DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS SERIES B
Volume **25**, Number **1**, January **2020**

doi:10.3934/dcdsb.2019181

pp. **259–286**

ANALYSIS OF TIME-DOMAIN MAXWELL'S EQUATIONS IN BIPERIODIC STRUCTURES

Gang Bao* and Bin Hu

School of Mathematical Sciences, Zhejiang University
Hangzhou 310027, China

Peijun Li

Department of Mathematics, Purdue University
West Lafayette, IN47907, USA

Jue Wang

School of Mathematical Sciences, Harbin Engineering University
Harbin 150001, China

(Communicated by Yuan Lou)

Abstract. This paper is devoted to the mathematical analysis of the diffraction of an electromagnetic plane wave by a biperiodic structure. The wave propagation is governed by the time-domain Maxwell equations in three dimensions. The method of a compressed coordinate transformation is proposed to reduce equivalently the diffraction problem into an initial-boundary value problem formulated in a bounded domain over a finite time interval. The reduced problem is shown to have a unique weak solution by using the constructive Galerkin method. The stability and a priori estimates with explicit time dependence are established for the weak solution.

1. **Introduction.** Consider the diffraction of an electromagnetic plane wave by a biperiodic structure, where the wave propagation is governed by the time-domain Maxwell equations in three dimensions. In optics, a biperiodic or doubly periodic structure is called a crossed grating or a two-dimensional grating. Scattering theory in periodic structures have many important applications in micro-optics, which include the design and fabrication of optical elements such as corrective lenses, antireflective interfaces, beam splitters, and sensors. The basic electromagnetic theory of gratings can be traced back to Rayleigh's time [36]. Recent advance has been greatly accelerated due to the development of new mathematical and numerical methods including analytical or approximation methods, differential, integral, or variational methods, and many others. The time-harmonic problems, where electromagnetic waves oscillate sinusoidally with respect to the time, have been well studied. A great deal of results are available. An introduction to the grating problems can be found in Petit [35]. We refer to [3, 7, 13, 17, 20, 21, 32, 34] for the mathematical studies on the well-posedness of the diffractive grating problems. Numerical methods can be

2010 *Mathematics Subject Classification.* 78A45, 35A15, 35Q60.
Key words and phrases. Time-domain Maxwell's equations, biperiodic structures, diffraction gratings, well-posedness and stability, a priori estimates.

* Corresponding author: Gang Bao.

found in [4, 5, 6, 8, 10, 11, 19, 42, 41] for various approaches including the integral equation method and the finite element method. A comprehensive review can be found in [9] on mathematical modeling, analysis, and computational methods for diffrative optics.

The time-domain scattering problems have received ever-increasing attention due to their capability of capturing wide-band signals and modeling more general material and nonlinearity [37, 38, 39, 40, 16]. Compared with the time-harmonic scattering problems, mathematical studies are much less done for the time-domain counterparts due to the challenge of the temporal dependence. The analysis can be found in [15, 25, 26, 27, 28, 33] for the time-domain acoustic, elastic, and electromagnetic scattering problems in different structures including bounded obstacles, open cavities, and unbounded rough surfaces. In these works, the idea was to utilize the Laplace transform as a bridge between the time-domain and the frequency domain. The exact time-domain transparent boundary conditions were developed to reduce the scattering problems into initial boundary value problems in bounded domains. Using the energy method, the authors showed the well-posedness and stability for these time-dependent problems. One of the key steps was to establish some desired properties of the time-domain transparent boundary conditions. There are even fewer results for the time-domain grating problems. The numerical solution and the mathematical analysis can be found in [24] and [25], respectively, for the one-dimensional grating problem, where the two-dimensional scalar wave equation was considered. It is left undone for the time-domain Maxwell equations in a biperiodic structure.

Besides the time dependence, there is another challenge of the problem: an unbounded domain. The unbounded domain needs to be truncated into a bounded one when doing analysis or numerical computation. An appropriate boundary condition needs to be imposed on the boundary of the truncated domain in order to avoid artificial wave reflection. Such a boundary condition is called a transparent boundary condition (TBC) or a non-reflecting boundary condition (NRBC). It has been an important and active research subject in the area of wave propagation [1, 18, 22, 29, 30]. It still remains the subject matter of much ongoing research, especially for the time-domain problems. We point out that the method proposed in [25] cannot be adopted to handle the three-dimensional Maxwell equations. It is too complicated, if not impossible, to show that the time-domain TBC has the desired properties in three dimensions.

This paper is devoted to the time-domain analysis of the electromagnetic scattering problem in a biperiodic structure by using a different method. It is known that the electromagnetic wave has a finite speed of propagation in the time-domain, i.e., it takes a certain amount of time for the wave to propagate over a distance. This feature differs from the infinite speed of propagation for the wave in the frequency domain. We make use of this fact and propose a compressed coordinate transformation to reduce the problem equivalently into an initial-boundary value problem in a bounded domain. Given any time T, we consider the problem in the finite time interval $(0, T]$. Unlike the time-harmonic problem, the solution does not have quasi-periodicity with respect to the spatial variable $\boldsymbol{r} = (x_1, x_2) \in \mathbb{R}^2$ in the time-domain. The method begins with change of variables to ensure that the solution of the new equation is periodic with respect to the spatial variable $\boldsymbol{r}$. Hence, the periodic boundary condition can be used in the x_1 and x_2 directions. In the x_3 direction, we first specify a rectangular slab which contains the biperiodic structure with a possible inhomogeneous medium. The medium is homogeneous in the region above and below the slab, respectively. Then we pick two plane surfaces, one is far away above and another one is far away below the rectangular slab, such that the diffracted wave and the transmitted wave cannot reach the top surface and the bottom surface at the time T. Therefore the homogeneous Dirichlet boundary condition can be imposed on these two surfaces. Although the problem is now restricted into a bounded domain, the whole physical domain in the x_3 direction may be too large for actual computation. To overcome this issue, we apply the second change of variables and compress those two surfaces towards to the specified rectangular slab. This is done by mapping the two far away surfaces into the two surfaces which are slightly above and below the rectangular slab. The reduced problem can be formulated into a much smaller domain. Based on the Galerkin method and energy estimates, we prove the existence and uniqueness of the weak solution for the corresponding variational problem. Furthermore, we obtain a priori estimates with explicit dependence on the time.

The paper is organized as follows. In Section 2, we introduce the model problem, and present the change of variables and the compressed coordinate transformation to reduce the problem equivalently into an initial boundary value problem formulated in a bounded domain over a finite time interval. Section 3 is devoted to the analysis of the initial boundary value problem. The well-posedness is addressed and a priori estimates are obtained. We conclude the paper with some general remarks and directions for future work in Section 4.

2. **Problem formulation.**

In this section, we introduce the mathematical model and define some notations on the time-domain scattering problem for Maxwell's equations in a biperiodic structure.

2.1. **Maxwell's equations.**

Let us first specify the problem geometry, which is shown in Figure 1. Since the structure is biperiodic in $\boldsymbol{r} = (x_1, x_2) \in \mathbb{R}^2$, the problem can be restricted into one periodic cell

$$R = \{\boldsymbol{x} = (\boldsymbol{r}, x_3) \in \mathbb{R}^3 : 0 < x_1 < \Lambda_1,\, 0 < x_2 < \Lambda_2\},$$

where $\Lambda_j, j = 1, 2$ is a positive constant. Let

$$\Omega_h = \{\boldsymbol{x} \in \mathbb{R}^3 : 0 < x_1 < \Lambda_1,\, 0 < x_2 < \Lambda_2,\, h_2 < x_3 < h_1\},$$

which may be filled with an inhomogeneous medium. Here $h_j, j = 1, 2$ is a constant. Denote

$$\Omega_h^+ = \{\boldsymbol{x} \in \mathbb{R}^3 |\; 0 < x_1 < \Lambda_1,\, 0 < x_2 < \Lambda_2,\, x_3 > h_1\},$$
$$\Omega_h^- = \{\boldsymbol{x} \in \mathbb{R}^3 |\; 0 < x_1 < \Lambda_1,\, 0 < x_2 < \Lambda_2,\, x_3 < h_2\},$$

which is assumed to be filled with a homogeneous medium, respectively. Define the boundary

$$\Gamma_{h_j} = \{\boldsymbol{x} \in \mathbb{R}^3 : 0 < x_1 < \Lambda_1,\, 0 < x_2 < \Lambda_2,\, x_3 = h_j\}.$$

Consider the time-domain Maxwell's equations

$$\begin{cases} \nabla \times \boldsymbol{E}(\boldsymbol{x}, t) + \mu \partial_t \boldsymbol{H}(\boldsymbol{x}, t) = 0, \\ \nabla \times \boldsymbol{H}(\boldsymbol{x}, t) - \varepsilon \partial_t \boldsymbol{E}(\boldsymbol{x}, t) = 0. \end{cases} \tag{1}$$

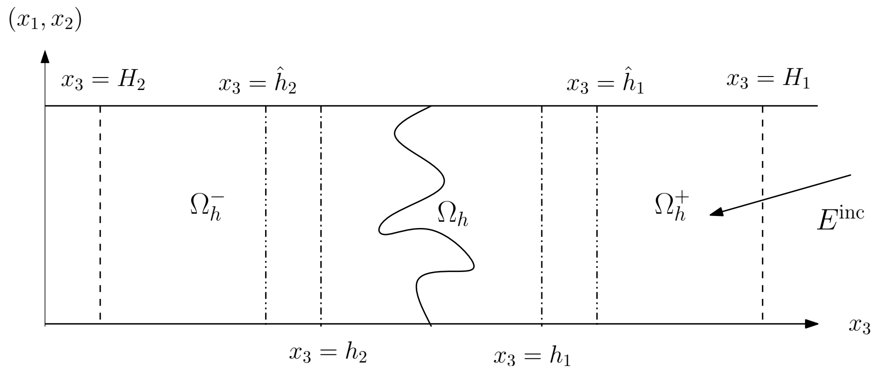

FIGURE 1. Problem geometry of the time-domain scattering by a biperiodic structure.

where $(\boldsymbol{x}, t) \in R \times \mathbb{R}^+$, $\boldsymbol{E}, \boldsymbol{H}$ are the electric and magnetic fields, and ε, μ are the dielectric permittivity and magnetic permeability and satisfy

$$\varepsilon(x_1 + n_1\Lambda_1, x_2 + n_2\Lambda_2, x_3) = \varepsilon(x_1, x_2, x_3),$$
$$\mu(x_1 + n_1\Lambda, x_2 + n_2\Lambda_2, x_3) = \mu(x_1, x_2, x_3),$$

for all $\boldsymbol{r} = (x_1, x_2) \in \mathbb{R}^2, \boldsymbol{n} = (n_1, n_2) \in \mathbb{Z}^2$. Since the medium is assumed to be homogeneous away from Ω_h, there exists constants $\varepsilon_j, \mu_j, j = 1, 2$ such that

$$\varepsilon(\boldsymbol{x}) = \varepsilon_1, \quad \mu(\boldsymbol{x}) = \mu_1, \quad \boldsymbol{x} \in \Omega_h^+,$$
$$\varepsilon(\boldsymbol{x}) = \varepsilon_2, \quad \mu(\boldsymbol{x}) = \mu_2, \quad \boldsymbol{x} \in \Omega_h^-.$$

Throughout we also assume that $\varepsilon\mu \geq \varepsilon_1\mu_1$ and

$$0 < \varepsilon_{\min} \leq \varepsilon \leq \varepsilon_{\max} < \infty, \quad 0 < \mu_{\min} \leq \mu \leq \mu_{\max} < \infty,$$

where $\varepsilon_{\min}$, $\varepsilon_{\max}$, $\mu_{\min}$, $\mu_{\max}$ are constants.

Let $(\boldsymbol{E}^{\rm inc}, \boldsymbol{H}^{\rm inc})$ be the electromagnetic plane waves that are incident upon the structure from Ω_h^+. Explicitly we have

$$\boldsymbol{E}^{\rm inc} = \boldsymbol{p} f(\boldsymbol{\alpha} \cdot \boldsymbol{r} - \beta x_3 - ct), \quad \boldsymbol{H}^{\rm inc} = \boldsymbol{q} f(\boldsymbol{\alpha} \cdot \boldsymbol{r} - \beta x_3 - ct), \tag{2}$$

where f is a smooth function and its regularity will be specified later, $c = 1/\sqrt{\varepsilon_1\mu_1}$ is the light speed in Ω_h^+, $\boldsymbol{\alpha} = (\alpha_1, \alpha_2), \alpha_1 = \sin\theta_1 \cos\theta_2, \alpha_2 = \sin\theta_1 \sin\theta_2, \beta = \cos\theta_1$, θ_1 and θ_2 are latitudinal and longitudinal incident angles satisfying $0 \leq \theta_1 < \frac{\pi}{2}, 0 \leq \theta_2 < 2\pi$. Denote by $\boldsymbol{d} = (\alpha_1, \alpha_2, -\beta)^\top$ the unit propagation direction vector. The polarization vectors $\boldsymbol{p} = (p_1, p_2, p_3)^\top$ and $\boldsymbol{q} = (q_1, q_2, q_3)^\top$ satisfy

$$\boldsymbol{p} \cdot \boldsymbol{d} = 0, \quad \boldsymbol{q} = \sqrt{\frac{\varepsilon_1}{\mu_1}} \boldsymbol{d} \times \boldsymbol{p}.$$

It is easy to verify that

$$\begin{cases} \nabla \times \boldsymbol{E}^{\rm inc}(\boldsymbol{x}, t) + \mu_1 \partial_t \boldsymbol{H}^{\rm inc}(\boldsymbol{x}, t) = 0 \\ \nabla \times \boldsymbol{H}^{\rm inc}(\boldsymbol{x}, t) - \varepsilon_1 \partial_t \boldsymbol{E}^{\rm inc}(\boldsymbol{x}, t) = 0. \end{cases}$$

To impose the initial conditions, we assume that the incident fields and total fields vanish for $t \leq 0$, i.e.,

$$\boldsymbol{E}^{\rm inc}|_{t=0} = \boldsymbol{E}|_{t=0} = 0, \quad \boldsymbol{H}^{\rm inc}|_{t=0} = \boldsymbol{H}|_{t=0} = 0, \quad \boldsymbol{x} \in R. \tag{3}$$

2.2. **Change of variables.** The incident fields $(\boldsymbol{E}^{\rm inc}, \boldsymbol{H}^{\rm inc})$ are not periodic functions in $\boldsymbol{r}$, but we can verify that

$$\begin{aligned}(\boldsymbol{E}^{\rm inc}, \boldsymbol{H}^{\rm inc})(\boldsymbol{r}+\boldsymbol{\Lambda}, x_3, t) &= (\boldsymbol{p}, \boldsymbol{q}) f(\boldsymbol{\alpha}\cdot(\boldsymbol{r}+\boldsymbol{\Lambda}) - \beta x_3 - ct) \\ &= (\boldsymbol{p}, \boldsymbol{q}) f(\boldsymbol{\alpha}\cdot\boldsymbol{r} - \beta x_3 - c(t - c^{-1}\boldsymbol{\alpha}\cdot\boldsymbol{\Lambda})) \\ &= (\boldsymbol{E}^{\rm inc}, \boldsymbol{H}^{\rm inc})(\boldsymbol{r}, x_3, t - c^{-1}\boldsymbol{\alpha}\cdot\boldsymbol{\Lambda}),\end{aligned}$$

where $\boldsymbol{\Lambda} = (\Lambda_1, 0), (0, \Lambda_2)$, or (Λ_1, Λ_2). Motivated by the uniqueness of the solution, we assume that the total fields satisfy the same translation property, i.e.,

$$(\boldsymbol{E}, \boldsymbol{H})(\boldsymbol{r}+\boldsymbol{\Lambda}, x_3, t) = (\boldsymbol{E}, \boldsymbol{H})(\boldsymbol{r}, x_3, t - c^{-1}\boldsymbol{\alpha}\cdot\boldsymbol{\Lambda}).$$

Define

$$(\boldsymbol{U}, \boldsymbol{V})(\boldsymbol{r}, x_3, t) = (\boldsymbol{E}, \boldsymbol{H})(\boldsymbol{r}, x_3, t + c^{-1}\boldsymbol{\alpha}\cdot(\boldsymbol{r}-\boldsymbol{\Lambda})). \tag{4}$$

A simple calculation yields that

$$\begin{aligned}(\boldsymbol{U}, \boldsymbol{V})(\boldsymbol{r}+\boldsymbol{\Lambda}, x_3, t) &= (\boldsymbol{E}, \boldsymbol{H})(\boldsymbol{r}+\boldsymbol{\Lambda}, x_3, t + c^{-1}\boldsymbol{\alpha}\cdot\boldsymbol{r}) \\ &= (\boldsymbol{E}, \boldsymbol{H})(\boldsymbol{r}, x_3, t + c^{-1}\boldsymbol{\alpha}\cdot(\boldsymbol{r}-\boldsymbol{\Lambda})) = (\boldsymbol{U}, \boldsymbol{V})(\boldsymbol{r}, x_3, t),\end{aligned}$$

which shows that $\boldsymbol{U}$ and $\boldsymbol{V}$ are periodic functions with a period (Λ_1, Λ_2) in $\boldsymbol{r}$. It is clear to note that

$$\begin{aligned}(\boldsymbol{U}^{\rm inc}, \boldsymbol{V}^{\rm inc})(\boldsymbol{r}, x_3, t) &= (\boldsymbol{E}^{\rm inc}, \boldsymbol{H}^{\rm inc})(\boldsymbol{r}, x_3, t + c^{-1}\boldsymbol{\alpha}\cdot(\boldsymbol{r}-\boldsymbol{\Lambda})) \\ &= (\boldsymbol{p}, \boldsymbol{q}) f(\boldsymbol{\alpha}\cdot\boldsymbol{r} - \beta x_3 - c(t + c^{-1}\boldsymbol{\alpha}\cdot(\boldsymbol{r}-\boldsymbol{\Lambda}))) \\ &= (\boldsymbol{p}, \boldsymbol{q}) f(-\beta x_3 - ct + \boldsymbol{\alpha}\cdot\boldsymbol{\Lambda}),\end{aligned} \tag{5}$$

which shows that $\boldsymbol{U}^{\rm inc}$ and $\boldsymbol{V}^{\rm inc}$ are also periodic functions of $\boldsymbol{r}$, since they do not depend on $\boldsymbol{r}$.

Using the change of variables, we have from straightforward calculations that

$$\begin{gathered}\partial_t(\boldsymbol{E}, \boldsymbol{H}) = \partial_t(\boldsymbol{U}, \boldsymbol{V}), \quad \partial_{x_3}(\boldsymbol{E}, \boldsymbol{H}) = \partial_{x_3}(\boldsymbol{U}, \boldsymbol{V}), \\ \partial_{x_j}(\boldsymbol{E}, \boldsymbol{H}) = \partial_{x_j}(\boldsymbol{U}, \boldsymbol{V}) - \frac{\alpha_j}{c}\partial_t(\boldsymbol{U}, \boldsymbol{V}), \quad j = 1, 2.\end{gathered}$$

It is easy to verify that

$$\nabla\times\boldsymbol{E} = \nabla\times\boldsymbol{U} - c^{-1}\tilde{\boldsymbol{\alpha}}\times\partial_t\boldsymbol{U}, \quad \nabla\times\boldsymbol{H} = \nabla\times\boldsymbol{V} - c^{-1}\tilde{\boldsymbol{\alpha}}\times\partial_t\boldsymbol{V},$$

where $\tilde{\boldsymbol{\alpha}} = (\alpha_1, \alpha_2, 0)$. Hence, the Maxwell equations (1) can be reduced to the system of coupled equations for $(\boldsymbol{U}, \boldsymbol{V})$ in $R\times\mathbb{R}^+$:

$$\begin{cases}\nabla\times\boldsymbol{U}(\boldsymbol{r}, x_3, t) - c^{-1}\tilde{\boldsymbol{\alpha}}\times\partial_t\boldsymbol{U}(\boldsymbol{r}, x_3, t) + \mu\partial_t\boldsymbol{V}(\boldsymbol{r}, x_3, t) = 0, \\ \nabla\times\boldsymbol{V}(\boldsymbol{r}, x_3, t) - c^{-1}\tilde{\boldsymbol{\alpha}}\times\partial_t\boldsymbol{V}(\boldsymbol{r}, x_3, t) - \varepsilon\partial_t\boldsymbol{U}(\boldsymbol{r}, x_3, t) = 0.\end{cases} \tag{6}$$

Clearly, $(\boldsymbol{U}^{\rm inc}, \boldsymbol{V}^{\rm inc})$ satisfy (6) in $R\times\mathbb{R}^+$ with $\varepsilon = \varepsilon_1, \mu = \mu_1$. By (3), we assume that the incident fields $(\boldsymbol{E}^{\rm inc}, \boldsymbol{H}^{\rm inc})$ vanish for $t \le 0$. Combining (2) and (5), we obtain that $\boldsymbol{U}^{\rm inc}|_{t=0} = \boldsymbol{V}^{\rm inc}|_{t=0} = 0$, which implies the initial condition

$$\boldsymbol{U}|_{t=0} = \boldsymbol{V}|_{t=0} = 0 \quad \text{in } R.$$

2.3. **Compressed coordinate transformation.** It is known that the scattered field $\boldsymbol{U}^{\rm s}=\boldsymbol{U}-\boldsymbol{U}^{\rm inc}$ has a finite speed of propagation in the time-domain. For any given time $T>0$, we may always pick a sufficiently large H_1 ($H_1\gg h_1$) and a sufficiently small H_2 ($H_2\ll h_2$) such that the scattered field $\boldsymbol{U}^{\rm s}$ cannot reach the surface $\Gamma_{H_j}=\{x\in\mathbb{R}^3: x_3=H_j\}, j=1,2$. Hence the following boundary conditions can be imposed:

$$\nu_1\times\boldsymbol{U}|_{\Gamma_{H_1}}=\nu_1\times\boldsymbol{U}^{\rm inc}|_{\Gamma_{H_1}},\quad \nu_2\times\boldsymbol{U}|_{\Gamma_{H_2}}=0,\quad t\in(0,T],$$

where $\nu_1=(0,0,1)^\top$ and $\nu_2=(0,0,-1)^\top$ is the unit normal vector on Γ_{H_1} and Γ_{H_2}, respectively. Pick another two constants $\hat h_1,\hat h_2$ satisfying $h_1<\hat h_1\ll H_1, H_2\ll \hat h_2<h_2$. Define $\Gamma_{\hat h_j}:=\{\boldsymbol{x}\in\mathbb{R}^3|\ 0<x_1<\Lambda_1,\ 0<x_2<\Lambda_2,\ x_3=\hat h_j\}$.

