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Stability on the one-dimensional inverse source scattering problem in a two-layered medium

Yue Zhao^a and Peijun Li^b

^aSchool of Mathematics and Statistics, Jiangsu Normal University, Xuzhou, China; ^bDepartment of Mathematics, Purdue University, West Lafayette, IN, USA

ABSTRACT

This paper concerns the stability on the inverse source scattering problem for the one-dimensional Helmholtz equation in a two-layered medium. We show that the increasing stability can be achieved using multi-frequency wave field at the two end points of the interval which contains the compact support of the source function.

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1. Introduction and problem formulation

We consider the one-dimensional Helmholtz equation in a two-layered medium:

$$u''(x, \omega) + \kappa^2(x)u(x, \omega) = f(x), \quad x \in (-1, 1), \quad (1.1)$$

where $\omega > 0$ is the angular frequency, the source function f has a compact support which is assumed to be contained in the interval $(-1, 1)$, and the wave number κ satisfies

$$\kappa(x) = \begin{cases} \kappa_1, & x > 0, \\ \kappa_2, & x < 0. \end{cases}$$

Here $\kappa_j = c_j\omega$, $j = 1, 2$, where $c_j > 0$ are constants. The wave field u is required to satisfy the outgoing wave conditions:

$$u'(-1, \omega) + i\kappa_2 u(-1, \omega) = 0, \quad u'(1, \omega) - i\kappa_1 u(1, \omega) = 0. \quad (1.2)$$

Given $f \in L^2(-1, 1)$, it is known that the problem (1.1)–(1.2) has a unique solution:

$$u(x, \omega) = \int_0^1 g(x, y)f(y)dy, \quad (1.3)$$

where g is the Green function given as follows

$$g(x, y) = \begin{cases} \frac{i}{2\kappa_1(\kappa_1+\kappa_2)} e^{i\kappa_1(x+y)} + \frac{i}{2\kappa_1} e^{i\kappa_1|x-y|}, & x > 0, \\ \frac{i}{\kappa_1+\kappa_2} e^{i(\kappa_1y-\kappa_2x)}, & x < 0, \end{cases} \quad \text{for } y > 0,$$

and

$$g(x, y) = \begin{cases} \frac{i}{2\kappa_2(\kappa_1+\kappa_2)} e^{-i\kappa_2(x+y)} + \frac{i}{2\kappa_2} e^{i\kappa_2|x-y|}, & x < 0, \\ \frac{i}{\kappa_1+\kappa_2} e^{i(-\kappa_2y+\kappa_1x)}, & x > 0, \end{cases} \quad \text{for } y < 0.$$

This paper concerns the inverse source problem: Let f be a complex function with a compact support contained in $(-1, 1)$. The inverse problem is to determine f using the boundary data $u(-1, \omega)$ and $u(1, \omega)$ with $\omega \in (0, K)$ where $K > 1$ is a positive constant.

The inverse source scattering problem has significant applications in antenna synthesis, medical imaging, and optical tomography [1,2]. They been extensively investigated by many researchers [3–8]. It is known that there is no uniqueness for the inverse source problems at a fixed frequency due to the existence of non-radiating sources [9,10]. Recently, it has been realized that the use of multi-frequency data cannot only overcome the difficulties of non-uniqueness, which are presented at a single frequency, but also achieve increasing stability [11–15]. These work assume that the medium is homogeneous in the whole space. In this work, we intend to establish the increasing stability on the inverse source problem for the one-dimensional Helmholtz equation in a two-layered medium.

2. Main result

Define a functional space:

$$\mathcal{F}_M = \{f \in H^n(-1, 1) : \|f\|_{H^n(-1,1)} \leq M, \text{supp}f \subset (-1, 1)\},$$

where $n \in \mathbb{N}$ and $M > 1$ is a constant. Hereafter, the notation " $a \lesssim b$ " stands for $a \leq Cb$, where C is a generic constant independent of n, ω, K, M , but may change step by step in the proofs.

The following stability estimate is the main result of this paper.

Theorem 2.1: *Let $f \in \mathcal{F}_M$ and let u be the solution (1.3) corresponding to f . Then we have*

$$\|f\|_{L^2(-1,1)}^2 \lesssim \epsilon^2 + \frac{M^2}{\left(\frac{K^{\frac{2}{3}}|\ln \epsilon|^{\frac{1}{4}}}{(6n-3)^3}\right)^{2n-1}}, \tag{2.1}$$

where

$$\epsilon = \left(\int_0^K \omega^2 (|u(-1, \omega)|^2 + |u(1, \omega)|^2) d\omega \right)^{\frac{1}{2}}. \tag{2.2}$$

Remark 2.2: The stability estimate (2.1) consists of two parts: the data discrepancy and the high frequency tail. The former is of the Lipschitz type. The latter decreases as K increases which makes the problem have an almost Lipschitz stability. The result explains that the problem becomes more stable when higher frequency data are used. The stability estimate (2.1) also implies the uniqueness of the inverse source problem.

