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INVERSE ELASTIC SCATTERING FOR A RANDOM POTENTIAL*

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Abstract. This paper is concerned with an inverse scattering problem for the time-harmonic elastic wave equation with a random potential. Interpreted as a distribution, the potential is assumed to be a microlocally isotropic generalized Gaussian random field with the covariance operator being described by a classical pseudodifferential operator. The goal is to determine the principal symbol of the covariance operator from the scattered wave measured in a bounded domain which has a positive distance from the domain of the potential. For such a rough potential, the well-posedness of the direct scattering problem in the distribution sense is established by studying an equivalent Lippmann–Schwinger integral equation. For the inverse scattering problem, it is shown with probability one that the principal symbol of the covariance operator can be uniquely determined by the amplitude of the scattered waves averaged over the frequency band from a single realization of the random potential. The analysis employs the Born approximation in high frequency, asymptotics of the Green tensor for the elastic wave equation, and microlocal analysis for the Fourier integral operators.

Key words. inverse scattering problem, elastic wave equation, generalized Gaussian random field, pseudodifferential operator, principal symbol, uniqueness

MSC codes. 35R30, 35R60, 60H15

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1. Introduction and statement of the main result.

The scattering problems for elastic waves have significant applications in diverse scientific areas such as geophysical exploration and nondestructive testing [18, 33]. In medical diagnostics, elastography is an emerging imaging modality that seeks to determine the mechanical properties of elastic media from their response to exciting forces [30]. By mapping the elastic properties and stiffness of soft tissues, it can give diagnostic information about the presence or status of disease [12]. Driven by these applications, the underlying inverse problems, which are to determine the medium properties based on the elastic wave equation, have been extensively studied and many mathematical results are available, especially for the uniqueness [9, 13, 14, 32]. We refer to [2] for a comprehensive account of mathematical methods in elasticity imaging.

Stochastic modeling has been widely adopted to handle problems involving uncertainties and randomness. In the research area of wave propagation, the wave fields may not be deterministic but rather are described by random fields due to the uncertainties for the media and/or the environments. Therefore, it is more appropriate to consider the stochastic wave equations to describe the wave motion in random settings. In addition to the ill-posedness and nonlinearity, stochastic inverse prob-

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lems have substantially more difficulties than their deterministic counterparts. The random parameters to be determined in stochastic inverse problems can be characterized not by a particular realization but instead by its statistics, such as expectation and covariance. Hence, the relationship between these statistics and the wave fields needs to be established. In general, the statistics of the data for the wave fields are required, which significantly increases the computational cost since a large number of realizations is needed. It is an important and challenging problem to determine the statistics of the random parameters through fewer realizations of the wave fields.

The paper is concerned with an inverse scattering problem for the time-harmonic elastic wave equation with a random potential in two dimensions. Specifically, we consider the stochastic elastic wave equation

$$\mu\Delta\boldsymbol{u} + (\lambda+\mu)\nabla\nabla\cdot\boldsymbol{u} + \omega^2\boldsymbol{u} - \rho\boldsymbol{u} = -\delta_y\boldsymbol{a} \quad \text{in } \mathbb{R}^2, \tag{1.1}$$

where $\omega > 0$ is the angular frequency, $\boldsymbol{a}$ is a unit polarization vector in $\mathbb{R}^2$, $\delta_y(\cdot) := \delta(\cdot - y)$ is the Dirac delta function concentrated at the source point $y \in \mathbb{R}^2$, and λ and μ are the Lamé parameters satisfying $\mu > 0$ and $\lambda + 2\mu > 0$ such that the second-order partial differential operator $\Delta^* := \mu\Delta + (\lambda+\mu)\nabla\nabla\cdot$ is strongly elliptic (cf. [29, section 10.4]). The potential ρ in (1.1) could be viewed as an unknown external linear load acting on a known homogeneous and isotropic elastic medium, or it could be interpreted as a random parameter coming from an unknown anisotropic mass density of the medium [9]. Throughout the paper, ρ is considered to be a generalized Gaussian random field satisfying the following assumption.

Assumption 1.1. Assume that the centered random potential ρ is a microlocally isotropic Gaussian random field of order $-m$ in D with $m \in (1, 2]$ and D being a bounded domain. More precisely, ρ has the principal symbol $\phi(x)|\xi|^{-m}$, where ϕ is called the microcorrelation strength of the potential ρ and satisfies $\phi \in C_0^\infty(D), \phi \geq 0$.

The displacement of the total field $\boldsymbol{u} \in \mathbb{C}^2$ in (1.1) can be decomposed into

$$\boldsymbol{u}(x, y) = \boldsymbol{u}^i(x, y) + \boldsymbol{u}^s(x, y),$$

where $\boldsymbol{u}^s$ represents the scattered field and $\boldsymbol{u}^i$ is the incident field given by

$$\boldsymbol{u}^i(x, y) = \boldsymbol{G}(x, y, \omega)\boldsymbol{a}, \quad x \neq y.$$

Here, $\boldsymbol{G}(x, y, \omega) \in \mathbb{C}^{2\times 2}$ denotes the Green tensor for the Navier equation. Explicitly,

$$\boldsymbol{G}(x, y, \omega) = \frac{1}{\mu}\Phi(x, y, \kappa_{\rm s})\boldsymbol{I} + \frac{1}{\omega^2}\nabla_x\nabla_x^\top\Big[\Phi(x, y, \kappa_{\rm s}) - \Phi(x, y, \kappa_{\rm p})\Big], \tag{1.2}$$

where $\boldsymbol{I}$ is the 2×2 identity matrix, $\nabla_x = (\partial_{x_1}, \partial_{x_2})^\top$ is the gradient operator, $\Phi(x, y, \kappa) = \frac{\rm i}{4}H_0^{(1)}(\kappa|x - y|)$ is the fundamental solution of the two-dimensional Helmholtz equation with $H_0^{(1)}$ being the Hankel function of the first kind with order zero, and $\kappa_{\rm p} := c_{\rm p}\omega$ and $\kappa_{\rm s} := c_{\rm s}\omega$ with $c_{\rm p} = (\lambda+2\mu)^{-\frac{1}{2}}$ and $c_{\rm s} = \mu^{-\frac{1}{2}}$ are known as the compressional and shear wavenumbers, respectively.

Since the elastic wave equation (1.1) is imposed in the whole space $\mathbb{R}^2$, an appropriate radiation condition is needed to complete the problem formulation. By the Helmholtz decomposition (cf. [8, Appendix B]), the scattered field $\boldsymbol{u}^s$ can be decomposed into the compressional wave component $\boldsymbol{u}^s_{\rm p}$ and the shear wave component $\boldsymbol{u}^s_{\rm s}$, i.e.,

$$\boldsymbol{u}^s = \boldsymbol{u}^s_{\rm p} + \boldsymbol{u}^s_{\rm s} \quad \text{in } \mathbb{R}^2 \setminus \overline{D}.$$

The Kupradze–Sommerfeld radiation condition requires that $\boldsymbol{u}_{\mathrm{p}}^s$ and $\boldsymbol{u}_{\mathrm{s}}^s$ satisfy the Sommerfeld radiation condition

$$\lim_{r\to\infty} r^{\frac{1}{2}}\left(\partial_r \boldsymbol{u}_{\mathrm{p}}^s - \mathrm{i}\kappa_{\mathrm{p}}\boldsymbol{u}_{\mathrm{p}}^s\right) = 0, \quad \lim_{r\to\infty} r^{\frac{1}{2}}\left(\partial_r \boldsymbol{u}_{\mathrm{s}}^s - \mathrm{i}\kappa_{\mathrm{s}}\boldsymbol{u}_{\mathrm{s}}^s\right) = 0, \quad r = |x|. \tag{1.3}$$

As is known, the inverse scattering problems are challenging due to the nonlinearity and ill-posedness. Apparently, the stochastic inverse scattering problems are even harder in order to handle the extra difficulties of randomness and uncertainties. There are very few results concerning the solutions of the stochastic inverse scattering problems. For the inverse random source problems, when the source is driven by an additive white noise, effective mathematical models and efficient computational methods have been proposed for the stochastic acoustic and elastic wave equations [4, 6, 25, 24, 5, 7]. To determine the unknown parameters in the above models, in general, the data of the expectation and variance for the measured wave field is needed, and hence a fairly large number of realizations of the random source are required. If the source is described as a generalized Gaussian random field whose covariance is a classical pseudodifferential operator, the results of uniqueness were established in [20, 21] for the stochastic acoustic and elastic wave equations. It was shown that the principal symbol of the covariance operator can be uniquely determined by the amplitude of the wave field averaged over the frequency band. It is worth mentioning that the methods in [20, 21] only require the scattering data corresponding to a single realization of the random source. For the random Schrödinger equation where the potential is a generalized Gaussian random field, it was proved in [10] and [19] that the principal symbol of the covariance operator can be uniquely determined by the backscattered far-field data associated with the plane wave and the scattered wave field associated with the point source, respectively. Similarly, the approach only needs a single realization of the random potential. A related work can be found in [16], where an inverse scattering problem in a half-space with an impedance boundary condition was studied where the impedance function is modeled as a generalized Gaussian random field.

In this work, we study both the direct and inverse scattering problems for the stochastic elastic wave equation (1.1) along with the radiation condition (1.3). Given the random potential ρ, which is rough and can only be interpreted as a distribution, the direct scattering problem is to determine the displacement $\boldsymbol{u}$ which satisfies (1.1) and (1.3) in an appropriate sense. Using Green's theorem and the Kupradze–Sommerfeld radiation condition, we deduce an equivalent Lippmann–Schwinger integral equation. Based on the Fredholm alternative theorem and the unique continuation principle, the Lippmann–Schwinger equation is shown to have a unique solution in the Sobolev space with a positive smoothness index. The inverse scattering problem is to determine the microcorrelation strength $\phi(x)$ from the scattered field measured in a bounded and convex domain $U \subset \mathbb{R}^2 \setminus \overline{D}$, which has a positive distance from D.

It is clear to note from the elastic wave equation (1.1) that the displacement $\boldsymbol{u}$ depends on the observation point x, the location of the source point y, the angular frequency ω, and the unit polarization vector $\boldsymbol{a}$. To express explicitly the dependence of $\boldsymbol{u}$ on these quantities, we write $\boldsymbol{u}(x, y, \omega, \boldsymbol{a})$, $\boldsymbol{u}^i(x, y, \omega, \boldsymbol{a})$, $\boldsymbol{u}^s(x, y, \omega, \boldsymbol{a})$, and $\boldsymbol{u}_j(x, y, \omega, \boldsymbol{a})$ in the Born series (cf. (4.1) and (4.2) for the definition of $\boldsymbol{u}_j$) for $\boldsymbol{u}(x, y)$, $\boldsymbol{u}^i(x, y)$, $\boldsymbol{u}^s(x, y)$, and $\boldsymbol{u}_j(x, y)$, respectively. Moreover, when the observation point x coincides the source point y, for simplicity, we write $\boldsymbol{u}^s(x, \omega, \boldsymbol{a})$ and $\boldsymbol{u}_j(x, \omega, \boldsymbol{a})$ for $\boldsymbol{u}^s(x, x, \omega, \boldsymbol{a})$ and $\boldsymbol{u}_j(x, x, \omega, \boldsymbol{a})$, respectively.

The following theorem concerns the uniqueness of the inverse scattering problem and is the main result of this paper.

THEOREM 1.2. *Let ρ satisfy Assumption* 1.1 *and additionally $m > \frac{5}{3}$. Let $U \subset \mathbb{R}^2 \backslash \overline{D}$ be a bounded and convex domain having a locally Lipschitz boundary and a positive distance from D. Then for all $x \in U$, it holds almost surely that*

$$\lim_{Q\to\infty} \frac{1}{Q-1} \int_1^Q \omega^{m+2} \sum_{j=1}^{2} |\boldsymbol{u}^s(x,\omega,\boldsymbol{a}_j)|^2 d\omega = C_m \int_{\mathbb{R}^2} \frac{1}{|x-\zeta|^2} \phi(\zeta) d\zeta, \tag{1.4}$$

where $C_m := \frac{c_s^{6-m}+c_p^{6-m}}{2^{m+6}\pi^2}$, and $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ are orthonormal vectors in $\mathbb{R}^2$. Moreover, the function ϕ can be uniquely determined from the integral equation (1.4) *for all $x \in U$.*

Since the scattered field $\boldsymbol{u}^s$ depends on the realization of the random potential ρ, the scattering data given on the left-hand side of (1.4) is random for any finite Q. However, (1.4) indicates that the scattering data is statistically stable when Q approaches to infinity, i.e., it is independent of the realization of the potential. The main result demonstrates that the function ϕ can be uniquely determined by the amplitude of two scattered fields averaged over the frequency band, which are generated by a single realization of the random potential. Here, the two scattered fields are excited by the incident waves $\boldsymbol{Ga}_1$ and $\boldsymbol{Ga}_2$. The proof of the main result is quite technical. The analysis employs the Born approximation in the high frequency regime, the asymptotics of Green's tensor for the elastic wave equation, and microlocal analysis for the Fourier integral operators.

For readability, we briefly sketch the steps of the proof for the main result. As mentioned above, the scattering problem (1.1) and (1.3) can be equivalently formulated as a Lippmann–Schwinger integral equation which admits a unique solution. A careful analysis shows that the Born series of the Lippmann–Schwinger integral equation $\sum_{j=0}^{\infty} \boldsymbol{u}_j$ (cf. (4.1) and (4.2) for the definition of $\boldsymbol{u}_j$) converges to the unique solution to the direct scattering problem when the angular frequency ω is sufficiently large. Hence, the scattered field $\boldsymbol{u}^s$ can be written as

$$\boldsymbol{u}^s = \boldsymbol{u}_1 + \boldsymbol{u}_2 + \boldsymbol{b}, \quad \boldsymbol{b} = \sum_{j=3}^{\infty} \boldsymbol{u}_j.$$

For the first item $\boldsymbol{u}_1$, by employing the asymptotic expansions of the Green tensor and microlocal analysis for the Fourier integral operators via multiple coordinate transformations, we show in Theorem 5.1 that

$$\lim_{Q\to\infty} \frac{1}{Q-1} \int_1^Q \omega^{m+2} \sum_{j=1}^{2} |\boldsymbol{u}_1(x,\omega,\boldsymbol{a}_j)|^2 d\omega = C_m \int_{\mathbb{R}^2} \frac{1}{|x-\zeta|^2} \phi(\zeta) d\zeta. \tag{1.5}$$

It is shown in Theorem 5.5 that the contribution of the second item $\boldsymbol{u}_2$ can be neglected, i.e.,

$$\lim_{Q\to\infty} \frac{1}{Q-1} \int_1^Q \omega^{m+2} |\boldsymbol{u}_2(x,\omega,\boldsymbol{a})|^2 d\omega = 0. \tag{1.6}$$

For the remaining term $\boldsymbol{b}$, by means of estimating the order with respect to the angular frequency ω, we deduce in section 5.3 that

$$\lim_{Q\to\infty} \frac{1}{Q-1} \int_1^Q \omega^{m+2} |\boldsymbol{b}(x,\omega,\boldsymbol{a})|^2 d\omega = 0. \tag{1.7}$$

The main result then follows from (1.5)–(1.7).

The rest of the paper is organized as follows. In section 2, we briefly introduce the microlocally isotropic generalized Gaussian random fields and present some of their properties. Section 3 concerns the well-posedness of the direct scattering problem. We show that the direct problem is equivalent to a Lippmann–Schwinger integral equation which is uniquely solvable for a distributional potential. In section 4, the Born series is studied for the Lippmann–Schwinger integral equation in the high frequency regime. Section 5 is devoted to the inverse scattering problem. The paper is concluded with some general remarks and directions for future work in section 6.

2. Generalized Gaussian random fields. In this section, we give a brief introduction to the microlocally isotropic generalized Gaussian random fields in $\mathbb{R}^d$, $d = 2$ or 3. Let $C_0^\infty(\mathbb{R}^d)$ be the set of smooth functions with compact support, and let $\mathcal{D} := \mathcal{D}(\mathbb{R}^d)$ be the space of test functions, which is $C_0^\infty(\mathbb{R}^d)$ equipped with a locally convex topology. The dual space $\mathcal{D}' := \mathcal{D}'(\mathbb{R}^d)$ of $\mathcal{D}$ is called the space of distributions on $\mathbb{R}^d$ and is equipped with a weak-star topology (cf. [1]). Denote by $(\Omega, \mathscr{F}, \mathbb{P})$ a complete probability space, where Ω is a sample space, $\mathscr{F}$ is a σ-algebra on Ω, and $\mathbb{P}$ is a probability measure on the measurable space $(\Omega, \mathscr{F})$.

A function ρ is said to be a generalized random field if, for each $\tilde{\omega} \in \Omega$, the realization $\rho(\tilde{\omega})$ belongs to $\mathcal{D}'(\mathbb{R}^d)$ and the mapping

$$\tilde{\omega} \in \Omega \longmapsto \langle \rho(\tilde{\omega}), \psi \rangle \in \mathbb{R} \tag{2.1}$$

is a random variable for all $\psi \in \mathcal{D}$, where $\langle \cdot, \cdot \rangle$ denotes the dual product between $\mathcal{D}'$ and $\mathcal{D}$. The distributional derivative of $\rho \in \mathcal{D}'$ is defined by

$$\langle \partial_{x_j} \rho, \psi \rangle = -\langle \rho, \partial_{x_j} \psi \rangle \quad \forall\, \psi \in \mathcal{D}, \quad j = 1, \ldots, d.$$

A generalized random field is said to be Gaussian if (2.1) defines a Gaussian random variable for all $\psi \in \mathcal{D}$.

For a generalized random field $\rho \in \mathcal{D}'$, we can define its expectation $\mathbb{E}\rho \in \mathcal{D}'$ and covariance operator $Q_\rho : \mathcal{D} \to \mathcal{D}'$ as follows:

$$\begin{aligned} \langle \mathbb{E}\rho, \psi \rangle &:= \mathbb{E}\langle \rho, \psi \rangle \quad \forall\, \psi \in \mathcal{D}, \\ \langle Q_\rho \psi_1, \psi_2 \rangle &:= \mathbb{E}\left[(\langle \rho, \psi_1 \rangle - \mathbb{E}\langle \rho, \psi_1 \rangle)(\langle \rho, \psi_2 \rangle - \mathbb{E}\langle \rho, \psi_2 \rangle) \right] \quad \forall\, \psi_1, \psi_2 \in \mathcal{D}. \end{aligned}$$

It follows from the continuity of Q_ρ and the Schwartz kernel theorem that there exists a unique kernel function $\mathcal{K}_\rho \in \mathcal{D}'(\mathbb{R}^d \times \mathbb{R}^d)$ satisfying

$$\langle Q_\rho \psi_1, \psi_2 \rangle = \langle \mathcal{K}_\rho, \psi_2 \otimes \psi_1 \rangle \quad \forall\, \psi_1, \psi_2 \in \mathcal{D}.$$

The following definition can be found in [19] on the microlocally isotropic generalized Gaussian random fields.