Consider the change of variables

$$x_3=\psi(\tilde x_3)=\begin{cases}\tilde x_3, & \tilde x_3\in(h_2,h_1),\\ \eta_1(\tilde x_3), & \tilde x_3\in[h_1,\hat h_1],\\ \eta_2(\tilde x_3), & \tilde x_3\in[\hat h_2,h_2],\end{cases}$$

where

$$\eta_j(\tilde x_3)=\frac{h_j^2(H_j-\hat h_j)+\tilde x_3[h_j^2+(\hat h_j-2h_j)H_j]}{(\hat h_j-\tilde x_3)(H_j-\hat h_j)+(\hat h_j-h_j)^2},\quad j=1,2.$$

A simple calculation yields that

$$\eta_j'(\tilde x_3)=\frac{(H_j-h_j)^2(\hat h_j-h_j)^2}{[(\hat h_j-\tilde x_3)(H_j-\hat h_j)+(\hat h_j-h_j)^2]^2}>0$$

and

$$\eta_j(h_j)=h_j,\quad \eta_j(\hat h_j)=H_j,\quad \eta_j'(h_j)=1,$$

which imply that the function $\psi\in C^1([\hat h_2,\hat h_1])$. Define

$$\Omega_{\hat h}=\{\boldsymbol{x}\in\mathbb{R}^3|\ 0<x_1<\Lambda_1,\ 0<x_2<\Lambda_2,\ \hat h_2<x_3<\hat h_1\},$$
$$\Omega_H=\{\boldsymbol{x}\in\mathbb{R}^3|\ 0<x_1<\Lambda_1,\ 0<x_2<\Lambda_2,\ H_2<x_3<H_1\}.$$

Clearly, the transform ψ keeps the domain Ω_h unchanged, while compresses the domain Ω_H into the domain $\Omega_{\hat h}$.

Let $(\widetilde{\boldsymbol{U}},\widetilde{\boldsymbol{V}})$ be the transformed total fields of $(\boldsymbol{U},\boldsymbol{V})$ under the change of variables, i.e.,

$$(\widetilde{\boldsymbol{U}},\widetilde{\boldsymbol{V}})(x_1,x_2,\tilde x_3,t)=(\boldsymbol{U},\boldsymbol{V})(x_1,x_2,\psi(\tilde x_3),t).$$

It follows from a straightforward calculation that

$$\nabla\times\boldsymbol{U}=\nabla_{\widetilde M}\times\widetilde{\boldsymbol{U}},\quad \nabla\times\boldsymbol{V}=\nabla_{\widetilde M}\times\widetilde{\boldsymbol{V}},\quad \partial_t\boldsymbol{U}=\partial_t\widetilde{\boldsymbol{U}},\quad \partial_t\boldsymbol{V}=\partial_t\widetilde{\boldsymbol{V}},$$

where

$$\nabla_{\widetilde M}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&\frac{1}{\psi'(\tilde x_3)}\end{bmatrix}\begin{bmatrix}\partial_{x_1}\\ \partial_{x_2}\\ \partial_{\tilde x_3}\end{bmatrix}=\begin{bmatrix}\partial_{x_1}\\ \partial_{x_2}\\ \frac{1}{\psi'(\tilde x_3)}\partial_{\tilde x_3}\end{bmatrix}.$$

With the aid of the function $\psi\in C^1([\hat h_2,\hat h_1])$ and the definition of the operator $\nabla_{\widetilde M}$, we assume that

$$\max_{\boldsymbol{x}\in\Omega_h}|\nabla_{\widetilde M}\varepsilon(\boldsymbol{x})|\le d_\varepsilon<\infty,\quad \max_{\boldsymbol{x}\in\Omega_h}|\nabla_{\widetilde M}\mu(\boldsymbol{x})|\le d_\mu<\infty,$$

where d_ε and d_μ are constants.

The system (6) can be written into a coupled system of equations for $(\widetilde{\boldsymbol{U}}, \widetilde{\boldsymbol{V}})$ in $\Omega_{\hat{h}} \times (0, T]$:

$$\begin{cases} \nabla_{\widetilde{M}} \times \widetilde{\boldsymbol{U}}(\boldsymbol{r}, \tilde{x}_3, t) - c^{-1}\tilde{\boldsymbol{\alpha}} \times \partial_t \widetilde{\boldsymbol{U}}(\boldsymbol{r}, \tilde{x}_3, t) + \mu \partial_t \widetilde{\boldsymbol{V}}(\boldsymbol{r}, \tilde{x}_3, t) = 0, \\ \nabla_{\widetilde{M}} \times \widetilde{\boldsymbol{V}}(\boldsymbol{r}, \tilde{x}_3, t) - c^{-1}\tilde{\boldsymbol{\alpha}} \times \partial_t \widetilde{\boldsymbol{V}}(\boldsymbol{r}, \tilde{x}_3, t) - \varepsilon \partial_t \widetilde{\boldsymbol{U}}(\boldsymbol{r}, \tilde{x}_3, t) = 0. \end{cases} \tag{7}$$

In addition, $(\widetilde{\boldsymbol{U}}, \widetilde{\boldsymbol{V}})$ satisfy the initial condition

$$\widetilde{\boldsymbol{U}}|_{t=0} = 0, \quad \widetilde{\boldsymbol{V}}|_{t=0} = 0 \quad \text{in } \Omega_{\hat{h}}, \tag{8}$$

the periodic boundary condition

$$\widetilde{\boldsymbol{U}}(0, x_2, \tilde{x}_3, t) = \widetilde{\boldsymbol{U}}(\Lambda_1, x_2, \tilde{x}_3, t), \quad \widetilde{\boldsymbol{U}}(x_1, 0, \tilde{x}_3, t) = \widetilde{\boldsymbol{U}}(x_1, \Lambda_2, \tilde{x}_3, t),$$
$$\widetilde{\boldsymbol{V}}(0, x_2, \tilde{x}_3, t) = \widetilde{\boldsymbol{V}}(\Lambda_1, x_2, \tilde{x}_3, t), \quad \widetilde{\boldsymbol{V}}(x_1, 0, \tilde{x}_3, t) = \widetilde{\boldsymbol{V}}(x_1, \Lambda_2, \tilde{x}_3, t),$$

and the boundary condition

$$\nu_1 \times \widetilde{\boldsymbol{U}}|_{\Gamma_{\hat{h}_1}} = \nu_1 \times \widetilde{\boldsymbol{U}}^{\text{inc}}|_{\Gamma_{\hat{h}_1}}, \quad \nu_2 \times \widetilde{\boldsymbol{U}}|_{\Gamma_{\hat{h}_2}} = 0, \quad t \in (0, T),$$

where

$$\widetilde{\boldsymbol{U}}^{\text{inc}}(\boldsymbol{r}, \tilde{x}_3, t) = \boldsymbol{U}^{\text{inc}}(\boldsymbol{r}, \psi(\tilde{x}_3), t) = \boldsymbol{p} f(-\beta\psi(\tilde{x}_3) - ct + \boldsymbol{\alpha} \cdot \boldsymbol{\Lambda}).$$

Eliminating the magnetic field $\widetilde{\boldsymbol{V}}$ from (7), we get the wave equation for the electric field $\widetilde{\boldsymbol{U}}$:

$$(\nabla_{\widetilde{M}} - c^{-1}\tilde{\boldsymbol{\alpha}}\partial_t) \times [\mu^{-1}(\nabla_{\widetilde{M}} - c^{-1}\tilde{\boldsymbol{\alpha}}\partial_t) \times \widetilde{\boldsymbol{U}}] + \varepsilon \partial_t^2 \widetilde{\boldsymbol{U}} = 0 \quad \text{in } \Omega_{\hat{h}} \times (0, T]. \tag{9}$$

It is also easy to get

$$\partial_t \widetilde{\boldsymbol{U}} = \varepsilon^{-1} \nabla_{\widetilde{M}} \times \widetilde{\boldsymbol{V}} + (c\varepsilon\mu)^{-1} \tilde{\boldsymbol{\alpha}} \times (\nabla_{\widetilde{M}} \times \widetilde{\boldsymbol{U}}) - (c^2\varepsilon\mu)^{-1} \tilde{\boldsymbol{\alpha}} \times (\tilde{\boldsymbol{\alpha}} \times \partial_t \widetilde{\boldsymbol{U}}). \tag{10}$$

Noting that $-(c^2\varepsilon\mu)^{-1}|\tilde{\boldsymbol{\alpha}}|^2 < 0$ and $1 - (c^2\varepsilon\mu)^{-1}|\tilde{\boldsymbol{\alpha}}|^2 > 1 - |\tilde{\boldsymbol{\alpha}}|^2 = 1 - \sin^2\theta_1 > 0$. Taking the dot product of (10) with $\tilde{\boldsymbol{\alpha}}$ and using (8), we obtain

$$[1 - (c^2\varepsilon\mu)^{-1}|\tilde{\boldsymbol{\alpha}}|^2][(\partial_t \widetilde{\boldsymbol{U}}|_{t=0}) \cdot \tilde{\boldsymbol{\alpha}}] = -(c^2\varepsilon\mu)^{-1}|\tilde{\boldsymbol{\alpha}}|^2[(\partial_t \widetilde{\boldsymbol{U}}|_{t=0}) \cdot \tilde{\boldsymbol{\alpha}}],$$

which shows that $(\partial_t \widetilde{\boldsymbol{U}}|_{t=0}) \cdot \tilde{\boldsymbol{\alpha}} = 0$ for any $\theta_1 \in [0, \frac{\pi}{2}), \theta_2 \in [0, 2\pi)$. Since θ_1 and θ_2 are arbitrary, we have the initial condition

$$\partial_t \widetilde{\boldsymbol{U}}|_{t=0} = 0.$$

Similarly, we may eliminate the electric field $\widetilde{\boldsymbol{U}}$ from (7) and obtain the wave equation for the magnetic field $\widetilde{\boldsymbol{V}}$:

$$(\nabla_{\widetilde{M}} - c^{-1}\tilde{\boldsymbol{\alpha}}\partial_t) \times [\varepsilon^{-1}(\nabla_{\widetilde{M}} - c^{-1}\tilde{\boldsymbol{\alpha}}\partial_t) \times \widetilde{\boldsymbol{V}}] + \mu \partial_t^2 \widetilde{\boldsymbol{V}} = 0 \quad \text{in } \Omega_{\hat{h}} \times (0, T]. \tag{11}$$

It is clear to note from (9) and (11) that these two model equations can be handled in a unified way by formally exchanging the roles of ε and μ. Hence we shall only present the results by using (9) in this paper.

2.4. **The reduced problem.** Let $H(\widetilde{\text{curl}};\Omega_{\hat h})$ and $H(\widetilde{\text{div}};\Omega_{\hat h})$ be the Sobolev spaces which are defined by

$$H(\widetilde{\text{curl}};\Omega_{\hat h}) = \{\boldsymbol{u}\in L^2(\Omega_{\hat h})^3 : \nabla_{\widetilde M}\times\boldsymbol{u}\in L^2(\Omega_{\hat h})^3\},$$
$$H(\widetilde{\text{div}};\Omega_{\hat h}) = \{\boldsymbol{u}\in L^2(\Omega_{\hat h})^3 : \nabla_{\widetilde M}\cdot\boldsymbol{u}\in L^2(\Omega_{\hat h})\}.$$

Define a biperiodic subspace of $H(\widetilde{\text{curl}};\Omega_{\hat h})$:

$$\begin{aligned}H_{\text{per},0}(\widetilde{\text{curl}};\Omega_{\hat h}) = \{\boldsymbol{u}\in H(\widetilde{\text{curl}};\Omega_{\hat h}) : \boldsymbol{u}(0,x_2,\tilde x_3) &= \boldsymbol{u}(\Lambda_1,x_2,\tilde x_3),\\ \boldsymbol{u}(x_1,0,\tilde x_3) &= \boldsymbol{u}(x_1,\Lambda_2,\tilde x_3),\\ \nu_1\times\boldsymbol{u}|_{\Gamma_{\hat h_1}} = \nu_2\times\boldsymbol{u}|_{\Gamma_{\hat h_2}} &= 0\}.\end{aligned}$$

Now we define a biperiodic function space

$$H_{\text{per},0}(\widetilde{\text{curl}},\widetilde{\text{div}};\Omega_{\hat h}) = H_{\text{per},0}(\widetilde{\text{curl}};\Omega_{\hat h})\cap H(\widetilde{\text{div}};\Omega_{\hat h}),$$

which is a Sobolev space with the norm given by

$$\|\boldsymbol{u}\|_{H_{\text{per},0}(\widetilde{\text{curl}},\widetilde{\text{div}};\Omega_{\hat h})} = \left[\|\boldsymbol{u}\|^2_{L^2(\Omega_{\hat h})^3} + \|\nabla_{\widetilde M}\times\boldsymbol{u}\|^2_{L^2(\Omega_{\hat h})^3} + \|\nabla_{\widetilde M}\cdot\boldsymbol{u}\|^2_{L^2(\Omega_{\hat h})}\right]^{1/2}.$$

For a given $\widetilde{\boldsymbol{U}}^{\text{inc}}$, there exists a smooth lifting $\widetilde{\boldsymbol{U}}_0$, which has a compact support contained in $\Omega_{\hat h}\times(0,T]$ and satisfies the boundary conditions:

$$\nu_1\times\widetilde{\boldsymbol{U}}_0|_{\Gamma_{\hat h_1}} = \nu_1\times\widetilde{\boldsymbol{U}}^{\text{inc}}|_{\Gamma_{\hat h_1}},\quad \nu_2\times\widetilde{\boldsymbol{U}}_0|_{\Gamma_{\hat h_2}} = 0.$$

Let $\widetilde{\boldsymbol{W}} = \widetilde{\boldsymbol{U}} - \widetilde{\boldsymbol{U}}_0$. We may consider an equivalent initial boundary value problem for $\widetilde{\boldsymbol{W}}$: to find a biperiodic function $\widetilde{\boldsymbol{W}}(\boldsymbol{r},\tilde x_3,t)$ such that

$$\begin{cases}\nabla^t_{\widetilde M,\tilde{\boldsymbol\alpha}}\times[\mu^{-1}\nabla^t_{\widetilde M,\tilde{\boldsymbol\alpha}}\times\widetilde{\boldsymbol{W}}] + \varepsilon\partial_t^2\widetilde{\boldsymbol{W}} = \tilde{\boldsymbol{f}}_1(\boldsymbol{r},\tilde x_3,t) & \text{in } \Omega_{\hat h}\times(0,T],\\ \widetilde{\boldsymbol{W}}|_{t=0} = \tilde{\boldsymbol{f}}_2(\boldsymbol{r},\tilde x_3),\quad \partial_t\widetilde{\boldsymbol{W}}|_{t=0} = \tilde{\boldsymbol{f}}_3(\boldsymbol{r},\tilde x_3) & \text{in } \Omega_{\hat h},\\ \nu_1\times\widetilde{\boldsymbol{W}}|_{\Gamma_{\hat h_1}} = \nu_2\times\widetilde{\boldsymbol{W}}|_{\Gamma_{\hat h_2}} = 0 & \text{in } (0,T],\end{cases}\tag{12}$$

where the operator

$$\nabla^t_{\widetilde M,\tilde{\boldsymbol\alpha}} = (\nabla_{\widetilde M} - c^{-1}\tilde{\boldsymbol\alpha}\partial_t)$$

and

$$\tilde{\boldsymbol{f}}_1\in L^2(\Omega_{\hat h})^3,\quad \tilde{\boldsymbol{f}}_2\in H(\widetilde{\text{curl}};\Omega_{\hat h})\cap H(\widetilde{\text{div}};\Omega_{\hat h}),\quad \tilde{\boldsymbol{f}}_3\in L^2(\Omega_{\hat h})^3.$$

The following results show that the operator $\nabla^t_{\widetilde M,\tilde{\boldsymbol\alpha}}$ satisfies the usual divergence free conditions for the electromagnetic fields in Maxwell's equations.

Lemma 2.1. *If $(\widetilde{\boldsymbol{U}},\widetilde{\boldsymbol{V}})$ satisfy (7), then the following divergence free conditions hold*

$$\nabla^t_{\widetilde M,\tilde{\boldsymbol\alpha}}\cdot(\varepsilon\widetilde{\boldsymbol{U}}) = 0,\quad \nabla^t_{\widetilde M,\tilde{\boldsymbol\alpha}}\cdot(\mu\widetilde{\boldsymbol{V}}) = 0\quad in\ \Omega_{\hat h}\times(0,T].$$

Proof. It follows from (7) that

$$\begin{aligned}\nabla^t_{\widetilde M,\tilde{\boldsymbol\alpha}}\cdot(\varepsilon\partial_t\widetilde{\boldsymbol{U}}) &= \nabla^t_{\widetilde M,\tilde{\boldsymbol\alpha}}\cdot(\nabla^t_{\widetilde M,\tilde{\boldsymbol\alpha}}\times\widetilde{\boldsymbol{V}})\\ &= (\nabla_{\widetilde M} - c^{-1}\tilde{\boldsymbol\alpha}\partial_t)\cdot[(\nabla_{\widetilde M} - c^{-1}\tilde{\boldsymbol\alpha}\partial_t)\times\widetilde{\boldsymbol{V}}]\end{aligned}$$

$$
\begin{aligned}
&= \nabla_{\widetilde{M}} \cdot (\nabla_{\widetilde{M}} \times \widetilde{\boldsymbol{V}}) - c^{-1}\nabla_{\widetilde{M}} \cdot (\tilde{\boldsymbol{\alpha}} \times \partial_t \widetilde{\boldsymbol{V}}) \\
&\quad - c^{-1}\tilde{\boldsymbol{\alpha}} \cdot (\nabla_{\widetilde{M}} \times \partial_t \widetilde{\boldsymbol{V}}) + c^{-2}\tilde{\boldsymbol{\alpha}} \cdot (\tilde{\boldsymbol{\alpha}} \times \partial_t^2 \widetilde{\boldsymbol{V}}) \\
&= \nabla_{\widetilde{M}} \cdot (\nabla_{\widetilde{M}} \times \widetilde{\boldsymbol{V}}) - c^{-1}\partial_t[\nabla_{\widetilde{M}} \cdot (\tilde{\boldsymbol{\alpha}} \times \widetilde{\boldsymbol{V}}) + \tilde{\boldsymbol{\alpha}} \cdot (\nabla_{\widetilde{M}} \times \widetilde{\boldsymbol{V}})] \\
&\quad + c^{-2}\partial_t^2[\tilde{\boldsymbol{\alpha}} \cdot (\tilde{\boldsymbol{\alpha}} \times \widetilde{\boldsymbol{V}})] \quad \text{in } \Omega_{\hat{h}} \times (0,T]. \qquad (13)
\end{aligned}
$$