3. Proof of theorem 2.1

Consider the following two functions:

$$f_1(x) = \begin{cases} f(x), & x > 0, \\ 0, & x < 0, \end{cases} \quad \text{and} \quad f_2(x) = \begin{cases} 0, & x > 0, \\ f(x), & x < 0. \end{cases} \tag{3.1}$$

Lemma 3.1: *Let $f \in L^2(-1, 1)$ with $\text{supp } f \subset (-1, 1)$. We have*

$$\|f\|_{L^2(-1,1)}^2 \lesssim \int_0^{+\infty} \omega^2 (|u(-1, \omega)|^2 + |u(1, \omega)|^2) d\omega.$$

Proof: Choosing $\xi_1 \in \mathbb{R}$ with $|\xi_1| = \kappa_1$, multiplying both sides of (1.1) by $e^{-i\xi_1 x}$, and integrating over $(0, 1)$ with respect to x , we have from the integration by parts that

$$e^{-i\xi_1} u'(1, \omega) + i\xi_1 e^{-i\xi_1} u(1, \omega) - u'(0, \omega) - i\xi_1 u(0, \omega) = \int_0^1 e^{-i\xi_1 x} f_1(x) dx. \tag{3.2}$$

Similarly, choosing $\xi_2 \in \mathbb{R}$ with $|\xi_2| = \kappa_2$, multiplying both sides of (1.1) by $e^{-i\xi_2 x}$, and integrating over $(-1, 0)$ with respect to x , we have from the integration by parts that

$$-e^{-i\xi_2} u'(-1, \omega) - i\xi_2 e^{i\xi_2} u(-1, \omega) + u'(0, \omega) + i\xi_2 u(0, \omega) = \int_{-1}^0 e^{-i\xi_2 x} f_2(x) dx. \tag{3.3}$$

It follows from (1.3) and (3.1) that

$$u(x, \omega) = \int_0^1 g(x, y) f(y) dy = \int_0^1 g(x, y) f_1(y) dy + \int_{-1}^0 g(x, y) f_2(y) dy,$$

which gives

$$\begin{aligned} u(0, \omega) &= \int_0^1 g(0, y) f_1(y) dy + \int_{-1}^0 g(0, y) f_2(y) dy \\ &= \int_0^1 \frac{i}{\kappa_1 + \kappa_2} e^{i\kappa_1 y} f_1(y) dy + \int_{-1}^0 \frac{i}{\kappa_1 + \kappa_2} e^{-i\kappa_2 y} f_2(y) dy. \end{aligned} \tag{3.4}$$

On the other hand, we have from a simple calculation that

$$\begin{aligned} u'(0, \omega) &= \int_0^1 g'(0, y) f_1(y) dy + \int_{-1}^0 g'(0, y) f_2(y) dy \\ &= \int_0^1 \frac{\kappa_2}{\kappa_1 + \kappa_2} e^{i\kappa_1 y} f_1(y) dy + \int_{-1}^0 \frac{-\kappa_1}{\kappa_1 + \kappa_2} e^{-i\kappa_2 y} f_2(y) dy. \end{aligned} \tag{3.5}$$

Letting $\xi_1 = -\kappa_1$, we have from (3.4) and (3.5) that

$$u'(0, \omega) - i\kappa_1 u(0, \omega) = \int_0^1 e^{i\kappa_1 y} f_1(y) dy. \tag{3.6}$$

Combining (3.6) and (3.2), we obtain

$$e^{i\kappa_1} u'(1, \omega) - i\kappa_1 e^{i\kappa_1} u(1, \omega) = 2 \int_0^1 e^{i\kappa_1 x} f_1(x) dx,$$

Using the outgoing radiation condition (1.2), we get from the above equation that

$$e^{i\kappa_1} i\kappa_1 u(1, \omega) - i\kappa_1 e^{i\kappa_1} u(1, \omega) = 2 \int_0^1 e^{i\kappa_1 x} f_1(x) dx,$$

which implies

$$|\hat{f}_1(-\kappa_1)|^2 \lesssim \omega^2 |u(\omega, 1)|^2. \tag{3.7}$$

Letting $\xi_2 = \kappa_2$, we have from (3.4) and (3.5) that

$$u'(0, \omega) + i\kappa_2 u(0, \omega) = - \int_{-1}^0 e^{-i\kappa_2 y} f_2(y) dy, \tag{3.8}$$