DEFINITION 2.1. *A generalized Gaussian random field ρ on $\mathbb{R}^d$ is called microlocally isotropic of order $-m$ in D with $m \geq 0$ if the realizations of ρ are almost surely supported in D and its covariance operator Q_ρ is a classical pseudodifferential operator having an isotropic principal symbol $\phi(x)|\xi|^{-m}$ with $\phi \in C_0^\infty(\mathbb{R}^d)$,* $\operatorname{supp}\phi \subset D$ *and $\phi \geq 0$.*

Without loss of generality, we choose the bounded domain D which not only contains the support of ρ almost surely but also has a locally Lipschitz boundary.

To have a better understanding of microlocally isotropic Gaussian random fields, we give an example by introducing the centered fractional Gaussian fields (cf. [26, 28]) defined by

$$f_m(x) := (-\Delta)^{-\frac{m}{4}} \dot{W}(x), \quad x \in \mathbb{R}^d, \tag{2.2}$$

where $(-\Delta)^{-\frac{m}{4}}$ is the fractional Laplacian and $\dot{W} \in \mathcal{D}'$ denotes the white noise. It is shown in [26] that the kernel $\mathcal{K}_{f_m}$ of f_m is isotropic since its value depends only on the distance between x and y, and f_m is a microlocally isotropic Gaussian random field of order $-m$ and satisfies Definition 2.1 with $\phi \equiv 1$. In particular, if $m \in (d, d+2)$, the fractional Gaussian field f_m defined above is a translation of the classical fractional Brownian motion. More precisely,

$$\tilde{f}_m(x) := \langle f_m, \delta_x - \delta_0 \rangle, \quad x \in \mathbb{R}^d,$$

has the same distribution as the classical fractional Brownian motion with Hurst parameter $H = \frac{m-d}{2} \in (0,1)$ up to a multiplicative constant.

The regularity and kernel functions can be obtained for the microlocally isotropic Gaussian random fields by using the relationship between them and the fractional Gaussian fields defined in (2.2). It is clear to note that the fractional Gaussian field f_m defined by (2.2) has the same regularity as the microlocally isotropic Gaussian random field ρ of order $-m$ in Definition 2.1. Hence, we have the following regularity results for the microlocally isotropic Gaussian random fields. The proof can be found in [26].

LEMMA 2.2. *Let ρ be a microlocally isotropic Gaussian random field of order $-m$ in D with $m \in [0, d+2)$.*

(i) *If $m \in (d, d+2)$, then $\rho \in C^{0,\alpha}(D)$ almost surely for all $\alpha \in (0, \frac{m-d}{2})$.*

(ii) *If $m \in [0, d]$, then $\rho \in W^{\frac{m-d}{2}-\epsilon,p}(D)$ almost surely for any $\epsilon > 0$ and $p \in (1, \infty)$.*

3. The direct scattering problem.

According to Lemma 2.2 with $d = 2$, if $m \in (2,4)$, the random potential ρ is a Hölder continuous function almost surely and has enough regularity such that the scattering problem (1.1) and (1.3) is well-posed in the traditional sense (cf. [8]). However, if $m \in [0,2]$, then the random potential $\rho \in W^{\frac{m-2}{2}-\epsilon,p}(D)$ is a distribution, and the elastic wave equation (1.1) should be considered in the distribution sense instead.

In this section, we study the well-posedness of the scattering problem (1.1) and (1.3) with $m \in [0,2]$ (cf. Assumption 1.1) by considering the equivalent Lippmann–Schwinger integral equation.

In what follows, we denote by $\boldsymbol{X} := X^2 = \{\boldsymbol{g} = (g_1, g_2)^\top : g_j \in X \ \forall j = 1, 2\}$ the Cartesian product vector space of X and use the notation $\boldsymbol{W}^{r,p} := (W^{r,p}(\mathbb{R}^2))^2$ and $\boldsymbol{H}^r := \boldsymbol{W}^{r,2}$ for simplicity. The notation $a \lesssim b$ or $a \gtrsim b$ stands for $a \le Cb$ or $a \ge Cb$, where C is a positive constant whose value is not required but should be clear from the context.

3.1. The Lippmann–Schwinger integral equation.

Based on the Green tensor $\boldsymbol{G}$ in (1.2) and given a source point $y \in \mathbb{R}^2$, the Lippmann–Schwinger integral equation takes the form

$$\boldsymbol{u}(x,y) + \int_D \boldsymbol{G}(x,z,\omega)\rho(z)\boldsymbol{u}(z,y)dz = \boldsymbol{G}(x,y,\omega)\boldsymbol{a}, \quad x \in \mathbb{R}^2, \quad x \neq y. \tag{3.1}$$

For a fixed $y \in \mathbb{R}^2$, define two scattering operators H_ω and K_ω by

$$(H_\omega \boldsymbol{u})(x) := [H_\omega \boldsymbol{u}(\cdot, y)](x) = \int_{\mathbb{R}^2} \boldsymbol{G}(x, z, \omega)\boldsymbol{u}(z, y)dz$$

and

$$(K_\omega \boldsymbol{u})(x) := [K_\omega \boldsymbol{u}(\cdot, y)](x) = \int_{\mathbb{R}^2} \boldsymbol{G}(x, z, \omega)\rho(z)\boldsymbol{u}(z, y)dz, \tag{3.2}$$

which have the following properties (cf. [27, Lemma 4.2]).

LEMMA 3.1. *Let ρ satisfy Assumption* 1.1, *$\mathcal{O} \subset \mathbb{R}^2$ be a bounded set, and $\mathcal{V} \subset \mathbb{R}^2$ be a bounded open set with a locally Lipschitz boundary.*

(i) *The operator $H_\omega : \boldsymbol{H}_0^{-\beta}(\mathcal{O}) \to \boldsymbol{H}^\beta(\mathcal{V})$ is bounded for any $\beta \in (0, 1]$.*

(ii) *The operator $H_\omega : \boldsymbol{W}_0^{-\gamma,p}(\mathcal{O}) \to \boldsymbol{W}^{\gamma,q}(\mathcal{V})$ is compact for any $q \in (2, \infty)$, $\gamma \in \left(0, \frac{2}{q}\right)$, and p satisfying $\frac{1}{p} + \frac{1}{q} = 1$.*

(iii) *The operator $K_\omega : \boldsymbol{W}^{\gamma,q}(\mathcal{V}) \to \boldsymbol{W}^{\gamma,q}(\mathcal{V})$ is compact for any $q \in \left(2, \frac{4}{2-m}\right)$ and $\gamma \in \left(\frac{2-m}{2}, \frac{2}{q}\right)$.*

The following result gives the well-posedness of the Lippmann–Schwinger integral equation (3.1).

THEOREM 3.2. *Let ρ satisfy Assumption* 1.1. *Then the Lippmann–Schwinger integral equation* (3.1) *admits a unique solution $\boldsymbol{u} \in \boldsymbol{W}^{\gamma,q}_{loc}$ almost surely with $q \in \left(2, \frac{2}{2-m}\right)$ and $\gamma \in \left(\frac{2-m}{2}, \frac{1}{q}\right)$.*

Proof. Let $\mathcal{V} \subset \mathbb{R}^2$ be any bounded open set with a locally Lipschitz boundary. By the definition of the operator K_ω, the Lippmann–Schwinger integral equation (3.1) can be written as

$$[(I + K_\omega)\boldsymbol{u}(\cdot, y)](x) = \boldsymbol{G}(x, y, \omega)\boldsymbol{a}, \quad x \in \mathbb{R}^2, \tag{3.3}$$

where $y \in \mathbb{R}^2$ is fixed and I is the identity operator. It follows from Lemma 3.1 that the operator $I + K_\omega : \boldsymbol{W}^{\gamma,q}(\mathcal{V}) \to \boldsymbol{W}^{\gamma,q}(\mathcal{V})$ is Fredholm. Moreover, it is shown in [20, Lemma 4.1] that $\boldsymbol{G}(\cdot, y, \omega) \in (W^{1,p'}(\mathcal{V}))^{2\times 2}$ with $p' \in (1, 2)$. Choosing $p' = 2 - \epsilon$ with $\epsilon > 0$ being sufficiently small, we obtain from the Kondrachov compact embedding theorem that the embedding $\boldsymbol{W}^{1,p'}(\mathcal{V}) \hookrightarrow \boldsymbol{W}^{\gamma,q}(\mathcal{V})$ is compact, which indicates that the right-hand side of (3.3) satisfies $\boldsymbol{G}(\cdot, y, \omega)\boldsymbol{a} \in \boldsymbol{W}^{\gamma,q}(\mathcal{V})$.

By the Fredholm alternative theorem, the Lippmann–Schwinger integral equation (3.3) has a unique solution $\boldsymbol{u} \in \boldsymbol{W}^{\gamma,q}(\mathcal{V})$ if

$$(I + K_\omega)\boldsymbol{u} = 0 \tag{3.4}$$

has only the trivial solution $\boldsymbol{u} \equiv 0$. This fact can be proved by the unique continuation principle (cf. [27]), which restricts the parameters q and γ to intervals $\left(2, \frac{2}{2-m}\right)$ and $\left(\frac{2-m}{2}, \frac{1}{q}\right)$, respectively. ∎

3.2. Well-posedness. Now we show the existence and uniqueness of the solution of (1.1) in the distribution sense by utilizing the Lippmann–Schwinger integral equation.

THEOREM 3.3. *Let ρ satisfy Assumption* 1.1. *The elastic wave equation* (1.1) *together with the radiation condition* (1.3) *is well-defined in the distribution sense and admits a unique solution $\boldsymbol{u} \in \boldsymbol{W}^{\gamma,q}_{loc}$ almost surely with $q \in \left(2, \frac{2}{2-m}\right)$ and $\gamma \in \left(\frac{2-m}{2}, \frac{1}{q}\right)$.*

Proof. First we show the existence of the solution to (1.1). It suffices to show that the solution of the Lippmann–Schwinger integral equation (3.1) is also a solution of (1.1) in the distribution sense.

Suppose that $\boldsymbol{u}^* \in \boldsymbol{W}_{loc}^{\gamma,q}$ is the solution of (3.1) and satisfies

$$\boldsymbol{u}^*(x,y) + \int_{\mathbb{R}^2} \boldsymbol{G}(x,z,\omega)\rho(z)\boldsymbol{u}^*(z,y)dz = \boldsymbol{G}(x,y,\omega)\boldsymbol{a}, \quad x \in \mathbb{R}^2.$$

Since the Green tensor $\boldsymbol{G}$ is the fundamental solution for the operator $\Delta^* + \omega^2$, we have

$$(\Delta^* + \omega^2)\boldsymbol{G}(\cdot,y,\omega) = -\delta_y \boldsymbol{I},$$

where $\boldsymbol{I}$ is the 2×2 identity matrix, and δ_y is a distribution, i.e., $\delta_y \in \mathcal{D}'$. Hence, we get for any $\boldsymbol{\psi} \in \boldsymbol{\mathcal{D}}$ that

$$\langle (\Delta^* + \omega^2)\boldsymbol{G}(\cdot,y,\omega), \boldsymbol{\psi}\rangle = -\langle \delta_y \boldsymbol{I}, \boldsymbol{\psi}\rangle = -\boldsymbol{\psi}(y). \tag{3.5}$$

For any $\boldsymbol{\psi} \in \boldsymbol{\mathcal{D}}$, it follows from (3.5) and a simple calculation that

$$\begin{aligned} &\langle \mu\Delta\boldsymbol{u}^* + (\lambda+\mu)\nabla\nabla\cdot\boldsymbol{u}^* + \omega^2\boldsymbol{u}^* - \rho\boldsymbol{u}^*, \boldsymbol{\psi}\rangle \\ = &-\Big\langle \int_{\mathbb{R}^2} \left(\Delta^* + \omega^2\right)\boldsymbol{G}(\cdot,z,\omega)\rho(z)\boldsymbol{u}^*(z,y)dz, \boldsymbol{\psi}\Big\rangle \\ &+ \big\langle \left(\Delta^* + \omega^2\right)\boldsymbol{G}(\cdot,y,\omega)\boldsymbol{a}, \boldsymbol{\psi}\big\rangle - \langle \rho\boldsymbol{u}^*, \boldsymbol{\psi}\rangle = -\langle \delta_y \boldsymbol{a}, \boldsymbol{\psi}\rangle, \end{aligned}$$

which implies that $\boldsymbol{u}^* \in \boldsymbol{W}_{loc}^{\gamma,q}$ is also a solution of (1.1) and shows the existence of the solution of (1.1) according to Theorem 3.2.

The uniqueness of the solution to (1.1) can be obtained by following the same procedure as that used in [27, Theorem 3.5], which completes the proof. ∎

4. The Born series.

The results in the previous section indicate that the scattering problem (1.1) and (1.3), which is interpreted in the distribution sense, is equivalent to the Lippmann–Schwinger integral equation (3.1). In what follows, we may just focus on the Lippmann–Schwinger integral equation (3.1).

To get an explicit expression of the solution, we consider the Born sequence of the Lippmann–Schwinger integral equation

$$\boldsymbol{u}_j(x,y) = [-K_\omega \boldsymbol{u}_{j-1}(\cdot,y)](x), \quad j \in \mathbb{N}, \tag{4.1}$$

where the leading term is

$$\boldsymbol{u}_0(x,y) = \boldsymbol{G}(x,y,\omega)\boldsymbol{a}. \tag{4.2}$$

The goal of this section is to prove that the Born series $\sum_{j=0}^{\infty} \boldsymbol{u}_j$ converges to the solution $\boldsymbol{u}$ for sufficiently large ω.

4.1. Estimates of the scattering operators.

Before showing the convergence of the Born series, we first introduce a weighted space which is equipped with a weighted L^p-norm (cf. [23]). For any $\delta \in \mathbb{R}$, let

$$L^p_\delta(\mathbb{R}^2) := \{f \in L^1_{loc}(\mathbb{R}^2) : \|f\|_{L^p_\delta} < \infty\},$$

which is denoted by L^p_δ for short and is equipped with the norm

$$\|f\|_{L^p_\delta} := \|(1+|\cdot|^2)^{\frac{\delta}{2}} f\|_{L^p} = \Big(\int_{\mathbb{R}^2} (1+|x|^2)^{\frac{\delta p}{2}} |f(x)|^p dx\Big)^{\frac{1}{p}}.$$

Define the space

$$H_\delta^{s,p}(\mathbb{R}^2) := \{f \in \mathcal{S}' : (I-\Delta)^{\frac{s}{2}} f \in L_\delta^p\},$$

which is denoted by $H_\delta^{s,p}$ for simplicity and is equipped with the norm

$$\|f\|_{H_\delta^{s,p}} = \|(I-\Delta)^{\frac{s}{2}} f\|_{L_\delta^p}.$$

Here $\mathcal{S}'$ denotes the dual space of $\mathcal{S}$ which is the space of all rapidly decreasing functions. When $\delta = 0$, the space $H_0^{s,p}$ can be identified with the classical Sobolev space $W^{s,p}$. When $p = 2$, let $H_\delta^s := H_\delta^{s,2}$. For any $s \in \mathbb{R}$ and $\delta \in [0,1]$, it is easy to verify that

$$\|f\|_{H_\delta^s} = \|(I-\Delta)^{\frac{\delta}{2}}(1+|\cdot|^2)^{\frac{s}{2}}\hat{f}\|_{L^2} \gtrsim \|(1+|\cdot|^2)^{\frac{s}{2}}\hat{f}\|_{L^2} = \|f\|_{H^s}, \tag{4.3}$$

where the inequality is obtained by using [11, Theorem 13.5].

Based on these weighted norms, the operators H_ω and K_ω can be estimated as follows.

Lemma 4.1. *Let $\mathcal{V} \subset \mathbb{R}^2$ be any bounded domain. For any $s \in (0, \frac{1}{2})$ and $\epsilon > 0$, the following estimates hold:*

$$\|H_\omega\|_{\mathcal{L}(\boldsymbol{H}_1^{-s}, \boldsymbol{H}_{-1}^{s})} \lesssim \omega^{-1+2s}, \tag{4.4}$$

$$\|H_\omega\|_{\mathcal{L}(\boldsymbol{H}_1^{-s}, \boldsymbol{L}^\infty(\mathcal{V}))} \lesssim \omega^{s+\epsilon}. \tag{4.5}$$

Proof. The Green tensor $\boldsymbol{G}(x,y) := \boldsymbol{G}(x,y,\omega)$ satisfies

$$\mu\Delta\boldsymbol{G}(x,y) + (\lambda+\mu)\nabla\nabla\cdot\boldsymbol{G}(x,y) + \omega^2\boldsymbol{G}(x,y) = -\delta(x-y)\boldsymbol{I} \quad \text{in } \mathbb{R}^2. \tag{4.6}$$

Taking the Fourier transform on both sides of (4.6) with respect to $x-y$ leads to

$$-\mu|\xi|^2\widehat{\boldsymbol{G}}(\xi) - (\lambda+\mu)\xi\cdot\xi^\top\widehat{\boldsymbol{G}}(\xi) + \omega^2\widehat{\boldsymbol{G}}(\xi) = -\boldsymbol{I},$$

where $\xi = (\xi_1, \xi_2)^\top$. A simple calculation gives

$$\widehat{\boldsymbol{G}}(\xi) = \frac{c_{\rm s}^2 c_{\rm p}^2}{(|\xi|^2 - c_{\rm s}^2\omega^2)(|\xi|^2 - c_{\rm p}^2\omega^2)} A(\xi), \tag{4.7}$$

where the matrix

$$A(\xi) := \begin{bmatrix} \mu|\xi|^2 - \omega^2 + (\lambda+\mu)\xi_2^2 & -(\lambda+\mu)\xi_1\xi_2 \\ -(\lambda+\mu)\xi_1\xi_2 & \mu|\xi|^2 - \omega^2 + (\lambda+\mu)\xi_1^2 \end{bmatrix}.$$

Let $\boldsymbol{f} = (f_1, f_2)^\top \in \boldsymbol{C}_0^\infty$ and $\boldsymbol{g} = (g_1, g_2)^\top \in \boldsymbol{C}_0^\infty$. We have from the Parseval identity that

$$\begin{aligned} \langle H_\omega \boldsymbol{f}, \boldsymbol{g}\rangle &= \int_{\mathbb{R}^2} \widehat{H_\omega \boldsymbol{f}}(\xi)\hat{\boldsymbol{g}}(\xi)d\xi = \int_{\mathbb{R}^2} \widehat{\boldsymbol{G}}(\xi)\hat{\boldsymbol{f}}(\xi)\hat{\boldsymbol{g}}(\xi)d\xi \\ &= \int_{\mathbb{R}^2} \Big(\big[\widehat{G_{11}}(\xi)\hat{f}_1(\xi) + \widehat{G_{12}}(\xi)\hat{f}_2(\xi)\big]\hat{g}_1(\xi) \\ &\quad + \big[\widehat{G_{21}}(\xi)\hat{f}_1(\xi) + \widehat{G_{22}}(\xi)\hat{f}_2(\xi)\big]\hat{g}_2(\xi)\Big)d\xi, \end{aligned} \tag{4.8}$$

where $\widehat{G_{ij}}$ denotes the (i,j)-entry of $\widehat{\boldsymbol{G}}$. Noting that each term in (4.8) has the same singularity at $|\xi| = c_{\rm s}\omega$ and $|\xi| = c_{\rm p}\omega$, we only need to estimate the terms

$$\int_{\mathbb{R}^2} \widehat{G_{11}}(\xi)\hat{f}_1(\xi)\hat{g}_1(\xi)d\xi = c_{\rm s}^2 c_{\rm p}^2 \int_{\mathbb{R}^2} \frac{\mu|\xi|^2 - \omega^2 + (\lambda+\mu)\xi_2^2}{(|\xi|^2 - c_{\rm s}^2\omega^2)(|\xi|^2 - c_{\rm p}^2\omega^2)} \hat{f}_1(\xi)\hat{g}_1(\xi)d\xi, \tag{4.9}$$

$$\int_{\mathbb{R}^2} \widehat{G_{12}}(\xi)\hat{f}_2(\xi)\hat{g}_1(\xi)d\xi = c_{\rm s}^2 c_{\rm p}^2 \int_{\mathbb{R}^2} \frac{-(\lambda+\mu)\xi_1\xi_2}{(|\xi|^2 - c_{\rm s}^2\omega^2)(|\xi|^2 - c_{\rm p}^2\omega^2)} \hat{f}_2(\xi)\hat{g}_1(\xi)d\xi, \tag{4.10}$$

and the other two terms can be estimated similarly.