A simple calculation yields that

$$
\begin{aligned}
&\nabla_{\widetilde{M}} \cdot (\nabla_{\widetilde{M}} \times \widetilde{\boldsymbol{V}}) \\
&= \nabla_{\widetilde{M}} \cdot \Big(\partial_{x_2}\widetilde{V}_3 - \frac{1}{\psi'(\tilde{x}_3)}\partial_{\tilde{x}_3}\widetilde{V}_2,\ \frac{1}{\psi'(\tilde{x}_3)}\partial_{\tilde{x}_3}\widetilde{V}_1 - \partial_{x_1}\widetilde{V}_3,\ \partial_{x_1}\widetilde{V}_2 - \partial_{x_2}\widetilde{V}_1\Big)^\top \\
&= \left(\partial_{x_1}\partial_{x_2}\widetilde{V}_3 - \frac{1}{\psi'(\tilde{x}_3)}\partial_{x_1}\partial_{\tilde{x}_3}\widetilde{V}_2\right) + \left(\frac{1}{\psi'(\tilde{x}_3)}\partial_{x_2}\partial_{\tilde{x}_3}\widetilde{V}_1 - \partial_{x_2}\partial_{x_1}\widetilde{V}_3\right) \\
&\quad + \frac{1}{\psi'(\tilde{x}_3)}(\partial_{\tilde{x}_3}\partial_{x_1}\widetilde{V}_2 - \partial_{\tilde{x}_3}\partial_{x_2}\widetilde{V}_1) = 0, \\
&\nabla_{\widetilde{M}} \cdot (\tilde{\boldsymbol{\alpha}} \times \widetilde{\boldsymbol{V}}) \\
&= \nabla_{\widetilde{M}} \cdot (\alpha_2\widetilde{V}_3, -\alpha_1\widetilde{V}_3, \alpha_1\widetilde{V}_2 - \alpha_2\widetilde{V}_1)^\top \\
&= \alpha_2\partial_{x_1}\widetilde{V}_3 - \alpha_1\partial_{x_2}\widetilde{V}_3 + \frac{1}{\psi'(\tilde{x}_3)}(\alpha_1\partial_{\tilde{x}_3}\widetilde{V}_2 - \alpha_2\partial_{\tilde{x}_3}\widetilde{V}_1), \\
&\tilde{\boldsymbol{\alpha}} \cdot (\nabla_{\widetilde{M}} \times \widetilde{\boldsymbol{V}}) \\
&= \tilde{\boldsymbol{\alpha}} \cdot \Big(\partial_{x_2}\widetilde{V}_3 - \frac{1}{\psi'(\tilde{x}_3)}\partial_{\tilde{x}_3}\widetilde{V}_2,\ \frac{1}{\psi'(\tilde{x}_3)}\partial_{\tilde{x}_3}\widetilde{V}_1 - \partial_{x_1}\widetilde{V}_3,\ \partial_{x_1}\widetilde{V}_2 - \partial_{x_2}\widetilde{V}_1\Big)^\top \\
&= -\left[\alpha_2\partial_{x_1}\widetilde{V}_3 - \alpha_1\partial_{x_2}\widetilde{V}_3 + \frac{1}{\psi'(\tilde{x}_3)}(\alpha_1\partial_{\tilde{x}_3}\widetilde{V}_2 - \alpha_2\partial_{\tilde{x}_3}\widetilde{V}_1)\right] \\
&= -\nabla_{\widetilde{M}} \cdot (\tilde{\boldsymbol{\alpha}} \times \widetilde{\boldsymbol{V}}).
\end{aligned}
$$

It is easy to note that

$$
\tilde{\boldsymbol{\alpha}} \cdot (\tilde{\boldsymbol{\alpha}} \times \widetilde{\boldsymbol{V}}) = \widetilde{\boldsymbol{V}} \cdot (\tilde{\boldsymbol{\alpha}} \times \tilde{\boldsymbol{\alpha}}) = 0.
$$

Substituting the above equations into (13), we get

$$
\nabla^t_{\widetilde{M},\tilde{\boldsymbol{\alpha}}} \cdot (\varepsilon \partial_t \widetilde{\boldsymbol{U}}) = 0 \quad \text{in } \Omega_{\hat{h}} \times (0,T]. \qquad (14)
$$

We deduce from (14) that

$$
\begin{aligned}
0 &= \nabla^t_{\widetilde{M},\tilde{\boldsymbol{\alpha}}} \cdot (\varepsilon \partial_t \widetilde{\boldsymbol{U}}) = \partial_t[\nabla^t_{\widetilde{M},\tilde{\boldsymbol{\alpha}}} \cdot (\varepsilon \widetilde{\boldsymbol{U}})] \\
&= \partial_t[(\nabla_{\widetilde{M}}\varepsilon) \cdot \widetilde{\boldsymbol{U}} + \varepsilon(\nabla^t_{\widetilde{M},\tilde{\boldsymbol{\alpha}}} \cdot \widetilde{\boldsymbol{U}})] \\
&= \partial_t[(\nabla_{\widetilde{M}}\varepsilon) \cdot \widetilde{\boldsymbol{U}} + \varepsilon(\nabla_{\widetilde{M}} \cdot \widetilde{\boldsymbol{U}} - c^{-1}\tilde{\boldsymbol{\alpha}} \cdot \partial_t \widetilde{\boldsymbol{U}})] \quad \text{in } \Omega_{\hat{h}} \times (0,T]. \qquad (15)
\end{aligned}
$$

Combining (15) and $\widetilde{\boldsymbol{U}}|_{t=0} = \partial_t \widetilde{\boldsymbol{U}}|_{t=0} = 0$, we obtain

$$
\nabla^t_{\widetilde{M},\tilde{\boldsymbol{\alpha}}} \cdot (\varepsilon \widetilde{\boldsymbol{U}}) = 0 \quad \text{in } \Omega_{\hat{h}} \times (0,T].
$$

Similarly, we may show that

$$
\nabla^t_{\widetilde{M},\tilde{\boldsymbol{\alpha}}} \cdot (\mu \widetilde{\boldsymbol{V}}) = 0 \quad \text{in } \Omega_{\hat{h}} \times (0,T],
$$

which completes the proof. □

Lemma 2.2. *For any function* $\boldsymbol{u} \in H_{\mathrm{per},0}(\widetilde{\mathrm{curl}}, \widetilde{\mathrm{div}}; \Omega_{\hat{h}})$ *with* $\partial_t \boldsymbol{u} \in L^2(\Omega_{\hat{h}})^3$*, there exist two positive constants* C_1, C_2 *such that*

$$C_1\left[\|\partial_t \boldsymbol{u}\|^2_{L^2(\Omega_{\hat{h}})^3} + \|\nabla_{\widetilde{M}} \times \boldsymbol{u}\|^2_{L^2(\Omega_{\hat{h}})^3} + \|\nabla_{\widetilde{M}} \cdot \boldsymbol{u}\|^2_{L^2(\Omega_{\hat{h}})}\right]$$
$$\leq \|\partial_t \boldsymbol{u}\|^2_{L^2(\Omega_{\hat{h}})^3} + \|\nabla^t_{\widetilde{M},\tilde{\boldsymbol{\alpha}}} \times \boldsymbol{u}\|^2_{L^2(\Omega_{\hat{h}})^3} + \|\nabla^t_{\widetilde{M},\tilde{\boldsymbol{\alpha}}} \cdot \boldsymbol{u}\|^2_{L^2(\Omega_{\hat{h}})}$$
$$\leq C_2\left[\|\partial_t \boldsymbol{u}\|^2_{L^2(\Omega_{\hat{h}})^3} + \|\nabla_{\widetilde{M}} \times \boldsymbol{u}\|^2_{L^2(\Omega_{\hat{h}})^3} + \|\nabla_{\widetilde{M}} \cdot \boldsymbol{u}\|^2_{L^2(\Omega_{\hat{h}})}\right],$$

where $C_1 = 1/\max\{4, 4c^{-2}+2\}$ *and* $C_2 = \max\left\{\frac{3}{2}, 3c^{-2} + \frac{1}{2}\right\}$.

Proof. For any function $\boldsymbol{u} \in H_{\mathrm{per},0}(\widetilde{\mathrm{curl}}, \widetilde{\mathrm{div}}; \Omega_{\hat{h}})$ with $\partial_t \boldsymbol{u} \in L^2(\Omega_{\hat{h}})^3$, we have

$$\|\tilde{\boldsymbol{\alpha}} \times \partial_t \boldsymbol{u}\|^2_{L^2(\Omega_{\hat{h}})^3} \leq |\tilde{\boldsymbol{\alpha}}|^2 \|\partial_t \boldsymbol{u}\|^2_{L^2(\Omega_{\hat{h}})^3} = \sin^2\theta_1 \|\partial_t \boldsymbol{u}\|^2_{L^2(\Omega_{\hat{h}})^3} \leq \|\partial_t \boldsymbol{u}\|^2_{L^2(\Omega_{\hat{h}})^3} \quad (16)$$

and

$$\begin{aligned}
&\|\nabla^t_{\widetilde{M},\tilde{\boldsymbol{\alpha}}} \times \boldsymbol{u}\|^2_{L^2(\Omega_{\hat{h}})^3} \\
&= \int_{\Omega_{\hat{h}}} [(\nabla_{\widetilde{M}} - c^{-1}\tilde{\boldsymbol{\alpha}}\partial_t) \times \boldsymbol{u}] \cdot [(\nabla_{\widetilde{M}} - c^{-1}\tilde{\boldsymbol{\alpha}}\partial_t) \times \boldsymbol{u}] \mathrm{d}\boldsymbol{x} \\
&= \int_{\Omega_{\hat{h}}} (\nabla_{\widetilde{M}} \times \boldsymbol{u}) \cdot (\nabla_{\widetilde{M}} \times \boldsymbol{u}) \mathrm{d}\boldsymbol{x} - 2c^{-1} \int_{\Omega_{\hat{h}}} (\nabla_{\widetilde{M}} \times \boldsymbol{u}) \cdot (\tilde{\boldsymbol{\alpha}} \times \partial_t \boldsymbol{u}) \mathrm{d}\boldsymbol{x} \\
&\quad + c^{-2} \int_{\Omega_{\hat{h}}} (\tilde{\boldsymbol{\alpha}} \times \partial_t \boldsymbol{u}) \cdot (\tilde{\boldsymbol{\alpha}} \times \partial_t \boldsymbol{u}) \mathrm{d}\boldsymbol{x} \\
&\geq \|\nabla_{\widetilde{M}} \times \boldsymbol{u}\|^2_{L^2(\Omega_{\hat{h}})^3} + c^{-2} \|\tilde{\boldsymbol{\alpha}} \times \partial_t \boldsymbol{u}\|^2_{L^2(\Omega_{\hat{h}})^3} \\
&\quad - \frac{1}{2} \|\nabla_{\widetilde{M}} \times \boldsymbol{u}\|^2_{L^2(\Omega_{\hat{h}})^3} - 2c^{-2} \|\tilde{\boldsymbol{\alpha}} \times \partial_t \boldsymbol{u}\|^2_{L^2(\Omega_{\hat{h}})^3} \\
&= \frac{1}{2} \|\nabla_{\widetilde{M}} \times \boldsymbol{u}\|^2_{L^2(\Omega_{\hat{h}})^3} - c^{-2} \|\tilde{\boldsymbol{\alpha}} \times \partial_t \boldsymbol{u}\|^2_{L^2(\Omega_{\hat{h}})^3} \\
&\geq \frac{1}{2} \|\nabla_{\widetilde{M}} \times \boldsymbol{u}\|^2_{L^2(\Omega_{\hat{h}})^3} - c^{-2} \|\partial_t \boldsymbol{u}\|^2_{L^2(\Omega_{\hat{h}})^3}.
\end{aligned} \quad (17)$$

From (16) and (17), it is easy to obtain that

$$\begin{aligned}
&\frac{1}{2} \|\nabla_{\widetilde{M}} \times \boldsymbol{u}\|^2_{L^2(\Omega_{\hat{h}})^3} + \frac{1}{2} \|\partial_t \boldsymbol{u}\|^2_{L^2(\Omega_{\hat{h}})^3} \\
&\leq \|\nabla^t_{\widetilde{M},\tilde{\boldsymbol{\alpha}}} \times \boldsymbol{u}\|^2_{L^2(\Omega_{\hat{h}})^3} + \left(c^{-2} + \frac{1}{2}\right) \|\partial_t \boldsymbol{u}\|^2_{L^2(\Omega_{\hat{h}})^3} \\
&\leq \max\left\{1, c^{-2} + \frac{1}{2}\right\} \left(\|\nabla^t_{\widetilde{M},\tilde{\boldsymbol{\alpha}}} \times \boldsymbol{u}\|^2_{L^2(\Omega_{\hat{h}})^3} + \|\partial_t \boldsymbol{u}\|^2_{L^2(\Omega_{\hat{h}})^3}\right),
\end{aligned}$$

which gives

$$\begin{aligned}
&2\|\nabla^t_{\widetilde{M},\tilde{\boldsymbol{\alpha}}} \times \boldsymbol{u}\|^2_{L^2(\Omega_{\hat{h}})^3} + \|\partial_t \boldsymbol{u}\|^2_{L^2(\Omega_{\hat{h}})^3} \\
&\geq \frac{1}{\max\{2, 2c^{-2}+1\}} \left(\|\nabla_{\widetilde{M}} \times \boldsymbol{u}\|^2_{L^2(\Omega_{\hat{h}})^3} + \|\partial_t \boldsymbol{u}\|^2_{L^2(\Omega_{\hat{h}})^3}\right).
\end{aligned}$$

On the other hand, we have

$$
\begin{aligned}
&\|\nabla^t_{\widetilde{M},\tilde{\boldsymbol{\alpha}}} \times \boldsymbol{u}\|^2_{L^2(\Omega_{\hat{h}})^3}\\
&= \int_{\Omega_{\hat{h}}} [(\nabla_{\widetilde{M}} - c^{-1}\tilde{\boldsymbol{\alpha}}\partial_t) \times \boldsymbol{u}] \cdot [(\nabla_{\widetilde{M}} - c^{-1}\tilde{\boldsymbol{\alpha}}\partial_t) \times \boldsymbol{u}]\mathrm{d}\boldsymbol{x}\\
&= \int_{\Omega_{\hat{h}}} (\nabla_{\widetilde{M}} \times u) \cdot (\nabla_{\widetilde{M}} \times \boldsymbol{u})\mathrm{d}\boldsymbol{x} - 2c^{-1} \int_{\Omega_{\hat{h}}} (\nabla_{\widetilde{M}} \times \boldsymbol{u}) \cdot (\tilde{\boldsymbol{\alpha}} \times \partial_t \boldsymbol{u})\mathrm{d}\boldsymbol{x}\\
&\quad + c^{-2} \int_{\Omega_{\hat{h}}} (\tilde{\boldsymbol{\alpha}} \times \partial_t \boldsymbol{u}) \cdot (\tilde{\boldsymbol{\alpha}} \times \partial_t \boldsymbol{u})\mathrm{d}\boldsymbol{x}\\
&\le \frac{3}{2}\|\nabla_{\widetilde{M}} \times \boldsymbol{u}\|^2_{L^2(\Omega_{\hat{h}})^3} + 3c^{-2}\|\tilde{\boldsymbol{\alpha}} \times \partial_t \boldsymbol{u}\|^2_{L^2(\Omega_{\hat{h}})^3}\\
&\le \frac{3}{2}\|\nabla_{\widetilde{M}} \times \boldsymbol{u}\|^2_{L^2(\Omega_{\hat{h}})^3} + 3c^{-2}\|\partial_t \boldsymbol{u}\|^2_{L^2(\Omega_{\hat{h}})^3} \qquad (18)
\end{aligned}
$$

and

$$
\begin{aligned}
&2\|\nabla^t_{\widetilde{M},\tilde{\boldsymbol{\alpha}}} \times \boldsymbol{u}\|^2_{L^2(\Omega_{\hat{h}})^3} + \|\partial_t \boldsymbol{u}\|^2_{L^2(\Omega_{\hat{h}})^3}\\
&\le 2\left(\frac{3}{2}\|\nabla_{\widetilde{M}} \times \boldsymbol{u}\|^2_{L^2(\Omega_{\hat{h}})^3} + 3c^{-2}\|\partial_t \boldsymbol{u}\|^2_{L^2(\Omega_{\hat{h}})^3}\right) + \|\partial_t \boldsymbol{u}\|^2_{L^2(\Omega_{\hat{h}})^3}\\
&= 3\|\nabla_{\widetilde{M}} \times \boldsymbol{u}\|^2_{L^2(\Omega_{\hat{h}})^3} + \left(6c^{-2} + 1\right)\|\partial_t \boldsymbol{u}\|^2_{L^2(\Omega_{\hat{h}})^3}\\
&\le \max\left\{3, 6c^{-2} + 1\right\}(\|\nabla_{\widetilde{M}} \times \boldsymbol{u}\|^2_{L^2(\Omega_{\hat{h}})^3} + \|\partial_t \boldsymbol{u}\|^2_{L^2(\Omega_{\hat{h}})^3}). \qquad (19)
\end{aligned}
$$

Combining (18) and (19) gives

$$
\begin{aligned}
&\frac{1}{\max\{2, 2c^{-2} + 1\}}(\|\nabla_{\widetilde{M}} \times \boldsymbol{u}\|^2_{L^2(\Omega_{\hat{h}})^3} + \|\partial_t \boldsymbol{u}\|^2_{L^2(\Omega_{\hat{h}})^3})\\
&\le 2\|\nabla^t_{\widetilde{M},\tilde{\boldsymbol{\alpha}}} \times \boldsymbol{u}\|^2_{L^2(\Omega_{\hat{h}})^3} + \|\partial_t \boldsymbol{u}\|^2_{L^2(\Omega_{\hat{h}})^3}\\
&\le \max\left\{3, 6c^{-2} + 1\right\}(\|\nabla_{\widetilde{M}} \times \boldsymbol{u}\|^2_{L^2(\Omega_{\hat{h}})^3} + \|\partial_t \boldsymbol{u}\|^2_{L^2(\Omega_{\hat{h}})^3}). \qquad (20)
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
&\frac{1}{\max\{2, 2c^{-2} + 1\}}\|\nabla_{\widetilde{M}} \cdot \boldsymbol{u}\|^2_{L^2(\Omega_{\hat{h}})} \le 2\|\nabla^t_{\widetilde{M},\tilde{\boldsymbol{\alpha}}} \cdot \boldsymbol{u}\|^2_{L^2(\Omega_{\hat{h}})} + \|\partial_t \boldsymbol{u}\|^2_{L^2(\Omega_{\hat{h}})^3}\\
&\le \max\left\{3, 6c^{-2} + 1\right\}(\|\nabla_{\widetilde{M}} \cdot \boldsymbol{u}\|^2_{L^2(\Omega_{\hat{h}})} + \|\partial_t \boldsymbol{u}\|^2_{L^2(\Omega_{\hat{h}})^3}). \qquad (21)
\end{aligned}
$$

The proof is completed by combining (20) and (21). □

Lemma 2.3. *For any function $\boldsymbol{u} \in L^2(\Omega_{\hat{h}})^3$ with $\partial_t \boldsymbol{u} \in L^2(\Omega_{\hat{h}})^3$, we have*

$$
\max_{t\in[0,T]} \|\boldsymbol{u}\|^2_{L^2(\Omega_{\hat{h}})^3} \le 2\|\boldsymbol{u}|_{t=0}\|^2_{L^2(\Omega_{\hat{h}})^3} + 2T^2 \max_{t\in[0,T]} \|\partial_t \boldsymbol{u}\|^2_{L^2(\Omega_{\hat{h}})^3}.
$$

Proof. For any $t \in (0, T]$, we have

$$
\boldsymbol{u}(\boldsymbol{x}, t) = \boldsymbol{u}(\boldsymbol{x}, 0) + \int_0^t \partial_t \boldsymbol{u}(\boldsymbol{x}, \tau)\mathrm{d}\tau, \quad \boldsymbol{x} \in \Omega_{\hat{h}}.
$$

It follows from the Cauchy–Schwarz inequality that

$$\begin{aligned}\|\boldsymbol{u}(\cdot,t)\|_{L^2(\Omega_{\hat{h}})^3} &\le \|\boldsymbol{u}|_{t=0}\|_{L^2(\Omega_{\hat{h}})^3} + \left\|\int_0^t \partial_t \boldsymbol{u}(\boldsymbol{x},\tau)\mathrm{d}\tau\right\|_{L^2(\Omega_{\hat{h}})^3} \\ &\le \|\boldsymbol{u}|_{t=0}\|_{L^2(\Omega_{\hat{h}})^3} + T^{\frac{1}{2}}\left(\int_0^T \|\partial_t \boldsymbol{u}(\cdot,t)\|^2_{L^2(\Omega_{\hat{h}})^3}\mathrm{d}t\right)^{\frac{1}{2}}.\end{aligned}$$

A simple calculation yields

$$\begin{aligned}\|\boldsymbol{u}(\cdot,t)\|^2_{L^2(\Omega_{\hat{h}})^3} &\le 2\|\boldsymbol{u}|^2_{t=0}\|_{L^2(\Omega_{\hat{h}})^3} + 2T\int_0^T \|\partial_t \boldsymbol{u}(\cdot,t)\|^2_{L^2(\Omega_{\hat{h}})^3}\mathrm{d}t \\ &\le 2\|\boldsymbol{u}|^2_{t=0}\|_{L^2(\Omega_{\hat{h}})^3} + 2T^2 \max_{t\in[0,T]} \|\partial_t \boldsymbol{u}\|^2_{L^2(\Omega_{\hat{h}})^3}, \quad \forall t \in (0,T],\end{aligned}$$

which completes the proof. □

3. **Well-posedness.** In this section, we present the main results of this work, which include the well-posedness and stability of the scattering problem and related a priori estimates.