Combining (3.8), (3.3), and (1.2), we obtain

$$e^{-i\kappa_2} i\kappa_2 u'(-1, \omega) - i\kappa_2 e^{i\kappa_2} u(-1, \omega) = 2 \int_{-1}^0 e^{-i\kappa_2 x} f_2(x) dx,$$

which shows

$$|\hat{f}_2(\kappa_2)|^2 \lesssim \omega^2 |u(\omega, -1)|^2. \tag{3.9}$$

Letting $\xi_1 = \kappa_1$, we get from (3.4) and (3.5) that

$$\begin{aligned} u'(0, \omega) + i\kappa_1 u(0, \omega) &= \int_0^1 \frac{\kappa_2 - \kappa_1}{\kappa_1 + \kappa_2} e^{i\kappa_1 y} f_1(y) dy - \int_{-1}^0 \frac{2\kappa_1}{\kappa_1 + \kappa_2} e^{-i\kappa_2 y} f_2(y) dy \\ &= \frac{\kappa_2 - \kappa_1}{\kappa_1 + \kappa_2} \hat{f}_1(-\kappa_1) - \frac{2\kappa_1}{\kappa_1 + \kappa_2} \hat{f}_2(\kappa_2). \end{aligned} \tag{3.10}$$

It follows from (3.10), (3.2), and (1.2) that we obtain

$$e^{-i\kappa_1} i\kappa_1 u(1, \omega) + i\kappa_1 e^{-i\kappa_1} u(1, \omega) - \frac{\kappa_2 - \kappa_1}{\kappa_1 + \kappa_2} \hat{f}_1(-\kappa_1) + \frac{2\kappa_1}{\kappa_1 + \kappa_2} \hat{f}_2(\kappa_2) = \hat{f}_1(\kappa_1),$$

which means

$$|\hat{f}_1(\kappa_1)|^2 \lesssim \omega^2 |u(1, \omega)| + |\hat{f}_1(-\kappa_1)|^2 + |\hat{f}_2(\kappa_2)|^2. \tag{3.11}$$

Finally, letting $\xi_2 = -\kappa_2$, we have from (3.4) and (3.5) that

$$\begin{aligned} u'(0, \omega) - i\kappa_2 u(0, \omega) &= \int_0^1 \frac{2\kappa_2}{\kappa_1 + \kappa_2} e^{i\kappa_1 y} f_1(y) dy + \int_{-1}^0 \frac{\kappa_2 - \kappa_1}{\kappa_1 + \kappa_2} e^{-i\kappa_2 y} f_2(y) dy \\ &= \frac{2\kappa_2}{\kappa_1 + \kappa_2} \hat{f}_1(-\kappa_1) + \frac{\kappa_2 - \kappa_1}{\kappa_1 + \kappa_2} \hat{f}_2(\kappa_2). \end{aligned} \tag{3.12}$$

Using (3.12), (3.3), and (1.2), we have

$$e^{i\kappa_2} i\kappa_2 u(-1, \omega) + i\kappa_2 e^{-i\kappa_2} u(-1, \omega) + \frac{2\kappa_2}{\kappa_1 + \kappa_2} \hat{f}_1(-\kappa_1) + \frac{\kappa_2 - \kappa_1}{\kappa_1 + \kappa_2} \hat{f}_2(\kappa_2) = \hat{f}_2(-\kappa_2),$$

which means

$$|\hat{f}_2(-\kappa_2)|^2 \lesssim \omega^2 |u(-1, \omega)|^2 + |\hat{f}_1(-\kappa_1)| + |\hat{f}_2(\kappa_2)|. \tag{3.13}$$

Therefore, it follows from (3.7) that we get

$$\int_0^\infty |\hat{f}_1(-\kappa_1)|^2 d\omega \lesssim \int_0^\infty \omega^2 |u(\omega, 1)|^2 d\omega. \tag{3.14}$$

Using (3.9) gives

$$\int_0^\infty |\hat{f}_2(\kappa_2)|^2 d\omega \lesssim \int_0^\infty \omega^2 |u(-1, \omega)|^2 d\omega. \tag{3.15}$$

It follows from (3.11), (3.14), and (3.15) that we have

$$\int_0^\infty |\hat{f}_1(\kappa_1)|^2 d\omega \lesssim \int_0^\infty \omega^2 |u(-1, \omega)|^2 d\omega + \int_0^\infty \omega^2 |u(1, \omega)|^2 d\omega; \tag{3.16}$$

Finally following from (3.13), (3.14), and (3.15), we obtain

$$\int_0^\infty |\hat{f}_2(-\kappa_2)|^2 d\omega \lesssim \int_0^\infty \omega^2 |u(-1, \omega)|^2 d\omega + \int_0^\infty \omega^2 |u(1, \omega)|^2 d\omega. \tag{3.17}$$