Define the Bessel potential operator $\mathcal{J}^s$ by

$$\mathcal{J}^s h(x) = \mathcal{F}^{-1}[(1+|\xi|^2)^{\frac{s}{2}}\hat{h}(\xi)](x) \quad \forall\, s \in \mathbb{R},\, h \in \mathcal{S},$$

where $\mathcal{F}^{-1}$ is the inverse Fourier transform. To deal with the singularity, we split the whole space $\mathbb{R}^2$ into three parts:

$$\begin{aligned} \Omega_1 &:= \{\xi \in \mathbb{R}^2 : \||\xi| - c_{\rm s}\omega| < \varepsilon_1\omega\}, \\ \Omega_2 &:= \{\xi \in \mathbb{R}^2 : \||\xi| - c_{\rm s}\omega| > \varepsilon_1\omega \ \text{ and } \ \||\xi| - c_{\rm p}\omega| < \varepsilon_2\omega\}, \\ \Omega_3 &:= \{\xi \in \mathbb{R}^2 : \||\xi| - c_{\rm s}\omega| > \varepsilon_1\omega \ \text{ and } \ \||\xi| - c_{\rm p}\omega| > \varepsilon_2\omega\}, \end{aligned}$$

where $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ are two constants.

First we estimate (4.9). Let

$$\begin{aligned} \mathrm{I}_j &= c_{\rm s}^2 c_{\rm p}^2 \int_{\Omega_j} \frac{\mu|\xi|^2 - \omega^2 + (\lambda+\mu)\xi_2^2}{(|\xi|^2 - c_{\rm s}^2\omega^2)(|\xi|^2 - c_{\rm p}^2\omega^2)} \hat{f}_1(\xi)\hat{g}_1(\xi)d\xi \\ &= c_{\rm s}^2 c_{\rm p}^2 \int_{\Omega_j} \frac{\mu|\xi|^2 - \omega^2 + (\lambda+\mu)\xi_2^2}{(|\xi|^2 - c_{\rm s}^2\omega^2)(|\xi|^2 - c_{\rm p}^2\omega^2)} (1+|\xi|^2)^s \widehat{\mathcal{J}^{-s}f_1}(\xi)\widehat{\mathcal{J}^{-s}g_1}(\xi)d\xi, \quad j = 1,2,3. \end{aligned}$$

For the term I_3, using the definition of Ω_3 and noting

$$|(\lambda+\mu)\xi_2^2| = |(\lambda+2\mu)\xi_2^2 - \mu\xi_2^2| \le (\lambda+2\mu)|\xi|^2 - \omega^2 + \mu|\xi|^2 - \omega^2 + 2\omega^2$$

and

$$\begin{aligned} \left| \frac{\mu|\xi|^2 - \omega^2 + (\lambda+\mu)\xi_2^2}{(|\xi|^2 - c_{\rm s}^2\omega^2)(|\xi|^2 - c_{\rm p}^2\omega^2)} \right| &\lesssim \frac{1}{||\xi|^2 - c_{\rm p}^2\omega^2|} + \frac{1}{||\xi|^2 - c_{\rm s}^2\omega^2|} \\ &\quad + \frac{\omega^2}{|(|\xi|^2 - c_{\rm s}^2\omega^2)(|\xi|^2 - c_{\rm p}^2\omega^2)|}, \end{aligned}$$

we get

$$\begin{aligned} |\mathrm{I}_3| &\lesssim \int_{\Omega_3} \left[\frac{(1+|\xi|^2)^s}{||\xi|^2 - c_{\rm p}^2\omega^2|} + \frac{(1+|\xi|^2)^s}{||\xi|^2 - c_{\rm s}^2\omega^2|} + \frac{\omega^2(1+|\xi|^2)^s}{|(|\xi|^2 - c_{\rm s}^2\omega^2)(|\xi|^2 - c_{\rm p}^2\omega^2)|} \right] \\ &\quad \times \big|\widehat{\mathcal{J}^{-s}f_1}(\xi)\big|\big|\widehat{\mathcal{J}^{-s}g_1}(\xi)\big|d\xi \\ &\lesssim \omega^{-2+2s} \int_{\Omega_3} \big|\widehat{\mathcal{J}^{-s}f_1}(\xi)\big|\big|\widehat{\mathcal{J}^{-s}g_1}(\xi)\big|d\xi \\ &\lesssim \omega^{-2+2s}\|f_1\|_{H^{-s}}\|g_1\|_{H^{-s}}, \end{aligned}$$

where in the second step we have used the following estimates: if $|\xi| < (c_{\rm p} - \varepsilon_2)\omega$, then

$$\frac{(1+|\xi|^2)^s}{||\xi|^2 - c_{\rm p}^2\omega^2|} \le \frac{(1+|\xi|^2)^s}{\varepsilon_2\omega(|\xi| + c_{\rm p}\omega)} \le \frac{(1+|(c_{\rm p}-\varepsilon_2)\omega|^2)^s}{\varepsilon_2 c_{\rm p}\omega^2} \lesssim \omega^{-2+2s};$$

if $|\xi| > (c_{\rm p} + \varepsilon_2)\omega$ in Ω_3, then

$$\frac{(1+|\xi|^2)^s}{\left||\xi|^2 - c_{\rm p}^2\omega^2\right|} \le \frac{(1+|\xi|^2)^s}{\varepsilon_2\omega(|\xi| + c_{\rm p}\omega)} \le \frac{(2|\xi|^2)^s}{\varepsilon_2\omega|\xi|} \lesssim \frac{1}{\omega|\xi|^{1-2s}} \lesssim \omega^{-2+2s}.$$

For the term I_1, we have

$$\begin{aligned}\mathrm{I}_1 &= c_{\rm p}^2 \int_{\Omega_1} \frac{1}{|\xi|^2 - c_{\rm p}^2\omega^2}(1+|\xi|^2)^s \widehat{\mathcal{J}^{-s}f_1}(\xi)\widehat{\mathcal{J}^{-s}g_1}(\xi)d\xi \\ &\quad + c_{\rm s}^2c_{\rm p}^2 \int_{\Omega_1} \frac{(\lambda+\mu)\xi_2^2}{(|\xi|^2 - c_{\rm s}^2\omega^2)(|\xi|^2 - c_{\rm p}^2\omega^2)}(1+|\xi|^2)^s \widehat{\mathcal{J}^{-s}f_1}(\xi)\widehat{\mathcal{J}^{-s}g_1}(\xi)d\xi \\ &=: \mathrm{I}_{11} + \mathrm{I}_{12}.\end{aligned}$$

For $\xi \in \Omega_1$, we can choose ε_1 small enough such that $\left||\xi| - c_{\rm p}\omega\right| \ge c\omega$ for some $c > 0$, and follow similarly the estimate of I_3 to get

$$|\mathrm{I}_{11}| \lesssim \omega^{-2+2s}\|f_1\|_{H^{-s}}\|g_1\|_{H^{-s}}.$$

To estimate I_{12}, we make the following change of variables:

$$\xi^* = \xi + 2(c_{\rm s}\omega - |\xi|)\tilde{\xi} = 2c_{\rm s}\omega\tilde{\xi} - \xi,$$

where $\tilde{\xi} := \xi/|\xi|$. It can be easily verified that the change of variables maps the domain $\Omega_{11} := \{\xi \in \mathbb{R}^2 : c_{\rm s}\omega - \varepsilon_1\omega < |\xi| < c_{\rm s}\omega\}$ to the domain $\Omega_{12} := \{\xi \in \mathbb{R}^2 : c_{\rm s}\omega < |\xi| < c_{\rm s}\omega + \varepsilon_1\omega\}$, and the Jacobian for the change of variables is

$$J(\xi) = \frac{2c_{\rm s}\omega}{|\xi|} - 1.$$

Using the fact $\Omega_1 = \Omega_{11} \cup \Omega_{12} \cup \{\xi \in \mathbb{R}^2 : |\xi| = c_{\rm s}\omega\}$ with $\{\xi \in \mathbb{R}^2 : |\xi| = c_{\rm s}\omega\}$ being a set of zero measure, we obtain

$$\begin{aligned}\mathrm{I}_{12} &= c_{\rm s}^2c_{\rm p}^2 \int_{\Omega_{11}\cup\Omega_{12}} \frac{(\lambda+\mu)\xi_2^2}{(|\xi|^2 - c_{\rm s}^2\omega^2)(|\xi|^2 - c_{\rm p}^2\omega^2)}(1+|\xi|^2)^s \widehat{\mathcal{J}^{-s}f_1}(\xi)\widehat{\mathcal{J}^{-s}g_1}(\xi)d\xi \\ &= c_{\rm s}^2c_{\rm p}^2 \int_{\Omega_{12}} \frac{(\lambda+\mu)\xi_2^2}{(|\xi|^2 - c_{\rm s}^2\omega^2)(|\xi|^2 - c_{\rm p}^2\omega^2)}(1+|\xi|^2)^s \widehat{\mathcal{J}^{-s}f_1}(\xi)\widehat{\mathcal{J}^{-s}g_1}(\xi)d\xi \\ &\quad + c_{\rm s}^2c_{\rm p}^2 \int_{\Omega_{12}} \frac{(\lambda+\mu){\xi_2^*}^2}{(|\xi^*|^2 - c_{\rm s}^2\omega^2)(|\xi^*|^2 - c_{\rm p}^2\omega^2)}(1+|\xi^*|^2)^s \widehat{\mathcal{J}^{-s}f_1}(\xi^*)\widehat{\mathcal{J}^{-s}g_1}(\xi^*)J(\xi)d\xi \\ &= c_{\rm s}^2c_{\rm p}^2 \int_{\Omega_{12}} m_1(\xi,\omega)(1+|\xi|^2)^s \widehat{\mathcal{J}^{-s}f_1}(\xi)\widehat{\mathcal{J}^{-s}g_1}(\xi)d\xi \\ &\quad + c_{\rm s}^2c_{\rm p}^2 \int_{\Omega_{12}} \Big[m_2(\xi,\omega)\big[(1+|\xi^*|^2)^s \widehat{\mathcal{J}^{-s}f_1}(\xi^*)\widehat{\mathcal{J}^{-s}g_1}(\xi^*) \\ &\quad - (1+|\xi|^2)^s \widehat{\mathcal{J}^{-s}f_1}(\xi)\widehat{\mathcal{J}^{-s}g_1}(\xi)\big]J(\xi)\Big]d\xi \\ &=: \mathrm{I}_{13} + \mathrm{I}_{14},\end{aligned}$$

where

$$m_1(\xi,\omega) = \frac{(\lambda+\mu)\xi_2^2}{(|\xi|^2 - c_{\rm s}^2\omega^2)(|\xi|^2 - c_{\rm p}^2\omega^2)} + \frac{(\lambda+\mu){\xi_2^*}^2}{(|\xi^*|^2 - c_{\rm s}^2\omega^2)(|\xi^*|^2 - c_{\rm p}^2\omega^2)}J(\xi),$$

$$m_2(\xi,\omega) = \frac{(\lambda+\mu)\xi_2^{*2}}{(|\xi^*|^2 - c_{\rm s}^2\omega^2)(|\xi^*|^2 - c_{\rm p}^2\omega^2)}.$$

For $\xi \in \Omega_{12}$, it is not difficult to show that $\xi^* \in \Omega_{11}$ with $|\xi^*|^2 = 4c_{\rm s}^2\omega^2 + |\xi|^2 - 4c_{\rm s}\omega|\xi|$. Then there exists a constant $C > 0$ such that

$$\left|\frac{(\lambda+\mu)\xi_2^2}{|\xi|^2 - c_{\rm p}^2\omega^2}\right| \le C$$

and

$$\begin{aligned}
&\left|\frac{1}{|\xi|^2 - c_{\rm s}^2\omega^2} + \frac{1}{|\xi^*|^2 - c_{\rm s}^2\omega^2}J(\xi)\right| \\
&= \left|\frac{1}{(|\xi| - c_{\rm s}\omega)(|\xi| + c_{\rm s}\omega)} + \frac{1}{(|\xi| - c_{\rm s}\omega)(|\xi| - 3c_{\rm s}\omega)}\frac{2c_{\rm s}\omega - |\xi|}{|\xi|}\right| \\
&= \frac{2c_{\rm s}^2\omega^2 - 2c_{\rm s}\omega|\xi|}{(|\xi| - c_{\rm s}\omega)(|\xi| + c_{\rm s}\omega)(|\xi| - 3c_{\rm s}\omega)|\xi|} \lesssim \omega^{-2},
\end{aligned}$$

which leads to

$$\begin{aligned}
|m_1(\xi,\omega)| \le &\left|\frac{(\lambda+\mu)\xi_2^2}{|\xi|^2 - c_{\rm p}^2\omega^2}\left[\frac{1}{|\xi|^2 - c_{\rm s}^2\omega^2} + \frac{1}{|\xi^*|^2 - c_{\rm s}^2\omega^2}J(\xi)\right]\right| \\
&+ \left|\frac{J(\xi)}{|\xi^*|^2 - c_{\rm s}^2\omega^2}\left[\frac{(\lambda+\mu)\xi_2^{*2}}{|\xi^*|^2 - c_{\rm p}^2\omega^2} - \frac{(\lambda+\mu)\xi_2^2}{|\xi|^2 - c_{\rm p}^2\omega^2}\right]\right| \lesssim \omega^{-2}.
\end{aligned}$$

Hence, the term I_{13} admits the estimate

$$|\mathrm{I}_{13}| \lesssim \omega^{-2+2s}\|f_1\|_{H^{-s}}\|g_1\|_{H^{-s}}.$$

Item I_{14} can be decomposed as

$$\begin{aligned}
\mathrm{I}_{14} = &\, c_{\rm s}^2c_{\rm p}^2\int_{\Omega_{12}} m_2(\xi,\omega)\left[(1+|\xi^*|^2)^s - (1+|\xi|^2)^s\right]\widehat{\mathcal{J}^{-s}f_1}(\xi)\widehat{\mathcal{J}^{-s}g_1}(\xi)J(\xi)d\xi \\
&+ c_{\rm s}^2c_{\rm p}^2\int_{\Omega_{12}} m_2(\xi,\omega)(1+|\xi^*|^2)^s\left[\widehat{\mathcal{J}^{-s}f_1}(\xi^*) - \widehat{\mathcal{J}^{-s}f_1}(\xi)\right]\widehat{\mathcal{J}^{-s}g_1}(\xi)J(\xi)d\xi \\
&+ c_{\rm s}^2c_{\rm p}^2\int_{\Omega_{12}} m_2(\xi,\omega)(1+|\xi^*|^2)^s\left[\widehat{\mathcal{J}^{-s}g_1}(\xi^*) - \widehat{\mathcal{J}^{-s}g_1}(\xi)\right]\widehat{\mathcal{J}^{-s}f_1}(\xi^*)J(\xi)d\xi \\
=:&\, \mathrm{I}_{15} + \mathrm{I}_{16} + \mathrm{I}_{17}.
\end{aligned}$$

By the mean value theorem, we get

$$\begin{aligned}
&\left|m_2(\xi,\omega)\left[(1+|\xi^*|^2)^s - (1+|\xi|^2)^s\right]J(\xi)\right| \\
&= \left|\frac{(\lambda+\mu)(\xi_2^*)^2\left(1+\theta|\xi^*|^2 + (1-\theta)|\xi|^2\right)^{s-1}s(|\xi^*|^2 - |\xi|^2)}{(|\xi^*|^2 - c_{\rm s}^2\omega^2)(|\xi^*|^2 - c_{\rm p}^2\omega^2)}\frac{2c_{\rm s}\omega - |\xi|}{|\xi|}\right| \\
&= \left|\frac{(\lambda+\mu)(\xi_2^*)^2\left(1+\theta|\xi^*|^2 + (1-\theta)|\xi|^2\right)^{s-1}4sc_{\rm s}\omega(c_{\rm s}\omega - |\xi|)}{(|\xi| - c_{\rm s}\omega)(|\xi| - 3c_{\rm s}\omega)(|\xi^*|^2 - c_{\rm p}^2\omega^2)}\frac{2c_{\rm s}\omega - |\xi|}{|\xi|}\right| \\
&\lesssim \omega^{-2+2s}
\end{aligned}$$

with some $\theta \in (0,1)$. It shows that

$$|\mathrm{I}_{15}| \lesssim \omega^{-2+2s}\|f_1\|_{H^{-s}}\|g_1\|_{H^{-s}}.$$

To estimate I_{16} and I_{17}, we employ the following characterization of $W^{1,p}(\mathbb{R}^d)$ introduced in [15].