3.1. **Existence and uniqueness.** Suppose that $\widetilde{\boldsymbol{W}} = \widetilde{\boldsymbol{W}}(\boldsymbol{r},\tilde{x}_3,t)$ is a smooth solution of (12) and define the associated mapping $\mathbf{W} : [0,T] \to H_{\mathrm{per},0}(\widetilde{\mathrm{curl}},\widetilde{\mathrm{div}};\Omega_{\hat{h}})$ by

$$[\mathbf{W}(t)](\boldsymbol{x}) := \widetilde{\boldsymbol{W}}(\boldsymbol{x},t), \quad \boldsymbol{x}\in\Omega_{\hat{h}},\ t\in[0,T].$$

Introduce the function $\mathbf{f}_1 : [0,T] \to L^2(\Omega_{\hat{h}})^3$ by

$$[\mathbf{f}_1(t)](\boldsymbol{x}) := \tilde{\boldsymbol{f}}_1(\boldsymbol{x},t), \quad \boldsymbol{x}\in\Omega_{\hat{h}},\ t\in[0,T].$$

Multiplying a test function $\boldsymbol{Q} \in H_{\mathrm{per},0}(\widetilde{\mathrm{curl}},\widetilde{\mathrm{div}};\Omega_{\hat{h}})$ on both sides of the first equation in (12) and using the integration by parts, we obtain

$$(\varepsilon\mathbf{W}'',\boldsymbol{Q}) + a[\mathbf{W},\boldsymbol{Q};t] = (\mathbf{f}_1,\boldsymbol{Q}), \quad t\in[0,T], \tag{22}$$

where

$$a[\mathbf{W},\boldsymbol{Q};t] = \int_{\Omega_{\hat{h}}} (\mu^{-1}\nabla^t_{\widetilde{M},\tilde{\boldsymbol{\alpha}}}\times\mathbf{W})\cdot(\nabla^t_{\widetilde{M},\tilde{\boldsymbol{\alpha}}}\times\boldsymbol{Q})\mathrm{d}\boldsymbol{x}, \quad t\in[0,T]. \tag{23}$$

Here $(\cdot,\cdot)$ in (22) is the inner product in $L^2(\Omega_{\hat{h}})^3$. We seek a weak solution $\mathbf{W}$ satisfying $\mathbf{W}'' \in H^{-1}_{\mathrm{per},0}(\widetilde{\mathrm{curl}},\widetilde{\mathrm{div}};\Omega_{\hat{h}})$ for a.e. $t\in[0,T]$, where $H^{-1}_{\mathrm{per},0}(\widetilde{\mathrm{curl}},\widetilde{\mathrm{div}};\Omega_{\hat{h}})$ is the dual space of $H_{\mathrm{per},0}(\widetilde{\mathrm{curl}},\widetilde{\mathrm{div}};\Omega_{\hat{h}})$. Hence the inner product $(\cdot,\cdot)$ can also be interpreted as the pairing $\langle\cdot,\cdot\rangle$, which is defined between $H^{-1}_{\mathrm{per},0}(\widetilde{\mathrm{curl}},\widetilde{\mathrm{div}};\Omega_{\hat{h}})$ and $H_{\mathrm{per},0}(\widetilde{\mathrm{curl}},\widetilde{\mathrm{div}};\Omega_{\hat{h}})$.

Definition 3.1. We say that a function $\mathbf{W} \in L^2[0,T;H_{\mathrm{per},0}(\widetilde{\mathrm{curl}},\widetilde{\mathrm{div}};\Omega_{\hat{h}})]$ with $\mathbf{W}' \in L^2[0,T;L^2(\Omega_{\hat{h}})^3]$ and $\mathbf{W}'' \in L^2[0,T;H^{-1}_{\mathrm{per},0}(\widetilde{\mathrm{curl}},\widetilde{\mathrm{div}};\Omega_{\hat{h}})]$ is a weak solution of the initial boundary value problem (12) if it satisfies

(i) $(\varepsilon\mathbf{W}'',\boldsymbol{Q}) + a[\mathbf{W},\boldsymbol{Q};t] = (\mathbf{f}_1,\boldsymbol{Q})$, $\forall\boldsymbol{Q}\in H_{\mathrm{per},0}(\widetilde{\mathrm{curl}},\widetilde{\mathrm{div}};\Omega_{\hat{h}})$ and a.e. $t\in[0,T]$;

(ii) $\mathbf{W}(0) = \tilde{\boldsymbol{f}}_2, \quad \mathbf{W}'(0) = \tilde{\boldsymbol{f}}_3$.

We adopt the Galerkin method to construct the weak solution of the initial boundary value problem (12) by solving a finite dimensional approximation. We refer to [23] for the method to construct the weak solutions of the general second order differential equations. The method begins with selecting smooth functions $\boldsymbol{\xi}_k = \boldsymbol{\xi}_k(\boldsymbol{x}), k \in \mathbb{N}$ by requiring that $\{\boldsymbol{\xi}_k\}_{k=1}^{\infty}$ is an orthogonal basis of $H_{\mathrm{per},0}(\widetilde{\mathrm{curl}}, \widetilde{\mathrm{div}}; \Omega_{\hat{h}})$ and $\{\boldsymbol{\xi}_k\}_{k=1}^{\infty}$ is an orthogonal basis of $L^2(\Omega_{\hat{h}})^3$. For a positive integer m, let

$$\mathbf{W}_m(t) := \sum_{k=1}^{m} w_m^k(t)\boldsymbol{\xi}_k, \tag{24}$$

where the coefficients $w_m^k(t)$ satisfy

$$w_m^k(0) = (\tilde{\boldsymbol{f}}_2, \boldsymbol{\xi}_k), \qquad \left.\frac{\mathrm{d}w_m^k}{\mathrm{d}t}\right|_{t=0} = (\tilde{\boldsymbol{f}}_3, \boldsymbol{\xi}_k), \tag{25}$$

and for $t \in (0, T]$, we have

$$(\varepsilon \mathbf{W}_m^{''}, \varepsilon^{-1}\boldsymbol{\xi}_k) + a[\mathbf{W}_m, \varepsilon^{-1}\boldsymbol{\xi}_k; t] = (\mathbf{f_1}, \varepsilon^{-1}\boldsymbol{\xi}_k), \quad k = 1, \ldots, m. \tag{26}$$

Theorem 3.2. *For each $m \in \mathbb{N}$, there exists a unique function $\mathbf{W}_m$, which is given in the form of* (24) *and satisfies* (25)–(26)*.*

Proof. Since $\{\boldsymbol{\xi}_k\}_{k=1}^{\infty}$ is an orthogonal basis of $H_{\mathrm{per},0}(\widetilde{\mathrm{curl}}, \widetilde{\mathrm{div}}; \Omega_{\hat{h}})$, we have from (24) that

$$(\varepsilon \mathbf{W}_m^{''}(t), \varepsilon^{-1}\boldsymbol{\xi}_k) = \sum_{l=1}^{m} \left\{ \int_{\Omega_{\hat{h}}} \boldsymbol{\xi}_l \cdot \boldsymbol{\xi}_k \mathrm{d}\boldsymbol{x} \right\} \frac{\mathrm{d}^2 w_m^l(t)}{\mathrm{d}t^2} = \frac{\mathrm{d}^2 w_m^k(t)}{\mathrm{d}t^2}. \tag{27}$$

It follows from (23) that

$$\begin{aligned}
&a[\mathbf{W}_m, \varepsilon^{-1}\boldsymbol{\xi}_k; t] \\
&= \int_{\Omega_{\hat{h}}} [\mu^{-1}\nabla^t_{\widetilde{M},\tilde{\boldsymbol{\alpha}}} \times \mathbf{W}_m] \cdot [\nabla^t_{\widetilde{M},\tilde{\boldsymbol{\alpha}}} \times (\varepsilon^{-1}\boldsymbol{\xi}_k)]\mathrm{d}\boldsymbol{x} \\
&= \int_{\Omega_{\hat{h}}} [\mu^{-1}(\nabla_{\widetilde{M}} - c^{-1}\tilde{\boldsymbol{\alpha}}\partial_t) \times \mathbf{W}_m] \cdot [\nabla_{\widetilde{M}} \times (\varepsilon^{-1}\boldsymbol{\xi}_k)]\mathrm{d}\boldsymbol{x} \\
&= \int_{\Omega_{\hat{h}}} [\mu^{-1}\nabla_{\widetilde{M}} \times \mathbf{W}_m] \cdot [\nabla_{\widetilde{M}} \times (\varepsilon^{-1}\boldsymbol{\xi}_k)]\mathrm{d}\boldsymbol{x} \\
&\qquad - c^{-1}\int_{\Omega_{\hat{h}}} [\mu^{-1}\tilde{\boldsymbol{\alpha}} \times \partial_t\mathbf{W}_m] \cdot [\nabla_{\widetilde{M}} \times (\varepsilon^{-1}\boldsymbol{\xi}_k)]\mathrm{d}\boldsymbol{x} \\
&= \sum_{l=1}^{m} \left\{ \int_{\Omega_{\hat{h}}} [\mu^{-1}\nabla_{\widetilde{M}} \times \boldsymbol{\xi}_l] \cdot [\nabla_{\widetilde{M}} \times (\varepsilon^{-1}\boldsymbol{\xi}_k)]\mathrm{d}\boldsymbol{x} \right\} w_m^l(t) \\
&\qquad + \sum_{l=1}^{m} \left\{ - c^{-1}\int_{\Omega_{\hat{h}}} [\mu^{-1}\tilde{\boldsymbol{\alpha}} \times \boldsymbol{\xi}_l] \cdot [\nabla_{\widetilde{M}} \times (\varepsilon^{-1}\boldsymbol{\xi}_k)]\mathrm{d}\boldsymbol{x} \right\} \frac{\mathrm{d}w_m^l(t)}{\mathrm{d}t} \\
&= \sum_{l=1}^{m} b_k^l \frac{\mathrm{d}w_m^l(t)}{\mathrm{d}t} + \sum_{l=1}^{m} c_k^l w_m^l(t),
\end{aligned} \tag{28}$$

where

$$b_k^l = -c^{-1}\int_{\hat{h}}[\mu^{-1}\tilde{\boldsymbol{\alpha}}\times\boldsymbol{\xi}_l]\cdot[\nabla_{\widetilde{M}}\times(\varepsilon^{-1}\boldsymbol{\xi}_k)]\mathrm{d}\boldsymbol{x},$$
$$c_k^l = \int_{\hat{h}}[\mu^{-1}\nabla_{\widetilde{M}}\times\boldsymbol{\xi}_l]\cdot[\nabla_{\widetilde{M}}\times(\varepsilon^{-1}\boldsymbol{\xi}_k)]\mathrm{d}\boldsymbol{x},$$

$k,\ l = 1,\ldots,m$. Denote

$$f^k(t) = (\mathbf{f}_1(t), \varepsilon^{-1}\boldsymbol{\xi}_k), \quad k = 1,\ldots,m. \tag{29}$$

Substituting (27)–(29) into (26), we obtain the matrix form of the linear system of ordinary differential equations

$$\frac{\mathrm{d}^2\mathbf{w}_m(t)}{\mathrm{d}t^2} + \mathbf{B}\frac{\mathrm{d}\mathbf{w}_m(t)}{\mathrm{d}t} + \mathbf{C}\mathbf{w}_m(t) = \mathbf{f}(t), \quad t\in(0,T], \tag{30}$$

where $\mathbf{w}_m(t) = (w_m^1(t),\ldots,w_m^m(t))^\top$, $\mathbf{f}(t) = (f^1(t),\ldots,f^m(t))^\top$, $\mathbf{B} = [b_k^l]_{m\times m}$, $\mathbf{C} = [c_k^l]_{m\times m}$. Subject to the initial conditions (25), it follows from the standard theory of ordinary differential equations that there exists a unique C^2 function $\mathbf{w}_m(t)$ satisfying (30) and (25) for $t\in(0,T]$. □

Theorem 3.3. *There exists a positive constant C depending only on Ω, T and the coefficients of the problem (12) such that*

$$\max_{t\in[0,T]}\left(\|\mathbf{W}_m(t)\|^2_{H_{\mathrm{per},0}(\widetilde{\mathrm{curl}},\widetilde{\mathrm{div}};\Omega_{\hat{h}})} + \|\mathbf{W}_m'(t)\|^2_{L^2(\Omega_{\hat{h}})^3}\right)$$
$$+ \|\mathbf{W}_m''(t)\|^2_{L^2[0,T;H^{-1}_{\mathrm{per},0}(\widetilde{\mathrm{curl}},\widetilde{\mathrm{div}};\Omega_{\hat{h}})]}$$
$$\le C\left(\|\tilde{\boldsymbol{f}}_1\|^2_{L^2[0,T;L^2(\Omega_{\hat{h}})^3]} + \|\tilde{\boldsymbol{f}}_2\|^2_{H_{\mathrm{per},0}(\widetilde{\mathrm{curl}},\widetilde{\mathrm{div}};\Omega_{\hat{h}})} + \|\tilde{\boldsymbol{f}}_3\|^2_{L^2(\Omega_{\hat{h}})^3}\right), \quad m = 1,2,\ldots.$$

Proof. For any $\mathbf{W}_m\in H_{\mathrm{per},0}(\widetilde{\mathrm{curl}},\widetilde{\mathrm{div}};\Omega_{\hat{h}})$ with $\mathbf{W}_m'\in L^2(\Omega_{\hat{h}})^3$, we have

$$(\varepsilon\mathbf{W}_m'', \mathbf{W}_m') + a[\mathbf{W}_m, \mathbf{W}_m'; t] = (\mathbf{f}_1, \mathbf{W}_m'), \quad \text{for a.e. } t\in[0,T]. \tag{31}$$

Observe that

$$(\varepsilon\mathbf{W}_m'', \mathbf{W}_m') = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\|\sqrt{\varepsilon}\mathbf{W}_m'\|^2_{L^2(\Omega_{\hat{h}})^3}\right) \tag{32}$$

and

$$a[\mathbf{W}_m, \mathbf{W}_m'; t] = \int_{\Omega_{\hat{h}}}(\mu^{-1}\nabla^t_{\widetilde{M},\tilde{\boldsymbol{\alpha}}}\times\mathbf{W}_m)\cdot(\nabla^t_{\widetilde{M},\tilde{\boldsymbol{\alpha}}}\times\mathbf{W}_m')\mathrm{d}\boldsymbol{x}$$
$$= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\|\sqrt{\mu^{-1}}\ \nabla^t_{\widetilde{M},\tilde{\boldsymbol{\alpha}}}\times\mathbf{W}_m\|^2_{L^2(\Omega_{\hat{h}})^3}\right). \tag{33}$$

Combining (31)–(33) and using the Cauchy–Schwarz inequality, we obtain

$$\min\{\varepsilon_{\min}, \mu^{-1}_{\max}\}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\mathbf{W}_m'\|^2_{L^2(\Omega_{\hat{h}})^3} + \|\ \nabla^t_{\widetilde{M},\tilde{\boldsymbol{\alpha}}}\times\mathbf{W}_m\|^2_{L^2(\Omega_{\hat{h}})^3}\right)$$
$$\le \frac{\mathrm{d}}{\mathrm{d}t}\left(\|\sqrt{\varepsilon}\mathbf{W}_m'\|^2_{L^2(\Omega_{\hat{h}})^3} + \|\sqrt{\mu^{-1}}\ \nabla^t_{\widetilde{M},\tilde{\boldsymbol{\alpha}}}\times\mathbf{W}_m\|^2_{L^2(\Omega_{\hat{h}})^3}\right)$$
$$= 2(\mathbf{f}_1, \mathbf{W}_m') \le 2|(\mathbf{f}_1, \mathbf{W}_m')| \le \|\mathbf{W}_m'\|^2_{L^2(\Omega_{\hat{h}})^3} + \|\mathbf{f}_1\|^2_{L^2(\Omega_{\hat{h}})^3}. \tag{34}$$

It follows from Lemma 2.1 and the Cauchy–Schwarz inequality that

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\nabla^t_{\widetilde{M},\tilde{\boldsymbol{\alpha}}}\cdot\mathbf{W}_m\|^2_{L^2(\Omega_{\hat h})}\right)\\
&=\int_{\Omega_{\hat h}}(\nabla^t_{\widetilde{M},\tilde{\boldsymbol{\alpha}}}\cdot\mathbf{W}_m)\cdot(\nabla^t_{\widetilde{M},\tilde{\boldsymbol{\alpha}}}\cdot\mathbf{W}^{'}_m)d\boldsymbol{x}\\
&\le\frac{1}{2}\|\nabla^t_{\widetilde{M},\tilde{\boldsymbol{\alpha}}}\cdot\mathbf{W}_m\|^2_{L^2(\Omega_{\hat h})}+\frac{1}{2}\|\nabla^t_{\widetilde{M},\tilde{\boldsymbol{\alpha}}}\cdot\mathbf{W}^{'}_m\|^2_{L^2(\Omega_{\hat h})}\\
&=\frac{1}{2}\|\nabla^t_{\widetilde{M},\tilde{\boldsymbol{\alpha}}}\cdot\mathbf{W}_m\|^2_{L^2(\Omega_{\hat h})}+\frac{1}{2}\|-\varepsilon^{-1}(\nabla_{\widetilde{M}}\varepsilon)\cdot\mathbf{W}^{'}_m\|^2_{L^2(\Omega_{\hat h})}\\
&\le\frac{1}{2}\|\nabla^t_{\widetilde{M},\tilde{\boldsymbol{\alpha}}}\cdot\mathbf{W}_m\|^2_{L^2(\Omega_{\hat h})}+\varepsilon^{-2}_{\min}\max_{\boldsymbol{x}\in\Omega_h}\{|\nabla_{\widetilde{M}}\varepsilon(\boldsymbol{x})|^2\}\|\mathbf{W}^{'}_m\|^2_{L^2(\Omega_{\hat h})^3}\\
&\le\frac{1}{2}\|\nabla^t_{\widetilde{M},\tilde{\boldsymbol{\alpha}}}\cdot\mathbf{W}_m\|^2_{L^2(\Omega_{\hat h})}+\varepsilon^{-2}_{\min}d^2_\varepsilon\|\mathbf{W}^{'}_m\|^2_{L^2(\Omega_{\hat h})^3}. \qquad (35)
\end{aligned}
$$

Combining (34)–(35) yields

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\mathbf{W}^{'}_m\|^2_{L^2(\Omega_{\hat h})^3}+\|\ \nabla^t_{\widetilde{M},\tilde{\boldsymbol{\alpha}}}\times\mathbf{W}_m\|^2_{L^2(\Omega_{\hat h})^3}+\|\nabla^t_{\widetilde{M},\tilde{\boldsymbol{\alpha}}}\cdot\mathbf{W}_m\|^2_{L^2(\Omega_{\hat h})}\right)\\
&\le[\min\{\varepsilon_{\min},\mu^{-1}_{\max}\}]^{-1}\left(\|\mathbf{W}^{'}_m\|^2_{L^2(\Omega_{\hat h})^3}+\|\mathbf{f}_1\|^2_{L^2(\Omega_{\hat h})^3}\right)\\
&\quad+\|\nabla^t_{\widetilde{M},\tilde{\boldsymbol{\alpha}}}\cdot\mathbf{W}_m\|^2_{L^2(\Omega_{\hat h})}+2\varepsilon^{-2}_{\min}d^2_\varepsilon\|\mathbf{W}^{'}_m\|^2_{L^2(\Omega_{\hat h})^3}\\
&\le([\min\{\varepsilon_{\min},\mu^{-1}_{\max}\}]^{-1}+2\varepsilon^{-2}_{\min}d^2_\varepsilon)\|\mathbf{W}^{'}_m\|^2_{L^2(\Omega_{\hat h})^3}+\|\nabla^t_{\widetilde{M},\tilde{\boldsymbol{\alpha}}}\cdot\mathbf{W}_m\|^2_{L^2(\Omega_{\hat h})}\\
&\quad+[\min\{\varepsilon_{\min},\mu^{-1}_{\max}\}]^{-1}\|\mathbf{f}_1\|^2_{L^2(\Omega_{\hat h})^3}\\
&\le C_3\left(\|\mathbf{W}^{'}_m\|^2_{L^2(\Omega_{\hat h})^3}+\|\nabla^t_{\widetilde{M},\tilde{\boldsymbol{\alpha}}}\times\mathbf{W}_m\|^2_{L^2(\Omega_{\hat h})^3}+\|\nabla^t_{\widetilde{M},\tilde{\boldsymbol{\alpha}}}\cdot\mathbf{W}^{'}_m\|^2_{L^2(\Omega_{\hat h})}\right)\\
&\quad+C_4\|\mathbf{f}_1\|^2_{L^2(\Omega_{\hat h})^3}, \qquad (36)
\end{aligned}
$$

where $C_3=\max\{1,C_4+2\varepsilon^{-2}_{\min}d^2_\varepsilon\}$ and $C_4=[\min\{\varepsilon_{\min},\mu^{-1}_{\max}\}]^{-1}$.