We obtain from the Plancherel theorem that

$$\|f\|_{L^2(-1,1)}^2 = \|f\|_{L^2(-\infty,\infty)}^2 = \|\hat{f}\|_{L^2(-\infty,\infty)}^2 \lesssim \|\hat{f}_1\|_{L^2(-\infty,\infty)}^2 + \|\hat{f}_2\|_{L^2(-\infty,\infty)}^2.$$

On the other hand, we have

$$\|\hat{f}_1\|_{L^2(-\infty,\infty)}^2 = \int_0^\infty |\hat{f}_1(\omega)|^2 d\omega + \int_0^\infty |\hat{f}_1(-\omega)|^2 d\omega.$$

Using (3.14) and (3.16) yields

$$\|\hat{f}_1\|_{L^2(-\infty,\infty)}^2 \lesssim \int_0^{+\infty} \omega^2 (|u(-1, \omega)|^2 + |u(1, \omega)|^2) d\omega.$$

Similarly, we have from (3.15) and (3.17) that

$$\|\hat{f}_2\|_{L^2(-\infty,\infty)}^2 \lesssim \int_0^{+\infty} \omega^2 (|u(-1, \omega)|^2 + |u(1, \omega)|^2) d\omega.$$

The proof is completed by combining the above estimates. □

Lemma 3.2: *Let $f \in L^2(-1, 1)$. We have*

$$\begin{aligned} \omega^2 |u(-1, \omega)|^2 &\lesssim \left| \int_0^1 e^{ic_1\omega y} f_1(y) dy \right|^2 + \left| \int_{-1}^0 e^{-ic_2\omega y} f_2(y) dy \right|^2 + \left| \int_{-1}^0 e^{ic_2\omega y} f_2(y) dy \right|^2, \\ \omega^2 |u(1, \omega)|^2 &\lesssim \left| \int_0^1 e^{ic_1\omega y} f_1(y) dy \right|^2 + \left| \int_0^1 e^{-ic_1\omega y} f_1(y) dy \right|^2 + \left| \int_{-1}^0 e^{-ic_2\omega y} f_2(y) dy \right|^2. \end{aligned}$$

Proof: It follows from (1.3) that we have

$$\begin{aligned} \omega u(-1, \omega) &= \int_0^1 \frac{i}{c_1 + c_2} e^{i(c_1 \omega y + c_2 \omega)} f_1(y) dy + \int_{-1}^0 \frac{i(c_2 - c_1)}{2c_2(c_1 + c_2)} e^{-ic_2 \omega(-1+y)} f_2(y) dy \\ &\quad + \int_{-1}^0 \frac{i}{2c_2} e^{-ic_2 \omega(-1-y)} f_2(y) dy \end{aligned}$$

and

$$\begin{aligned} \omega u(1, \omega) &= \int_0^1 \frac{i(c_1 - c_2)}{2c_1(c_1 + c_2)} e^{ic_1 \omega(1+y)} f_1(y) dy + \int_0^1 \frac{i}{2c_1} e^{ic_1 \omega(1-y)} f_1(y) dy \\ &\quad + \int_{-1}^0 \frac{i}{c_1 + c_2} e^{i(-c_2 \omega y + c_1 \omega)} f_2(y) dy. \end{aligned}$$

The proof is done by taking square of the amplitudes on both sides of the above equations. □

Next, let

$$I(s) = I_1(s) + I_2(s),$$

where

$$I_1(s) = \omega^2 \int_0^s |u(-1, \omega)|^2 d\omega, \quad I_2(s) = \omega^2 \int_0^s |u(1, \omega)|^2 d\omega.$$

We have the following explicit representations for $I_1(s)$ and $I_2(s)$:

$$\begin{aligned} I_1(s) &= \int_0^s \left| \int_0^1 \frac{1}{c_1 + c_2} e^{i(c_1 \omega y + c_2 \omega)} f_1(y) dy + \int_{-1}^0 \frac{c_2 - c_1}{2c_2(c_1 + c_2)} e^{-ic_2 \omega(-1+y)} f_2(y) dy \right. \\ &\quad \left. + \int_{-1}^0 \frac{1}{2c_2} e^{-ic_2 \omega(-1-y)} f_2(y) dy \right|^2 d\omega \end{aligned} \tag{3.18}$$

and

$$\begin{aligned} I_2(s) &= \int_0^s \left| \int_0^1 \frac{c_1 - c_2}{2c_1(c_1 + c_2)} e^{ic_1 \omega(1+y)} f_1(y) dy + \int_0^1 \frac{1}{2c_1} e^{ic_1 \omega(1-y)} f_1(y) dy \right. \\ &\quad \left. + \int_{-1}^0 \frac{1}{c_1 + c_2} e^{i(-c_2 \omega y + c_1 \omega)} f_2(y) dy \right|^2 d\omega. \end{aligned} \tag{3.19}$$