LEMMA 4.2. *For $1 < p \le \infty$, the function $u \in W^{1,p}(\mathbb{R}^d)$ if and only if there exist $g \in L^p(\mathbb{R}^d)$ and $C > 0$ such that*

$$|u(x) - u(y)| \le C|x - y|(g(x) + g(y)).$$

Moreover, we can choose $g = M(|\nabla u|)$, where M is defined by

$$M(f)(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)| dy$$

and is called the Hardy–Littlewood maximal function of f.

For $f_1 \in C_0^\infty$, we have $\widehat{\mathcal{J}^{-s} f_1} \in \mathcal{S} \subset H^1$. An application of Lemma 4.2 gives

$$\left|\widehat{\mathcal{J}^{-s} f_1}(\xi^*) - \widehat{\mathcal{J}^{-s} f_1}(\xi)\right| \lesssim \left|c_{\mathrm{s}}\omega - |\xi|\right| \left[M(|\nabla \widehat{\mathcal{J}^{-s} f_1}|)(\xi^*) + M(|\nabla \widehat{\mathcal{J}^{-s} f_1}|)(\xi)\right]. \tag{4.11}$$

By [31, Theorem 2.1], we get

$$\begin{aligned} \|M(|\nabla \widehat{\mathcal{J}^{-s} f_1}|)\|_{L^2} &\lesssim \|\nabla \widehat{\mathcal{J}^{-s} f_1}\|_{L^2} \lesssim \|(I - \Delta)^{\frac{1}{2}} \widehat{\mathcal{J}^{-s} f_1}\|_{L^2} \\ &= \|(I - \Delta)^{\frac{1}{2}} (1 + |\cdot|^2)^{-\frac{s}{2}} \hat{f}_1(\cdot)\|_{L^2} = \|f_1\|_{H_1^{-s}}, \end{aligned} \tag{4.12}$$

where (4.3) is used in the last step. Combining (4.11) and (4.12) gives

$$\begin{aligned} |\mathrm{I}_{16}| &\lesssim \omega^{-1+2s} \int_{\Omega_{12}} \left[M(|\nabla \widehat{\mathcal{J}^{-s} f_1}|)(\xi^*) + M(|\nabla \widehat{\mathcal{J}^{-s} f_1}|)(\xi)\right] |\widehat{\mathcal{J}^{-s} g_1}(\xi)| d\xi \\ &\lesssim \omega^{-1+2s} \|f_1\|_{H_1^{-s}} \|g_1\|_{H^{-s}}. \end{aligned}$$

Item I_{17} can be similarly estimated and satisfies

$$|\mathrm{I}_{17}| \lesssim \omega^{-1+2s} \|f_1\|_{H^{-s}} \|g_1\|_{H_1^{-s}}.$$

Hence we conclude from the above estimates that

$$|\mathrm{I}_1| \lesssim \omega^{-1+2s} \|f_1\|_{H_1^{-s}} \|g_1\|_{H_1^{-s}}.$$

Similarly, we may repeat the steps for the estimate of I_1 and show that

$$|\mathrm{I}_2| \lesssim \omega^{-1+2s} \|f_1\|_{H_1^{-s}} \|g_1\|_{H_1^{-s}}.$$

It follows from the estimates of I_j, $j = 1, 2, 3$, that (4.9) satisfies the estimate

$$\left|\int_{\mathbb{R}^2} \widehat{G_{11}}(\xi) \hat{f}_1(\xi) \hat{g}_1(\xi) d\xi\right| \lesssim \omega^{-1+2s} \|f_1\|_{H_1^{-s}} \|g_1\|_{H_1^{-s}}.$$

Next is to estimate (4.10). Let

$$\begin{aligned} \mathrm{II}_j &= -c_{\mathrm{s}}^2 c_{\mathrm{p}}^2 \int_{\Omega_j} \frac{(\lambda + \mu)\xi_1 \xi_2}{(|\xi|^2 - c_{\mathrm{s}}^2\omega^2)(|\xi|^2 - c_{\mathrm{p}}^2\omega^2)} \hat{f}_2(\xi) \hat{g}_1(\xi) d\xi \\ &= -c_{\mathrm{s}}^2 c_{\mathrm{p}}^2 \int_{\Omega_j} \frac{(\lambda + \mu)\xi_1 \xi_2}{(|\xi|^2 - c_{\mathrm{s}}^2\omega^2)(|\xi|^2 - c_{\mathrm{p}}^2\omega^2)} \\ &\quad \times (1 + |\xi|^2)^s \widehat{\mathcal{J}^{-s} f_2}(\xi) \widehat{\mathcal{J}^{-s} g_1}(\xi) d\xi, \quad j = 1, 2, 3. \end{aligned}$$

Following the same estimate as that of I_3 and noting

$$|(\lambda+\mu)\xi_1\xi_2| = |(\lambda+2\mu)\xi_1\xi_2 - \mu\xi_1\xi_2| \le \frac{(\lambda+2\mu)|\xi|^2-\omega^2}{2} + \frac{\mu|\xi|^2-\omega^2}{2} + \omega^2,$$

we have

$$|\mathrm{II}_3| \lesssim \omega^{-2+2s}\|f_2\|_{H^{-s}}\|g_1\|_{H^{-s}}.$$

As for the estimate of

$$\mathrm{II}_1 = -c_{\mathrm{s}}^2 c_{\mathrm{p}}^2 \int_{\Omega_1} \frac{(\lambda+\mu)\xi_1\xi_2}{(|\xi|^2-c_{\mathrm{s}}^2\omega^2)(|\xi|^2-c_{\mathrm{p}}^2\omega^2)}(1+|\xi|^2)^s \widehat{\mathcal{J}^{-s}f_2}(\xi)\widehat{\mathcal{J}^{-s}g_1}(\xi)d\xi,$$

it is similar to that of I_{12} and admits

$$|\mathrm{II}_1| \lesssim \omega^{-1+2s}\|f_2\|_{H_1^{-s}}\|g_1\|_{H_1^{-s}},$$

which may further lead to the estimate

$$|\mathrm{II}_2| \lesssim \omega^{-1+2s}\|f_2\|_{H_1^{-s}}\|g_1\|_{H_1^{-s}}.$$

Combining the above estimates yields

$$\left|\int_{\mathbb{R}^2} \widehat{G_{11}}(\xi)\hat{f}_1(\xi)\hat{g}_1(\xi)d\xi\right| \lesssim \omega^{-1+2s}\|f_2\|_{H_1^{-s}}\|g_1\|_{H_1^{-s}}.$$

It follows from (4.9)–(4.10) that (4.8) has the following estimate:

$$|\langle H_\omega \boldsymbol{f}, \boldsymbol{g}\rangle| \lesssim \omega^{-1+2s}\|\boldsymbol{f}\|_{\boldsymbol{H}_1^{-s}}\|\boldsymbol{g}\|_{\boldsymbol{H}_1^{-s}} \quad \forall \boldsymbol{f}, \boldsymbol{g} \in \boldsymbol{C}_0^\infty.$$

This result can be extended for any $\boldsymbol{f}, \boldsymbol{g} \in \boldsymbol{H}_1^{-s}$ since $\boldsymbol{C}_0^\infty$ is dense in $\boldsymbol{H}_1^{-s}$. The density argument can be found in [23, Theorem 2.2]. It then completes the proof of (4.4).

To prove (4.5), let $\boldsymbol{f} = (f_1, f_2)^\top \in \boldsymbol{C}_0^\infty$. We have

$$\begin{aligned}(H_\omega \boldsymbol{f})(x) &= \int_{\mathbb{R}^2} \boldsymbol{G}(x,y)\boldsymbol{f}(y)dy \\ &= \int_{\mathbb{R}^2} (1+|\xi|^2)^{\frac{s}{2}} \widehat{\boldsymbol{G}}(x,\xi)\widehat{\mathcal{J}^{-s}\boldsymbol{f}}(\xi)d\xi \\ &= \int_{\mathbb{R}^2} (1+|\xi|^2)^{\frac{s}{2}} \begin{bmatrix} \widehat{G_{11}}(x,\xi)\widehat{\mathcal{J}^{-s}f_1}(\xi) + \widehat{G_{12}}(x,\xi)\widehat{\mathcal{J}^{-s}f_2}(\xi) \\ \widehat{G_{21}}(x,\xi)\widehat{\mathcal{J}^{-s}f_1}(\xi) + \widehat{G_{22}}(x,\xi)\widehat{\mathcal{J}^{-s}f_2}(\xi)\end{bmatrix} d\xi, \end{aligned} \tag{4.13}$$

where $\widehat{\boldsymbol{G}}(x,\xi)$, different from $\widehat{\boldsymbol{G}}(\xi)$, denotes the Fourier transform of $\boldsymbol{G}(x,y)$ obtained by taking the Fourier transform on both sides of (4.6) with respect to y and satisfies

$$-\mu|\xi|^2\widehat{\boldsymbol{G}}(x,\xi) - (\lambda+\mu)\xi\cdot\xi^\top\widehat{\boldsymbol{G}}(x,\xi) + \omega^2\widehat{\boldsymbol{G}}(x,\xi) = -e^{-\mathrm{i}x\cdot\xi}\boldsymbol{I}.$$

Comparing the above equation with (4.7), we get $\widehat{G_{ij}}(x,\xi) = e^{-\mathrm{i}x\cdot\xi}\widehat{G_{ij}}(\xi)$. It follows from the same arguments as those for item (4.9) that

$$\left|\int_{\mathbb{R}^2} (1+|\xi|^2)^{\frac{s}{2}}\widehat{G_{11}}(x,\xi)\widehat{\mathcal{J}^{-s}f_1}(\xi)d\xi\right|$$

$$= \left| \int_{\mathbb{R}^2} \widehat{G_{11}}(\xi)(1+|\xi|^2)^{\frac{s+\epsilon+1}{2}} \widehat{\mathcal{J}^{-s} f_1}(\xi) e^{-\mathrm{i}x\cdot\xi}(1+|\xi|^2)^{\frac{-1-\epsilon}{2}} d\xi \right|$$
$$\lesssim \frac{1}{\omega^{1-2\frac{s+\epsilon+1}{2}}} \|f_1\|_{H_1^{-s}} \lesssim \omega^{s+\epsilon} \|f_1\|_{H_1^{-s}},$$

where we have used the fact that the function

$$g(\xi) := e^{-\mathrm{i}x\cdot\xi}(1+|\xi|^2)^{\frac{-1-\epsilon}{2}}$$

satisfies $g \in H^1$ for any $x \in \mathcal{V}$. The estimates can be similarly obtained for the other three items in (4.13). Therefore we have

$$\|H_\omega \boldsymbol{f}\|_{\boldsymbol{L}^\infty(\mathcal{V})} \lesssim \omega^{s+\epsilon} \|\boldsymbol{f}\|_{\boldsymbol{H}_1^{-s}},$$

which completes the proof of (4.5). ∎

Based on the estimates for the operator H_ω, the following results present the estimates for the operator K_ω.

LEMMA 4.3. *Let $\mathcal{V} \subset \mathbb{R}^2$ be a bounded domain and ρ satisfy Assumption* 1.1. *For any $s \in (\frac{2-m}{2}, \frac{1}{2})$ and $\epsilon > 0$, the following estimates hold almost surely:*

$$\|K_\omega\|_{\mathcal{L}(\boldsymbol{H}_{-1}^s)} \lesssim \omega^{-1+2s},$$
$$\|K_\omega\|_{\mathcal{L}(\boldsymbol{H}_{-1}^s, \boldsymbol{L}^\infty(\mathcal{V}))} \lesssim \omega^{s+\epsilon}.$$

Proof. For any $\boldsymbol{u} \in \boldsymbol{H}_{-1}^s$, it holds that $K_\omega \boldsymbol{u} = H_\omega(\rho \boldsymbol{u})$ with H_ω being a bounded operator from $\boldsymbol{H}_1^{-s}$ to $\boldsymbol{H}_{-1}^s$ according to Lemma 4.1.

We first claim that $\rho\boldsymbol{u} \in \boldsymbol{H}_1^{-s}$ for any $\boldsymbol{u} \in \boldsymbol{H}_{-1}^s$. Note that $\rho \in W_0^{-\gamma,p}(D)$ for any $\gamma > \frac{2-m}{2}$ and $p > 1$ according to Lemma 2.2. For any $\boldsymbol{u}, \boldsymbol{v} \in \mathcal{S}$, define $\langle \rho\boldsymbol{u}, \boldsymbol{v}\rangle := \langle \rho, \boldsymbol{u}\cdot\boldsymbol{v}\rangle$ and a cutoff function $\vartheta \in C_0^\infty$ whose support $\tilde{D}$ has a locally Lipschitz boundary and $\vartheta(x) = 1$ if $x \in D \subset \tilde{D}$. It is easy to verify that

$$\begin{aligned} |\langle \rho\boldsymbol{u}, \boldsymbol{v}\rangle| &= |\langle \rho, (\vartheta\boldsymbol{u})\cdot(\vartheta\boldsymbol{v})\rangle| \\ &= \left|\langle (I-\Delta)^{-\gamma}\rho, (I-\Delta)^{\gamma}[(\vartheta\boldsymbol{u})\cdot(\vartheta\boldsymbol{v})]\rangle\right| \\ &\le \|\rho\|_{W^{-\gamma,p}} \|(I-\Delta)^\gamma[(\vartheta\boldsymbol{u})\cdot(\vartheta\boldsymbol{v})]\|_{L^q} \end{aligned}$$

with $q > 1$ satisfying $\frac{1}{p} + \frac{1}{q} = 1$. It follows from the fractional Leibniz principle with $\tilde{q}$ satisfying $\frac{1}{q} = \frac{1}{2} + \frac{1}{\tilde{q}}$ that

$$\|(I-\Delta)^\gamma[(\vartheta\boldsymbol{u})\cdot(\vartheta\boldsymbol{v})]\|_{L^q} \le \|\vartheta\boldsymbol{u}\|_{\boldsymbol{L}^2(\tilde{D})} \|\vartheta\boldsymbol{v}\|_{\boldsymbol{W}^{\gamma,\tilde{q}}(\tilde{D})} + \|\vartheta\boldsymbol{v}\|_{\boldsymbol{L}^2(\tilde{D})} \|\vartheta\boldsymbol{u}\|_{\boldsymbol{W}^{\gamma,\tilde{q}}(\tilde{D})}.$$

For any $s \in (\frac{2-m}{2}, \frac{1}{2})$, there exist $\gamma \in (\frac{2-m}{2}, s)$ and $q > 1$ such that $\gamma \le s$ and $\frac{1}{q} - \frac{1}{2} = \frac{1}{\tilde{q}} > \frac{1}{2} - \frac{s-\gamma}{2}$, which implies $\boldsymbol{H}^s(\tilde{D}) \hookrightarrow \boldsymbol{W}^{\gamma,\tilde{q}}(\tilde{D})$. Hence

$$|\langle \rho\boldsymbol{u}, \boldsymbol{v}\rangle| \le \|\rho\|_{W^{-\gamma,p}} \|\vartheta\boldsymbol{u}\|_{\boldsymbol{W}^{\gamma,\tilde{q}}(\tilde{D})} \|\vartheta\boldsymbol{v}\|_{\boldsymbol{W}^{\gamma,\tilde{q}}(\tilde{D})} \lesssim \|\rho\|_{W^{-\gamma,p}} \|\vartheta\boldsymbol{u}\|_{\boldsymbol{H}^s(\tilde{D})} \|\vartheta\boldsymbol{v}\|_{\boldsymbol{H}^s(\tilde{D})}.$$

Using the facts that $\|\vartheta\boldsymbol{u}\|_{\boldsymbol{H}^s(\tilde{D})} \lesssim \|\boldsymbol{u}\|_{\boldsymbol{H}_{-2}^s} \lesssim \|\boldsymbol{u}\|_{\boldsymbol{H}_{-1}^s}$ and that $\mathcal{S}$ is dense in $\boldsymbol{H}_{-1}^s$ proved in [23, Theorem 2.2], we get almost surely that

$$|\langle \rho\boldsymbol{u}, \boldsymbol{v}\rangle| \lesssim \|\boldsymbol{u}\|_{\boldsymbol{H}_{-1}^s} \|\boldsymbol{v}\|_{\boldsymbol{H}_{-1}^s} \quad \forall \boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{H}_{-1}^s,$$

which completes the claim. Then the following two estimates hold almost surely:

$$\|K_\omega\boldsymbol{u}\|_{\boldsymbol{H}_{-1}^s} \le \|H_\omega\|_{\mathcal{L}(\boldsymbol{H}_1^{-s}, \boldsymbol{H}_{-1}^s)} \|\rho\boldsymbol{u}\|_{\boldsymbol{H}_1^{-s}} \lesssim \omega^{-1+2s} \|\boldsymbol{u}\|_{\boldsymbol{H}_{-1}^s}$$

and

$$\|K_\omega\boldsymbol{u}\|_{\boldsymbol{L}^\infty(\mathcal{V})} \le \|H_\omega\|_{\mathcal{L}(\boldsymbol{H}_1^{-s}, \boldsymbol{L}^\infty(\mathcal{V}))} \|\rho\boldsymbol{u}\|_{\boldsymbol{H}_1^{-s}} \lesssim \omega^{s+\epsilon} \|\boldsymbol{u}\|_{\boldsymbol{H}_{-1}^s},$$

which complete the proof. ∎

4.2. Convergence of the Born series. Let assumptions in Lemma 4.3 hold and $U \subset \mathbb{R}^2 \backslash \overline{D}$ be a bounded and convex measurement domain which has a locally Lipschitz boundary and a positive distance from D. This section shows the convergence of the Born series defined in (4.1).

It follows from (4.1) that

$$\left[(I + K_\omega) \sum_{j=0}^{N} \boldsymbol{u}_j(\cdot, y) \right](x) = \boldsymbol{u}_0(x, y) + (-1)^N [K_\omega^{N+1} \boldsymbol{u}_0(\cdot, y)](x). \tag{4.14}$$

Note that

$$[K_\omega \boldsymbol{u}_0(\cdot, y)](x) = \int_D \boldsymbol{G}(x, z, \omega)\rho(z)\boldsymbol{u}_0(z, y) dz \quad \forall\, x, y \in U,$$

where $\boldsymbol{u}_0(z, y) = \boldsymbol{G}(z, y, \omega)\boldsymbol{a}$ and $\boldsymbol{G}(z, y, \omega)$ is smooth for any $z \in D$ and $y \in U$.