Let

$$
\begin{aligned}
\alpha(t)&=\|\mathbf{W}^{'}_m\|^2_{L^2(\Omega_{\hat h})^3}+\|\nabla^t_{\widetilde{M},\tilde{\boldsymbol{\alpha}}}\times\mathbf{W}_m\|^2_{L^2(\Omega_{\hat h})^3}+\|\nabla^t_{\widetilde{M},\tilde{\boldsymbol{\alpha}}}\cdot\mathbf{W}_m\|^2_{L^2(\Omega_{\hat h})}\\
\delta(t)&=\|\mathbf{f}_1\|^2_{L^2(\Omega_{\hat h})^3}.
\end{aligned}
$$

Then (36) implies

$$
\alpha'(t)\le C_3\alpha(t)+C_4\delta(t),\quad t\in[0,T].
$$

Integrating the above inequality from 0 to t gives

$$
\begin{aligned}
\alpha(t)&\le C_3\int_0^t\alpha(s)\mathrm{d}s+\left(\alpha(0)+C_4\int_0^t\delta(s)\mathrm{d}s\right)\\
&\le C_3\int_0^t\alpha(s)\mathrm{d}s+\left(\alpha(0)+C_4\int_0^T\delta(s)\mathrm{d}s\right),\quad t\in[0,T],
\end{aligned}
$$

which gives after applying the Gronwall inequality that

$$
\begin{aligned}
\alpha(t)&\le e^{C_3t}\left(\alpha(0)+C_4\int_0^T\delta(s)\mathrm{d}s\right)\\
&\le e^{C_3T}\left(\alpha(0)+C_4\int_0^T\|\tilde{\boldsymbol{f}}_1\|^2_{L^2(\Omega_{\hat h})^3}\mathrm{d}t\right)
\end{aligned}
$$

$$= e^{C_3T}\left(\alpha(0) + C_4\|\tilde{\boldsymbol{f}}_1\|^2_{L^2[0,T;L^2(\Omega_{\hat{h}})^3]}\right), \quad t \in [0,T]. \tag{37}$$

It follows from Lemma 2.2 that

$$\begin{aligned}\alpha(0) &= \|\mathbf{W}'_m|_{t=0}\|^2_{L^2(\Omega_{\hat{h}})^3} + \|\nabla^t_{\widetilde{M},\tilde{\boldsymbol{\alpha}}} \times \mathbf{W}_m|_{t=0}\|^2_{L^2(\Omega_{\hat{h}})^3} + \|\nabla^t_{\widetilde{M},\tilde{\boldsymbol{\alpha}}} \cdot \mathbf{W}_m|_{t=0}\|^2_{L^2(\Omega_{\hat{h}})} \\ &\leq C_2\left(\|\mathbf{W}'_m|_{t=0}\|^2_{L^2(\Omega_{\hat{h}})^3} + \|\nabla_{\widetilde{M}} \times \mathbf{W}_m|_{t=0}\|^2_{L^2(\Omega_{\hat{h}})^3} + \|\nabla_{\widetilde{M}} \cdot \mathbf{W}_m|_{t=0}\|^2_{L^2(\Omega_{\hat{h}})}\right) \\ &= C_2\left(\|\tilde{\boldsymbol{f}}_3\|^2_{L^2(\Omega_{\hat{h}})^3} + \|\nabla_{\widetilde{M}} \times \tilde{\boldsymbol{f}}_2\|^2_{L^2(\Omega_{\hat{h}})^3} + \|\nabla_{\widetilde{M}} \cdot \tilde{\boldsymbol{f}}_2\|^2_{L^2(\Omega_{\hat{h}})}\right),\end{aligned} \tag{38}$$

where $C_2 = \max\left\{\frac{3}{2}, 3c^{-2} + \frac{1}{2}\right\}$. Combining (37) and (38), we obtain

$$\begin{aligned}\alpha(t) &= \|\mathbf{W}'_m\|^2_{L^2(\Omega_{\hat{h}})^3} + \|\nabla^t_{\widetilde{M},\tilde{\boldsymbol{\alpha}}} \times \mathbf{W}_m\|^2_{L^2(\Omega_{\hat{h}})^3} + \|\nabla^t_{\widetilde{M},\tilde{\boldsymbol{\alpha}}} \cdot \mathbf{W}_m\|^2_{L^2(\Omega_{\hat{h}})} \\ &\leq e^{C_3T}\Bigg[C_2\left(\|\tilde{\boldsymbol{f}}_3\|^2_{L^2(\Omega_{\hat{h}})^3} + \|\nabla_{\widetilde{M}} \times \tilde{\boldsymbol{f}}_2\|^2_{L^2(\Omega_{\hat{h}})^3} + \|\nabla_{\widetilde{M}} \cdot \tilde{\boldsymbol{f}}_2\|^2_{L^2(\Omega_{\hat{h}})}\right) \\ &\quad + C_4\|\tilde{\boldsymbol{f}}_1\|^2_{L^2[0,T;L^2(\Omega_{\hat{h}})^3]}\Bigg] \\ &\leq C\left(\|\tilde{\boldsymbol{f}}_1\|^2_{L^2[0,T;L^2(\Omega_{\hat{h}})^3]} + \|\nabla_{\widetilde{M}} \times \tilde{\boldsymbol{f}}_2\|^2_{L^2(\Omega_{\hat{h}})^3} + \|\nabla_{\widetilde{M}} \cdot \tilde{\boldsymbol{f}}_2\|^2_{L^2(\Omega_{\hat{h}})} \right. \\ &\quad \left. + \|\tilde{\boldsymbol{f}}_3\|^2_{L^2(\Omega_{\hat{h}})^3}\right).\end{aligned} \tag{39}$$

On the other hand, it follows from Lemma 2.2 that

$$\begin{aligned}\alpha(t) &= \|\mathbf{W}'_m\|^2_{L^2(\Omega_{\hat{h}})^3} + \|\nabla^t_{\widetilde{M},\tilde{\boldsymbol{\alpha}}} \times \mathbf{W}_m\|^2_{L^2(\Omega_{\hat{h}})^3} + \|\nabla^t_{\widetilde{M},\tilde{\boldsymbol{\alpha}}} \cdot \mathbf{W}_m\|^2_{L^2(\Omega_{\hat{h}})} \\ &\geq C_1\left(\|\mathbf{W}'_m\|^2_{L^2(\Omega_{\hat{h}})^3} + \|\nabla_{\widetilde{M}} \times \mathbf{W}_m\|^2_{L^2(\Omega_{\hat{h}})^3} + \|\nabla_{\widetilde{M}} \cdot \mathbf{W}_m\|^2_{L^2(\Omega_{\hat{h}})}\right).\end{aligned} \tag{40}$$

Noting that $t \in [0,T]$ is arbitrary and using (39)–(40), we have

$$\begin{aligned}&\max_{t\in[0,T]}\{\|\mathbf{W}'_m\|^2_{L^2(\Omega_{\hat{h}})^3} + \|\nabla_{\widetilde{M}} \times \mathbf{W}_m\|^2_{L^2(\Omega_{\hat{h}})^3} + \|\nabla_{\widetilde{M}} \cdot \mathbf{W}_m\|^2_{L^2(\Omega_{\hat{h}})}\} \\ &\quad \leq C\Big(\|\tilde{\boldsymbol{f}}_1\|^2_{L^2[0,T;L^2(\Omega_{\hat{h}})^3]} + \|\nabla_{\widetilde{M}} \times \tilde{\boldsymbol{f}}_2\|^2_{L^2(\Omega_{\hat{h}})^3} \\ &\qquad + \|\nabla_{\widetilde{M}} \cdot \tilde{\boldsymbol{f}}_2\|^2_{L^2(\Omega_{\hat{h}})} + \|\tilde{\boldsymbol{f}}_3\|^2_{L^2(\Omega_{\hat{h}})^3}\Big).\end{aligned} \tag{41}$$

We conclude from (41) and Lemma 2.3 that

$$\begin{aligned}&\max_{t\in[0,T]}\{\|\mathbf{W}'_m\|^2_{L^2(\Omega_{\hat{h}})^3} + \|\mathbf{W}_m\|^2_{H_{\mathrm{per},0}(\widetilde{\mathrm{curl}},\widetilde{\mathrm{div}};\Omega_{\hat{h}})}\} \\ &= \max_{t\in[0,T]}\{\|\mathbf{W}'_m\|^2_{L^2(\Omega_{\hat{h}})^3} + \|\mathbf{W}_m\|^2_{L^2(\Omega_{\hat{h}})^3} + \|\nabla_{\widetilde{M}} \times \mathbf{W}_m\|^2_{L^2(\Omega_{\hat{h}})^3} \\ &\quad + \|\nabla_{\widetilde{M}} \cdot \mathbf{W}_m\|^2_{L^2(\Omega_{\hat{h}})}\} \\ &\leq C\left(\|\tilde{\boldsymbol{f}}_1\|^2_{L^2[0,T;L^2(\Omega_{\hat{h}})^3]} + \|\nabla_{\widetilde{M}} \times \tilde{\boldsymbol{f}}_2\|^2_{L^2(\Omega_{\hat{h}})^3} + \|\nabla_{\widetilde{M}} \cdot \tilde{\boldsymbol{f}}_2\|^2_{L^2(\Omega_{\hat{h}})} + \|\tilde{\boldsymbol{f}}_3\|^2_{L^2(\Omega_{\hat{h}})^3}\right) \\ &\quad + \max_{t\in[0,T]}\|\mathbf{W}_m\|^2_{L^2(\Omega_{\hat{h}})^3} \\ &\leq C\left(\|\tilde{\boldsymbol{f}}_1\|^2_{L^2[0,T;L^2(\Omega_{\hat{h}})^3]} + \|\nabla_{\widetilde{M}} \times \tilde{\boldsymbol{f}}_2\|^2_{L^2(\Omega_{\hat{h}})^3} + \|\nabla_{\widetilde{M}} \cdot \tilde{\boldsymbol{f}}_2\|^2_{L^2(\Omega_{\hat{h}})} + \|\tilde{\boldsymbol{f}}_3\|^2_{L^2(\Omega_{\hat{h}})^3}\right) \\ &\quad + 2\|\tilde{\boldsymbol{f}}_2\|^2_{L^2(\Omega_{\hat{h}})^3} + 2T^2\max_{t\in[0,T]}\|\mathbf{W}'_m\|^2_{L^2(\Omega_{\hat{h}})^3}\end{aligned}$$

$$\leq C\Big(\|\tilde{\boldsymbol{f}}_1\|^2_{L^2[0,T;L^2(\Omega_{\hat{h}})^3]} + \|\tilde{\boldsymbol{f}}_2\|^2_{L^2(\Omega_{\hat{h}})^3} + \|\nabla_{\widetilde{M}} \times \tilde{\boldsymbol{f}}_2\|^2_{L^2(\Omega_{\hat{h}})^3} + \|\nabla_{\widetilde{M}} \cdot \tilde{\boldsymbol{f}}_2\|^2_{L^2(\Omega_{\hat{h}})}$$
$$+\|\tilde{\boldsymbol{f}}_3\|^2_{L^2(\Omega_{\hat{h}})^3}\Big) + 2T^2C\Big(\|\tilde{\boldsymbol{f}}_1\|^2_{L^2[0,T;L^2(\Omega_{\hat{h}})^3]} + \|\nabla_{\widetilde{M}} \times \tilde{\boldsymbol{f}}_2\|^2_{L^2(\Omega_{\hat{h}})^3}$$
$$+\|\nabla_{\widetilde{M}} \cdot \tilde{\boldsymbol{f}}_2\|^2_{L^2(\Omega_{\hat{h}})} + \|\tilde{\boldsymbol{f}}_3\|^2_{L^2(\Omega_{\hat{h}})^3}\Big)$$
$$\leq C\Big(\|\tilde{\boldsymbol{f}}_1\|^2_{L^2[0,T;L^2(\Omega_{\hat{h}})^3]} + \|\tilde{\boldsymbol{f}}_2\|^2_{H_{\mathrm{per},0}(\widetilde{\mathrm{curl}},\widetilde{\mathrm{div}};\Omega_{\hat{h}})} + \|\tilde{\boldsymbol{f}}_3\|^2_{L^2(\Omega_{\hat{h}})^3}\Big). \tag{42}$$

For any $\boldsymbol{v} \in H_{\mathrm{per},0}(\widetilde{\mathrm{curl}},\widetilde{\mathrm{div}};\Omega_{\hat{h}})$ with $\|\boldsymbol{v}\|_{H_{\mathrm{per},0}(\widetilde{\mathrm{curl}},\widetilde{\mathrm{div}};\Omega_{\hat{h}})} \leq 1$, let $\boldsymbol{v} = \boldsymbol{v}_1 + \boldsymbol{v}_2$, where $\boldsymbol{v}_1 \in \mathrm{span}\{\boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_m\}$ and $(\boldsymbol{v}_2, \boldsymbol{\xi}_k) = 0, k = 1, \ldots, m$. Thus we have $(\boldsymbol{v}_1, \boldsymbol{v}_2) = 0$ and

$$\|\boldsymbol{v}_1\|^2_{H_{\mathrm{per},0}(\widetilde{\mathrm{curl}},\widetilde{\mathrm{div}};\Omega_{\hat{h}})} = \|\boldsymbol{v}\|^2_{H_{\mathrm{per},0}(\widetilde{\mathrm{curl}},\widetilde{\mathrm{div}};\Omega_{\hat{h}})} - \|\boldsymbol{v}_2\|^2_{H_{\mathrm{per},0}(\widetilde{\mathrm{curl}},\widetilde{\mathrm{div}};\Omega_{\hat{h}})} \leq 1.$$

It follows from (24) and (26) that

$$\langle \varepsilon \mathbf{W}''_m, \boldsymbol{v}\rangle = \langle \varepsilon \mathbf{W}''_m, \boldsymbol{v}_1\rangle + \langle \varepsilon \mathbf{W}''_m, \boldsymbol{v}_2\rangle = \langle \varepsilon \mathbf{W}''_m, \boldsymbol{v}_1\rangle = (\mathbf{f}_1, \boldsymbol{v}_1) - a[\mathbf{W}_m, \boldsymbol{v}_1; t],$$

which gives

$$\begin{aligned}
&\varepsilon_{\min}|\langle \mathbf{W}''_m, \boldsymbol{v}\rangle| \\
&= \varepsilon_{\min}|\langle \mathbf{W}''_m, \boldsymbol{v}_1\rangle| = |\langle \varepsilon_{\min}\mathbf{W}''_m, \boldsymbol{v}_1\rangle| \\
&\leq |\langle \varepsilon \mathbf{W}''_m, \boldsymbol{v}_1\rangle| = |(\mathbf{f}_1, \boldsymbol{v}_1) - a[\mathbf{W}_m, \boldsymbol{v}_1; t]| \\
&\leq |(\mathbf{f}_1, \boldsymbol{v}_1)| + |a[\mathbf{W}_m, \boldsymbol{v}_1; t]| \\
&\leq \|\tilde{\boldsymbol{f}}_1\|_{L^2(\Omega_{\hat{h}})^3}\|\boldsymbol{v}_1\|_{L^2(\Omega_{\hat{h}})^3} + \mu_{\min}^{-1}\|\nabla^t_{\widetilde{M},\tilde{\boldsymbol{\alpha}}} \times \mathbf{W}_m\|_{L^2(\Omega_{\hat{h}})^3}\|\nabla^t_{\widetilde{M},\tilde{\boldsymbol{\alpha}}} \times \boldsymbol{v}_1\|_{L^2(\Omega_{\hat{h}})^3} \\
&\leq \max\{1, \mu_{\min}^{-1}\}\Big(\|\tilde{\boldsymbol{f}}_1\|_{L^2(\Omega_{\hat{h}})^3} + \|\nabla^t_{\widetilde{M},\tilde{\boldsymbol{\alpha}}} \times \mathbf{W}_m\|_{L^2(\Omega_{\hat{h}})^3}\Big)\|\boldsymbol{v}_1\|_{H_{\mathrm{per},0}(\widetilde{\mathrm{curl}},\widetilde{\mathrm{div}};\Omega_{\hat{h}})} \\
&\leq \max\{1, \mu_{\min}^{-1}\}\Big(\|\tilde{\boldsymbol{f}}_1\|_{L^2(\Omega_{\hat{h}})^3} + \|\nabla^t_{\widetilde{M},\tilde{\boldsymbol{\alpha}}} \times \mathbf{W}_m\|_{L^2(\Omega_{\hat{h}})^3}\Big).
\end{aligned} \tag{43}$$

By (39) and (43), we have

$$\begin{aligned}
&\int_0^T \|\mathbf{W}''_m\|^2_{H^{-1}_{\mathrm{per},0}(\widetilde{\mathrm{curl}},\widetilde{\mathrm{div}};\Omega_{\hat{h}})}\mathrm{d}t = \int_0^T \sup_{\|\boldsymbol{v}\|_{H_{\mathrm{per},0}(\widetilde{\mathrm{curl}},\widetilde{\mathrm{div}};\Omega_{\hat{h}})}=1} |\langle \mathbf{W}''_m, \boldsymbol{v}\rangle|^2\mathrm{d}t \\
&\leq C\int_0^T \Big(\|\tilde{\boldsymbol{f}}_1\|_{L^2(\Omega_{\hat{h}})^3} + C(\|\tilde{\boldsymbol{f}}_1\|^2_{L^2[0,T;L^2(\Omega_{\hat{h}})^3]} + \|\tilde{\boldsymbol{f}}_2\|^2_{H_{\mathrm{per},0}(\widetilde{\mathrm{curl}},\widetilde{\mathrm{div}};\Omega_{\hat{h}})} \\
&\quad + \|\tilde{\boldsymbol{f}}_3\|^2_{L^2(\Omega_{\hat{h}})^3})^{\frac{1}{2}}\Big)^2\mathrm{d}t \\
&\leq C\Big(\|\tilde{\boldsymbol{f}}_1\|^2_{L^2[0,T;L^2(\Omega_{\hat{h}})^3]} + \|\tilde{\boldsymbol{f}}_2\|^2_{H_{\mathrm{per},0}(\widetilde{\mathrm{curl}},\widetilde{\mathrm{div}};\Omega_{\hat{h}})} + \|\tilde{\boldsymbol{f}}_3\|^2_{L^2(\Omega_{\hat{h}})^3}\Big).
\end{aligned} \tag{44}$$

The proof is completed after combining (42) and (44). □

Now we pass to limits in the Galerkin approximations to obtain the existence of a weak solution.

Theorem 3.4. *There exists a weak solution of the initial boundary value problem* (12).

Proof. It follows from the energy estimate in Theorem 3.3 that

$$\{\mathbf{W}_m\}_{m=1}^{\infty} \text{ is bounded in } L^2[0,T;H_{\rm per,0}(\widetilde{\rm curl},\widetilde{\rm div};\Omega_{\hat h})],$$
$$\{\mathbf{W}_m^{'}\}_{m=1}^{\infty} \text{ is bounded in } L^2[0,T;L^2(\Omega_{\hat h})^3],$$
$$\{\mathbf{W}_m^{''}\}_{m=1}^{\infty} \text{ is bounded in } L^2[0,T;H^{-1}_{\rm per,0}(\widetilde{\rm curl},\widetilde{\rm div};\Omega_{\hat h})].$$

Therefore, there exists a subsequence still denoted as $\{\mathbf{W}_m\}_{m=1}^{\infty}$ and $\mathbf{W}\in L^2[0,T;H_{\rm per,0}(\widetilde{\rm curl},\widetilde{\rm div};\Omega_{\hat h})]$ with $\mathbf{W}^{'}\in L^2[0,T;L^2(\Omega_{\hat h})^3]$ and $\mathbf{W}^{''}\in L^2[0,T;H^{-1}_{\rm per,0}(\widetilde{\rm curl},\widetilde{\rm div};\Omega_{\hat h})]$ such that

$$\begin{cases}\mathbf{W}_m\rightharpoonup\mathbf{W} & \text{weakly in } L^2[0,T;\ H_{\rm per,0}(\widetilde{\rm curl},\widetilde{\rm div};\Omega_{\hat h})],\\ \mathbf{W}_m^{'}\rightharpoonup\mathbf{W}^{'} & \text{weakly in } L^2[0,T;L^2(\Omega_{\hat h})^3],\\ \mathbf{W}_m^{''}\rightharpoonup\mathbf{W}^{''} & \text{weakly in } L^2[0,T;H^{-1}_{\rm per,0}(\widetilde{\rm curl},\widetilde{\rm div};\Omega_{\hat h})].\end{cases} \tag{45}$$

Next we fix an integer N and choose a function $\mathbf{u}\in C^1([0,T];\ H_{\rm per,0}(\widetilde{\rm curl},\widetilde{\rm div};\Omega_{\hat h}))$ of the form

$$\mathbf{u}(t)=\sum_{k=1}^{N}u^k(t)\boldsymbol{\xi}_k, \tag{46}$$

where $u^k, k=1,\ldots,N$ are smooth functions. Selecting $m\ge N$, then we have

$$\int_0^T\Big(\langle\varepsilon\mathbf{W}_m^{''},\mathbf{u}\rangle+a[\mathbf{W}_m,\mathbf{u};t]\Big)\,{\rm d}t=\int_0^T(\mathbf{f}_1,\mathbf{u}){\rm d}t. \tag{47}$$

Using (45) and taking the limits $m\to\infty$ in (47) yields

$$\int_0^T\Big(\langle\varepsilon\mathbf{W}^{''},\mathbf{u}\rangle+a[\mathbf{W},\mathbf{u};t]\Big)\,{\rm d}t=\int_0^T(\mathbf{f}_1,\mathbf{u}){\rm d}t, \tag{48}$$

which holds for any function $\mathbf{u}\in L^2[0,T;\ H_{\rm per,0}(\widetilde{\rm curl},\widetilde{\rm div};\Omega_{\hat h})]$ since functions of the form (46) are dense in this space. Moreover, we have from (48) that

$$\langle\varepsilon\mathbf{W}^{''},\mathbf{u}\rangle+a[\mathbf{W},\mathbf{u};t]=(\mathbf{f}_1,\mathbf{u}),\quad\forall\mathbf{u}\in H_{\rm per,0}(\widetilde{\rm curl},\widetilde{\rm div};\Omega_{\hat h}),\ \text{a.e. } t\in[0,T]$$

and

$$\mathbf{W}\in C([0,T];H_{\rm per,0}(\widetilde{\rm curl},\widetilde{\rm div};\Omega_{\hat h})),\quad \mathbf{W}^{'}\in C([0,T];L^2(\Omega_{\hat h})^3).$$

Next is to verify

$$\mathbf{W}|_{t=0}=\tilde{\boldsymbol{f}}_2,\quad \mathbf{W}^{'}|_{t=0}=\tilde{\boldsymbol{f}}_3. \tag{49}$$

Choose any function $\mathbf{u}\in C^2([0,T];H_{\rm per,0}(\widetilde{\rm curl},\widetilde{\rm div};\Omega_{\hat h}))$ with $\mathbf{u}(T)=\mathbf{u}^{'}(T)=0$. Using the integration by parts twice with respect to t in (48) gives

$$\begin{aligned}&\int_0^T\Big((\varepsilon\mathbf{u}^{''},\mathbf{W})+a[\mathbf{W},\mathbf{u};t]\Big)\,{\rm d}t\\&=\int_0^T(\mathbf{f}_1,\mathbf{u}){\rm d}t-(\varepsilon\mathbf{W}(0),\mathbf{u}^{'}(0))+(\varepsilon\mathbf{W}^{'}(0),\mathbf{u}(0)).\end{aligned} \tag{50}$$