Lemma 3.3: Let $f \in L^2(-1, 1)$ and $c_{\max} = \max\{c_1, c_2\}$. We have for any $s = s_1 + is_2, s_1, s_2 \in \mathbb{R}$ that

$$\begin{aligned} |I_1(s)| &\lesssim |s| e^{4c_{\max}|s_2|} \int_0^1 |f(y)|^2 dy, \\ |I_2(s)| &\lesssim |s| e^{4c_{\max}|s_2|} \int_0^1 |f(y)|^2 dy. \end{aligned}$$

Proof: Let $\omega = st, t \in (0, 1)$. A simple calculation yields

$$\begin{aligned} I_1(s) &= s \int_0^1 \left| \int_0^1 \frac{1}{c_1 + c_2} e^{i(c_1 sty + c_2 st)} f_1(y) dy + \int_{-1}^0 \frac{c_2 - c_1}{2c_2(c_1 + c_2)} e^{-ic_2 st(-1+y)} f_2(y) dy \right. \\ &\quad \left. + \int_{-1}^0 \frac{1}{2c_2} e^{-ic_2 st(-1-y)} f_2(y) dy \right|^2 dt \end{aligned}$$

and

$$I_2(s) = s \int_0^1 \left| \int_0^1 \frac{c_1 - c_2}{2c_1(c_1 + c_2)} e^{ic_1st(1+y)} f_1(y) dy + \int_0^1 \frac{1}{2c_1} e^{ic_1st(1-y)} f_1(y) dy + \int_{-1}^0 \frac{1}{c_1 + c_2} e^{i(-c_2sty+c_1st)} f_2(y) dy \right|^2 dt.$$

Noting

$$|e^{\pm ic_1sty+c_2st}| \leq e^{2c_{\max}|s_2|}, \quad |e^{\pm ic_2st(-1\pm y)}| \leq e^{2c_{\max}|s_2|}, \quad \forall y \in (-1, 1),$$

we have from the Schwartz inequality that

$$|I_1(s)| \lesssim |s| e^{4c_{\max}|s_2|} \int_{-1}^1 |f(y)|^2 dy.$$

Similarly noting

$$|e^{\pm ic_1st(1\pm y)}| \leq e^{2c_{\max}|s_2|}, \quad |e^{\pm i(-c_2sty+c_1st)}| \leq e^{2c_{\max}|s_2|}, \quad \forall y \in (-1, 1),$$

we get from the Schwartz inequality that

$$|I_2(s)| \lesssim |s| e^{4c_{\max}|s_2|} \int_{-1}^1 |f(y)|^2 dy,$$

which completes the proof. □

Lemma 3.4: *Let $f \in H^n(-1, 1)$, $\text{supp} f \subset (-1, 1)$. We have for any $s > 0$ that*

$$\int_s^\infty \omega^2 (|u(-1, \omega)|^2 + |u(1, \omega)|^2) d\omega \lesssim s^{-(2n-1)} \|f\|_{H^n(-1,1)}^2.$$

Proof: It follows from Lemma 3.2 that we have

$$\begin{aligned} & \int_s^\infty \omega^2 |u(-1, \omega)|^2 d\omega + \int_s^\infty \omega^2 |u(1, \omega)|^2 d\omega \\ & \lesssim \int_s^\infty \left| \int_0^1 e^{ic_1\omega y} f_1(y) dy \right|^2 d\omega + \int_s^\infty \left| \int_0^1 e^{-ic_1\omega y} f_1(y) dy \right|^2 d\omega \\ & \quad + \int_s^\infty \left| \int_{-1}^0 e^{ic_2\omega y} f_2(y) dy \right|^2 d\omega + \int_s^\infty \left| \int_{-1}^0 e^{-ic_2\omega y} f_2(y) dy \right|^2 d\omega. \end{aligned}$$

Using the integration by parts and noting $\text{supp} f_1 \subset (0, 1)$ and $\text{supp} f_2 \subset (-1, 0)$, we obtain

$$\int_0^1 e^{\pm ic_1\omega y} f_1(y) dy = \frac{1}{(\pm ic_1\omega)^n} \int_0^1 e^{\pm ic_1\omega y} f_1^{(n)}(y) dy$$

and

$$\int_{-1}^0 e^{\pm ic_2\omega y} f_2(y) dy = \frac{1}{(\pm ic_2\omega)^n} \int_{-1}^0 e^{\pm ic_2\omega y} f_2^{(n)}(y) dy,$$

which give

$$\left| \int_0^1 e^{\pm ic_1\omega y} f_1(y) dy \right|^2 \lesssim c_1^{-2n} \omega^{-2n} \|f_1\|_{H^n(0,1)}^2 \lesssim c_1^{-2n} \omega^{-2n} \|f\|_{H^n(-1,1)}^2$$

and

$$\left| \int_{-1}^0 e^{\pm ic_2 \omega y} f_2(y) dy \right|^2 \lesssim c_2^{-2n} \omega^{-2n} \|f_2\|_{H^n(-1,0)}^2 \lesssim c_2^{-2n} \omega^{-2n} \|f\|_{H^n(-1,1)}^2.$$

