) \quad \text{in} \quad \Omega_2^{\mathrm{PML}}.$$

Thus it can be easily verified that

$$\int_{\Omega_2} c^2\rho_1^2\Lambda_1\Lambda_2(\tilde{\sigma}_0 I - \Lambda_2)\boldsymbol{\psi}^* \cdot \partial_t\boldsymbol{\Psi}\mathrm{d}\boldsymbol{r} = 0. \tag{4.29}$$

Substituting (4.26)–(4.29) into (4.25) and integrating from 0 to t, we obtain

$$\begin{aligned}
&\frac{c^2\rho_1^2}{2}\left(\|\partial_t\boldsymbol{\Psi}(\cdot,t)\|^2_{L^2(\Omega_2)^2} + \|\sigma_0\Psi_1(\cdot,t)\|^2_{L^2(\Omega_1^{\mathrm{PML}})} + \|\sigma_0\Psi_2(\cdot,t)\|^2_{L^2(\Omega_2^{\mathrm{PML}})}\right)\\
&\quad + \int_0^t\int_{\Omega_2} c^2\rho_1\nabla(\partial_t\phi + \tilde{\sigma}_0\phi)\cdot\partial_t\boldsymbol{\Psi}\mathrm{d}\boldsymbol{r}\mathrm{d}\tau\\
&\leqq \frac{c^2\rho_1^2}{2}\|\partial_t\boldsymbol{\Psi}(\cdot,0)\|^2_{L^2(\Omega_2)^2}\\
&\quad + \int_0^t\int_{\Omega_2} c^2\rho_1^2(\partial_t\boldsymbol{f}_2 + \tilde{\sigma}_0\boldsymbol{f}_2)\cdot\partial_t\boldsymbol{\Psi}\mathrm{d}\boldsymbol{r}\mathrm{d}\tau.
\end{aligned} \tag{4.30}$$

Adding (4.23) and (4.30) yields

$$\begin{aligned}
&\frac{1}{2}\|\partial_t\phi(\cdot,t) + \tilde{\sigma}_0\phi(\cdot,t)\|^2_{L^2(\Omega_2)} - \int_0^t\int_{\partial D} c^2\rho_1\partial_t\phi(\boldsymbol{n}_D\cdot\partial_t^2\boldsymbol{u})\mathrm{d}\gamma\mathrm{d}\tau\\
&\quad + \frac{c^2\rho_1^2}{2}\Big(\|\partial_t\boldsymbol{\Psi}(\cdot,t)\|^2_{L^2(\Omega_2)^2} + \|\sigma_0\Psi_1(\cdot,t)\|^2_{L^2(\Omega_1^{\mathrm{PML}})}\\
&\quad + \|\sigma_0\Psi_2(\cdot,t)\|^2_{L^2(\Omega_2^{\mathrm{PML}})}\Big)\\
&\leqq \frac{1}{2}\|\partial_t\phi(0,\cdot)\|^2_{L^2(\Omega_2)} + \frac{c^2\rho_1^2}{2}\|\partial_t\boldsymbol{\Psi}(\cdot,0)\|^2_{L^2(\Omega_2)^2}\\
&\quad + \int_0^t\int_{\Omega_2}\partial_t f_1(\partial_t\phi + \tilde{\sigma}_0\phi)\mathrm{d}\boldsymbol{r}\mathrm{d}\tau\\
&\quad + \int_0^t\int_{\Omega_2} c^2\rho_1^2(\partial_t\boldsymbol{f}_2 + \tilde{\sigma}_0\boldsymbol{f}_2)\cdot\partial_t\boldsymbol{\Psi}\mathrm{d}\boldsymbol{r}\mathrm{d}\tau.
\end{aligned} \tag{4.31}$$

Let

$$\begin{aligned}
G_1(t) = &\frac{1}{2}\|\partial_t\phi(\cdot,t) + \tilde{\sigma}_0\phi(\cdot,t)\|^2_{L^2(\Omega_2)}\\
&+ \frac{c^2\rho_1^2}{2}\Big(\|\partial_t\boldsymbol{\Psi}(\cdot,t)\|^2_{L^2(\Omega_2)^2} + \|\sigma_0\Psi_1(\cdot,t)\|^2_{L^2(\Omega_1^{\mathrm{PML}})}\\
&+ \|\sigma_0\Psi_2(\cdot,t)\|^2_{L^2(\Omega_2^{\mathrm{PML}})}\Big).
\end{aligned}$$

It follows from (4.31) that

$$G_1(t) \leqq G_1(0) + \int_0^t \int_{\Omega_2} \partial_t f_1 (\partial_t \phi + \tilde{\sigma}_0 \phi) \mathrm{d}\boldsymbol{r} \mathrm{d}\tau$$
$$+ \int_0^t \int_{\Omega_2} c^2 \rho_1^2 (\partial_t \boldsymbol{f}_2 + \tilde{\sigma}_0 \boldsymbol{f}_2) \cdot \partial_t \boldsymbol{\Psi} \mathrm{d}\boldsymbol{r} \mathrm{d}\tau$$
$$+ \int_0^t \int_{\partial D} c^2 \rho_1 \partial_t \phi (\boldsymbol{n}_D \cdot \partial_t^2 \boldsymbol{u}) \mathrm{d}\gamma \mathrm{d}\tau. \tag{4.32}$$