We begin with the estimate for $\boldsymbol{u}_0$.

LEMMA 4.4. *Let $U \subset \mathbb{R}^2 \backslash \overline{D}$ be a bounded and convex domain having a locally Lipschitz boundary and a positive distance to D. For any $s \in [0, 1]$, $p \in (1, \infty)$ and any fixed $y \in U$, the following estimate holds:*

$$\|\boldsymbol{u}_0(\cdot, y)\|_{\boldsymbol{W}^{s,p}(D)} \lesssim \omega^{-\frac{1}{2}+s}.$$

Proof. For any $y \in U$, it is easy to check that

$$\|\boldsymbol{u}_0(\cdot, y)\|_{\boldsymbol{L}^p(D)} = \|\boldsymbol{G}(\cdot, y, \omega)\boldsymbol{a}\|_{\boldsymbol{L}^p(D)} \lesssim \omega^{-\frac{1}{2}},$$
$$\|\boldsymbol{u}_0(\cdot, y)\|_{\boldsymbol{W}^{1,p}(D)} = \|\boldsymbol{G}(\cdot, y, \omega)\boldsymbol{a}\|_{\boldsymbol{W}^{1,p}(D)} \lesssim \omega^{\frac{1}{2}}.$$

Utilizing the interpolation inequality [17], we get

$$\|\boldsymbol{u}_0(\cdot, y)\|_{\boldsymbol{W}^{s,p}(D)} \lesssim \|\boldsymbol{u}_0(\cdot, y)\|^{1-s}_{\boldsymbol{L}^p(D)} \|\boldsymbol{u}_0(\cdot, y)\|^{s}_{\boldsymbol{W}^{1,p}(D)} \lesssim \omega^{-\frac{1}{2}+s},$$

which completes the proof. ∎

By Lemmas 4.3 and 4.4, we have for $s \in (\frac{2-m}{2}, \frac{1}{2})$ that

$$\|K_\omega^{N+1} \boldsymbol{u}_0\|_{\boldsymbol{H}^s_{-1}(U)} \lesssim \omega^{(-1+2s)(N+1)} \|\boldsymbol{u}_0(\cdot, y)\|_{\boldsymbol{H}^s_{-1}(D)} \lesssim \omega^{(-1+2s)(N+1)-\frac{1}{2}+s} \to 0$$

as $N \to \infty$, where we have used the fact

$$\|\boldsymbol{u}_0(\cdot, y)\|_{\boldsymbol{H}^s_{-1}(D)} = \|(1 + |\cdot|^2)^{-\frac{1}{2}} (I - \Delta)^{\frac{s}{2}} \boldsymbol{u}_0(\cdot, y)\|_{\boldsymbol{L}^2(D)} \lesssim \|\boldsymbol{u}_0(\cdot, y)\|_{\boldsymbol{H}^s(D)}.$$

Combining the above estimate with (4.14) leads to

$$(I + K_\omega) \sum_{j=0}^{\infty} \boldsymbol{u}_j = \boldsymbol{u}_0 \quad \text{in} \quad \boldsymbol{H}^s_{-1}(U).$$

Note also that $\boldsymbol{G}(\cdot, y, \omega) \in (L^2_{\rm loc} \cap W^{1,p'}_{\rm loc})^{2\times 2}$ for any $p' \in (1, 2)$. Choosing $p' = 2-\epsilon$ for sufficient small $\epsilon > 0$, we may follow the same proof as that of Theorem 3.2 and get $\boldsymbol{W}^{1,p'}(U) \hookrightarrow \boldsymbol{H}^s(U)$, which implies that $\boldsymbol{u}_0(\cdot, y) \in \boldsymbol{H}^s(U) \hookrightarrow \boldsymbol{W}^{\gamma,q}(U)$ and $(I + K_\omega)^{-1}\boldsymbol{u}_0 = \boldsymbol{u}$ in $\boldsymbol{W}^{\gamma,q}(U)$. Hence, the Born series converges to the unique solution $\boldsymbol{u}$ of (1.1) in $\boldsymbol{W}^{\gamma,q}(U)$ and

$$\boldsymbol{u} = \sum_{j=0}^{\infty} \boldsymbol{u}_j. \tag{4.15}$$

Moreover,

$$\begin{aligned} \|\boldsymbol{u} - \sum_{j=0}^{N} \boldsymbol{u}_j\|_{\boldsymbol{L}^\infty(U)} &\lesssim \sum_{j=N+1}^{\infty} \|K_\omega^j \boldsymbol{u}_0\|_{\boldsymbol{L}^\infty(U)} \\ &\leq \sum_{j=N+1}^{\infty} \|K_\omega\|_{\mathcal{L}(\boldsymbol{H}_{-1}^s, \boldsymbol{L}^\infty(U))} \|K_\omega\|_{\mathcal{L}(\boldsymbol{H}_{-1}^s)}^{j-1} \|\boldsymbol{u}_0(\cdot, y)\|_{\boldsymbol{H}_{-1}^s(D)} \\ &\lesssim \omega^{s+\epsilon+(-1+2s)N-\frac{1}{2}+s} \to 0 \end{aligned} \tag{4.16}$$

as $N \to \infty$, which implies that (4.15) also holds in $\boldsymbol{L}^\infty(U)$.

5. The inverse scattering problem. In this section, we study the inverse scattering problem which is to reconstruct the microcorrelation strength ϕ of the random potential ρ.

We consider the case $y = x$ and recall that notations $\boldsymbol{u}^s(x, \omega, \boldsymbol{a})$ and $\boldsymbol{u}_j(x, \omega, \boldsymbol{a})$ stand for $\boldsymbol{u}^s(x, x, \omega, \boldsymbol{a})$ and $\boldsymbol{u}_j(x, x, \omega, \boldsymbol{a})$, respectively. Then we rewrite (4.15) in terms of the scattered field

$$\boldsymbol{u}^s(x, \omega, \boldsymbol{a}) = \boldsymbol{u}_1(x, \omega, \boldsymbol{a}) + \boldsymbol{u}_2(x, \omega, \boldsymbol{a}) + \boldsymbol{b}(x, \omega, \boldsymbol{a}),$$

where

$$\boldsymbol{b}(x, \omega, \boldsymbol{a}) = \sum_{j=3}^{\infty} \boldsymbol{u}_j(x, \omega, \boldsymbol{a}).$$

5.1. The analysis of $\boldsymbol{u}_1$. This subsection is devoted to the analysis of the leading term $\boldsymbol{u}_1$. Explicitly, we have

$$\boldsymbol{u}_1(x, \omega, \boldsymbol{a}) = -\int_D \rho(z) \boldsymbol{G}(x, z, \omega)^2 \boldsymbol{a} dz. \tag{5.1}$$

THEOREM 5.1. *Let ρ satisfy Assumption* 1.1 *and $U \subset \mathbb{R}^2 \backslash \overline{D}$ be a bounded and convex domain having a locally Lipschitz boundary and a positive distance to D. Then for all $x \in U$, it holds that*

$$\lim_{Q \to \infty} \frac{1}{Q-1} \int_1^Q \omega^{m+2} \sum_{j=1}^{2} |\boldsymbol{u}_1(x, \omega, \boldsymbol{a}_j)|^2 d\omega = C_m \int_{\mathbb{R}^2} \frac{1}{|x - \zeta|^2} \phi(\zeta) d\zeta \quad a.s.,$$

where $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ are two orthonormal vectors in $\mathbb{R}^2$, and C_m is given in Theorem 1.2.

Before giving the proof of Theorem 5.1, we first introduce the truncation of the Green tensor $\boldsymbol{G}$ and some a priori estimates. Let $H_n^{(1)}$ be the Hankel function of the first kind with order n, which has the following asymptotic expansion (cf. [3]):

$$H_n^{(1)}(c) = \sum_{j=0}^{N} b_j^{(n)} c^{-(j+\frac{1}{2})} e^{\mathrm{i}(c - \frac{n\pi}{2})} + O(|c|^{-N-\frac{3}{2}}), \quad c \in \mathbb{C}, \ |c| \to \infty, \tag{5.2}$$

where $b_0^{(n)} = \frac{1+\mathrm{i}}{\sqrt{\pi}}$ and

$$b_j^{(n)} = \frac{(1+\mathrm{i})\mathrm{i}^j}{\sqrt{\pi}8^j j!}\prod_{l=1}^{j}\left(4n^2-(2l-1)^2\right), \quad j \geq 1.$$

For the sufficiently large argument $c = \kappa|z|$, define the truncated Hankel function

$$H_{n,N}^{(1)}(c) := \sum_{j=0}^{N} b_j^{(n)} c^{-(j+\frac{1}{2})} e^{\mathrm{i}(c-\frac{n\pi}{2})}.$$

It follows from (5.2) that

$$\left|H_n^{(1)}(\kappa|z|) - H_{n,N}^{(1)}(\kappa|z|)\right| \lesssim \kappa^{-N-\frac{3}{2}}|z|^{-N-\frac{3}{2}}, \tag{5.3}$$

$$\left|\nabla_z\left[H_n^{(1)}(\kappa|z|) - H_{n,N}^{(1)}(\kappa|z|)\right]\right| \lesssim \kappa^{-N-\frac{1}{2}}|z|^{-N-\frac{3}{2}}. \tag{5.4}$$

By (1.2), a straightforward calculation shows that $\boldsymbol{G}$ can be written as

$$\begin{aligned}\boldsymbol{G}(x,y,\omega) = \mathrm{i}&\left\{\frac{1}{4\mu}H_0^{(1)}(\kappa_{\mathrm{s}}|x-y|) - \frac{\kappa_{\mathrm{s}}H_1^{(1)}(\kappa_{\mathrm{s}}|x-y|) - \kappa_{\mathrm{p}}H_1^{(1)}(\kappa_{\mathrm{p}}|x-y|)}{4\omega^2|x-y|}\right\}\boldsymbol{I}\\ &+\frac{\mathrm{i}}{4\omega^2}\frac{1}{|x-y|^2}\left[\kappa_{\mathrm{s}}^2H_2^{(1)}(\kappa_{\mathrm{s}}|x-y|) - \kappa_{\mathrm{p}}^2H_2^{(1)}(\kappa_{\mathrm{p}}|x-y|)\right](x-y)(x-y)^\top,\end{aligned} \tag{5.5}$$

where $x-y = (x_1-y_1, x_2-y_2)^\top$. Denote by $\boldsymbol{G}^{(N)}$ the truncation of the Green tensor $\boldsymbol{G}$. Explicitly,

$$\begin{aligned}\boldsymbol{G}^{(N)}(x,y,\omega) = \mathrm{i}&\left\{\frac{1}{4\mu}H_{0,N}^{(1)}(\kappa_{\mathrm{s}}|x-y|) - \frac{\kappa_{\mathrm{s}}H_{1,N}^{(1)}(\kappa_{\mathrm{s}}|x-y|) - \kappa_{\mathrm{p}}H_{1,N}^{(1)}(\kappa_{\mathrm{p}}|x-y|)}{4\omega^2|x-y|}\right\}\boldsymbol{I}\\ &+\frac{\mathrm{i}}{4\omega^2}\frac{\kappa_{\mathrm{s}}^2H_{2,N}^{(1)}(\kappa_{\mathrm{s}}|x-y|) - \kappa_{\mathrm{p}}^2H_{2,N}^{(1)}(\kappa_{\mathrm{p}}|x-y|)}{|x-y|^2}(x-y)(x-y)^\top.\end{aligned} \tag{5.6}$$

Let G_{ij} and $G_{ij}^{(N)}$ be the (i,j)-entry of $\boldsymbol{G}$ and $\boldsymbol{G}^{(N)}$, respectively. Using (5.2)–(5.4), we have the following asymptotic estimates:

$$\begin{aligned}&|G_{ij}(x,y,\omega)| \lesssim \omega^{-\frac{1}{2}}|x-y|^{-\frac{1}{2}}, \qquad |\nabla_x G_{ij}(x,y,\omega)| \lesssim \omega^{\frac{1}{2}}|x-y|^{-\frac{1}{2}},\\ &|G_{ij}^{(N)}(x,y,\omega)| \lesssim \omega^{-\frac{1}{2}}|x-y|^{-\frac{1}{2}}, \quad |\nabla_x G_{ij}^{(N)}(x,y,\omega)| \lesssim \omega^{\frac{1}{2}}|x-y|^{-\frac{1}{2}},\\ &|G_{ij}(x,y,\omega) - G_{ij}^{(N)}(x,y,\omega)| \lesssim \omega^{-N-\frac{3}{2}}|x-y|^{-N-\frac{3}{2}},\\ &|\nabla_x(G_{ij}(x,y,\omega) - G_{ij}^{(N)}(x,y,\omega))| \lesssim \omega^{-N-\frac{1}{2}}|x-y|^{-N-\frac{3}{2}}.\end{aligned} \tag{5.7}$$

Replacing $\boldsymbol{G}$ by $\boldsymbol{G}^{(2)}$ in (5.1), we define

$$\boldsymbol{u}_1^{(2)}(x,\omega,\boldsymbol{a}) = -\int_D \rho(z)\boldsymbol{G}^{(2)}(x,z,\omega)^2\boldsymbol{a}dz, \quad x \in U. \tag{5.8}$$

For the difference $\boldsymbol{u}_1 - \boldsymbol{u}_1^{(2)}$, we have the following estimate.

LEMMA 5.2. *Under the assumptions in Theorem* 5.1*, it holds for* $x \in U$ *that*

$$|\boldsymbol{u}_1(x,\omega,\boldsymbol{a}) - \boldsymbol{u}_1^{(2)}(x,\omega,\boldsymbol{a})| \leq C\omega^{-3} \quad a.s.,$$

where the constant C *depends on the distance between* U *and* D.

Proof. Using (5.1) and (5.8), for $x, y \in U$ and $z \in D$, we obtain

$$\begin{aligned}|\boldsymbol{u}_1(x,\omega,\boldsymbol{a}) - \boldsymbol{u}_1^{(2)}(x,\omega,\boldsymbol{a})| &\le \left|\int_D \rho(z)(\boldsymbol{G}(x,z,\omega) - \boldsymbol{G}^{(2)}(x,z,\omega))\boldsymbol{G}(x,z,\omega)\boldsymbol{a}dz\right| \\ &\quad + \left|\int_D \rho(z)\boldsymbol{G}^{(2)}(x,z,\omega)(\boldsymbol{G}(x,z,\omega) - \boldsymbol{G}^{(2)}(x,z,\omega))\boldsymbol{a}dz\right| \\ &=: \mathrm{J}_1 + \mathrm{J}_2.\end{aligned}$$

For J_1, we have from (5.7) that

$$\begin{aligned}\mathrm{J}_1 &\le \|\rho\|_{H_0^{-1}(D)} \|(\boldsymbol{G}(x,\cdot,\omega) - \boldsymbol{G}^{(2)}(x,\cdot,\omega))\boldsymbol{G}(x,\cdot,\omega)\boldsymbol{a}\|_{\boldsymbol{H}^1(D)} \\ &\le \|\rho\|_{H_0^{-1}(D)} \big[\|\nabla(\boldsymbol{G}(x,\cdot,\omega) - \boldsymbol{G}^{(2)}(x,\cdot,\omega))\|_{(L^2(D))^{2\times 2}} \|\boldsymbol{G}(x,\cdot,\omega)\boldsymbol{a}\|_{\boldsymbol{L}^\infty(D)} \\ &\quad + \|\boldsymbol{G}(x,\cdot,\omega) - \boldsymbol{G}^{(2)}(x,\cdot,\omega)\|_{(L^2(D))^{2\times 2}} \\ &\quad \times \big(\|\boldsymbol{G}(x,\cdot,\omega)\boldsymbol{a}\|_{\boldsymbol{L}^\infty(D)} + \|\nabla \boldsymbol{G}(x,\cdot,\omega)\boldsymbol{a}\|_{\boldsymbol{L}^\infty(D)}\big)\big] \\ &\lesssim \big[\omega^{-\frac{5}{2}}\omega^{-\frac{1}{2}} + \omega^{-\frac{7}{2}}(\omega^{-\frac{1}{2}} + \omega^{\frac{1}{2}})\big]\Big(\int_D |x-z|^{-7}dz\Big)^{\frac{1}{2}} \sup_{z\in D}|x-z|^{-\frac{1}{2}} \lesssim \omega^{-3},\end{aligned}$$

where we have used the facts $\rho \in W_0^{\frac{m-2}{2}-\epsilon,p}(D) \subset H_0^{-1}(D)$ by choosing a sufficiently small $\epsilon > 0$ and that there is a positive distance between U and D. Similarly, we can prove that $\mathrm{J}_2 \lesssim \omega^{-3}$, which completes the proof. ∎

Let $\boldsymbol{u}_1^{(2)} = \big(u_{1,1}^{(2)}, u_{1,2}^{(2)}\big)^\top$, where

$$u_{1,k}^{(2)} = -\sum_{i,j=1}^{2} \int_D \rho(z) G_{ki}^{(2)}(x,z,\omega) G_{ij}^{(2)}(x,z,\omega) a_j dz, \quad k = 1,2,$$

and a_j is the component of the vector $\boldsymbol{a}$. A straightforward calculation gives

$$\begin{aligned}\mathbb{E}\left[\boldsymbol{u}_1^{(2)}(x,\omega_1,\boldsymbol{a}) \cdot \overline{\boldsymbol{u}_1^{(2)}(x,\omega_2,\boldsymbol{a})}\right] &= \sum_{k,i,j,\tilde{i},\tilde{j}=1}^{2} a_j a_{\tilde{j}} \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} G_{ki}^{(2)}(x,z,\omega_1) G_{ij}^{(2)}(x,z,\omega_1) \\ &\quad \times \overline{G_{k\tilde{i}}^{(2)}(x,z',\omega_2) G_{\tilde{i}\tilde{j}}^{(2)}(x,z',\omega_2)} \mathbb{E}[\rho(z)\rho(z')] dz dz', \end{aligned} \tag{5.9}$$

where the entries $G_{kl}^{(2)}$ in $\boldsymbol{G}^{(2)}$ can be expressed by

$$\begin{aligned}G_{kl}^{(2)}(x,z,\omega) &= \frac{\mathrm{i}}{4}\sum_{j=0}^{2}\left[\frac{b_j^{(0)} c_{\mathrm{s}}^{-j+\frac{3}{2}}\delta_{kl}}{\omega^{j+\frac{1}{2}}|x-z|^{j+\frac{1}{2}}} + \frac{\mathrm{i} b_j^{(1)} c_{\mathrm{s}}^{-j+\frac{1}{2}}\delta_{kl}}{\omega^{j+\frac{3}{2}}|x-z|^{j+\frac{3}{2}}}\right. \\ &\quad \left. - \frac{b_j^{(2)} c_{\mathrm{s}}^{-j+\frac{3}{2}}(x_k - z_k)(x_l - z_l)}{\omega^{j+\frac{1}{2}}|x-z|^{j+\frac{5}{2}}}\right] e^{\mathrm{i}c_{\mathrm{s}}\omega|x-z|} \\ &\quad - \frac{\mathrm{i}}{4}\sum_{j=0}^{2}\left[\frac{\mathrm{i} b_j^{(1)} c_{\mathrm{p}}^{-j+\frac{1}{2}}\delta_{kl}}{\omega^{j+\frac{3}{2}}|x-z|^{j+\frac{3}{2}}} - \frac{b_j^{(2)} c_{\mathrm{p}}^{-j+\frac{3}{2}}(x_k - z_k)(x_l - z_l)}{\omega^{j+\frac{1}{2}}|x-z|^{j+\frac{5}{2}}}\right] e^{\mathrm{i}c_{\mathrm{p}}\omega|x-z|}\end{aligned}$$

and δ_{kl} is the Kronecker delta function. Substituting the expression of $G_{kl}^{(2)}$ into (5.9), we get that $\mathbb{E}(\boldsymbol{u}_1^{(2)}(x,\omega_1,\boldsymbol{a}) \cdot \overline{\boldsymbol{u}_1^{(2)}(x,\omega_2,\boldsymbol{a})})$ is a linear combination of the following type of integral:

$$I(x,\omega_1,\omega_2) := \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{\mathrm{i}(c_1\omega_1|x-z|-c_2\omega_2|x-z'|)}F(z,z',x)\mathbb{E}[\rho(z)\rho(z')]dzdz',$$

where $c_1, c_2 \in \{2c_{\rm s}, c_{\rm s}+c_{\rm p}, 2c_{\rm p}\}$ and

$$F(z,z',x) := \frac{(x_1-z_1)^{d_{11}}(x_2-z_2)^{d_{12}}(x_1-z_1')^{d_{21}}(x_2-z_2')^{d_{22}}}{|x-z|^{d_1}|x-z'|^{d_2}}$$

with $d_i, d_{ij} \in \mathbb{N}_+$ for $i,j=1,2$.