Hence we have

$$\int_s^\infty \left| \int_0^1 e^{\pm ic_1 \omega y} f_1(y) dy \right|^2 d\omega \lesssim c_1^{-2n} \|f_1\|_{H^n(0,1)}^2 \int_s^\infty \omega^{-2n} d\omega \lesssim c_1^{-2n} \frac{s^{-(2n-1)}}{(2n-1)} \|f\|_{H^n(-1,1)}^2$$

and

$$\int_s^\infty \left| \int_{-1}^0 e^{\pm ic_2 \omega y} f_2(y) dy \right|^2 d\omega \lesssim c_2^{-2n} \|f_2\|_{H^n(-1,0)}^2 \int_s^\infty \omega^{-2n} d\omega \lesssim c_2^{-2n} \frac{s^{-(2n-1)}}{(2n-1)} \|f\|_{H^n(-1,1)}^2,$$

which completes the proof. □

The following lemma is proved in [13].

Lemma 3.5: Denote $S = \{z = x + iy \in \mathbb{C} : -\frac{\pi}{4} < \arg z < \frac{\pi}{4}\}$. Let $J(z)$ be analytic in S and continuous in \bar{S} satisfying

$$\begin{cases} |J(z)| \leq \epsilon, & z \in (0, L], \\ |J(z)| \leq V, & z \in S, \\ |J(0)| = 0. \end{cases}$$

Then there exists a function $\mu(z)$ satisfying

$$\begin{cases} \mu(z) \geq \frac{1}{2}, & z \in (L, 2^{\frac{1}{4}}L), \\ \mu(z) \geq \frac{1}{\pi} \left(\left(\frac{z}{L} \right)^4 - 1 \right)^{-\frac{1}{2}}, & z \in (2^{\frac{1}{4}}L, \infty) \end{cases}$$

such that

$$|J(z)| \leq V \epsilon^{\mu(z)}, \quad \forall z \in (L, \infty).$$

Lemma 3.6: Let $f \in \mathcal{F}_M$. Then there exists a function $\mu(s)$ satisfying

$$\begin{cases} \mu(s) \geq \frac{1}{2}, & s \in (K, 2^{\frac{1}{4}}K), \\ \mu(s) \geq \frac{1}{\pi} \left(\left(\frac{s}{K} \right)^4 - 1 \right)^{-\frac{1}{2}}, & s \in (2^{\frac{1}{4}}K, \infty), \end{cases} \tag{3.20}$$

such that

$$|I(s)| \lesssim M^2 e^{as} \epsilon^{2\mu(s)}, \quad \forall s \in (K, \infty),$$

where $a = \max\{5c, 3\}$.

Proof: It follows from Lemma 3.3 that

$$|I_1(s)e^{-as}| \lesssim M^2, \quad |I_2(s)e^{-as}| \lesssim M^2, \quad s \in S.$$

Recalling (2.2), (3.18), and (3.19), we have

$$|I_1(s)e^{-as}| \lesssim \epsilon^2, \quad |I_2(s)e^{-as}| \lesssim \epsilon^2, \quad s \in [0, K].$$

A direct application of Lemma 3.5 shows that there exists a function $\mu(s)$ satisfying (3.20) such that

$$|I_1(s)e^{-as}| \lesssim M^2 \epsilon^{2\mu}, \quad |I_2(s)e^{-as}| \lesssim M^2 \epsilon^{2\mu}, \quad s \in (K, \infty),$$

then we have

$$|I(s)e^{-as}| = |I_1(s)e^{-as} + I_2(s)e^{-as}| \lesssim M^2 \epsilon^{2\mu}, \quad s \in (K, \infty),$$

which completes the proof. □

Now we show the proof of Theorem 2.1.