Next, taking the derivative of (4.16c), (4.16e), and (4.16g) with respect to t, we obtain

$$\begin{cases} \mu \Delta(\partial_t \boldsymbol{u}) + (\lambda + \mu) \nabla \nabla \cdot (\partial_t \boldsymbol{u}) - \rho_2 \partial_t^2 (\partial_t \boldsymbol{u}) = 0 & \text{in } D \times (0, T), \\ \partial_t \boldsymbol{u}\big|_{t=0} = 0, \quad \partial_t^2 \boldsymbol{u}\big|_{t=0} = 0 & \text{in } D, \\ -(\partial_t \phi) \boldsymbol{n}_D = \mu \partial_{\boldsymbol{n}_D} (\partial_t \boldsymbol{u}) + (\lambda + \mu)(\nabla \cdot \partial_t \boldsymbol{u}) \boldsymbol{n}_D & \text{on } \partial D \times (0, T). \end{cases} \tag{4.33}$$

Define

$$G_2(t) = \frac{c^2 \rho_1}{2} \Big(\|\rho_2^{1/2} \partial_t^2 \boldsymbol{u}\|_{L^2(D)^2}^2 + \|(\lambda + \mu)^{1/2} \nabla \cdot (\partial_t \boldsymbol{u})\|_{L^2(D)}^2$$
$$+ \|\mu^{1/2} \nabla (\partial_t \boldsymbol{u})\|_{L^2(D)^{2\times 2}}^2 \Big). \tag{4.34}$$

Note that $G_2(0) = 0$ due to the initial conditions in (4.33). It follows from (4.33) and (4.34) that

$$G_2(t) = \int_0^t G_2'(\tau) \mathrm{d}\tau = \int_0^t \int_D c^2 \rho_1 \rho_2 \partial_t^2 (\partial_t \boldsymbol{u}) \cdot \partial_t^2 \boldsymbol{u} \mathrm{d}x \mathrm{d}\tau$$
$$+ \int_0^t \int_D c^2 \rho_1 \Big((\lambda + \mu)(\nabla \cdot (\partial_t^2 \boldsymbol{u}))(\nabla \cdot (\partial_t \boldsymbol{u}))$$
$$+ \mu (\nabla (\partial_t^2 \boldsymbol{u})) : (\nabla (\partial_t \boldsymbol{u})) \Big) \mathrm{d}\boldsymbol{r} \mathrm{d}\tau$$
$$= \int_0^t \int_D c^2 \rho_1 \Big(-\mu (\nabla (\partial_t \boldsymbol{u})) : (\nabla (\partial_t^2 \boldsymbol{u}))$$
$$- (\lambda + \mu)(\nabla \cdot (\partial_t^2 \boldsymbol{u}))(\nabla \cdot (\partial_t \boldsymbol{u})) \Big) \mathrm{d}\boldsymbol{r} \mathrm{d}\tau$$
$$+ \int_0^t \int_{\partial D} c^2 \rho_1 (\mu \partial_{\boldsymbol{n}_D} (\partial_t \boldsymbol{u}) + (\lambda + \mu)(\nabla \cdot \partial_t \boldsymbol{u}) \boldsymbol{n}_D) \cdot \partial_t^2 \boldsymbol{u} \mathrm{d}\gamma \mathrm{d}\tau$$
$$+ \int_0^t \int_D c^2 \rho_1 \Big((\lambda + \mu)(\nabla \cdot (\partial_t^2 \boldsymbol{u}))(\nabla \cdot (\partial_t \boldsymbol{u}))$$
$$+ \mu (\nabla (\partial_t^2 \boldsymbol{u})) : (\nabla (\partial_t \boldsymbol{u})) \Big) \mathrm{d}\boldsymbol{r} \mathrm{d}\tau$$
$$= - \int_0^t \int_{\partial D} c^2 \rho_1 (\partial_t \phi \boldsymbol{n}_D) \cdot \partial_t^2 \boldsymbol{u} \mathrm{d}\gamma \mathrm{d}\tau. \tag{4.35}$$

Combining (4.32) and (4.35) gives

$$\|\partial_t^2 \phi(\cdot, t) + \tilde{\sigma}_0 \phi(\cdot, t)\|_{L^2(\Omega_2)}^2 + \|\partial_t \boldsymbol{\Psi}(\cdot, t)\|_{L^2(\Omega_2)^2}^2 + \sum_j^2 \|\sigma_0 \Psi_j(\cdot, t)\|_{L^2(\Omega_j^{\mathrm{PML}})}^2$$
$$\lesssim G_1(t) + G_2(t)$$
$$\lesssim \|\partial_t \phi(\cdot, 0)\|_{L^2(\Omega_2)}^2 + \|\partial_t \boldsymbol{\Psi}(\cdot, 0)\|_{L^2(\Omega_2)^2}^2$$

$$+\int_0^t\int_{\Omega_2}\partial_t f_1(\partial_t\phi+\tilde\sigma_0\phi)\mathrm{d}\boldsymbol{r}\mathrm{d}\tau$$
$$+\int_0^t\int_{\Omega_2}c^2\rho_1^2(\partial_t\boldsymbol{f}_2+\tilde\sigma_0\boldsymbol{f}_2)\cdot\partial_t\boldsymbol{\Psi}\mathrm{d}\boldsymbol{r}\mathrm{d}\tau.$$