As studied in [20, Lemma 3.4] and [26, Proposition 4.1], the estimate of the above integral I can be similarly made by examining the behavior of the kernel $\mathbb{E}[\rho(z)\rho(z')]$ under coordinate transformations. The result is stated as follows and the proof is omitted for brevity.

Lemma 5.3. *For $\omega_1,\omega_2 \geq 1$, the following estimate holds uniformly for $x\in U$:*

$$|I(x,\omega_1,\omega_2)| \leq C_M(\omega_1+\omega_2)^{-m}(1+|\omega_1-\omega_2|)^{-M}, \tag{5.10}$$

where $M\in\mathbb{N}$ is an arbitrary integer and the positive constant C_M depends only on M. Moreover, if $\omega_1=\omega_2=\omega$, then the following identity holds:

$$I(x,\omega,\omega) = R(x,\omega)\omega^{-m} + O(\omega^{-(m+1)}), \tag{5.11}$$

where

$$R(x,\omega) := \frac{2^m}{(c_1+c_2)^m}\int_{\mathbb{R}^2} e^{\mathrm{i}(c_1-c_2)\omega|x-\zeta|}\frac{(x_1-\zeta_1)^{d_{11}+d_{21}}(x_2-\zeta_2)^{d_{12}+d_{22}}}{|x-\zeta|^{d_1+d_2}}\phi(\zeta)d\zeta.$$

Corollary 5.4. *For $\omega_1,\omega_2\geq 1$, the following estimates hold uniformly for $x\in U$:*

$$|\mathbb{E}(\boldsymbol{u}_1^{(2)}(x,\omega_1,\boldsymbol{a})\cdot\overline{\boldsymbol{u}_1^{(2)}(x,\omega_2,\boldsymbol{a})})| \leq C_M(\omega_1\omega_2)^{-1}(\omega_1+\omega_2)^{-m}(1+|\omega_1-\omega_2|)^{-M}, \tag{5.12}$$

$$|\mathbb{E}(\boldsymbol{u}_1^{(2)}(x,\omega_1,\boldsymbol{a})\cdot\boldsymbol{u}_1^{(2)}(x,\omega_2,\boldsymbol{a}))| \leq C_M(\omega_1\omega_2)^{-1}(\omega_1+\omega_2)^{-M}(1+|\omega_1-\omega_2|)^{-m}, \tag{5.13}$$

where $M\in\mathbb{N}$ is arbitrary and C_M is a constant depending on M.

Proof. Since $\mathbb{E}(\boldsymbol{u}_1^{(2)}(x,\omega_1,\boldsymbol{a})\cdot\overline{\boldsymbol{u}_1^{(2)}(x,\omega_2,\boldsymbol{a})})$ is a linear combination of the integral I, where the coefficient of the highest order is $(\omega_1\omega_2)^{-1}$, it follows from Lemma 5.3 that the estimate (5.12) holds. A simple calculation shows that $\mathbb{E}(\boldsymbol{u}_1^{(2)}(x,\omega_1,\boldsymbol{a})\cdot\boldsymbol{u}_1^{(2)}(x,\omega_2,\boldsymbol{a}))$ is a linear combination of the following type of integral:

$$\tilde{I} = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{\mathrm{i}(c_1\omega_1|x-z|+c_2\omega_2|x-z'|)}F(z,z',x)\mathbb{E}[\rho(z)\rho(z')]dzdz'$$

with the coefficient $(\omega_1\omega_2)^{-1}$. Clearly, $\tilde{I}$ is analogous to I except that ω_2 in I is replaced by $-\omega_2$ in $\tilde{I}$. Following the same proof as that for the estimate of I, we may show that

$$|\tilde{I}(x,\omega_1,\omega_2)| \leq C_M(\omega_1+\omega_2)^{-M}(1+|\omega_1-\omega_2|)^{-m},$$

which implies (5.13) and completes the proof. ∎

Proof of Theorem 5.1. Rewriting $\boldsymbol{u}_1 = \boldsymbol{u}_1^{(2)} + (\boldsymbol{u}_1 - \boldsymbol{u}_1^{(2)})$, we only need to show

$$\lim_{Q\to\infty} \frac{1}{Q-1}\int_1^Q \omega^{m+2}\sum_{j=1}^2 |\boldsymbol{u}_1^{(2)}(x,\omega,\boldsymbol{a}_j)|^2 d\omega = C_m \int_{\mathbb{R}^2} \frac{1}{|x-\zeta|^2}\phi(\zeta)d\zeta, \tag{5.14}$$

$$\lim_{Q\to\infty} \frac{1}{Q-1}\int_1^Q \omega^{m+2}|\boldsymbol{u}_1(x,\omega,\boldsymbol{a}) - \boldsymbol{u}_1^{(2)}(x,\omega,\boldsymbol{a})|^2 d\omega = 0, \tag{5.15}$$

$$\lim_{Q\to\infty} \frac{2}{Q-1}\int_1^Q \omega^{m+2}\Re\big[\overline{\boldsymbol{u}_1^{(2)}(x,\omega,\boldsymbol{a})}(\boldsymbol{u}_1(x,\omega,\boldsymbol{a}) - \boldsymbol{u}_1^{(2)}(x,\omega,\boldsymbol{a}))\big] d\omega = 0. \tag{5.16}$$

For (5.14), it follows from a straightforward calculation that

$$\begin{aligned}
\mathbb{E}|\boldsymbol{u}_1^{(2)}(x,\omega,\boldsymbol{a})|^2 &= \left|\frac{\mathrm{i}-1}{4\sqrt{\pi}}\right|^4 \omega^{-2}\sum_{j,\tilde{j}=1}^2 a_j a_{\tilde{j}} \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \mathbb{E}[\rho(z)\rho(z')] \sum_{k,i,\tilde{i}=1}^2 \\
&\left[c_{\mathrm{s}}^{\frac32}\left(\frac{\delta_{ki}}{|x-z|^{\frac12}} - \frac{(x_k-z_k)(x_i-z_i)}{|x-z|^{\frac52}}\right)e^{\mathrm{i}c_{\mathrm{s}}|x-z|} + c_{\mathrm{p}}^{\frac32}\frac{(x_k-z_k)(x_i-z_i)}{|x-z|^{\frac52}}e^{\mathrm{i}c_{\mathrm{p}}|x-z|}\right] \\
&\times\left[c_{\mathrm{s}}^{\frac32}\left(\frac{\delta_{ij}}{|x-z|^{\frac12}} - \frac{(x_i-z_i)(x_j-z_j)}{|x-z|^{\frac52}}\right)e^{\mathrm{i}c_{\mathrm{s}}|x-z|} + c_{\mathrm{p}}^{\frac32}\frac{(x_i-z_i)(x_j-z_j)}{|x-z|^{\frac52}}e^{\mathrm{i}c_{\mathrm{p}}|x-z|}\right] \\
&\times\left[c_{\mathrm{s}}^{\frac32}\left(\frac{\delta_{k\tilde{i}}}{|x-z'|^{\frac12}} - \frac{(x_k-z'_k)(x_{\tilde{i}}-z'_{\tilde{i}})}{|x-z'|^{\frac52}}\right)e^{-\mathrm{i}c_{\mathrm{s}}|x-z'|} + c_{\mathrm{p}}^{\frac32}\frac{(x_k-z'_k)(x_{\tilde{i}}-z'_{\tilde{i}})}{|x-z'|^{\frac52}}e^{-\mathrm{i}c_{\mathrm{p}}|x-z'|}\right] \\
&\times\left[c_{\mathrm{s}}^{\frac32}\left(\frac{\delta_{\tilde{i}\tilde{j}}}{|x-z'|^{\frac12}} - \frac{(x_{\tilde{i}}-z'_{\tilde{i}})(x_{\tilde{j}}-z'_{\tilde{j}})}{|x-z'|^{\frac52}}\right)e^{-\mathrm{i}c_{\mathrm{s}}|x-z'|}\right. \\
&\left.+c_{\mathrm{p}}^{\frac32}\frac{(x_{\tilde{i}}-z'_{\tilde{i}})(x_{\tilde{j}}-z'_{\tilde{j}})}{|x-z'|^{\frac52}}e^{-\mathrm{i}c_{\mathrm{p}}|x-z'|}\right] dzdz' + O(\omega^{-(m+3)}).
\end{aligned}$$

Together with Lemma 5.3, it gives

$$\mathbb{E}|\boldsymbol{u}_1^{(2)}(x,\omega,\boldsymbol{a})|^2 = T_2(x,\omega,\boldsymbol{a})\omega^{-(m+2)} + O(\omega^{-(m+3)}), \tag{5.17}$$

where

$$\begin{aligned}
T_2(x,\omega,\boldsymbol{a}) &= 2^{-6-m}\pi^{-2}c_{\mathrm{s}}^{6-m}\int_{\mathbb{R}^2}\left[\frac{1}{|x-\zeta|^2} - \sum_{i,j=1}^2 \frac{(x_i-\zeta_i)(x_j-\zeta_j)}{|x-\zeta|^4}a_i a_j\right]\phi(\zeta)d\zeta \\
&\quad + 2^{-6-m}\pi^{-2}c_{\mathrm{p}}^{6-m}\int_{\mathbb{R}^2}\left[\sum_{i,j=1}^2 \frac{(x_i-\zeta_i)(x_j-\zeta_j)}{|x-\zeta|^4}a_i a_j\right]\phi(\zeta)d\zeta
\end{aligned}$$

with a_i being the ith component of the vector $\boldsymbol{a}$. Let $\boldsymbol{a}_1 = (a_{11}, a_{12})^\top$ and $\boldsymbol{a}_2 = (a_{21}, a_{22})^\top$ be two orthonormal vectors, i.e., there exists some angle α such that $\boldsymbol{a}_1 = (\cos\alpha, \sin\alpha)^\top$ and $\boldsymbol{a}_2 = (-\sin\alpha, \cos\alpha)^\top$. It then holds that $a_{11}^2 + a_{21}^2 = 1$, $a_{12}^2 + a_{22}^2 = 1$ and $a_{11}a_{12} + a_{21}a_{22} = 0$, which lead to

$$\sum_{j=1}^2 T_2(x,\omega,\boldsymbol{a}_j) = C_m\int_{\mathbb{R}^2}\frac{1}{|x-\zeta|^2}\phi(\zeta)d\zeta.$$

It follows from (5.17) and the above equation that

$$\sum_{j=1}^2 \mathbb{E}|\boldsymbol{u}_1^{(2)}(x,\omega,\boldsymbol{a}_j)|^2 = C_m\int_{\mathbb{R}^2}\frac{1}{|x-\zeta|^2}\phi(\zeta)d\zeta\omega^{-(m+2)} + O(\omega^{-(m+3)}),$$

which gives

$$\lim_{Q\to\infty}\frac{1}{Q-1}\int_1^Q\omega^{m+2}\sum_{j=1}^2\mathbb{E}|\boldsymbol{u}_1^{(2)}(x,\omega,\boldsymbol{a}_j)|^2d\omega=C_m\int_{\mathbb{R}^2}\frac{1}{|x-\zeta|^2}\phi(\zeta)d\zeta.$$

To prove (5.14), based on the above equation, it suffices to prove

$$\lim_{Q\to\infty}\frac{1}{Q-1}\int_1^Q\omega^{m+2}\left[|\boldsymbol{u}_1^{(2)}(x,\omega,\boldsymbol{a})|^2-\mathbb{E}|\boldsymbol{u}_1^{(2)}(x,\omega,\boldsymbol{a})|^2\right]d\omega=0, \tag{5.18}$$

which can be shown by following the same procedure as that in the proof of [20, Theorem 3.10].

For (5.15), by Lemma 5.2, we obtain from the fact $m\le d=2$ that

$$\begin{aligned}&\frac{1}{Q-1}\int_1^Q\omega^{m+2}|\boldsymbol{u}_1(x,\omega,\boldsymbol{a})-\boldsymbol{u}_1^{(2)}(x,\omega,\boldsymbol{a})|^2d\omega\\&\lesssim\frac{1}{Q-1}\int_1^Q\omega^{m-4}d\omega=\frac{1}{m-3}\frac{Q^{m-3}-1}{Q-1}\to0\quad\text{as } Q\to\infty.\end{aligned}$$

Combining (5.14)–(5.15) and the Hölder inequality, we may easily verify (5.16) and complete the proof. ∎

5.2. The analysis of $\boldsymbol{u}_2$. This subsection is devoted to analyzing the term $\boldsymbol{u}_2$ in the Born approximation (4.1), which is given by

$$\boldsymbol{u}_2(x,\omega,\boldsymbol{a})=\int_D\int_D\boldsymbol{G}(x,z,\omega)\rho(z)\boldsymbol{G}(z,z',\omega)\rho(z')\boldsymbol{G}(z',x,\omega)\boldsymbol{a}dz'dz$$

for $x\in U$. The purpose is to show that the contribution of $\boldsymbol{u}_2$ can also be ignored, which is presented in the following theorem.

THEOREM 5.5. *Under the assumptions in Theorem* 1.2*, for all $x\in U$, it holds almost surely that*

$$\lim_{Q\to\infty}\frac{1}{Q-1}\int_1^Q\omega^{m+2}|\boldsymbol{u}_2(x,\omega,\boldsymbol{a})|^2d\omega=0.$$

To prove Theorem 5.5, motivated by [19] in the acoustic wave case, we decompose $\boldsymbol{u}_2$ into several terms by defining the following auxiliary functions:

$$\boldsymbol{u}_{2,l}(x,\omega,\boldsymbol{a})=\int_D\int_D\boldsymbol{G}^{(0)}(x,z,\omega)\rho(z)\boldsymbol{G}(z,z',\omega)\rho(z')\boldsymbol{G}(z',x,\omega)\boldsymbol{a}dz'dz, \tag{5.19}$$

$$\boldsymbol{u}_{2,r}(x,\omega,\boldsymbol{a})=\int_D\int_D\boldsymbol{G}^{(0)}(x,z,\omega)\rho(z)\boldsymbol{G}(z,z',\omega)\rho(z')\boldsymbol{G}^{(0)}(z',x,\omega)\boldsymbol{a}dz'dz, \tag{5.20}$$

$$\boldsymbol{v}(x,\omega,\boldsymbol{a})=\int_D\int_D\boldsymbol{G}^{(0)}(x,z,\omega)\rho(z)\boldsymbol{G}^{(0)}(z,z',\omega)\rho(z')\boldsymbol{G}^{(0)}(z',x,\omega)\boldsymbol{a}dz'dz, \tag{5.21}$$

where $\boldsymbol{G}^{(0)}$ is defined in (5.6). It is clear to note that $\boldsymbol{u}_2=(\boldsymbol{u}_2-\boldsymbol{u}_{2,l})+(\boldsymbol{u}_{2,l}-\boldsymbol{u}_{2,r})+(\boldsymbol{u}_{2,r}-\boldsymbol{v})+\boldsymbol{v}$. To estimate these terms, the following preliminary results on $\boldsymbol{G}$, $\boldsymbol{G}^{(0)}$ and their difference are needed.

LEMMA 5.6. *Let $s\in[0,1]$ and $U\subset\mathbb{R}^2\backslash\overline{D}$ be a bounded and convex domain with a positive distance from D.*

(i) *For any $p \in (1,\infty)$ and $x \in U$, it holds that*

$$\|\boldsymbol{G}(x,\cdot,\omega)\|_{(W^{s,p}(D))^{2\times 2}} \lesssim \omega^{-\frac{1}{2}+s},$$
$$\|\boldsymbol{G}^{(0)}(x,\cdot,\omega)\|_{(W^{s,p}(D))^{2\times 2}} \lesssim \omega^{-\frac{1}{2}+s},$$
$$\|\boldsymbol{G}(x,\cdot,\omega) - \boldsymbol{G}^{(0)}(x,\cdot,\omega)\|_{(W^{s,p}(D))^{2\times 2}} \lesssim \omega^{-\frac{3}{2}+s}.$$

(ii) *For any $p \in (1,\frac{4}{3})$, it holds that*

$$\|\boldsymbol{G}(\cdot,\cdot,\omega) - \boldsymbol{G}^{(0)}(\cdot,\cdot,\omega)\|_{(W^{s,p}(D\times D))^{2\times 2}} \lesssim \omega^{-\frac{3}{2}+s}.$$

Proof. Results in (i) can be easily obtained by (5.7) and the interpolation between the spaces $L^p(D)$ and $W^{1,p}(D)$. Next is to show (ii).