Similarly, we have from (47) that

$$\begin{aligned}\int_0^T\Big((\varepsilon\mathbf{u}^{''},\mathbf{W}_m)+a[\mathbf{W}_m,\mathbf{u};t]\Big)\,{\rm d}t=\int_0^T(\mathbf{f}_1,\mathbf{u}){\rm d}t-(\varepsilon\mathbf{W}_m(0),\mathbf{u}^{'}(0))\\+(\varepsilon\mathbf{W}_m^{'}(0),\mathbf{u}(0)).\end{aligned} \tag{51}$$

Taking the limits $m \to \infty$ in (51), using (25) and (45) yields

$$\int_0^T \Big((\varepsilon \mathbf{u}^{''}, \mathbf{W}) + a[\mathbf{W}, \mathbf{u}; t] \Big) \, \mathrm{d}t = \int_0^T (\mathbf{f}_1, \mathbf{u}) \mathrm{d}t - (\varepsilon \tilde{\boldsymbol{f}}_2, \mathbf{u}^{'}(0)) + (\varepsilon \tilde{\boldsymbol{f}}_3, \mathbf{u}(0)). \quad (52)$$

Comparing (50) and (52), we conclude (49) since $\mathbf{u}(0)$ and $\mathbf{u}^{'}(0)$ are arbitrary. Hence $\mathbf{W}$ is a weak solution of the initial boundary value problem (12). □

Theorem 3.5. *The initial boundary value problem* (12) *has a unique weak solution.*

Proof. It suffices to show that $\mathbf{W} = 0$ if $\tilde{\boldsymbol{f}}_1 = \tilde{\boldsymbol{f}}_2 = \tilde{\boldsymbol{f}}_3 = 0$. Fix $0 \le s \le T$ and let

$$\mathbf{v}(t) := \begin{cases} \int_t^s \mathbf{W}(\tau) \mathrm{d}\tau & \text{if } 0 \le t \le s, \\ 0 & \text{if } s \le t \le T. \end{cases}$$

Then $\mathbf{v}(t) \in H_{\mathrm{per},0}(\widetilde{\mathrm{curl}}, \widetilde{\mathrm{div}}; \Omega_{\hat{h}})$ for each $t \in [0, T]$, and satisfies

$$\int_0^s \Big(\langle \varepsilon \mathbf{W}^{''}, \mathbf{v} \rangle + a[\mathbf{W}, \mathbf{v}; t] \Big) \, \mathrm{d}t = 0.$$

Since $\mathbf{W}^{'}(0) = 0 = \mathbf{v}(s)$, from the integration by parts, we obtain

$$\int_0^s \Big(-(\varepsilon \mathbf{W}^{'}, \mathbf{v}^{'}) + a[\mathbf{W}, \mathbf{v}; t] \Big) \, \mathrm{d}t = 0. \quad (53)$$

It is easy to note that $\mathbf{v}^{'}(t) = -\mathbf{W}(t), \ 0 \le t \le s$, from (53), we have

$$\begin{aligned} 0 &= \int_0^s \Big((\varepsilon \mathbf{W}^{'}, \mathbf{W}) - a[\mathbf{v}^{'}, \mathbf{v}; t] \Big) \, \mathrm{d}t \\ &= \frac{1}{2} \int_0^s \frac{\mathrm{d}}{\mathrm{d}t} \Big(\|\sqrt{\varepsilon} \mathbf{W}\|^2_{L^2(\Omega_{\hat{h}})^3} - a[\mathbf{v}, \mathbf{v}; t] \Big) \, \mathrm{d}t \\ &= \frac{1}{2} \int_0^s \frac{\mathrm{d}}{\mathrm{d}t} \Big(\|\sqrt{\varepsilon} \mathbf{W}\|^2_{L^2(\Omega_{\hat{h}})^3} - \|\sqrt{\mu^{-1}} \nabla^t_{\widetilde{M}, \tilde{\boldsymbol{\alpha}}} \times \mathbf{v}\|^2_{L^2(\Omega_{\hat{h}})^3} \Big) \, \mathrm{d}t, \end{aligned}$$

which gives

$$\|\sqrt{\varepsilon} \mathbf{W}(s)\|^2_{L^2(\Omega_{\hat{h}})^3} + \|\sqrt{\mu^{-1}} \nabla^t_{\widetilde{M}, \tilde{\boldsymbol{\alpha}}} \times \mathbf{v}(0)\|^2_{L^2(\Omega_{\hat{h}})^3} = 0, \quad s \in [0, T]. \quad (54)$$

Since $\|\sqrt{\mu^{-1}} \nabla^t_{\widetilde{M}, \tilde{\boldsymbol{\alpha}}} \times \mathbf{v}(0)\|^2_{L^2(\Omega_{\hat{h}})^3} \ge 0$, we obtain from (54) that $\mathbf{W} = 0$, which completes the proof. □

3.2. **Stability.** In this section we discuss the stability estimate for the unique weak solution of the initial boundary value problem (12) and present a priori estimates with an explicit dependence on the time.

Theorem 3.6. *Let $\widetilde{\boldsymbol{W}}$ be the unique weak solution of the initial boundary value problem* (12)*. Given $\tilde{\boldsymbol{f}}_1 \in L^1[0, T; L^2(\Omega_{\hat{h}})^3]$, $\tilde{\boldsymbol{f}}_2 \in H(\widetilde{\mathrm{curl}}; \Omega_{\hat{h}})$, $\tilde{\boldsymbol{f}}_3 \in L^2(\Omega_{\hat{h}})^3$, then there exists a positive constant C_5 such that*

$$\begin{aligned} &\max_{t \in [0,T]} \Big\{ \|\partial_t \widetilde{\boldsymbol{W}}(\cdot, t)\|^2_{L^2(\Omega_{\hat{h}})^3} + \|\nabla^t_{\widetilde{M}, \tilde{\boldsymbol{\alpha}}} \times \widetilde{\boldsymbol{W}}(\cdot, t)\|^2_{L^2(\Omega_{\hat{h}})^3} \Big\} \\ &\le C_5 \Big(\|\tilde{\boldsymbol{f}}_1\|^2_{L^1[0,T; L^2(\Omega_{\hat{h}})^3]} + \|\nabla_{\widetilde{M}} \times \tilde{\boldsymbol{f}}_2\|^2_{L^2(\Omega_{\hat{h}})^3} + \|\tilde{\boldsymbol{f}}_3\|^2_{L^2(\Omega_{\hat{h}})^3} \Big), \end{aligned}$$

where $C_5 = \max\{2\varepsilon^{-1}_{\min}, 2\mu^{-1}_{\min}, \varepsilon_{\max} + 4c^{-2}\mu^{-1}_{\min}\} / \min\{\frac{\varepsilon_{\min}}{2}, \mu^{-1}_{\max}\}$.

Proof. It follows from the discussion in the previous section that the initial boundary value problem (12) has a unique weak solution

$$\widetilde{\boldsymbol{W}}(\boldsymbol{x},t) \in L^2[0,T;H_{\rm per,0}(\widetilde{\rm curl},\widetilde{\rm div};\Omega_{\hat h})] \cap H^1[0,T;L^2(\Omega_{\hat h})^3].$$

For any $t\in[0,T]$, consider the energy function

$$E(t) = \|\sqrt{\varepsilon}\partial_t\widetilde{\boldsymbol{W}}\|^2_{L^2(\Omega_{\hat h})^3} + \|\sqrt{\mu^{-1}}\ \nabla^t_{\widetilde{M},\tilde{\boldsymbol{\alpha}}}\times\widetilde{\boldsymbol{W}}\|^2_{L^2(\Omega_{\hat h})^3}.$$

A simple calculation yields that

$$\begin{aligned}
&\int_0^t E'(\tau)\mathrm{d}\tau\\
&= E(t)-E(0)\\
&= \Big(\|\sqrt{\varepsilon}\partial_t\widetilde{\boldsymbol{W}}\|^2_{L^2(\Omega_{\hat h})^3} + \|\sqrt{\mu^{-1}}\ \nabla^t_{\widetilde{M},\tilde{\boldsymbol{\alpha}}}\times\widetilde{\boldsymbol{W}}\|^2_{L^2(\Omega_{\hat h})^3}\Big)\\
&\quad - \Big(\|\sqrt{\varepsilon}\partial_t\widetilde{\boldsymbol{W}}|_{t=0}\|^2_{L^2(\Omega_{\hat h})^3} + \|\sqrt{\mu^{-1}}\ \nabla^t_{\widetilde{M},\tilde{\boldsymbol{\alpha}}}\times\widetilde{\boldsymbol{W}}|_{t=0}\|^2_{L^2(\Omega_{\hat h})^3}\Big)\\
&= \Big(\|\sqrt{\varepsilon}\partial_t\widetilde{\boldsymbol{W}}\|^2_{L^2(\Omega_{\hat h})^3} + \|\sqrt{\mu^{-1}}\ \nabla^t_{\widetilde{M},\tilde{\boldsymbol{\alpha}}}\times\widetilde{\boldsymbol{W}}\|^2_{L^2(\Omega_{\hat h})^3}\Big)\\
&\quad - \Big(\|\sqrt{\varepsilon}\partial_t\widetilde{\boldsymbol{W}}|_{t=0}\|^2_{L^2(\Omega_{\hat h})^3} + \|\sqrt{\mu^{-1}}\ (\nabla_{\widetilde{M}}\times\widetilde{\boldsymbol{W}}|_{t=0} - c^{-1}\tilde{\boldsymbol{\alpha}}\times\partial_t\widetilde{\boldsymbol{W}}|_{t=0})\|^2_{L^2(\Omega_{\hat h})^3}\Big)\\
&= \Big(\|\sqrt{\varepsilon}\partial_t\widetilde{\boldsymbol{W}}\|^2_{L^2(\Omega_{\hat h})^3} + \|\sqrt{\mu^{-1}}\ \nabla^t_{\widetilde{M},\tilde{\boldsymbol{\alpha}}}\times\widetilde{\boldsymbol{W}}\|^2_{L^2(\Omega_{\hat h})^3}\Big)\\
&\quad - \Big(\|\sqrt{\varepsilon}\tilde{\boldsymbol{f}}_3\|^2_{L^2(\Omega_{\hat h})^3} + \|\sqrt{\mu^{-1}}\ (\nabla_{\widetilde{M}}\times\tilde{\boldsymbol{f}}_2 - c^{-1}\tilde{\boldsymbol{\alpha}}\times\tilde{\boldsymbol{f}}_3)\|^2_{L^2(\Omega_{\hat h})^3}\Big).
\end{aligned}\tag{55}$$

On the other hand, it follows from (12) and the integration by parts that

$$\begin{aligned}
\int_0^t E'(\tau)\mathrm{d}\tau &= 2\int_0^t\int_{\Omega_{\hat h}}\Big[\varepsilon\partial_t^2\widetilde{\boldsymbol{W}}\cdot\partial_t\widetilde{\boldsymbol{W}} + \mu^{-1}(\nabla^t_{\widetilde{M},\tilde{\boldsymbol{\alpha}}}\times\widetilde{\boldsymbol{W}})\cdot(\nabla^t_{\widetilde{M},\tilde{\boldsymbol{\alpha}}}\times\partial_t\widetilde{\boldsymbol{W}})\Big]\mathrm{d}\boldsymbol{x}\mathrm{d}t\\
&= 2\int_0^t\int_{\Omega_{\hat h}}\Big\{[\varepsilon\partial_t^2\widetilde{\boldsymbol{W}} + \nabla^t_{\widetilde{M},\tilde{\boldsymbol{\alpha}}}\times(\mu^{-1}(\nabla^t_{\widetilde{M},\tilde{\boldsymbol{\alpha}}}\times\widetilde{\boldsymbol{W}}))]\cdot\partial_t\widetilde{\boldsymbol{W}}\Big\}\mathrm{d}\boldsymbol{x}\mathrm{d}t\\
&= 2\int_0^t\int_{\Omega_{\hat h}}\tilde{\boldsymbol{f}}_1\cdot\partial_t\widetilde{\boldsymbol{W}}\mathrm{d}\boldsymbol{x}\mathrm{d}t\\
&\le 2\int_0^T\|\partial_t\widetilde{\boldsymbol{W}}\|_{L^2(\Omega_{\hat h})^3}\|\tilde{\boldsymbol{f}}_1\|_{L^2(\Omega_{\hat h})^3}\mathrm{d}t\\
&\le 2\max_{t\in[0,T]}\{\|\partial_t\widetilde{\boldsymbol{W}}(\cdot,t)\|_{L^2(\Omega_{\hat h})^3}\}\|\tilde{\boldsymbol{f}}_1\|_{L^1[0,T;L^2(\Omega_{\hat h})^3]}.
\end{aligned}\tag{56}$$

Combining (55)–(56) and using Young's inequality, we obtain

$$\begin{aligned}
&\|\sqrt{\varepsilon}\partial_t\widetilde{\boldsymbol{W}}\|^2_{L^2(\Omega_{\hat h})^3} + \|\sqrt{\mu^{-1}}\ \nabla^t_{\widetilde{M},\tilde{\boldsymbol{\alpha}}}\times\widetilde{\boldsymbol{W}}\|^2_{L^2(\Omega_{\hat h})^3}\\
&\le \max_{t\in[0,T]}\{\|\partial_t\widetilde{\boldsymbol{W}}(\cdot,t)\|_{L^2(\Omega_{\hat h})^3}\}\Big(2\|\tilde{\boldsymbol{f}}_1\|_{L^1[0,T;L^2(\Omega_{\hat h})^3]}\Big)\\
&\quad + \Big(\|\sqrt{\varepsilon}\tilde{\boldsymbol{f}}_3\|^2_{L^2(\Omega_{\hat h})^3} + \|\sqrt{\mu^{-1}}\ (\nabla_{\widetilde{M}}\times\tilde{\boldsymbol{f}}_2 - c^{-1}\tilde{\boldsymbol{\alpha}}\times\tilde{\boldsymbol{f}}_3)\|^2_{L^2(\Omega_{\hat h})^3}\Big)\\
&\le \frac{\varepsilon_{\min}}{2}\max_{t\in[0,T]}\{\|\partial_t\widetilde{\boldsymbol{W}}(\cdot,t)\|^2_{L^2(\Omega_{\hat h})^3}\} + \frac{2}{\varepsilon_{\min}}\|\tilde{\boldsymbol{f}}_1\|^2_{L^1[0,T;L^2(\Omega_{\hat h})^3]}\\
&\quad + \Big(\varepsilon_{\max}\|\tilde{\boldsymbol{f}}_3\|^2_{L^2(\Omega_{\hat h})^3} + 2\mu^{-1}_{\min}\|\nabla_{\widetilde{M}}\times\tilde{\boldsymbol{f}}_2\|^2_{L^2(\Omega_{\hat h})^3} + 2c^{-2}\mu^{-1}_{\min}\|\tilde{\boldsymbol{\alpha}}\times\tilde{\boldsymbol{f}}_3\|^2_{L^2(\Omega_{\hat h})^3}\Big)\\
&\le \frac{\varepsilon_{\min}}{2}\max_{t\in[0,T]}\{\|\partial_t\widetilde{\boldsymbol{W}}(\cdot,t)\|^2_{L^2(\Omega_{\hat h})^3}\} + \frac{2}{\varepsilon_{\min}}\|\tilde{\boldsymbol{f}}_1\|^2_{L^1[0,T;L^2(\Omega_{\hat h})^3]}
\end{aligned}$$

$$+ \left(\varepsilon_{\max}\|\tilde{\boldsymbol{f}}_3\|^2_{L^2(\Omega_{\hat{h}})^3} + 2\mu^{-1}_{\min}\|\nabla_{\widetilde{M}} \times \tilde{\boldsymbol{f}}_2\|^2_{L^2(\Omega_{\hat{h}})^3} + 4c^{-2}\mu^{-1}_{\min}\|\tilde{\boldsymbol{f}}_3\|^2_{L^2(\Omega_{\hat{h}})^3}\right)$$
$$\leq \frac{\varepsilon_{\min}}{2}\max_{t\in[0,T]}\{\|\partial_t\widetilde{\boldsymbol{W}}(\cdot,t)\|^2_{L^2(\Omega_{\hat{h}})^3}\} + \max\left\{2\varepsilon^{-1}_{\min}, 2\mu^{-1}_{\min}, \varepsilon_{\max} + 4c^{-2}\mu^{-1}_{\min}\right\}$$
$$\times\left(\|\tilde{\boldsymbol{f}}_1\|^2_{L^1[0,T;L^2(\Omega_{\hat{h}})^3]} + \|\nabla_{\widetilde{M}} \times \tilde{\boldsymbol{f}}_2\|^2_{L^2(\Omega_{\hat{h}})^3} + \|\tilde{\boldsymbol{f}}_3\|^2_{L^2(\Omega_{\hat{h}})^3}\right). \tag{57}$$

It is clear that

$$\|\sqrt{\varepsilon}\partial_t\widetilde{\boldsymbol{W}}\|^2_{L^2(\Omega_{\hat{h}})^3} + \|\sqrt{\mu^{-1}}\ \nabla^t_{\widetilde{M},\tilde{\boldsymbol{\alpha}}} \times \widetilde{\boldsymbol{W}}\|^2_{L^2(\Omega_{\hat{h}})^3}$$
$$\geq \varepsilon_{\min}\|\partial_t\widetilde{\boldsymbol{W}}\|^2_{L^2(\Omega_{\hat{h}})^3} + \mu^{-1}_{\max}\|\nabla^t_{\widetilde{M},\tilde{\boldsymbol{\alpha}}} \times \widetilde{\boldsymbol{W}}\|^2_{L^2(\Omega_{\hat{h}})^3}. \tag{58}$$

It follows from (57)–(58) that

$$\min\left\{\frac{\varepsilon_{\min}}{2}, \mu^{-1}_{\max}\right\}\max_{t\in[0,T]}\left\{\|\partial_t\widetilde{\boldsymbol{W}}(\cdot,t)\|^2_{L^2(\Omega_{\hat{h}})^3} + \|\nabla^t_{\widetilde{M},\tilde{\boldsymbol{\alpha}}} \times \widetilde{\boldsymbol{W}}\|^2_{L^2(\Omega_{\hat{h}})^3}\right\}$$
$$\leq \max\{2\varepsilon^{-1}_{\min}, 2\mu^{-1}_{\min}, \varepsilon_{\max} + 4c^{-2}\mu^{-1}_{\min}\}$$
$$\times\left(\|\tilde{\boldsymbol{f}}_1\|^2_{L^1[0,T;L^2(\Omega_{\hat{h}})^3]} + \|\nabla_{\widetilde{M}} \times \tilde{\boldsymbol{f}}_2\|^2_{L^2(\Omega_{\hat{h}})^3} + \|\tilde{\boldsymbol{f}}_3\|^2_{L^2(\Omega_{\hat{h}})^3}\right),$$

which completes the proof. □

Next we derive a priori estimates for the electric field with a minimum regularity requirement for the data and an explicit dependence on the time.

The variational problem of (12) is to find $\widetilde{\boldsymbol{W}} \in H_{\mathrm{per},0}(\widetilde{\mathrm{curl}}, \widetilde{\mathrm{div}}; \Omega_{\hat{h}})$ for $t \in [0,T]$ such that

$$\int_{\Omega_{\hat{h}}} \varepsilon\partial^2_t\widetilde{\boldsymbol{W}} \cdot \widetilde{\boldsymbol{Q}}\mathrm{d}\boldsymbol{x} + \int_{\Omega_{\hat{h}}} \mu^{-1}(\nabla^t_{\widetilde{M},\tilde{\boldsymbol{\alpha}}} \times \widetilde{\boldsymbol{W}}) \cdot (\nabla^t_{\widetilde{M},\tilde{\boldsymbol{\alpha}}} \times \widetilde{\boldsymbol{Q}})\mathrm{d}\boldsymbol{x}$$
$$= \int_{\Omega_{\hat{h}}} \tilde{\boldsymbol{f}}_1 \cdot \widetilde{\boldsymbol{Q}}\mathrm{d}\boldsymbol{x}, \quad \forall\widetilde{\boldsymbol{Q}} \in H_{\mathrm{per},0}(\widetilde{\mathrm{curl}}, \widetilde{\mathrm{div}}; \Omega_{\hat{h}}). \tag{59}$$

Theorem 3.7. *Let $\widetilde{\boldsymbol{W}}$ be the unique weak solution of the initial boundary value problem (12). Given $\tilde{\boldsymbol{f}}_1 \in L^1[0,T;L^2(\Omega_{\hat{h}})^3]$, $\tilde{\boldsymbol{f}}_2 \in H(\widetilde{\mathrm{curl}}; \Omega_{\hat{h}})$, $\tilde{\boldsymbol{f}}_3 \in L^2(\Omega_{\hat{h}})^3$, there exist positive constants C_7, C_8 such that*

$$\|\widetilde{\boldsymbol{W}}\|^2_{L^\infty[0,T;L^2(\Omega_{\hat{h}})^3]} \leq C_7\left(\|\tilde{\boldsymbol{f}}_1\|^2_{L^1(0,T;L^2[\Omega_{\hat{h}})^3]} + \|\tilde{\boldsymbol{f}}_2\|^2_{L^2(\Omega_{\hat{h}})^3} + \|\tilde{\boldsymbol{f}}_3\|^2_{L^2(\Omega_{\hat{h}})^3}\right),$$

and

$$\|\widetilde{\boldsymbol{W}}\|^2_{L^2[0,T;L^2(\Omega_{\hat{h}})^3]} \leq C_8\left(\|\tilde{\boldsymbol{f}}_1\|^2_{L^1(0,T;L^2[\Omega_{\hat{h}})^3]} + \|\tilde{\boldsymbol{f}}_2\|^2_{L^2(\Omega_{\hat{h}})^3} + \|\tilde{\boldsymbol{f}}_3\|^2_{L^2(\Omega_{\hat{h}})^3}\right),$$

where $C_7 = C_6\max\{8C_6T^2, 2C_6\varepsilon_{\max}, 8C_6\varepsilon^2_{\max}T^2\}, C_8 = C_7T$, and $C_6 = \left(\varepsilon_{\min} - \frac{|\tilde{\boldsymbol{\alpha}}|^2}{c^2\mu_{\min}}\right)^{-1}$.