It follows from the compatibility conditions in (4.16a), (4.16b) and the initial conditions (4.16d) that

$$\partial_t\phi(\cdot,0)=f_1(\cdot,0),\quad \partial_t\boldsymbol{\Psi}\big|_{t=0}=\boldsymbol{f}_2(\cdot,0). \tag{4.36}$$

Using the Young inequality and (4.36), we have

$$\begin{aligned}
&\max_{t\in[0,T]}\Bigg(\|\partial_t^2\phi(\cdot,t)+\tilde\sigma_0\phi(\cdot,t)\|_{L^2(\Omega_2)}+\|\partial_t\boldsymbol{\Psi}(\cdot,t)\|_{L^2(\Omega_2)^2}\\
&\quad+\sum_j^2\|\sigma_0\Psi_j(\cdot,t)\|_{L^2(\Omega_j^{\mathrm{PML}})}\Bigg)\\
&\lesssim\|f_1(\cdot,0)\|_{L^2(\Omega_2)}+\|\boldsymbol{f}_2(\cdot,0)\|_{L^2(\Omega_2)^2}\\
&\quad+\int_0^T\Big(\|\partial_t f_1\|_{L^2(\Omega_2)}+\|\partial_t\boldsymbol{f}_2+\tilde\sigma_0\boldsymbol{f}_2\|_{L^2(\Omega_2)^2}\Big)\mathrm{d}t.
\end{aligned}$$

For the elastic field $\boldsymbol{u}$, we can also derive from (4.32) and (4.35) that

$$\begin{aligned}
&\|\partial_t^2\boldsymbol{u}\|^2_{L^2(D)^2}+\|\nabla\cdot(\partial_t\boldsymbol{u})\|^2_{L^2(D)}+\|\nabla(\partial_t\boldsymbol{u})\|^2_{L^2(D)^{2\times2}}\\
&\quad+\|\partial_t^2\phi(\cdot,t)+\tilde\sigma_0\phi(\cdot,t)\|^2_{L^2(\Omega_2)}+\|\partial_t\boldsymbol{\Psi}(\cdot,t)\|^2_{L^2(\Omega_2)^2}\\
&\lesssim G_1(t)+G_2(t)\lesssim\|f_1(\cdot,0)\|^2_{L^2(\Omega_2)}+\|\partial_t\boldsymbol{f}_2(\cdot,0)\|^2_{L^2(\Omega_2)^2}\\
&\quad+\int_0^t\int_{\Omega_2}\partial_t f_1(\partial_t\phi+\tilde\sigma_0\phi)\mathrm{d}\boldsymbol{r}\mathrm{d}\tau\\
&\quad+\int_0^t\int_{\Omega_2}c^2\rho_1^2(\partial_t\boldsymbol{f}_2+\tilde\sigma_0\boldsymbol{f}_2)\cdot\partial_t\boldsymbol{\Psi}\mathrm{d}\boldsymbol{r}\mathrm{d}\tau.
\end{aligned}$$

Using the Young inequality again, we can obtain

$$\begin{aligned}
&\max_{t\in[0,T]}\Big(\|\partial_t^2\boldsymbol{u}\|^2_{L^2(D)^2}+\|\nabla\cdot(\partial_t\boldsymbol{u})\|^2_{L^2(D)}+\|\nabla(\partial_t\boldsymbol{u})\|^2_{L^2(D)^{2\times2}}\Big)\\
&\qquad\lesssim\|f_1(\cdot,0)\|^2_{L^2(\Omega_2)}+\|\boldsymbol{f}_2(\cdot,0)\|^2_{L^2(\Omega_2)^2}\\
&\qquad\quad+\int_0^T\Big(\|\partial_t f_1\|^2_{L^2(\Omega_2)}+\|\partial_t\boldsymbol{f}_2+\tilde\sigma_0\boldsymbol{f}_2\|^2_{L^2(\Omega_2)^2}\Big)\mathrm{d}t.
\end{aligned}$$