Proof: We can assume that $\epsilon < e^{-1}$, otherwise the estimate is obvious. Let

$$s = \begin{cases} \frac{1}{(3\pi)^{\frac{1}{3}}} K^{\frac{2}{3}} |\ln \epsilon|^{\frac{1}{4}}, & 2^{\frac{1}{4}} (3\pi)^{\frac{1}{3}} K^{\frac{1}{3}} < |\ln \epsilon|^{\frac{1}{4}}, \\ K, & |\ln \epsilon| \leq 2^{\frac{1}{4}} (3\pi)^{\frac{1}{3}} K^{\frac{1}{3}}. \end{cases}$$

If $2^{\frac{1}{4}} (3\pi)^{\frac{1}{3}} K^{\frac{1}{3}} < |\ln \epsilon|^{\frac{1}{4}}$, then we have

$$\begin{aligned} |I(s)| &\lesssim M^2 e^{as} e^{-\frac{2|\ln \epsilon|}{\pi} ((\frac{a}{K})^4 - 1)^{-\frac{1}{2}}} \lesssim M^2 e^{\frac{-a}{1} K^{\frac{2}{3}} |\ln \epsilon|^{\frac{1}{4}} - \frac{2|\ln \epsilon|}{\pi} (\frac{K}{s})^2} \\ &= M^2 e^{-2(\frac{a^3}{3\pi})^{\frac{1}{3}} K^{\frac{2}{3}} |\ln \epsilon|^{\frac{1}{2}} \left(1 - \frac{1}{2} |\ln \epsilon|^{-\frac{1}{4}}\right)}. \end{aligned}$$

Noting $\frac{1}{2} |\ln \epsilon|^{-\frac{1}{4}} < \frac{1}{2}$ and $a \geq 3$, we have $\left(\frac{a^3}{3\pi}\right)^{\frac{1}{3}} \geq \left(\frac{3^3}{3\pi}\right)^{\frac{1}{3}} > 1$ and

$$|I(s)| \lesssim M^2 e^{-K^{\frac{2}{3}} |\ln \epsilon|^{\frac{1}{2}}}.$$

Using the elementary inequality

$$e^{-x} \leq \frac{(6n-3)!}{x^{3(2n-1)}}, \quad x > 0,$$

we get

$$|I(s)| \lesssim \frac{M^2}{\left(\frac{K^2 |\ln \epsilon|^{\frac{3}{2}}}{(6n-3)^3}\right)^{2n-1}}.$$

If $|\ln \epsilon| \leq 2^{\frac{1}{4}} (3\pi)^{\frac{1}{3}} K^{\frac{1}{3}}$, then $s = K$. We have from (2.2) and Lemma 3.2 that

$$|I(s)| \leq \epsilon^2.$$

Hence we obtain from Lemma 3.4 that

$$\begin{aligned} &\int_0^\infty \omega^2 (|u(-1, \omega)|^2 + |u(1, \omega)|^2) d\omega \\ &\lesssim \epsilon^2 + \frac{M^2}{\left(\frac{K^2 |\ln \epsilon|^{\frac{3}{2}}}{(6n-3)^3}\right)^{2n-1}} + \frac{\|f\|_{H^n(-1,1)}^2}{\left(2^{-\frac{1}{4}} (3\pi)^{-\frac{1}{3}} K^{\frac{2}{3}} |\ln \epsilon|^{\frac{1}{4}}\right)^{2n-1}}. \end{aligned}$$

By Lemma 3.1, we have

$$\|f\|_{L^2(-1,1)}^2 \lesssim \epsilon^2 + \frac{M^2}{\left(\frac{K^2 |\ln \epsilon|^{\frac{3}{2}}}{(6n-3)^3}\right)^{2n-1}} + \frac{M^2}{\left(\frac{K^{\frac{2}{3}} |\ln \epsilon|^{\frac{1}{4}}}{(6n-3)^3}\right)^{2n-1}}.$$

Since $K^{\frac{2}{3}} |\ln \epsilon|^{\frac{1}{4}} \leq K^2 |\ln \epsilon|^{\frac{3}{2}}$ when $K > 1$ and $|\ln \epsilon| > 1$, we obtain the stability estimate. □

4. Conclusion

In this paper, we show that the increasing stability can be obtained for the inverse source scattering problem of the one-dimensional Helmholtz equation in a two-layered medium using multi-frequency Dirichlet data at the two end points of an interval which contains the compact support of the source. The stability estimate consists of the data discrepancy and the high frequency tail of the source function. We believe that the proposed method can be extended to handle a multi-layered medium. Another possible future work is to investigate the higher dimensional problem.