According to (5.7), we get

$$\|\boldsymbol{G}(\cdot,\cdot,\omega) - \boldsymbol{G}^{(0)}(\cdot,\cdot,\omega)\|_{(L^p(D\times D))^{2\times 2}} \lesssim \omega^{-\frac{3}{2}} \left(\int_D\int_D |z-z'|^{-\frac{3}{2}p} dz dz'\right)^{\frac{1}{p}} \lesssim \omega^{-\frac{3}{2}},$$

where we use the facts that there exists a constant $R>0$ such that $|z-z'|<R$ for any $z,z'\in D$, and

$$\int_D\int_D |z-z'|^{-\frac{3}{2}p} dz dz' \lesssim \int_0^R r^{-\frac{3}{2}p+1} dr < \infty$$

for any $p\in(1,\frac{4}{3})$. Similarly,

$$\|\boldsymbol{G}(\cdot,\cdot,\omega) - \boldsymbol{G}^{(0)}(\cdot,\cdot,\omega)\|_{(W^{1,p}(D\times D))^{2\times 2}} \lesssim \omega^{-\frac{1}{2}}.$$

Finally, the result in (ii) is obtained by the interpolation. ∎

The operator K_ω, defined by (3.2), satisfies the following estimates when restricted to bounded domains, where the proof is given for a more general case $m\in(1,2]$. We also refer to [19, Lemma 5] for the acoustic wave case with $m=2$.

LEMMA 5.7. *Let ρ satisfy Assumption* 1.1. *For any $s \in (\frac{2-m}{2},\frac{1}{2})$, $q\in(1,\infty)$, and $\omega\ge 1$, the following estimate holds:*

$$\|K_\omega\|_{\mathcal{L}(\boldsymbol{W}^{s,2q}(D))} \lesssim \omega^{-1+2s}.$$

Proof. For any $\boldsymbol{f},\boldsymbol{g}\in\boldsymbol{C}_0^\infty(D)$, denote by $\tilde{\boldsymbol{f}},\tilde{\boldsymbol{g}}$ the zero extensions of $\boldsymbol{f},\boldsymbol{g}$ in $\mathbb{R}^2$ such that $\tilde{\boldsymbol{f}},\tilde{\boldsymbol{g}}\in\boldsymbol{C}_0^\infty$. Using Lemma 4.1 leads to

$$|\langle H_\omega \boldsymbol{f},\boldsymbol{g}\rangle| = |\langle H_\omega \tilde{\boldsymbol{f}},\tilde{\boldsymbol{g}}\rangle| \lesssim \omega^{-1+2s+s_1+s_2}\|\tilde{\boldsymbol{f}}\|_{\boldsymbol{H}_1^{-s-s_1}}\|\tilde{\boldsymbol{g}}\|_{\boldsymbol{H}_1^{-s-s_2}},$$

where

$$\|\tilde{\boldsymbol{f}}\|_{\boldsymbol{H}_1^{-s-s_1}} \lesssim \|\mathcal{J}^{-s}\boldsymbol{f}\|_{\boldsymbol{H}^{-s_1}(D)} \lesssim \|\mathcal{J}^{-s}\boldsymbol{f}\|_{\boldsymbol{L}^{\tilde{p}}(D)} \lesssim \|\boldsymbol{f}\|_{\boldsymbol{W}^{-s,\tilde{p}}(D)},$$
$$\|\tilde{\boldsymbol{g}}\|_{\boldsymbol{H}_1^{-s-s_2}} \lesssim \|\mathcal{J}^{-s}\boldsymbol{g}\|_{\boldsymbol{H}^{-s_2}(D)} \lesssim \|\mathcal{J}^{-s}\boldsymbol{g}\|_{\boldsymbol{L}^{\tilde{q}'}(D)} \lesssim \|\boldsymbol{g}\|_{\boldsymbol{W}^{-s,\tilde{q}'}(D)}$$

according to the Sobolev embeddings

$$\boldsymbol{L}^{\tilde{p}}(D) \hookrightarrow \boldsymbol{H}^{-s_1}(D) \quad \text{for} \quad s_1 \ge \frac{2}{\tilde{p}} - 1,$$

$$\boldsymbol{L}^{\tilde{q}'}(D) \hookrightarrow \boldsymbol{H}^{-s_2}(D) \quad \text{for} \quad s_2 \geq 1 - \frac{2}{\tilde{q}}$$

with $1 < \tilde{p} \leq 2 \leq \tilde{q} < \infty$ and $\tilde{q}'$ satisfying $\frac{1}{\tilde{q}} + \frac{1}{\tilde{q}'} = 1$ (cf. [31, Theorem 3.1]). By choosing $s_1 = \frac{2}{\tilde{p}} - 1$ and $s_2 = 1 - \frac{2}{\tilde{q}}$, we have

$$|\langle H_\omega \boldsymbol{f}, \boldsymbol{g}\rangle| \lesssim \omega^{-1+2s+2(\frac{1}{\tilde{p}}-\frac{1}{\tilde{q}})} \|\boldsymbol{f}\|_{\boldsymbol{W}^{-s,\tilde{p}}(D)} \|\boldsymbol{g}\|_{\boldsymbol{W}^{-s,\tilde{q}'}(D)},$$

and hence

$$\|H_\omega\|_{\mathcal{L}(\boldsymbol{W}^{-s,\tilde{p}}(D),\boldsymbol{W}^{s,\tilde{q}}(D))} \lesssim \omega^{-1+2s+2(\frac{1}{\tilde{p}}-\frac{1}{\tilde{q}})}.$$

Note that $K_\omega \boldsymbol{u} = H_\omega(\rho \boldsymbol{u})$. For any $\boldsymbol{u} \in \boldsymbol{C}_0^\infty(D) \subset \boldsymbol{W}^{s,2q}(D)$, we obtain from [19, Lemma 2] and Lemma 2.2 that

$$\begin{aligned}
\|K_\omega \boldsymbol{u}\|_{\boldsymbol{W}^{s,2q}(D)} &\lesssim \|H_\omega\|_{\mathcal{L}(\boldsymbol{W}^{-s,(2q)'}(D),\boldsymbol{W}^{s,2q}(D))} \|\rho \boldsymbol{u}\|_{\boldsymbol{W}^{-s,(2q)'}(D)} \\
&\lesssim \|H_\omega\|_{\mathcal{L}(\boldsymbol{W}^{-s,(2q)'}(D),\boldsymbol{W}^{s,2q}(D))} \|\rho\|_{W^{-s,p}(D)} \|\boldsymbol{u}\|_{\boldsymbol{W}^{s,2q}(D)} \\
&\lesssim \omega^{-1+2s+2(1-\frac{1}{q})} \|\boldsymbol{u}\|_{\boldsymbol{W}^{s,2q}(D)},
\end{aligned}$$

where p satisfies $\frac{1}{p} + \frac{1}{q} = 1$ and $(2q)'$ satisfies $\frac{1}{2q} + \frac{1}{(2q)'} = 1$. The proof is then completed based on the fact that $\boldsymbol{C}_0^\infty(D)$ is dense in $\boldsymbol{L}^{2q}(D)$ (cf. [1, section 2.30]). $\square$

According to Lemmas 4.4, 5.6, and 5.7, we get for any $x \in U$ and $\omega \geq 1$ that

$$\begin{aligned}
&|\boldsymbol{u}_2(x,\omega,\boldsymbol{a}) - \boldsymbol{u}_{2,l}(x,\omega,\boldsymbol{a})| \\
&= \left| \left\langle \rho, \left[\boldsymbol{G}(x,\cdot,\omega) - \boldsymbol{G}^{(0)}(x,\cdot,\omega)\right] \int_D \boldsymbol{G}(\cdot,z',\omega)\rho(z')\boldsymbol{G}(z',x,\omega)\boldsymbol{a}dz' \right\rangle \right| \\
&\leq \|\rho\|_{W_0^{-s,p}(D)} \left\| \left[\boldsymbol{G}(x,\cdot,\omega) - \boldsymbol{G}^{(0)}(x,\cdot,\omega)\right] K_\omega \boldsymbol{u}_0(\cdot,x) \right\|_{\boldsymbol{W}^{s,q}(D)} \\
&\leq \|\rho\|_{W_0^{-s,p}(D)} \|\boldsymbol{G}(x,\cdot,\omega) - \boldsymbol{G}^{(0)}(x,\cdot,\omega)\|_{(W^{s,2q}(D))^{2\times 2}} \\
&\quad \times \|K_\omega\|_{\mathcal{L}(\boldsymbol{W}^{s,2q}(D))} \|\boldsymbol{u}_0(\cdot,x)\|_{\boldsymbol{W}^{s,2q}(D)} \\
&\lesssim \omega^{-\frac{3}{2}+s-1+2s+2(1-\frac{1}{q})-\frac{1}{2}+s} = \omega^{-3+4s+2(1-\frac{1}{q})}
\end{aligned}$$

for any $s \in (\frac{2-m}{2}, \frac{1}{2})$, $q \in (1,\infty)$, and p satisfying $\frac{1}{p} + \frac{1}{q} = 1$. Taking $q = 1+s$, we then deduce

(5.22)
$$\lim_{Q\to\infty} \frac{1}{Q-1} \int_1^Q \omega^{m+2} |\boldsymbol{u}_2(x,\omega,\boldsymbol{a}) - \boldsymbol{u}_{2,l}(x,\omega,\boldsymbol{a})|^2 d\omega \lesssim \lim_{Q\to\infty} \frac{Q^{m-3+8s+\frac{4s}{1+s}} - 1}{Q-1} = 0$$

almost surely by choosing $s \in (\frac{2-m}{2}, \frac{1}{2})$ such that $m-3+8s+\frac{4s}{1+s} < m-3+12s < 1$. Note that such an s can be chosen in the interval $(\frac{2-m}{2}, \frac{4-m}{12})$, which is not empty due to the fact $m \in (\frac{5}{3}, 2]$ under assumptions in Theorem 1.2.

Similarly, for the term $\boldsymbol{u}_{2,l} - \boldsymbol{u}_{2,r}$, we get

$$\begin{aligned}
&|\boldsymbol{u}_{2,l}(x,\omega,\boldsymbol{a}) - \boldsymbol{u}_{2,r}(x,\omega,\boldsymbol{a})| \\
&= \left| \left\langle \rho, \boldsymbol{G}^{(0)}(x,\cdot,\omega) \int_D \boldsymbol{G}(\cdot,z',\omega)\rho(z')\left[\boldsymbol{G}(z',x,\omega) - \boldsymbol{G}^{(0)}(z',x,\omega)\right]\boldsymbol{a}dz' \right\rangle \right| \\
&\leq \|\rho\|_{W_0^{-s,p}(D)} \left\| \boldsymbol{G}^{(0)}(x,\cdot,\omega) K_\omega\big((\boldsymbol{G} - \boldsymbol{G}^{(0)})\boldsymbol{a}\big)(\cdot,x) \right\|_{\boldsymbol{W}^{s,q}(D)}
\end{aligned}$$

$$
\begin{aligned}
&\le \|\rho\|_{W_0^{-s,p}(D)}\|\boldsymbol{G}^{(0)}(x,\cdot,\omega)\|_{(W^{s,2q}(D))^{2\times 2}}\\
&\quad\times \|K_\omega\|_{\mathcal{L}(\boldsymbol{W}^{s,2q}(D))}\|(\boldsymbol{G}-\boldsymbol{G}^{(0)})(\cdot,x,\omega)\boldsymbol{a}\|_{\boldsymbol{W}^{s,2q}(D)}\\
&\lesssim \omega^{-\frac{1}{2}+s-1+2s+2(1-\frac{1}{q})-\frac{3}{2}+s} = \omega^{-3+4s+2(1-\frac{1}{q})}
\end{aligned}
$$

for any $s\in(\frac{2-m}{2},\frac{1}{2})$, $q\in(1,\infty)$ and p satisfying $\frac{1}{p}+\frac{1}{q}=1$, which leads to

$$
\lim_{Q\to\infty}\frac{1}{Q-1}\int_1^Q \omega^{m+2}|\boldsymbol{u}_{2,l}(x,\omega,\boldsymbol{a})-\boldsymbol{u}_{2,r}(x,\omega,\boldsymbol{a})|^2 d\omega = 0 \tag{5.23}
$$

in the almost surely sense by taking $q=1+s$.

The term $\boldsymbol{u}_{2,r}-\boldsymbol{v}$ satisfies

$$
\begin{aligned}
&|\boldsymbol{u}_{2,r}(x,\omega,\boldsymbol{a})-\boldsymbol{v}(x,\omega,\boldsymbol{a})|\\
&=\left|\int_D\int_D \boldsymbol{G}^{(0)}(x,z,\omega)\rho(z)\big[\boldsymbol{G}(z,z',\omega)-\boldsymbol{G}^{(0)}(z,z',\omega)\big]\rho(z')\boldsymbol{G}^{(0)}(z',x,\omega)\boldsymbol{a}dz'dz\right|\\
&\lesssim \|\boldsymbol{G}(\cdot,\cdot,\omega)-\boldsymbol{G}^{(0)}(\cdot,\cdot,\omega)\|_{(W^{2s,p}(D\times D))^{2\times 2}}\|\boldsymbol{f}\|_{\boldsymbol{W}_0^{-2s,q}(D\times D)},
\end{aligned} \tag{5.24}
$$

where $p\in(1,\frac{4}{3})$, q satisfies $\frac{1}{p}+\frac{1}{q}=1$, and

$$
\boldsymbol{f}(z,z') := \rho(z)\rho(z')\boldsymbol{G}^{(0)}(x,z,\omega)\boldsymbol{G}^{(0)}(z',x,\omega)\boldsymbol{a}.
$$

Note that for any $\boldsymbol{g}\in\boldsymbol{W}^{2s,p}(D\times D)$ and $s\in(\frac{2-m}{2},\frac{1}{2})$,

$$
\begin{aligned}
|\langle\boldsymbol{f},\boldsymbol{g}\rangle| &\lesssim \|\rho\otimes\rho\|_{W_0^{-2s,q}(D\times D)}\|(\boldsymbol{G}^{(0)}(x,\cdot,\omega)\boldsymbol{G}^{(0)}(\cdot,x,\omega)\boldsymbol{a})\cdot\boldsymbol{g}\|_{W^{2s,p}(D\times D)}\\
&\lesssim \|\rho\|^2_{W^{-s,\infty}(D)}\|\boldsymbol{G}^{(0)}(x,\cdot,\omega)\boldsymbol{G}^{(0)}(\cdot,x,\omega)\boldsymbol{a}\|_{\boldsymbol{W}^{2s,\infty}(D\times D)}\|\boldsymbol{g}\|_{\boldsymbol{W}^{2s,p}(D\times D)}\\
&\lesssim \omega^{-1+2s}\|\boldsymbol{g}\|_{\boldsymbol{W}^{2s,p}(D\times D)}
\end{aligned}
$$

according to (5.7), Lemma 2.2, and the fact that $\rho\otimes\rho\in W_0^{-2s,\infty}(D\times D)\subset W_0^{-2s,q}(D\times D)$ for any $\rho\in W_0^{-s,\infty}(D)$ (cf. [19]). As a result,

$$
\|\boldsymbol{f}\|_{\boldsymbol{W}_0^{-2s,q}(D\times D)}\lesssim\omega^{-1+2s}.
$$

Hence, (5.24) turns out to be

$$
|\boldsymbol{u}_{2,r}(x,\omega,\boldsymbol{a})-\boldsymbol{v}(x,\omega,\boldsymbol{a})|\lesssim\omega^{-\frac{5}{2}+4s},
$$

which leads to

$$
\lim_{Q\to\infty}\frac{1}{Q-1}\int_1^Q\omega^{m+2}|\boldsymbol{u}_{2,r}(x,\omega,\boldsymbol{a})-\boldsymbol{v}(x,\omega,\boldsymbol{a})|^2d\omega\lesssim\lim_{Q\to\infty}\frac{Q^{m-2+8s}-1}{Q-1}=0 \tag{5.25}
$$

almost surely by choosing $s\in(\frac{2-m}{2},\frac{1}{2})$ with $m\in(\frac{5}{3},2]$ such that $m-2+8s<1$.

Finally, the result in Theorem 5.5 is obtained by combining (5.22), (5.23), (5.25), and the following lemma, whose proof is rather technical and is given in the appendix.

LEMMA 5.8. *Under assumptions in Theorem* 1.2*, for all $x\in U$, the auxiliary function $\boldsymbol{v}$ defined in* (5.21) *satisfies*

$$
\lim_{Q\to\infty}\frac{1}{Q-1}\int_1^Q\omega^{m+2}|\boldsymbol{v}(x,\omega,\boldsymbol{a})|^2d\omega=0\quad a.s. \tag{5.26}
$$

5.3. The analysis of $\boldsymbol{b}$. For any $x, y \in U$ and $s \in (\frac{2-m}{2}, \frac{1}{2})$, it follows from Lemmas 4.3 and 4.4 and (5.7) that

$$
\begin{aligned}
|\boldsymbol{b}(x,y)| &= \left|\sum_{j=3}^{\infty} \boldsymbol{u}_j(x,y)\right| = \left|\sum_{j=3}^{\infty} \langle \rho, \boldsymbol{G}(x,\cdot,\omega)\boldsymbol{u}_{j-1}(\cdot,y)\rangle\right| \\
&\lesssim \sum_{j=2}^{\infty} \|\rho\|_{W^{-s,\infty}(D)} \|\boldsymbol{G}(x,\cdot,\omega)\boldsymbol{u}_j(\cdot,y)\|_{\boldsymbol{W}^{s,1}(D)} \\
&\lesssim \sum_{j=2}^{\infty} \|\boldsymbol{G}(x,\cdot,\omega)\|_{(W^{s,\infty}(D))^{2\times 2}} \|\boldsymbol{u}_j(\cdot,y)\|_{\boldsymbol{H}^s(D)} \\
&\lesssim \sum_{j=2}^{\infty} \|\boldsymbol{G}(x,\cdot,\omega)\|_{(W^{s,\infty}(D))^{2\times 2}} \|K_\omega\|^j_{\mathcal{L}(\boldsymbol{H}^s_{-1})} \|\boldsymbol{u}_0(\cdot,y)\|_{\boldsymbol{H}^s_{-1}} \\
&\lesssim \sum_{j=2}^{\infty} \omega^{-\frac{1}{2}+s} \left(\omega^{-1+2s}\right)^j \omega^{-\frac{1}{2}+s} \lesssim \omega^{-3+6s}.
\end{aligned}
$$

Then it holds for $\boldsymbol{b}(x,\omega,\boldsymbol{a}_j) := \boldsymbol{b}(x,x)$ that

$$
\begin{aligned}
\frac{1}{Q-1}\int_1^Q \omega^{m+2} |\boldsymbol{b}(x,\omega,\boldsymbol{a}_j)|^2 d\omega &\lesssim \frac{1}{Q-1}\int_1^Q \omega^{m+2+(-3+6s)2} d\omega \\
&\lesssim \frac{Q^{m-3+12s}-1}{Q-1} \to 0 \quad \text{as } Q \to \infty
\end{aligned}
$$

by choosing $s \in (\frac{2-m}{2}, \frac{1}{2})$ and $s < \frac{4-m}{12}$. We mention that such an s exists since $m > \frac{5}{3}$ under assumptions in Theorem 1.2.