Using Young's inequality as in (3.27) and (3.28), we can obtain the estiamte

$$\begin{aligned}
&\max_{t\in[0,T]}\Big(\|\partial_t\boldsymbol{u}\|_{L^2(D)^2}+\|\nabla\cdot\boldsymbol{u}\|_{L^2(D)}+\|\nabla\boldsymbol{u}\|_{L^2(D)^{2\times2}}\Big)\\
&\qquad\lesssim\|f_1(\cdot,0)\|_{L^2(\Omega_2)}+\|\boldsymbol{f}_2(\cdot,0)\|_{L^2(\Omega_2)}\\
&\qquad\quad+\int_0^T\Big(\|\partial_t f_1\|_{L^2(\Omega_2)}+\|\partial_t\boldsymbol{f}_2+\tilde\sigma_0\boldsymbol{f}_2\|_{L^2(\Omega_2)}\Big)\mathrm{d}t.
\end{aligned}$$

□

The following lemma is directly obtained from the Lemma 3.2 of [8]:

Lemma 4.6. *It holds that*

$$\max_{t\in[0,T]} \|\tilde{\sigma}_0\phi\|^2_{L^2(\Omega_2)} \leqq \max_{t\in[0,T]} \|\partial_t\phi + \tilde{\sigma}_0\phi\|_{L^2(\Omega_2)}.$$

Using Lemma 4.6, we have following stability of the PML problem:

Theorem 4.7. *Let $(\phi, \boldsymbol{\Psi}, \phi^*, \boldsymbol{\psi}^*, \boldsymbol{u})$ be the solution of the PML system (4.16). Given $f_1 \in H^1(0, T; L^2(\Omega_2))$, $\boldsymbol{f}_2 \in H^1(0, T; L^2(\Omega_2)^2)$, for any $T > 0$, it holds that*

$$\begin{aligned}
\max_{t\in[0,T]} \Big(&\|\partial_t\phi\|_{L^2(\Omega_2)} + \|\partial_t\boldsymbol{\Psi}\|_{L^2(\Omega_2)^2} + \|\partial_t\phi^*\|_{L^2(\Omega_2)} + \|\partial_t\boldsymbol{\psi}^*\|_{L^2(\Omega_2)^2} \\
&+ \|\partial_t\boldsymbol{u}\|_{L^2(D)^2} + \|\nabla\cdot\boldsymbol{u}\|_{L^2(D)} + \|\nabla\boldsymbol{u}\|_{L^2(D)^{2\times 2}}\Big) \\
\lesssim &\|f_1(\cdot,0)\|_{L^2(\Omega_2)} + \|\boldsymbol{f}_2(\cdot,0)\|_{L^2(\Omega_2)^2} \\
&+ \int_0^T \Big(\|\partial_t f_1\|_{L^2(\Omega_2)} + \|\partial_t\boldsymbol{f}_2 + \tilde{\sigma}_0\boldsymbol{f}_2\|_{L^2(\Omega_2)^2}\Big)\,\mathrm{d}t. \qquad (4.37)
\end{aligned}$$

Proof. It follows from Lemma 4.6 that

$$\begin{aligned}
\max_{t\in[0,T]} \|\partial_t\phi\|_{L^2(\Omega_2)} &\leqq \max_{t\in[0,T]} \|\partial_t\phi + \tilde{\sigma}_0\phi\|_{L^2(\Omega_2)} + \max_{t\in[0,T]} \|\tilde{\sigma}_0\phi\|_{L^2(\Omega_2)} \\
&\leqq 2\max_{t\in[0,T]} \|\partial_t\phi + \tilde{\sigma}_0\phi\|_{L^2(\Omega_2)}. \qquad (4.38)
\end{aligned}$$

By the second equation in (4.16a), we have

$$\|\partial_t\phi^*\|_{L^2(\Omega_2)} \lesssim \|\tilde{\sigma}_0\phi\| \leqq \max_{t\in[0,T]} \|\partial_t\phi + \tilde{\sigma}_0\phi\|_{L^2(\Omega_2)}. \qquad (4.39)$$

Since in $R_1\backslash\bar{D}$, $\partial_t\boldsymbol{\psi}^* = \boldsymbol{\Psi}$, we only have to consider the estimate of $\partial_t\boldsymbol{\psi}^*$ in Ω^{PML}. In Ω_1^{PML}, by the second equation in (4.16b), we have

$$\partial_t\psi_1^* = \Psi_1, \quad \partial_t\psi_2^* = \Psi_2 - \sigma_0\psi_2^*.$$

The estimate of $\|\partial_t\psi_1^*\|_{L^2(\Omega_1^{\mathrm{PML}})}$ can be derived directly from (4.17). The estimate $\|\partial_t\psi_2^*\|_{L^2(\Omega_1^{\mathrm{PMl}})}$ can be derived from (4.17) and the similar proof in Lemma 4.6. Similarly, we can get the estimate $\|\partial_t\boldsymbol{\psi}^*\|_{L^2(\Omega_2^{\mathrm{PML}})^2}$. In $\Omega_{\mathrm{c}}^{\mathrm{PML}}$, we have

$$\partial_t\psi_1^* = \Psi_1 - \sigma_0\psi_1^*, \quad \partial_t\psi_2^* = \Psi_2 - \sigma_0\psi_2^*.$$

The estimates of $\|\partial_t\psi_1^*\|_{L^2(\Omega_{\mathrm{c}}^{\mathrm{PML}})}$ and $\|\partial_t\psi_2^*\|_{L^2(\Omega_{\mathrm{c}}^{\mathrm{PML}})}$ can be derived from (4.17) and the similar proof in Lemma 4.6.