Proof. Let $0 < s < T$ and define an auxiliary function

$$\Phi(\boldsymbol{x},t) = \int_t^s \widetilde{\boldsymbol{W}}(\boldsymbol{x},\tau)\mathrm{d}\tau, \quad \boldsymbol{x} \in \Omega_{\hat{h}}, \quad 0 \leq t \leq s.$$

It is clear to note that

$$\Phi(\boldsymbol{x},s) = 0, \quad \partial_t\Phi(\boldsymbol{x},t) = -\widetilde{\boldsymbol{W}}(\boldsymbol{x},t). \tag{60}$$

For any $\Psi(x,t) \in L^2[0,s;L^2(\Omega_{\hat{h}})^3]$, using integration by parts and (60), we have

$$\begin{aligned}\int_0^s \Psi(\boldsymbol{x},t)\cdot\Phi(\boldsymbol{x},t)\mathrm{d}t &= \int_0^s \left(\Psi(\boldsymbol{x},t)\cdot\int_t^s \widetilde{\boldsymbol{W}}(\boldsymbol{x},\tau)\mathrm{d}\tau\right)\mathrm{d}t\\
&= \int_0^s \left[\left(\int_0^t \Psi(\boldsymbol{x},\tau)\mathrm{d}\tau\right)'\cdot\left(\int_t^s \widetilde{\boldsymbol{W}}(\boldsymbol{x},\tau)\mathrm{d}\tau\right)\right]\mathrm{d}t\\
&= \left[\left(\int_0^t \Psi(\boldsymbol{x},\tau)\mathrm{d}\tau\right)\cdot\left(\int_t^s \widetilde{\boldsymbol{W}}(\boldsymbol{x},\tau)\mathrm{d}\tau\right)\right]\Bigg|_0^s\\
&\quad -\int_0^s \left[\left(\int_0^t \Psi(\boldsymbol{x},\tau)\mathrm{d}\tau\right)\cdot\left(\int_t^s \widetilde{\boldsymbol{W}}(\boldsymbol{x},\tau)\mathrm{d}\tau\right)'\right]\mathrm{d}t\\
&= -\int_0^s \left[\left(\int_0^t \Psi(\boldsymbol{x},\tau)\mathrm{d}\tau\right)\cdot\left(\int_t^s \widetilde{\boldsymbol{W}}(\boldsymbol{x},\tau)\mathrm{d}\tau\right)'\right]\mathrm{d}t\\
&= -\int_0^s \left[\left(\int_0^t \Psi(\boldsymbol{x},\tau)\mathrm{d}\tau\right)\cdot\left(-\widetilde{\boldsymbol{W}}(\boldsymbol{x},t)\right)\right]\mathrm{d}t\\
&= \int_0^s \left(\int_0^t \Psi(\boldsymbol{x},\tau)\mathrm{d}\tau\right)\cdot\widetilde{\boldsymbol{W}}(\boldsymbol{x},t)\mathrm{d}t. \qquad (61)\end{aligned}$$

Next, we take the test function $\widetilde{\boldsymbol{Q}} = \Phi$ in (59) and integrating from $t=0$ to $t=s$ yields that

$$\begin{aligned}&\int_0^s \left(\int_{\Omega_{\hat{h}}} \varepsilon\partial_t^2\widetilde{\boldsymbol{W}}\cdot\Phi\mathrm{d}\boldsymbol{x}\right)\mathrm{d}t + \int_0^s \left(\int_{\Omega_{\hat{h}}} \mu^{-1}(\nabla^t_{\widetilde{M},\tilde{\boldsymbol{\alpha}}}\times\widetilde{\boldsymbol{W}})\cdot(\nabla^t_{\widetilde{M},\tilde{\boldsymbol{\alpha}}}\times\Phi)\mathrm{d}\boldsymbol{x}\right)\mathrm{d}t\\
&= \int_0^s \left(\int_{\Omega_{\hat{h}}} \tilde{\boldsymbol{f}}_1\cdot\Phi\mathrm{d}\boldsymbol{x}\right)\mathrm{d}t. \qquad (62)\end{aligned}$$

From (60), we now derive that

$$\begin{aligned}&\int_0^s \left(\int_{\Omega_{\hat{h}}} \varepsilon\partial_t^2\widetilde{\boldsymbol{W}}\cdot\Phi\mathrm{d}\boldsymbol{x}\right)\mathrm{d}t = \int_{\Omega_{\hat{h}}} \varepsilon\left(\int_0^s \partial_t^2\widetilde{\boldsymbol{W}}\cdot\Phi\mathrm{d}t\right)\mathrm{d}\boldsymbol{x}\\
&= \int_{\Omega_{\hat{h}}} \varepsilon\left(\partial_t\widetilde{\boldsymbol{W}}\cdot\Phi\Big|_0^s - \int_0^s \partial_t\widetilde{\boldsymbol{W}}\cdot\partial_t\Phi\mathrm{d}t\right)\mathrm{d}\boldsymbol{x}\\
&= \int_{\Omega_{\hat{h}}} \varepsilon\left(\partial_t\widetilde{\boldsymbol{W}}(\boldsymbol{x},s)\cdot\Phi(\boldsymbol{x},s) - \partial_t\widetilde{\boldsymbol{W}}(\boldsymbol{x},0)\cdot\Phi(\boldsymbol{x},0) - \int_0^s \partial_t\widetilde{\boldsymbol{W}}\cdot(-\widetilde{\boldsymbol{W}})\mathrm{d}t\right)\mathrm{d}\boldsymbol{x}\\
&= \int_{\Omega_{\hat{h}}} \varepsilon\left(\int_0^s \partial_t\widetilde{\boldsymbol{W}}\cdot\widetilde{\boldsymbol{W}}\mathrm{d}t - \tilde{\boldsymbol{f}}_3(\boldsymbol{x})\cdot\Phi(\boldsymbol{x},0)\right)\mathrm{d}\boldsymbol{x}\\
&= \int_{\Omega_{\hat{h}}} \varepsilon\left(\int_0^s (\frac{1}{2}|\widetilde{\boldsymbol{W}}|^2)'\mathrm{d}t - \tilde{\boldsymbol{f}}_3(\boldsymbol{x})\cdot\Phi(\boldsymbol{x},0)\right)\mathrm{d}\boldsymbol{x}\\
&= \int_{\Omega_{\hat{h}}} \varepsilon\left(\frac{1}{2}|\widetilde{\boldsymbol{W}}(\boldsymbol{x},s)|^2 - \frac{1}{2}|\widetilde{\boldsymbol{W}}(\boldsymbol{x},0)|^2 - \tilde{\boldsymbol{f}}_3(\boldsymbol{x})\cdot\Phi(\boldsymbol{x},0)\right)\mathrm{d}\boldsymbol{x}\\
&= \frac{1}{2}\int_{\Omega_{\hat{h}}} |\sqrt{\varepsilon}\widetilde{\boldsymbol{W}}(\boldsymbol{x},s)|^2\mathrm{d}\boldsymbol{x} - \frac{1}{2}\int_{\Omega_{\hat{h}}} |\sqrt{\varepsilon}\tilde{\boldsymbol{f}}_2(\boldsymbol{x})|^2\mathrm{d}\boldsymbol{x} - \int_{\Omega_{\hat{h}}} \varepsilon\tilde{\boldsymbol{f}}_3(\boldsymbol{x})\cdot\Phi(\boldsymbol{x},0)\mathrm{d}\boldsymbol{x}\\
&= \frac{1}{2}\|\sqrt{\varepsilon}\widetilde{\boldsymbol{W}}(\cdot,s)\|^2_{L^2(\Omega_{\hat{h}})^3} - \frac{1}{2}\|\sqrt{\varepsilon}\tilde{\boldsymbol{f}}_2\|^2_{L^2(\Omega_{\hat{h}})^3} - \int_{\Omega_{\hat{h}}} \varepsilon\tilde{\boldsymbol{f}}_3(\boldsymbol{x})\cdot\Phi(\boldsymbol{x},0)\mathrm{d}\boldsymbol{x} \qquad (63)\end{aligned}$$

and

$$\begin{aligned}
&\int_0^s \left(\int_{\Omega_{\hat{h}}} \mu^{-1} (\nabla^t_{\widetilde{M},\tilde{\boldsymbol{\alpha}}} \times \widetilde{\boldsymbol{W}}) \cdot (\nabla^t_{\widetilde{M},\tilde{\boldsymbol{\alpha}}} \times \Phi) \mathrm{d}\boldsymbol{x} \right) \mathrm{d}t \\
&= \int_{\Omega_{\hat{h}}} \mu^{-1} \left(\int_0^s (\nabla_{\widetilde{M}} \times \widetilde{\boldsymbol{W}} - c^{-1} \tilde{\boldsymbol{\alpha}} \times \partial_t \widetilde{\boldsymbol{W}}) \cdot (\nabla_{\widetilde{M}} \times \Phi - c^{-1} \tilde{\boldsymbol{\alpha}} \times \partial_t \Phi) \mathrm{d}t \right) \mathrm{d}\boldsymbol{x} \\
&= \int_{\Omega_{\hat{h}}} \mu^{-1} \left(\int_0^s (\nabla_{\widetilde{M}} \times \widetilde{\boldsymbol{W}} - c^{-1} \tilde{\boldsymbol{\alpha}} \times \partial_t \widetilde{\boldsymbol{W}}) \cdot (\nabla_{\widetilde{M}} \times \Phi + c^{-1} \tilde{\boldsymbol{\alpha}} \times \widetilde{\boldsymbol{W}}) \mathrm{d}t \right) \mathrm{d}\boldsymbol{x} \\
&= \int_{\Omega_{\hat{h}}} \mu^{-1} \left(\int_0^s (\nabla_{\widetilde{M}} \times \widetilde{\boldsymbol{W}}) \cdot (\nabla_{\widetilde{M}} \times \Phi) \mathrm{d}t \right) \mathrm{d}\boldsymbol{x} \\
&\quad + c^{-1} \int_{\Omega_{\hat{h}}} \mu^{-1} \left(\int_0^s (\nabla_{\widetilde{M}} \times \widetilde{\boldsymbol{W}}) \cdot (\tilde{\boldsymbol{\alpha}} \times \widetilde{\boldsymbol{W}}) \mathrm{d}t \right) \mathrm{d}\boldsymbol{x} \\
&\quad - c^{-1} \int_{\Omega_{\hat{h}}} \mu^{-1} \left(\int_0^s (\tilde{\boldsymbol{\alpha}} \times \partial_t \widetilde{\boldsymbol{W}}) \cdot (\nabla_{\widetilde{M}} \times \Phi) \mathrm{d}t \right) \mathrm{d}\boldsymbol{x} \\
&\quad - c^{-2} \int_{\Omega_{\hat{h}}} \mu^{-1} \left(\int_0^s (\tilde{\boldsymbol{\alpha}} \times \partial_t \widetilde{\boldsymbol{W}}) \cdot (\tilde{\boldsymbol{\alpha}} \times \widetilde{\boldsymbol{W}}) \mathrm{d}t \right) \mathrm{d}\boldsymbol{x}. \qquad (64)
\end{aligned}$$

With the aid of (62) and (63), we have

$$\begin{aligned}
&\frac{1}{2} \|\sqrt{\varepsilon} \widetilde{\boldsymbol{W}}(\cdot, s)\|^2_{L^2(\Omega_{\hat{h}})^3} + \int_0^s \left(\int_{\Omega_{\hat{h}}} \mu^{-1} (\nabla^t_{\widetilde{M},\tilde{\boldsymbol{\alpha}}} \times \widetilde{\boldsymbol{W}}) \cdot (\nabla^t_{\widetilde{M},\tilde{\boldsymbol{\alpha}}} \times \Phi) \mathrm{d}\boldsymbol{x} \right) \mathrm{d}t \\
&= \int_0^s \left(\int_{\Omega_{\hat{h}}} \tilde{\boldsymbol{f}}_1 \cdot \Phi \mathrm{d}\boldsymbol{x} \right) \mathrm{d}t + \frac{1}{2} \|\sqrt{\varepsilon} \tilde{\boldsymbol{f}}_2\|^2_{L^2(\Omega_{\hat{h}})^3} + \int_{\Omega_{\hat{h}}} \varepsilon \tilde{\boldsymbol{f}}_3(\boldsymbol{x}) \cdot \Phi(\boldsymbol{x}, 0) \mathrm{d}\boldsymbol{x}. \qquad (65)
\end{aligned}$$

In what follows, we estimate the three terms of the right-hand side of (64) separately. By the property of integration, we have

$$\begin{aligned}
&\int_{\Omega_{\hat{h}}} \mu^{-1} \left(\int_0^s (\nabla_{\widetilde{M}} \times \widetilde{\boldsymbol{W}}) \cdot (\nabla_{\widetilde{M}} \times \Phi) \mathrm{d}t \right) \mathrm{d}\boldsymbol{x} \\
&= \int_{\Omega_{\hat{h}}} \mu^{-1} \left(\int_0^s \int_t^s (\nabla_{\widetilde{M}} \times \widetilde{\boldsymbol{W}}(\boldsymbol{x}, t)) \cdot (\nabla_{\widetilde{M}} \times \widetilde{\boldsymbol{W}}(\boldsymbol{x}, \tau)) \mathrm{d}\tau \mathrm{d}t \right) \mathrm{d}\boldsymbol{x} \\
&= \frac{1}{2} \int_{\Omega_{\hat{h}}} \mu^{-1} \left(\left(\nabla_{\widetilde{M}} \times \int_0^s \widetilde{\boldsymbol{W}}(\boldsymbol{x}, t) \mathrm{d}t \right) \cdot \left(\nabla_{\widetilde{M}} \times \int_0^s \widetilde{\boldsymbol{W}}(\boldsymbol{x}, \tau) \mathrm{d}\tau \right) \right) \mathrm{d}\boldsymbol{x} \\
&= \frac{1}{2} \int_{\Omega_{\hat{h}}} \mu^{-1} (\nabla_{\widetilde{M}} \times \Phi(\boldsymbol{x}, 0)) \cdot (\nabla_{\widetilde{M}} \times \Phi(\boldsymbol{x}, 0)) \mathrm{d}\boldsymbol{x} \\
&= \frac{1}{2} \|\sqrt{\mu^{-1}} \nabla_{\widetilde{M}} \times \Phi(\cdot, 0)\|^2_{L^2(\Omega_{\hat{h}})^3}. \qquad (66)
\end{aligned}$$

We obtain from (60) and using integration by parts that

$$\begin{aligned}
&c^{-1} \int_{\Omega_{\hat{h}}} \mu^{-1} \left(\int_0^s (\nabla_{\widetilde{M}} \times \widetilde{\boldsymbol{W}}) \cdot (\tilde{\boldsymbol{\alpha}} \times \widetilde{\boldsymbol{W}}) \mathrm{d}t \right) \mathrm{d}\boldsymbol{x} \\
&= c^{-1} \int_{\Omega_{\hat{h}}} \mu^{-1} \left(\int_0^s (\nabla_{\widetilde{M}} \times (-\partial_t \Phi)) \cdot (\tilde{\boldsymbol{\alpha}} \times \widetilde{\boldsymbol{W}}) \mathrm{d}t \right) \mathrm{d}\boldsymbol{x} \\
&= c^{-1} \int_{\Omega_{\hat{h}}} \mu^{-1} \Big((\nabla_{\widetilde{M}} \times \Phi(\boldsymbol{x}, 0)) \cdot (\tilde{\boldsymbol{\alpha}} \times \widetilde{\boldsymbol{W}}(\boldsymbol{x}, 0)) \Big) \mathrm{d}\boldsymbol{x}
\end{aligned}$$

$$+ c^{-1}\int_{\Omega_{\hat{h}}}\mu^{-1}\left(\int_0^s (\nabla_{\widetilde{M}}\times\Phi)\cdot(\tilde{\boldsymbol{\alpha}}\times\partial_t\widetilde{\boldsymbol{W}})\mathrm{d}t\right)\mathrm{d}\boldsymbol{x} \tag{67}$$

and

$$\begin{aligned}
&c^{-2}\int_{\Omega_{\hat{h}}}\mu^{-1}\left(\int_0^s (\tilde{\boldsymbol{\alpha}}\times\partial_t\widetilde{\boldsymbol{W}})\cdot(\tilde{\boldsymbol{\alpha}}\times\widetilde{\boldsymbol{W}})\mathrm{d}t\right)\mathrm{d}\boldsymbol{x}\\
&=\frac{c^{-2}}{2}\int_{\Omega_{\hat{h}}}\mu^{-1}\left(\int_0^s \frac{\mathrm{d}}{\mathrm{d}t}|\tilde{\boldsymbol{\alpha}}\times\widetilde{\boldsymbol{W}}|^2\mathrm{d}t\right)\mathrm{d}\boldsymbol{x}\\
&=\frac{c^{-2}}{2}|\tilde{\boldsymbol{\alpha}}|^2\int_{\Omega_{\hat{h}}}\mu^{-1}|\widetilde{\boldsymbol{W}}(\boldsymbol{x},s)|^2\mathrm{d}\boldsymbol{x}-\frac{c^{-2}}{2}\int_{\Omega_{\hat{h}}}\mu^{-1}|\tilde{\boldsymbol{\alpha}}\cdot\widetilde{\boldsymbol{W}}(\boldsymbol{x},s)|^2\mathrm{d}\boldsymbol{x}\\
&\quad-\frac{c^{-2}}{2}\int_{\Omega_{\hat{h}}}\mu^{-1}|\tilde{\boldsymbol{\alpha}}\times\widetilde{\boldsymbol{W}}(\boldsymbol{x},0)|^2\mathrm{d}\boldsymbol{x}.
\end{aligned}\tag{68}$$