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Appendix 1. Green's function in a two-layered medium

Consider the equation

$$\frac{d^2 g(x, y)}{dx^2} + \kappa^2(x)g(x, y) = -\delta(x - y), \quad (\text{A1})$$

where δ is the Dirac delta function and the wavenumber κ is a piecewise constant, i.e.

$$\kappa(x) = \begin{cases} \kappa_1, & x > 0, \\ \kappa_2, & x < 0. \end{cases}$$

If $y > 0$, the solution of (A1) has the following form

$$g(x, y) = \begin{cases} Ae^{i\kappa_1 x}, & x > y, \\ Be^{i\kappa_1 x} + Ce^{-i\kappa_1 x}, & 0 < x < y, \\ De^{-i\kappa_2 x}, & x < 0, \end{cases}$$

where A, B, C, D are to be determined. Using the continuity conditions

$$\begin{cases} g(x, y)|_{x=y^+} = g(x, y)|_{x=y^-}, \\ \frac{dg(x, y)}{dx}|_{x=y^+} - \frac{dg(x, y)}{dx}|_{x=y^-} = -1, \\ g(x, y)|_{x=0^+} = g(x, y)|_{x=0^-}, \\ \frac{dg(x, y)}{dx}|_{x=0^+} = \frac{dg(x, y)}{dx}|_{x=0^-}, \end{cases}$$

we get a linear system:

$$\begin{cases} Ae^{i\kappa_1 y} = Be^{i\kappa_1 y} + Ce^{-i\kappa_1 y}, \\ i\kappa_1 Ae^{i\kappa_1 y} - i\kappa_1 Be^{i\kappa_1 y} + i\kappa_1 Ce^{-i\kappa_1 y} = -1, \\ B + C = D, \\ i\kappa_1 B - i\kappa_1 C = -i\kappa_2 D. \end{cases}$$

A simple calculation yields that

$$\begin{cases} A = i \frac{\kappa_1 - \kappa_2}{2\kappa_1(\kappa_1 + \kappa_2)} e^{i\kappa_1 y} + \frac{i}{2\kappa_1} e^{-i\kappa_1 y}, \\ B = i \frac{\kappa_1 - \kappa_2}{2\kappa_1(\kappa_1 + \kappa_2)} e^{i\kappa_1 y}, \\ C = \frac{i}{2\kappa_1} e^{i\kappa_1 y}, \\ D = \frac{i}{\kappa_1 + \kappa_2} e^{i\kappa_1 y}, \end{cases}$$

which gives

$$g(x, y) = \begin{cases} i \frac{\kappa_1 - \kappa_2}{2\kappa_1(\kappa_1 + \kappa_2)} e^{i\kappa_1(x+y)} + \frac{i}{2\kappa_1} e^{i\kappa_1|x-y|}, & x > 0, \\ \frac{i}{\kappa_1 + \kappa_2} e^{i(\kappa_1 y - \kappa_2 x)}, & x < 0. \end{cases}$$

If $y < 0$, the solution has the following form

$$g(x, y) = \begin{cases} Ae^{-i\kappa_2 x}, & x < y, \\ Be^{-i\kappa_2 x} + Ce^{i\kappa_2 x}, & y < x < 0, \\ De^{i\kappa_1 x}, & x > 0. \end{cases}$$

Using the continuity conditions

$$\begin{cases} g(x, y)|_{x=y^+} = g(x, y)|_{x=y^-}, \\ \frac{dg(x, y)}{dx}|_{x=y^+} - \frac{dg(x, y)}{dx}|_{x=y^-} = -1, \\ g(x, y)|_{x=0^+} = g(x, y)|_{x=0^-}, \\ \frac{dg(x, y)}{dx}|_{x=0^+} = \frac{dg(x, y)}{dx}|_{x=0^-}, \end{cases}$$

we obtain

$$\begin{cases} Ae^{-i\kappa_2 y} = Be^{-i\kappa_2 y} + Ce^{i\kappa_2 y}, \\ -i\kappa_2 Ae^{-i\kappa_2 y} + i\kappa_2 Be^{-i\kappa_2 y} - i\kappa_2 Ce^{i\kappa_2 y} = 1, \\ B + C = D, \\ -i\kappa_2 B + i\kappa_2 C = i\kappa_1 D. \end{cases}$$

It follows from solving the above linear system that

$$\begin{cases} A = i \frac{\kappa_2 - \kappa_1}{2\kappa_2(\kappa_1 + \kappa_2)} e^{-i\kappa_2 y} + \frac{i}{2\kappa_2} e^{i\kappa_2 y}, \\ B = i \frac{\kappa_2 - \kappa_1}{2\kappa_2(\kappa_1 + \kappa_2)} e^{-i\kappa_2 y}, \\ C = \frac{i}{2\kappa_2} e^{-i\kappa_2 y}, \\ D = \frac{i}{\kappa_1 + \kappa_2} e^{-i\kappa_2 y}, \end{cases}$$

which yields

$$g(x, y) = \begin{cases} i \frac{\kappa_2 - \kappa_1}{2\kappa_2(\kappa_1 + \kappa_2)} e^{-i\kappa_2(x+y)} + \frac{i}{2\kappa_2} e^{i\kappa_2|x-y|}, & x < 0, \\ \frac{i}{\kappa_1 + \kappa_2} e^{i(-\kappa_2 y + \kappa_1 x)}, & x > 0. \end{cases}$$