Combining (4.17), (4.38), (4.39), and the arguments for $\partial_t\boldsymbol{\psi}^*$, we deduce the estimate (4.37). □

The proof of the following trace lemma can be found in [38, Lemma 2.5.3 and Lemma 2.5.4]:

Lemma 4.8. *Let $\xi \in H^2(0, T; H^{1/2}(\partial R_2))$. Then there exists a function $\eta \in H^2(0, T; H^1(\Omega_2))$ such that $\eta = 0$ on $\partial D \times (0, T)$, $\eta = \xi$ on $\partial R_2 \times (0, T)$, and*

$$\max_{t\in[0,T]} \|\partial_t^2\eta\|_{H^1(\Omega_2)} \lesssim \max_{t\in[0,T]} \|\partial_t^2\xi\|_{H^{1/2}(\partial R_2)},$$
$$\max_{t\in[0,T]} \|\partial_t\eta\|_{H^1(\Omega_2)} \lesssim \max_{t\in[0,T]} \|\partial_t\xi\|_{H^{1/2}(\partial R_2)},$$
$$\max_{t\in[0,T]} \|\eta\|_{H^1(\Omega_2)} \lesssim \max_{t\in[0,T]} \|\xi\|_{H^{1/2}(\partial R_2)}.$$

Now we turn to the proof Theorem 4.4.

Proof. Let η be a lifting function in $\Omega_2 \times (0, T)$ such that $\eta = 0$ on ∂D and $\eta = p^{\rm inc}$ on ∂R_2. In fact, we can get a mollified lifting function η in $\Omega_2 \times (0, T)$ such that $\eta = 0$, $\nabla\eta = 0$ on ∂D and $\eta = p^{\rm inc}$ on ∂R_2. We can choose $\eta(\cdot, 0) = 0$. Let

$$\tilde{\phi} = \hat{p} - \eta, \quad \boldsymbol{\Psi} = \boldsymbol{V}, \quad \tilde{\phi}^* = \hat{p}^* - \int_0^t \sigma_1\sigma_2\eta \mathrm{d}\tau, \quad \boldsymbol{\psi}^* = \boldsymbol{v}^* \quad \text{in } \Omega_2 \times (0, T). \tag{4.40}$$

It can be verified that $\left(\tilde{\phi}, \boldsymbol{V}, \tilde{\phi}^*, \boldsymbol{v}^*, \boldsymbol{u}\right)$ satisfies (4.16) with homogeneous initial conditions in $\mathbb{R}^2$, homogeneous Dirichlet boundary conditions on ∂R_2 and the continuity conditions on ∂D, and

$$f_1 = -\left(\partial_t\eta + (\sigma_1 + \sigma_2)\eta + \int_0^t \sigma_1\sigma_2\eta \mathrm{d}\tau\right), \quad \boldsymbol{f}_2 = -\rho_1^{-1}\nabla\eta \quad \text{in } \Omega_2 \times (0, T). \tag{4.41}$$

Clearly, if $\eta \in H^2(0, T; L^2(\Omega_2)) \cap H^1(0, T; H^1(\Omega_2))$, then we get $f_1 \in H^1(0, T; L^2(\Omega_2))$ and $\boldsymbol{f}_2 \in H^1(0, T; L^2(\Omega_2)^2)$. Furthermore, we have $\boldsymbol{f}_2|_{\partial D} = 0$, $f_1(\cdot, 0) = 0$, $\boldsymbol{f}_2(\cdot, 0) = 0$ in Ω_2.

Using Theorem 4.7, we have

$$\begin{aligned}
&\|\partial_t\tilde{\phi}\|_{L^2(\Omega_2)} + \|\partial_t\boldsymbol{V}\|_{L^2(\Omega_2)^2} + \|\partial_t\tilde{\phi}^*\|_{L^2(\Omega_2)} + \|\partial_t\boldsymbol{v}^*\|_{L^2(\Omega_2)^2} \\
&\quad + \|\partial_t\boldsymbol{u}\|_{L^2(D)^2} + \|\nabla\cdot\boldsymbol{u}\|_{L^2(D)} + \|\nabla\boldsymbol{u}\|_{L^2(D)^{2\times 2}} \\
&\lesssim \int_0^T \left(\left\|-\partial_t(\partial_t\eta + (\sigma_1+\sigma_2)\eta + \int_0^t \sigma_1\sigma_2\eta \mathrm{d}\tau)\right\|_{L^2(\Omega_2)}\right. \\
&\quad \left. + \|\nabla\partial_t\eta + \tilde{\sigma}_0\nabla\eta\|_{L^2(\Omega_2)^2}\right) \mathrm{d}t \\
&\lesssim \int_0^T \left(\|\partial_t^2\eta\|_{L^2(\Omega_2)} + \|\partial_t\eta\|_{L^2(\Omega_2)} + \|\eta\|_{L^2(\Omega_2)}\right. \\
&\quad \left. + \|\nabla\eta\|_{L^2(\Omega_2)^2} + \|\nabla\partial_t\eta\|_{L^2(\Omega_2)^2}\right) \mathrm{d}t.
\end{aligned}$$