Substituting (66)–(68) into (64) yields that

$$\begin{aligned}
&\int_0^s\left(\int_{\Omega_{\hat{h}}}\mu^{-1}(\nabla^t_{\widetilde{M},\tilde{\boldsymbol{\alpha}}}\times\widetilde{\boldsymbol{W}})\cdot(\nabla^t_{\widetilde{M},\tilde{\boldsymbol{\alpha}}}\times\Phi)\mathrm{d}\boldsymbol{x}\right)\mathrm{d}t\\
&=\frac{1}{2}\|\sqrt{\mu^{-1}}\nabla_{\widetilde{M}}\times\Phi(\cdot,0)\|^2_{L^2(\Omega_{\hat{h}})^3}\\
&\quad+c^{-1}\int_{\Omega_{\hat{h}}}\mu^{-1}\Big((\nabla_{\widetilde{M}}\times\Phi(\boldsymbol{x},0))\cdot(\tilde{\boldsymbol{\alpha}}\times\widetilde{\boldsymbol{W}}(\boldsymbol{x},0))\Big)\mathrm{d}\boldsymbol{x}\\
&\quad+c^{-1}\int_{\Omega_{\hat{h}}}\mu^{-1}\left(\int_0^s(\nabla_{\widetilde{M}}\times\Phi)\cdot(\tilde{\boldsymbol{\alpha}}\times\partial_t\widetilde{\boldsymbol{W}})\mathrm{d}t\right)\mathrm{d}\boldsymbol{x}\\
&\quad-c^{-1}\int_{\Omega_{\hat{h}}}\mu^{-1}\left(\int_0^s(\tilde{\boldsymbol{\alpha}}\times\partial_t\widetilde{\boldsymbol{W}})\cdot(\nabla_{\widetilde{M}}\times\Phi)\mathrm{d}t\right)\mathrm{d}\boldsymbol{x}\\
&\quad-\frac{c^{-2}}{2}|\tilde{\boldsymbol{\alpha}}|^2\int_{\Omega_{\hat{h}}}\mu^{-1}|\widetilde{\boldsymbol{W}}(\boldsymbol{x},s)|^2\mathrm{d}\boldsymbol{x}+\frac{c^{-2}}{2}\int_{\Omega_{\hat{h}}}\mu^{-1}|\tilde{\boldsymbol{\alpha}}\times\widetilde{\boldsymbol{W}}(\boldsymbol{x},s)|^2\mathrm{d}\boldsymbol{x}\\
&\quad+\frac{c^{-2}}{2}\int_{\Omega_{\hat{h}}}\mu^{-1}|\tilde{\boldsymbol{\alpha}}\times\widetilde{\boldsymbol{W}}(\boldsymbol{x},0)|^2\mathrm{d}\boldsymbol{x}\\
&=\frac{1}{2}\|\sqrt{\mu^{-1}}\nabla_{\widetilde{M}}\times\Phi(\cdot,0)\|^2_{L^2(\Omega_{\hat{h}})^3}\\
&\quad+c^{-1}\int_{\Omega_{\hat{h}}}\mu^{-1}\Big((\nabla_{\widetilde{M}}\times\Phi(\boldsymbol{x},0))\cdot(\tilde{\boldsymbol{\alpha}}\times\widetilde{\boldsymbol{W}}(\boldsymbol{x},0))\Big)\mathrm{d}\boldsymbol{x}\\
&\quad-\frac{c^{-2}}{2}|\tilde{\boldsymbol{\alpha}}|^2\int_{\Omega_{\hat{h}}}\mu^{-1}|\widetilde{\boldsymbol{W}}(\boldsymbol{x},s)|^2\mathrm{d}\boldsymbol{x}+\frac{c^{-2}}{2}\int_{\Omega_{\hat{h}}}\mu^{-1}|\tilde{\boldsymbol{\alpha}}\cdot\widetilde{\boldsymbol{W}}(\boldsymbol{x},s)|^2\mathrm{d}\boldsymbol{x}\\
&\quad+\frac{c^{-2}}{2}\int_{\Omega_{\hat{h}}}\mu^{-1}|\tilde{\boldsymbol{\alpha}}\times\widetilde{\boldsymbol{W}}(\boldsymbol{x},0)|^2\mathrm{d}\boldsymbol{x}\\
&\geq\frac{1}{2}\|\sqrt{\mu^{-1}}\nabla_{\widetilde{M}}\times\Phi(\cdot,0)\|^2_{L^2(\Omega_{\hat{h}})^3}\\
&\quad-\left(\frac{1}{2}\|\sqrt{\mu^{-1}}\nabla_{\widetilde{M}}\times\Phi(\cdot,0)\|^2_{L^2(\Omega_{\hat{h}})^3}+\frac{c^{-2}}{2}\|\sqrt{\mu^{-1}}\tilde{\boldsymbol{\alpha}}\times\widetilde{\boldsymbol{W}}(\cdot,0)\|^2_{L^2(\Omega_{\hat{h}})^3}\right)\\
&\quad-\frac{c^{-2}}{2}|\tilde{\boldsymbol{\alpha}}|^2\|\sqrt{\mu^{-1}}\widetilde{\boldsymbol{W}}(\cdot,s)\|^2_{L^2(\Omega_{\hat{h}})^3}+\frac{c^{-2}}{2}\|\sqrt{\mu^{-1}}\tilde{\boldsymbol{\alpha}}\cdot\widetilde{\boldsymbol{W}}(\cdot,s)\|^2_{L^2(\Omega_{\hat{h}})^3}
\end{aligned}$$

$$+\frac{c^{-2}}{2}\|\sqrt{\mu^{-1}}\tilde{\boldsymbol{\alpha}}\times\widetilde{\boldsymbol{W}}(\cdot,0)\|^2_{L^2(\Omega_{\hat{h}})^3}$$
$$=-\frac{c^{-2}}{2}|\tilde{\boldsymbol{\alpha}}|^2\|\sqrt{\mu^{-1}}\widetilde{\boldsymbol{W}}(\cdot,s)\|^2_{L^2(\Omega_{\hat{h}})^3}+\frac{c^{-2}}{2}\|\sqrt{\mu^{-1}}\tilde{\boldsymbol{\alpha}}\cdot\widetilde{\boldsymbol{W}}(\cdot,s)\|^2_{L^2(\Omega_{\hat{h}})^3}$$
$$\geq-\frac{|\tilde{\boldsymbol{\alpha}}|^2}{2c^2\mu_{\min}}\|\widetilde{\boldsymbol{W}}(\cdot,s)\|^2_{L^2(\Omega_{\hat{h}})^3}, \tag{69}$$

Combining (65) and (69), we obtain

$$\int_0^s\left(\int_{\Omega_{\hat{h}}}\tilde{\boldsymbol{f}}_1(\boldsymbol{x},t)\cdot\Phi(\boldsymbol{x},t)\mathrm{d}\boldsymbol{x}\right)\mathrm{d}t+\frac{1}{2}\|\sqrt{\varepsilon}\tilde{\boldsymbol{f}}_2\|^2_{L^2(\Omega_{\hat{h}})^3}+\int_{\Omega_{\hat{h}}}\varepsilon\tilde{\boldsymbol{f}}_3(\boldsymbol{x})\cdot\Phi(\boldsymbol{x},0)\mathrm{d}\boldsymbol{x}$$
$$\geq\frac{1}{2}\|\sqrt{\varepsilon}\widetilde{\boldsymbol{W}}(\cdot,s)\|^2_{L^2(\Omega_{\hat{h}})^3}-\frac{|\tilde{\boldsymbol{\alpha}}|^2}{2c^2\mu_{\min}}\|\widetilde{\boldsymbol{W}}(\cdot,s)\|^2_{L^2(\Omega_{\hat{h}})^3}$$
$$\geq\frac{1}{2}\left(\varepsilon_{\min}-\frac{|\tilde{\boldsymbol{\alpha}}|^2}{c^2\mu_{\min}}\right)\|\widetilde{\boldsymbol{W}}(\cdot,s)\|^2_{L^2(\Omega_{\hat{h}})^3}. \tag{70}$$

Since by assumption $\varepsilon_{\min}\mu_{\min}\geq\varepsilon_1\mu_1$ and $|\tilde{\boldsymbol{\alpha}}|^2=\sin^2\theta_1$ $(0\leq\theta_1<\frac{\pi}{2})$, we can conclude that $\frac{\varepsilon_{\min}\mu_{\min}}{\varepsilon_1\mu_1}=\varepsilon_{\min}\mu_{\min}c^2\geq 1>|\tilde{\boldsymbol{\alpha}}|^2$. This implies that $\left(\varepsilon_{\min}-\frac{|\tilde{\boldsymbol{\alpha}}|^2}{c^2\mu_{\min}}\right)>0$.

In what follows, we estimate the two terms on the left-hand side of (70) separately. It follows from the Cauchy–Schwarz inequality that

$$\int_{\Omega_{\hat{h}}}\varepsilon\tilde{\boldsymbol{f}}_3(\boldsymbol{x})\cdot\Phi(\boldsymbol{x},0)\mathrm{d}\boldsymbol{x}=\int_{\Omega_{\hat{h}}}\varepsilon\tilde{\boldsymbol{f}}_3(\boldsymbol{x})\cdot\left(\int_0^s\widetilde{\boldsymbol{W}}(\boldsymbol{x},t)\mathrm{d}t\right)\mathrm{d}\boldsymbol{x}$$
$$=\int_0^s\left(\int_{\Omega_{\hat{h}}}\varepsilon\tilde{\boldsymbol{f}}_3(\boldsymbol{x})\cdot\widetilde{\boldsymbol{W}}(\boldsymbol{x},t)\mathrm{d}\boldsymbol{x}\right)\mathrm{d}t$$
$$\leq\varepsilon_{\max}\int_0^s\|\tilde{\boldsymbol{f}}_3\|_{L^2(\Omega_{\hat{h}})^3}\|\widetilde{\boldsymbol{W}}(\cdot,t)\|_{L^2(\Omega_{\hat{h}})^3}\mathrm{d}t$$
$$\leq\left(\varepsilon_{\max}\|\tilde{\boldsymbol{f}}_3\|_{L^2(\Omega_{\hat{h}})^3}\right)\int_0^s\|\widetilde{\boldsymbol{W}}(\cdot,t)\|_{L^2(\Omega_{\hat{h}})^3}\mathrm{d}t. \tag{71}$$

For $0\leq t\leq s\leq T$, we have from (61) that

$$\int_0^s\left(\int_{\Omega_{\hat{h}}}\tilde{\boldsymbol{f}}_1(\boldsymbol{x},t)\cdot\Phi(\boldsymbol{x},t)\mathrm{d}\boldsymbol{x}\right)\mathrm{d}t$$
$$=\int_{\Omega_{\hat{h}}}\left(\int_0^s\left(\int_0^t\tilde{\boldsymbol{f}}_1(\boldsymbol{x},\tau)\mathrm{d}\tau\right)\cdot\Phi(\boldsymbol{x},t)\mathrm{d}t\right)\mathrm{d}\boldsymbol{x}$$
$$\leq\int_0^s\int_0^t\|\tilde{\boldsymbol{f}}_1(\cdot,\tau)\|_{L^2(\Omega_{\hat{h}})^3}\cdot\|\widetilde{\boldsymbol{W}}(\cdot,t)\|_{L^2(\Omega_{\hat{h}})^3}\mathrm{d}\tau\mathrm{d}t$$
$$\leq\left(\int_0^s\|\tilde{\boldsymbol{f}}_1(\cdot,t)\|_{L^2(\Omega_{\hat{h}})^3}\mathrm{d}t\right)\left(\int_0^s\|\widetilde{\boldsymbol{W}}(\cdot,t)\|_{L^2(\Omega_{\hat{h}})^3}\mathrm{d}t\right)$$
$$\leq\|\tilde{\boldsymbol{f}}_1\|_{L^1[0,T;L^2(\Omega_{\hat{h}})^3]}\int_0^s\|\widetilde{\boldsymbol{W}}(\cdot,t)\|_{L^2(\Omega_{\hat{h}})^3}\mathrm{d}t. \tag{72}$$

Substituting (71)–(72) into (70), we have for any $s \in [0,T]$ that

$$
\begin{aligned}
&\|\widetilde{\boldsymbol{W}}(\cdot,s)\|^2_{L^2(\Omega_{\hat{h}})^3} \\
&\le C_6\|\sqrt{\varepsilon}\tilde{\boldsymbol{f}}_2\|^2_{L^2(\Omega_{\hat{h}})^3} \\
&\quad + 2C_6\left(\|\tilde{\boldsymbol{f}}_1\|_{L^1[0,T;L^2(\Omega_{\hat{h}})^3]} + \varepsilon_{\max}\|\tilde{\boldsymbol{f}}_3\|_{L^2(\Omega_{\hat{h}})^3}\right)\int_0^s \|\widetilde{\boldsymbol{W}}(\cdot,t)\|_{L^2(\Omega_{\hat{h}})^3}\mathrm{d}t, \qquad (73)
\end{aligned}
$$

where $C_6 = \left(\varepsilon_{\min} - \frac{|\tilde{\boldsymbol{\alpha}}|^2}{c^2\mu_{\min}}\right)^{-1}$.

Taking the L^∞-norm with respect to s on both sides of (73), we get by applying the Young inequality that

$$
\begin{aligned}
&\|\widetilde{\boldsymbol{W}}(\cdot,s)\|^2_{L^\infty[0,T;L^2(\Omega_{\hat{h}})^3]} \\
&\le 2C_6\varepsilon_{\max}\|\tilde{\boldsymbol{f}}_2\|^2_{L^2(\Omega_{\hat{h}})^3} + [2C_6T(\|\tilde{\boldsymbol{f}}_1\|_{L^1[0,T;L^2(\Omega_{\hat{h}})^3]} + \varepsilon_{\max}\|\tilde{\boldsymbol{f}}_3\|_{L^2(\Omega_{\hat{h}})^3})]^2 \\
&\le \max\{8C_6^2T^2, 2C_6\varepsilon_{\max}, 8C_6^2T^2\varepsilon_{\max}^2\} \\
&\quad \left(\|\tilde{\boldsymbol{f}}_1\|^2_{L^1[0,T;L^2(\Omega_{\hat{h}})^3]} + \|\tilde{\boldsymbol{f}}_2\|^2_{L^2(\Omega_{\hat{h}})^3} + \|\tilde{\boldsymbol{f}}_3\|^2_{L^2(\Omega_{\hat{h}})^3}\right).
\end{aligned}
$$

Similarly, we can get

$$
\begin{aligned}
&\|\widetilde{\boldsymbol{W}}(\cdot,s)\|^2_{L^2[0,T;L^2(\Omega_{\hat{h}})^3]} \\
&\le 2C_6\varepsilon_{\max}T\|\tilde{\boldsymbol{f}}_2\|^2_{L^2(\Omega_{\hat{h}})^3} + [2C_6T^{\frac{3}{2}}(\|\tilde{\boldsymbol{f}}_1\|_{L^1[0,T;L^2(\Omega_{\hat{h}})^3]} + \varepsilon_{\max}\|\tilde{\boldsymbol{f}}_3\|_{L^2(\Omega_{\hat{h}})^3})]^2 \\
&\le \max\{8C_6^2T^3, 2C_6\varepsilon_{\max}T, 8C_6^2\varepsilon_{\max}^2T^3\} \\
&\quad \left(\|\tilde{\boldsymbol{f}}_1\|^2_{L^1[0,T;L^2(\Omega_{\hat{h}})^3]} + \|\tilde{\boldsymbol{f}}_2\|^2_{L^2(\Omega_{\hat{h}})^3} + \|\tilde{\boldsymbol{f}}_3\|^2_{L^2(\Omega_{\hat{h}})^3}\right),
\end{aligned}
$$

which completes the proof. □

4. **Conclusion.** In this paper, we have studied the time-domain electromagnetic scattering problem in a biperiodic structure. The three-dimensional Maxwell equations are considered. By developing the method of a compressed coordinate transformation, the scattering problem is reduced equivalently into an initial boundary value problem in a bounded domain. The well-posedness of the corresponding variational problem is proved by using the constructive Galerkin method. Moreover, by directly considering the variational problem of the time-domain Maxwell equation, we obtain the a priori estimates with explicit dependence on the time. The main ingredients of the proofs are the change of variables, the Galerkin method, and the energy method.

The method does not introduce any approximation or truncation error. It avoids the complicated error or convergence analysis which is needed for the TBC method. The reduced model problem is particularly suitable for numerical simulations due to its simplicity and small computational domain. We believe that the method is applicable to many other time-domain scattering problems imposed in open domains. Another possible research direction is to study the time-domain inverse diffractive grating problem, which is to determine the periodic structures giving rise to the measured wave patterns [2, 12, 14, 31]. We hope to report the work on the numerical analysis and computation, as well as the inverse problem elsewhere in the future.

Acknowledgments. The work of GB is supported in part by an NSFC Innovative Group Fund (No.11621101), an Integrated Project of the Major Research Plan of NSFC (No. 91630309), and an NSFC A3 Project (No.11421110002), and the Fundamental Research Funds for the Central Universities. The research of JW was supported in part by a National Natural Science Foundation of China (No.11801116) and Fundamental Research Funds for the Central Universities.

REFERENCES

[1] B. Alpert, L. Greengard and T. Hagstrom, Nonreflecting boundary conditions for the time-dependent wave equation, *J. Comput. Phys.*, **180** (2002), 270–296.
[2] H. Ammari, Uniqueness theorems for an inverse problem in a doubly periodic structure, *Inverse Problems*, **11** (1995), 823–833.
[3] H. Ammari and G. Bao, Maxwell's equations in periodic chiral structures, *Math.Nachr.*, **251** (2003), 3–18.
[4] G. Bao, Finite element approximation of time harmonic waves in periodic structures, *SIAM J. Numer. Anal.*, **32** (1995), 1155–1169.
[5] G. Bao, Numerical analysis of diffraction by periodic structures: TM polarization, *Numer. Math.*, **75** (1996), 1–16.
[6] G. Bao, Variational approximation of Maxwell's equations in biperiodic structures, *SIAM J. Appl. Math.*, **57** (1997), 364–381.
[7] G. Bao, D. Dobson and J. A. Cox, Mathematical studies in rigorous grating theory, *J. Opt. Soc. Am. A*, **12** (1995), 1029–1042.
[8] G. Bao, Z. Chen, and H. Wu, Adaptive finite-element method for diffraction gratings, *J. Opt. Soc. Amer. A*, **22** (2005), 1106–1114.
[9] G. Bao, L. Cowsar and W. Masters, Eds., *Mathematical Modeling in Optical Science*, Frontiers in Applied Mathematics, vol. 22, SIAM, Philadelphia, PA, 2001.
[10] G. Bao, P. Li and H. Wu, An adaptive edge element method with perfectly matched absorbing layers for wave scattering by biperiodic structures, *Math. Comp.*, **79** (2010), 1–34.
[11] G. Bao and H. Yang, A least-squares finite element analysis for diffraction problems, *SIAM J. Numer. Anal.*, **37** (2000), 665–682.
[12] G. Bao, T. Cui, and P. Li, Inverse diffraction grating of Maxwell's equations in biperiodic structures, *Opt. Express*, **22** (2014), 4799–4816.
[13] G. Bao and D. Dobson, On the scattering by a biperiodic structure, *Proc. Amer. Math. Soc.*, **128** (2000), 2715–2723.
[14] G. Bao and A. Friedman, Inverse problems for scattering by periodic structure, *Arch. Rational Mech. Anal.*, **132** (1995), 49–72.
[15] G. Bao, Y. Gao and P. Li, Time-domain analysis of an acoustic-elastic interaction problem, *Arch. Ration. Mech. Anal.*, **292** (2018), 835–884.
[16] Q. Chen and P. Monk, Discretization of the time domain CFIE for acoustic scattering problems using convolution quadrature, *SIAM J. Math. Anal.*, **46** (2014), 3107–3130.
[17] X. Chen and A. Friedman, Maxwell's equations in a periodic structure, *Trans. Amer. Math. Soc.*, **323** (1991), 465–507.
[18] Z. Chen and J.-C. Nédélec, On Maxwell equations with the transparent boundary condition, *J. Comput. Math.*, **26** (2008), 284–296.
[19] Z. Chen and H. Wu, An adaptive finite element method with perfectly matched absorbing layers for the wave scattering by periodic structures, *SIAM J. Numer. Anal.*, **41** (2003), 799–826.
[20] D. Dobson, A variational method for electromagnetic diffraction in biperiodic structures, *Math. Modelling Numer. Anal.*, **28** (1994), 419–439.
[21] D. Dobson and A. Friedman, The time-harmonic Maxwell equations in a doubly periodic structure, *J. Math. Anal. Appl.*, **166** (1992), 507–528.
[22] B. Engquist and A. Majda, Absorbing boundary conditions for the numerical simulation of waves, *Math. Comp.*, **31** (1977), 629–651.
[23] L. C. Evans, *Partial Differential Equations*, 2^{nd} edition, vol. 19, Graduate Studies in Mathematics, AMS, Providence, RI, 2010.
[24] L. Fan and P. Monk, Time dependent scattering from a grating, *J. Comput. Phys.*, **302** (2015), 97–113.

[25] Y. Gao and P. Li, Analysis of time-domain scattering by periodic structures, *J. Differential Equations*, **261** (2016), 5094–5118.

[26] Y. Gao and P. Li, Electromagnetic scattering for time-domain Maxwell's equations in an unbounded structure, *Math. Models Methods Appl. Sci.*, **27** (2017), 1843–1870.

[27] Y. Gao, P. Li and Y. Li, Analysis of time-domain elastic scattering by an unbounded structure, *Math. Meth. Appl. Sci.*, **41** (2018), 7032–7054.

[28] Y. Gao, P. Li and B. Zhang, Analysis of transient acoustic-elastic interaction in an unbounded structure, *SIAM J. Math. Anal.*, **49** (2017), 3951–3972.

[29] M. J. Grote and J. B. Keller, Exact nonreflecting boundary conditions for the time dependent wave equation, *SIAM J. Appl. Math.*, **55** (1995), 280–297.

[30] T. Hagstrom, Radiation boundary conditions for the numerical simulation of waves, *Acta Numer.*, **8** (1999), 47–106.

[31] X. Jiang and P. Li, Inverse electromagnetic diffraction by biperiodic dielectric gratings, *Inverse Probl.*, **33** (2017), 085004, 29pp.

[32] A. Lechleiter and D. L. Nguyen. On uniqueness in electromagnetic scattering from biperiodic structures, *ESAIM: M2AN*, **47** (2013), 1167–1184.

[33] P. Li, L.-L. Wang and A. Wood, Analysis of transient electromagentic scattering from a three-dimensional open cavity, *SIAM J. Appl. Math.*, **75** (2015), 1675–1699.

[34] J.-C. Nedelec and F. Starling, Integral equation methods in a quasi-periodic diffraction problem for the time-harmonic Maxwell's equations, *SIAM J. Math. Anal.*, **22** (1991), 1679–1701.

[35] R. Petit, ed., *Electromagnetic Theory of Gratings*, Springer, 1980.

[36] P. Rayleigh, On the dynamical theory of gratings, *R. Soc. London Ser. A*, **79** (1907), 399–416.

[37] D. J. Riley and J.-M. Jin, Finite-element time-domain analysis of electrically and magnetically dispersive periodic structures, *IEEE Trans. Antennas and Propagation*, **56** (2008), 3501–3509.

[38] M. Veysoglu, R. Shin and J. A. Kong, A finite-difference time-domain analysis of wave scattering from periodic surfaces: oblique incidence case, *J. Electromagn. Waves Appl.*, **7** (1993), 1595–1607.

[39] B. Wang and L.-L. Wang, On L^2-stability analysis of time-domain acoustic scattering problems with exact nonreflecting boundary conditions, *J. Math. Study*, **47** (2014), 65–84.

[40] L.-L. Wang, B. Wang and X. Zhao, Fast and accurate computation of time-domain acoustic scattering problems with exact nonreflecting boundary conditions, *SIAM J. Appl. Math.*, **72** (2012), 1869–1898.

[41] Z. Wang, G. Bao, J. Li, P. Li and H. Wu, An adaptive finite element method for the diffraction grating problem with transparent boundary conditions, *SIAM J. Numer. Anal.*, **53** (2015), 1585–1607.

[42] Y. Wu and Y. Y. Lu, Analyzing diffraction gratings by a boundary integral equation Neumann-to-Dirichlet map method, *J. Opt. Soc. Am. A*, **26** (2009), 2444–2451.

Received January 2019; revised March 2019.

E-mail address: baog@zju.edu.cn
E-mail address: binh@zju.edu.cn
E-mail address: lipeijun@math.purdue.edu
E-mail address: wangjue@hrbeu.edu.cn