It follows from (4.40)–(4.41) that we obtain the following estimate for the solution $(\hat{p}, \hat{p}^*, \boldsymbol{V}, \boldsymbol{v}^*, \boldsymbol{u})$ of (4.15):

$$\begin{aligned}
&\max_{t\in[0,T]}\left(\|\partial_t\hat{p}\|_{L_2(\Omega_2)} + \|\partial_t\boldsymbol{V}\|_{L^2(\Omega_2)^2} + \|\partial_t\hat{p}^*\|_{L_2(\Omega_2)} + \|\partial_t\boldsymbol{v}^*\|_{L^2(\Omega_2)^2}\right. \\
&\quad \left. + \|\partial_t\boldsymbol{u}\|_{L^2(D)^2} + \|\nabla\cdot\boldsymbol{u}\|_{L^2(D)} + \|\nabla\boldsymbol{u}\|_{L^2(D)^{2\times 2}}\right) \\
&\lesssim (1+T)\max_{t\in[0,T]}\left(\|\partial_t^2\eta\|_{H^1(\Omega_2)} + \|\partial_t\eta\|_{H^1(\Omega_2)} + \|\eta\|_{H^1(\Omega_2)}\right).
\end{aligned}$$

Using Lemma 4.8 gives

$$\max_{t\in[0,T]}\Big(\|\partial_t \hat{p}\|_{L_2(\Omega_2)} + \|\partial_t \boldsymbol{V}\|_{L^2(\Omega_2)^2} + \|\partial_t \hat{p}^*\|_{L_2(\Omega_2)} + \|\partial_t \boldsymbol{v}^*\|_{L^2(\Omega_2)^2}$$
$$+ \|\partial_t \boldsymbol{u}\|_{L^2(D)^2} + \|\nabla\cdot\boldsymbol{u}\|_{L^2(D)} + \|\nabla\boldsymbol{u}\|_{L^2(D)^{2\times 2}}\Big)$$
$$\lesssim (1+T)\max_{t\in[0,T]}\left(\|\partial_t^2 p^{\rm inc}\|_{H^{1/2}(\partial R_2)} + \|\partial_t p^{\rm inc}\|_{H^{1/2}(\partial R_2)} + \|p^{\rm inc}\|_{H^{1/2}(\partial R_2)}\right),$$

which completes the proof. □

5. Conclusion

We have studied the time-domain acoustic–elastic interaction problem in two dimensions where the acoustic wave equation and the elastic wave equation are coupled on the surface of the elastic obstacle. Using the exact TBC, we reduce the scattering problem from an open domain into an initial-boundary value problem in a bounded domain. We study the well-posedness and the stability for the variational problems in both the s-domain and the time-domain. The main ingredients of the proofs are the Laplace transform, the Lax–Milgram lemma, and the Parseval identity. We also obtain a priori estimates with explicit time dependence for the acoustic pressure and elastic displacement by taking special test functions to the time-domain variational problem. In addition, we introduce the PML formulation for the scattering problem. Computationally, the PML problem is more attractive than the original scattering problem because the nonlocal TBC is replaced by the local Dirichlet boundary condition. We derive a first order symmetric hyperbolic system for the truncated PML problem and show that the PML system is strongly well-posed. The stability of the truncted PML problem is also achieved by considering special test functions for the coupled acoustic and elastic wave equations.

The paper concerns only the two-dimensional problem. We believe that the method can be extended to solve the three-dimensional problem where the spherical harmonics need to be considered when deriving the TBC. The domain-time PML results are expected to be useful in the computational air/fluid–solid interaction problems. In particular, the first order symmetric hyperbolic system is readily to be solved numerically. It is also interesting to investigate the error of the solutions between the original PML problem and the truncated PML problem. We will report the work on the three-dimensional problem and numerical analysis and computation elsewhere in the future.

Acknowledgements. The authors wish to thank the anonymous reviewers for their constructive suggestions and comments on improving the presentation of the paper. The work of G. Bao is supported in part by a NSFC Innovative Group Fun (No. 11621101), an Integrated Project of the Major Research Plan of NSFC (No. 91630309), and an NSFC A3 Project (No. 11421110002), and the Fundamental Research Funds for the Central Universities. The research of Y. Gao was supported in part by NSFC Grants 11571065, 11671071, National Research program of China Grant 2013CB834100 and FRFCU2412017FZ005. The research of P. Li was supported in part by the NSF Grant DMS-1151308.

References

1. Appelö, D., Kreiss, G.: A new absorbing layer for elastic waves. *J. Comput. Phys.* **215**(2), 642–660, 2006

2. BERENGER, J.-P.: A perfectly matched layer for the absorption of electromagnetic waves. *J. Comput. Phys.* **114**(2), 185–200, 1994
3. BRAMBLE, J.H., PASCIAK, J.E.: Analysis of a finite PML approximation for the three dimensional time-harmonic Maxwell and acoustic scattering problems. *Math. Comp.* **76**, 597–614, 2007
4. CHEN, Q., MONK, P.: Discretization of the time domain CFIE for acoustic scattering problems using convolution quadrature. *SIAM J. Math. Anal.* **46**(5), 3107–3130, 2014
5. CHEN, S.: *Introduction to Modern Partial Differential Equations*. Science Press, Beijing, 2005
6. CHEN, Z.: Convergence of the time-domain perfectly matched layer method for acoustic scattering problems. *Int. J. Numer. Anal. Model.* **6**(1), 124–146, 2009
7. CHEN, Z., NÉDÉLEC, J.-C.: On Maxwell's equations with the transparent boundary condition. *J. Comput. Math.* **26**(3), 284–296, 2008
8. CHEN, Z., WU, X.: Long-time stability and convergence of the uniaxial perfectly matched layer method for time-domain acoustic scattering problems. *SIAM J. Numer. Anal.* **50**(5), 2632–2655, 2012
9. CHEN, Z., ZHENG, W.: Convergence of the uniaxial perfectly matched layer method for time-harmonic scattering problems in two-layered media. *SIAM J. Numer. Anal.* **48**(6), 2158–2185, 2010
10. CHEW, W., LIU, Q.: Perfectly matched layers for elastodynamics: a new absorbing boundary condition. *J. Comput. Acoust.* **4**(4), 341–359, 1996
11. CHEW, W., WEEDON, W.: A 3D perfectly matched medium for modified Maxwells equations with stretched coordinates. *Microw. Opt. Technol. Lett.* **13**, 599–604, 1994
12. COHEN, A.M.: *Numerical Methods for Laplace Transform Inversion*, vol. 5, Numerical Methods and Algorithms. Springer, New York, 2007
13. COLLINO, F., MONK, P.: The perfectly matched layer in curvilinear coordinates. *SIAM J. Sci. Comput.* **19**, 1061–1090, 1998
14. COLTON, D., KRESS, R.: *Inverse Acoustic and Electromagnetic Scattering Theory*, vol. 93, Applied Mathematical Sciences, 3rd edn. Springer, New York, 2013
15. COURJON, D.: *Near-Field Microscopy and Near-Field Optics*, vol. 317. World Scientific, Singapore, 2003
16. DALLAS, A.G.: *Analysis of a Limiting-Amplitude Problem in Acousto–Elastic Interactions*, Technical report, DTIC Document, 1989
17. DE HOOP, A.T., VAN DEN BERG, P.M., REMIS, R.F.: Absorbing boundary conditions and perfectly matched layers-analytic time-domain performance analysis. *IEEE Trans. Magn.* **38**(2), 657–660, 2002
18. DIAZ, J., JOLY, P.: A time domain analysis of PML models in acoustics. *Comput. Methods Appl. Mech. Eng.* **195**(29), 3820–3853, 2006
19. DONEA, J., GIULIANI, S., HALLEUX, J.-P.: An arbitrary Lagrangian-Eulerian finite element method for transient dynamic fluid-structure interactions. *Comput. Methods Appl. Mech. Eng.* **33**(1–3), 689–723, 1982
20. ESTORFF, O.V., ANTES, H.: On FEM-BEM coupling for fluid-structure interaction analyses in the time domain. *Int. J. Numer. Methods Eng.* **31**(6), 1151–1168, 1991
21. EVANS, L.C.: *Partial Differential Equations*, vol. 19, Graduate Studies in Mathematics, 2nd edn. American Mathematical Society, Providence, 2010
22. FAHY, F.J., GARDONIO, P.: *Sound and Structural Vibration: Radiation, Transmission and Response*. Academic Press, London, 2007
23. FATEMI, M., GREENLEAF, J.F.: Ultrasound-Stimulated Vibro-Acoustic Spectrography. *Science*, **280**(5360), 82–85, 1998

24. Flemisch, B., Kaltenbacher, M., Wohlmuth, B.I.: Elasto-acoustic and acoustic-acoustic coupling on non-matching grids. *Int. J. Numer. Methods Eng.* **67**(13), 1791–1810, 2006
25. Gao, Y., Li, P.: Analysis of time-domain scattering by periodic structures. *J. Differ. Equ.* **261**(9), 5094–5118, 2016
26. Gustafsson, B., Kreiss, H.O., Oliger, J.: *Time Dependent Problems and Difference Methods*. Wiley, New York, 1995
27. Hagstrom, T.: Radiation boundary conditions for the numerical simulation of waves. *Acta Numer.* **8**, 47–106, 1999
28. Hamdi, M., Jean, P.: *A Mixed Functional for the Numerical Resolution of Fluid-Structure Interaction Problems, Aero-and Hydro-Acoustics*, pp. 269–276. Springer, Berlin, 1986
29. Hsiao, G.C.: *On the Boundary-Field Equation Methods for Fluid-Structure Interactions, Problems and Methods in Mathematical Physics*, vol. 134, Teubner-Texte Mathematics, pp. 79–88. Teubner, Stuttgart, 1994
30. Hsiao, G.C., Kleinman, R.E., Schuetz, L.S.: *On Variational Formulations of Boundary Value Problems for Fluid–Solid Interactions, Elastic Wave Propagation*, vol. 35, North-Holland Series Applied Mathematics and Mechanics, pp. 321–326. North-Holland, Amsterdam, 1989
31. Hu, G., Kirsch, A., Yin, T.: Factorization method in inverse interaction problems with bi-periodic interfaces between acoustic and elastic waves. *Inverse Probl. Imaging* **10**(1), 103–129, 2016
32. Jin, J.-M., Riley, D.J.: *Finite Element Analysis of Antennas and Arrays*. Wiley, New York, 2009
33. Kucukcoban, S., Kallivokas, L.: Mixed perfectly-matched-layers for direct transient analysis in 2D elastic heterogeneous media. *Comput. Methods Appl. Mech. Eng.* **200**(1), 57–76, 2011
34. Li, J., Huang, Y.: *Time-Domain Finite Element Methods for Maxwell's Equations in Metamaterials*, vol. 43, Springer Series in Computational Mathematics. Springer, Heidelberg, 2013
35. Li, P., Wang, L.-L., Wood, A.: Analysis of transient electromagnetic scattering from a three-dimensional open cavity. *SIAM J. Appl. Math.* **75**(4), 1675–1699, 2015
36. Luke, C.J., Martin, P.A.: Fluid–solid interaction: acoustic scattering by a smooth elastic obstacle. *SIAM J. Appl. Math.* **55**(4), 904–922, 1995
37. Morand, H.J.-P., Ohayon, R.: *Fluid Structure Interaction*. Wiley, New York, 1995
38. Nédélec, J.-C.: *Acoustic and Electromagnetic Equations Integral Representations for Harmonic Problems*, vol. 144, Applied Mathematical Sciences. Springer, New York, 2001
39. Ohayon, R., Soize, C.: *Structural Acoustics and Vibration: Mechanical Models, Variational Formulations and Discretization*. Elsevier, Amsterdam, 1997
40. Qi, Q., Geers, T.L.: Evaluation of the perfectly matched layer for computational acoustics. *J. Comput. Phys.*, **139**(1), 166–183, 1998
41. Riley, D.J., Jin, J.-M.: Finite-element time-domain analysis of electrically and magnetically dispersive periodic structures. *IEEE Trans. Antennas Propag.* **56**(11), 3501–3509, 2008
42. Shirron, J.J., Giddings, T.E.: A finite element model for acoustic scattering from objects near a fluid–fluid interface. *Comput. Methods Appl. Mech. Eng.* **196**(1), 279–288, 2006
43. Soares, D., Mansur, W.: Dynamic analysis of fluid–soil-structure interaction problems by the boundary element method. *J. Comput. Phys.* **219**(2), 498–512, 2006
44. Trèves, F.: *Basic Linear Partial Differential Equations*, vol. 62, Pure and Applied Mathematics. Academic Press, New York, 1975
45. Turkel, E., Yefet, A.: Absorbing PML boundary layers for wave-like equations. *Appl. Numer. Math.* **27**, 533–557, 1998

46. UHLMANN, G.: *Inside Out: Inverse Problems and Applications*, vol. 47. Cambridge University Press, Cambridge, 2003
47. WANG, B., WANG, L.: On L^2-stability analysis of time-domain acoustic scattering problems with exact nonreflecting boundary conditions. *J. Math. Study* **1**(1), 65–84, 2014
48. WANG, L., WANG, B., ZHAO, X.: Fast and accurate computation of time-domain acoustic scattering problems with exact nonreflecting boundary conditions. *SIAM J. Appl. Math.* **72**(6), 1869–1898, 2012
49. YIN, T., HU, G., XU, L., ZHANG, B.: Near-field imaging of obstacles with the factorization method: fluid–solid interaction. *Inverse Probl.* **32**(1), 01500329, 2016
50. ZHANG, W., SHEN, Y.: Unsplit complex frequency-shifted pml implementation using auxiliary differential equations for seismic wave modeling. *Geophysics* **75**(4), T141–T154, 2010

GANG BAO
School of Mathematical Sciences,
Zhejiang University,
Hangzhou
310027,
China.
e-mail: baog@zju.edu.cn

and

YIXIAN GAO
School of Mathematics and Statistics, Center for Mathematics and Interdisciplinary Sciences,
Northeast Normal University,
Changchun,
130024
Jilin,
China.
e-mail: gaoyx643@nenu.edu.cn

and

PEIJUN LI
Department of Mathematics,
Purdue University,
West Lafayette,
IN
47907,
USA.
e-mail: lipeijun@math.purdue.edu

(*Received October 12, 2016 / Accepted February 14, 2018*)