5.4. The proof of Theorem 1.2. Based on the analysis of $\boldsymbol{u}_1$, $\boldsymbol{u}_2$, and $\boldsymbol{b}$, we are now able to prove the main result: the strength ϕ in the principal symbol of the covariance operator Q_ρ can be uniquely determined by the amplitude of two scattered fields averaged over the frequency band with probability one. Here, the two scattered fields are associated with the incident waves given by $\boldsymbol{G}(x,y)\boldsymbol{a}_1$ and $\boldsymbol{G}(x,y)\boldsymbol{a}_2$ for any two orthonormal vectors $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$.

Recall that the scattered field $\boldsymbol{u}^s$ can be written as

$$
\boldsymbol{u}^s(x,\omega,\boldsymbol{a}_j) = \boldsymbol{u}_1(x,\omega,\boldsymbol{a}_j) + \boldsymbol{u}_2(x,\omega,\boldsymbol{a}_j) + \boldsymbol{b}(x,\omega,\boldsymbol{a}_j), \quad j = 1,2,
$$

where $\boldsymbol{u}_1$, $\boldsymbol{u}_2$, and $\boldsymbol{b}$ satisfy for $x \in U$ that

$$
\begin{aligned}
&\lim_{Q\to\infty} \frac{1}{Q-1}\int_1^Q \omega^{m+2} \sum_{j=1}^{2} |\boldsymbol{u}_1(x,\omega,\boldsymbol{a}_j)|^2 d\omega = C_m \int_{\mathbb{R}^2} \frac{1}{|x-\zeta|^2}\phi(\zeta) d\zeta, \\
&\lim_{Q\to\infty} \frac{1}{Q-1}\int_1^Q \omega^{m+2} |\boldsymbol{u}_2(x,\omega,\boldsymbol{a}_j)|^2 d\omega = 0, \\
&\lim_{Q\to\infty} \frac{1}{Q-1}\int_1^Q \omega^{m+2} |\boldsymbol{b}(x,\omega,\boldsymbol{a}_j)|^2 d\omega = 0.
\end{aligned}
$$

Using the Hölder inequality gives

$$
\left|\frac{1}{Q-1}\int_1^Q \omega^{m+2} \Re\left[\boldsymbol{u}_i(x,\omega,\boldsymbol{a}_j)\overline{\boldsymbol{b}(x,\omega,\boldsymbol{a}_j)}\right] d\omega\right|
$$

$$\lesssim \frac{1}{Q-1}\int_1^Q \omega^{m+2}|\boldsymbol{u}_i(x,\omega,\boldsymbol{a}_j)||\boldsymbol{b}(x,\omega,\boldsymbol{a}_j)|d\omega$$
$$\lesssim \left[\frac{1}{Q-1}\int_1^Q \omega^{m+2}|\boldsymbol{u}_i(x,\omega,\boldsymbol{a}_j)|^2\right]^{\frac{1}{2}}\left[\frac{1}{Q-1}\int_1^Q \omega^{m+2}|\boldsymbol{b}(x,\omega,\boldsymbol{a}_j)|^2\right]^{\frac{1}{2}} \to 0$$

as $Q\to\infty$ for $i,j=1,2$ and similarly

$$\left|\frac{1}{Q-1}\int_1^Q \omega^{m+2}\Re\left[\boldsymbol{u}_1(x,\omega,\boldsymbol{a}_j)\overline{\boldsymbol{u}_2(x,\omega,\boldsymbol{a}_j)}\right]d\omega\right| \to 0$$

as $Q\to\infty$ for $j=1,2$. Hence, we obtain

$$\lim_{Q\to\infty}\frac{1}{Q-1}\int_1^Q \omega^{m+2}\sum_{j=1}^2|\boldsymbol{u}^s(x,\omega,\boldsymbol{a}_j)|^2 d\omega$$
$$=\lim_{Q\to\infty}\frac{1}{Q-1}\int_1^Q \omega^{m+2}\sum_{j=1}^2\left|\boldsymbol{u}_1(x,\omega,\boldsymbol{a}_j)+\boldsymbol{u}_2(x,\omega,\boldsymbol{a}_j)+\boldsymbol{b}(x,\omega,\boldsymbol{a}_j)\right|^2 d\omega$$
$$=\lim_{Q\to\infty}\frac{1}{Q-1}\int_1^Q \omega^{m+2}\sum_{j=1}^2|\boldsymbol{u}_1(x,\omega,\boldsymbol{a}_j)|^2 d\omega$$
$$=C_m\int_{\mathbb{R}^2}\frac{1}{|x-\zeta|^2}\phi(\zeta)d\zeta.$$

It follows from [20, Lemma 3.6] that the function ϕ can be uniquely determined from the integral equation (1.4) for all $x\in U$, which completes the proof of Theorem 1.2.

6. Conclusion. We have studied the direct and inverse scattering problems for the time-harmonic elastic wave equation with a random potential in two dimensions. The potential is assumed to be a microlocally isotropic generalized Gaussian random field whose covariance is a classical pseudodifferential operator. For such a distribution potential, we deduce the equivalence between the direct scattering problem and the Lippmann–Schwinger integral equation which is shown to have a unique solution. Employing the Born approximation in the high frequency regime and microlocal analysis for the Fourier integral operators, we establish the connection between the principal symbol of the covariance operator for the random potential and the amplitude of the scattered field generated by a single realization of the random potential. Based on the identity, we obtain the uniqueness for the recovery of the microcorrelation strength of the random potential.

For the three-dimensional case, the well-posedness of the direct scattering problem can be obtained based on the same procedure as the two-dimensional case. The convergence of the Born series and the estimate for $\boldsymbol{u}_1$ obtained in Theorem 5.1 can also be extended to the three-dimensional case. However, what is different from the two-dimensional case is that the Green tensor in three dimensions does not decay with respect to the frequency ω. It is unclear whether the contribution of higher-order terms can be neglected in three dimensions. Hence, a frequency-dependence assumption of the potential ρ, e.g., $\rho(x,\omega)=\rho(x)\omega^{-\theta}$ with $\theta>\frac{m-1}{2}$, might be required to uniquely recover the microcorrelation strength by using near-field data [16]. A possible way to overcome this difficulty in the three-dimensional case is to recover the strength by using the far-field patterns as the data [22].

Another interesting and challenging problem is to investigate the inverse random medium problem which is to replace $(\omega^2-\rho)\boldsymbol{u}$ in (1.1) by $\omega^2\rho\boldsymbol{u}$. In this case, the unknown parameter ρ describes the density of the random medium. The direct problem has been studied in [27]. However, the inverse problem is open. The method used in this paper cannot be applied to the inverse random medium problem, since ρ is involved in the high frequency term and the Born series defined by the corresponding Lippmann–Schwinger equation does not converge for large frequency ω. We hope to report the progress on these problems elsewhere in the future.

Appendix A. Proof of Lemma 5.8.

For simplicity, we first introduce the notations $\bar\sigma := -\frac{\mathrm{i}}{64}\left(\frac{2}{\pi}\right)^{\frac{3}{2}}$, $\boldsymbol{J}(\zeta):=\zeta\zeta^\top$,

$$\psi_1(\zeta):=c_{\mathrm{s}}^{-\frac12}e^{\mathrm{i}\kappa_{\mathrm{s}}|\zeta|},\quad \psi_2(\zeta):=c_{\mathrm{s}}^{\frac12}e^{\mathrm{i}\kappa_{\mathrm{s}}|\zeta|}-c_{\mathrm{p}}^{\frac12}e^{\mathrm{i}\kappa_{\mathrm{p}}|\zeta|},\quad \psi_3(\zeta):=c_{\mathrm{s}}^{\frac32}e^{\mathrm{i}\kappa_{\mathrm{s}}|\zeta|}-c_{\mathrm{p}}^{\frac32}e^{\mathrm{i}\kappa_{\mathrm{p}}|\zeta|}$$

for $\zeta\in\mathbb{R}^2$, and the integral

$$\begin{aligned}A(\beta_1,\beta_2,\beta_3,p_1,p_2,p_3,\boldsymbol{M}_1,\boldsymbol{M}_2,\boldsymbol{M}_3):=&\int_D\int_D\frac{\beta_1(x-z)\beta_2(z-z')\beta_3(z'-x)}{|x-z|^{p_1}|z-z'|^{p_2}|z'-x|^{p_3}}\\&\times\rho(z)\rho(z')\boldsymbol{M}_1(x-z)\boldsymbol{M}_2(z-z')\boldsymbol{M}_3(z'-x)\boldsymbol{a}dzdz',\end{aligned}$$

where $\beta_i\in\{\psi_1,\psi_2,\psi_3\}$ and $\boldsymbol{M}_i\in\{\boldsymbol{I},\boldsymbol{J}\}$ for $i=1,2,3$. Substituting (5.6) into (5.22) shows that

$$\boldsymbol{v}(x,\omega,\boldsymbol{a})=\sum_{k=1}^4\boldsymbol{v}_k(x,\omega,\boldsymbol{a})\omega^{-(k+\frac12)},$$

where

$$\begin{aligned}\boldsymbol{v}_1(x,\omega,\boldsymbol{a})=&\,e^{-\frac34\pi\mathrm{i}}\mu^{-3}\bar\sigma A(\psi_1,\psi_1,\psi_1,\frac12,\frac12,\frac12,\boldsymbol{I},\boldsymbol{I},\boldsymbol{I})\\&+e^{-\frac74\pi\mathrm{i}}\mu^{-2}\bar\sigma A(\psi_1,\psi_1,\psi_3,\frac12,\frac12,\frac52,\boldsymbol{I},\boldsymbol{I},\boldsymbol{J})+e^{-\frac74\pi\mathrm{i}}\mu^{-2}\bar\sigma A(\psi_1,\psi_3,\psi_1,\frac12,\frac52,\frac12,\boldsymbol{I},\boldsymbol{J},\boldsymbol{I})\\&+e^{-\frac{11}{4}\pi\mathrm{i}}\mu^{-1}\bar\sigma A(\psi_1,\psi_3,\psi_3,\frac12,\frac52,\frac52,\boldsymbol{I},\boldsymbol{J},\boldsymbol{J})+e^{-\frac74\pi\mathrm{i}}\mu^{-2}\bar\sigma A(\psi_3,\psi_1,\psi_1,\frac52,\frac12,\frac12,\boldsymbol{J},\boldsymbol{I},\boldsymbol{I})\\&+e^{-\frac{11}{4}\pi\mathrm{i}}\mu^{-1}\bar\sigma A(\psi_3,\psi_1,\psi_3,\frac52,\frac12,\frac52,\boldsymbol{J},\boldsymbol{I},\boldsymbol{J})+e^{-\frac{11}{4}\pi\mathrm{i}}\mu^{-1}\bar\sigma A(\psi_3,\psi_3,\psi_1,\frac52,\frac52,\frac12,\boldsymbol{J},\boldsymbol{J},\boldsymbol{I})\\&+e^{-\frac{15}{4}\pi\mathrm{i}}\bar\sigma A(\psi_3,\psi_3,\psi_3,\frac52,\frac52,\frac52,\boldsymbol{J},\boldsymbol{J},\boldsymbol{J}),\end{aligned}$$

$$\begin{aligned}\boldsymbol{v}_2(x,\omega,\boldsymbol{a})=&\,e^{-\frac54\pi\mathrm{i}}\mu^{-2}\bar\sigma A(\psi_1,\psi_1,\psi_2,\frac12,\frac12,\frac32,\boldsymbol{I},\boldsymbol{I},\boldsymbol{I})\\&+e^{-\frac94\pi\mathrm{i}}\mu^{-1}\bar\sigma A(\psi_1,\psi_3,\psi_2,\frac12,\frac52,\frac32,\boldsymbol{I},\boldsymbol{J},\boldsymbol{I})+e^{-\frac94\pi\mathrm{i}}\mu^{-1}\bar\sigma A(\psi_3,\psi_1,\psi_2,\frac52,\frac12,\frac32,\boldsymbol{J},\boldsymbol{I},\boldsymbol{I})\\&+e^{-\frac{13}{4}\pi\mathrm{i}}\bar\sigma A(\psi_3,\psi_3,\psi_2,\frac52,\frac52,\frac32,\boldsymbol{J},\boldsymbol{J},\boldsymbol{I})+e^{-\frac54\pi\mathrm{i}}\mu^{-2}\bar\sigma A(\psi_1,\psi_2,\psi_1,\frac12,\frac32,\frac12,\boldsymbol{I},\boldsymbol{I},\boldsymbol{I})\\&+e^{-\frac94\pi\mathrm{i}}\mu^{-1}\bar\sigma A(\psi_1,\psi_2,\psi_3,\frac12,\frac32,\frac52,\boldsymbol{I},\boldsymbol{I},\boldsymbol{J})+e^{-\frac94\pi\mathrm{i}}\mu^{-1}\bar\sigma A(\psi_3,\psi_2,\psi_1,\frac52,\frac32,\frac12,\boldsymbol{J},\boldsymbol{I},\boldsymbol{I})\\&+e^{-\frac{13}{4}\pi\mathrm{i}}\bar\sigma A(\psi_3,\psi_2,\psi_3,\frac52,\frac32,\frac52,\boldsymbol{J},\boldsymbol{I},\boldsymbol{J})+e^{-\frac54\pi\mathrm{i}}\mu^{-2}\bar\sigma A(\psi_2,\psi_1,\psi_1,\frac32,\frac12,\frac12,\boldsymbol{I},\boldsymbol{I},\boldsymbol{I})\\&+e^{-\frac94\pi\mathrm{i}}\mu^{-1}\bar\sigma A(\psi_2,\psi_1,\psi_3,\frac32,\frac12,\frac52,\boldsymbol{I},\boldsymbol{I},\boldsymbol{J})+e^{-\frac94\pi\mathrm{i}}\mu^{-1}\bar\sigma A(\psi_2,\psi_3,\psi_1,\frac32,\frac52,\frac12,\boldsymbol{I},\boldsymbol{J},\boldsymbol{I})\\&+e^{-\frac{13}{4}\pi\mathrm{i}}\bar\sigma A(\psi_2,\psi_3,\psi_3,\frac32,\frac52,\frac52,\boldsymbol{I},\boldsymbol{J},\boldsymbol{J}),\end{aligned}$$

$$
\begin{aligned}
&\boldsymbol{v}_3(x,\omega,\boldsymbol{a}) = e^{-\frac{7}{4}\pi\mathrm{i}}\mu^{-1}\bar{\sigma}A(\psi_1,\psi_2,\psi_2,\frac{1}{2},\frac{3}{2},\frac{3}{2},\boldsymbol{I},\boldsymbol{I},\boldsymbol{I})\\
&+ e^{-\frac{11}{4}\pi\mathrm{i}}\bar{\sigma}A(\psi_3,\psi_2,\psi_2,\frac{5}{2},\frac{3}{2},\frac{3}{2},\boldsymbol{J},\boldsymbol{I},\boldsymbol{I}) + e^{-\frac{7}{4}\pi\mathrm{i}}\mu^{-1}\bar{\sigma}A(\psi_2,\psi_1,\psi_2,\frac{3}{2},\frac{1}{2},\frac{3}{2},\boldsymbol{I},\boldsymbol{I},\boldsymbol{I})\\
&+ e^{-\frac{11}{4}\pi\mathrm{i}}\bar{\sigma}A(\psi_2,\psi_3,\psi_2,\frac{3}{2},\frac{5}{2},\frac{3}{2},\boldsymbol{I},\boldsymbol{J},\boldsymbol{I}) + e^{-\frac{7}{4}\pi\mathrm{i}}\mu^{-1}\bar{\sigma}A(\psi_2,\psi_2,\psi_1,\frac{3}{2},\frac{3}{2},\frac{1}{2},\boldsymbol{I},\boldsymbol{I},\boldsymbol{I})\\
&+ e^{-\frac{11}{4}\pi\mathrm{i}}\bar{\sigma}A(\psi_2,\psi_2,\psi_3,\frac{3}{2},\frac{3}{2},\frac{5}{2},\boldsymbol{I},\boldsymbol{I},\boldsymbol{J}),
\end{aligned}
$$

and

$$\boldsymbol{v}_4(x,\omega,\boldsymbol{a}) = e^{-\frac{9}{4}\pi\mathrm{i}}\bar{\sigma}A(\psi_2,\psi_2,\psi_2,\frac{3}{2},\frac{3}{2},\frac{3}{2},\boldsymbol{I},\boldsymbol{I},\boldsymbol{I}).$$

Then applying the Cauchy–Schwarz inequality leads to

$$\text{(A.1)}\quad \frac{1}{Q-1}\int_1^Q \omega^{m+2}|\boldsymbol{v}(x,\omega,\boldsymbol{a})|^2 d\omega \lesssim \frac{1}{Q-1}\sum_{k=1}^{4}\int_1^Q \omega^{m-(2k-1)}|\boldsymbol{v}_k(x,\omega,\boldsymbol{a})|^2 d\omega.$$

Noting that $\omega^{-\frac{1}{2}}\frac{\beta_2(z-z')}{|z-z'|^{p_2}}\boldsymbol{M}_2(z-z')$ involved in $\boldsymbol{v}_k$ has the same singularity as $\boldsymbol{G}^{(0)}(z,z',\omega)$, we get

$$
\begin{aligned}
\left|\omega^{-\frac{3}{2}}\boldsymbol{v}_k(x,\omega,\boldsymbol{a})\right| &\lesssim \Bigg|\omega^{-\frac{3}{2}}\int_D\int_D \frac{\beta_1(x-z)\beta_2(z-z')\beta_3(z'-x)}{|x-z|^{p_1}|z-z'|^{p_2}|z'-x|^{p_3}}\\
&\qquad \times \rho(z)\rho(z')\boldsymbol{M}_1(x-z)\boldsymbol{M}_2(z-z')\boldsymbol{M}_3(z'-x)\boldsymbol{a}dzdz'\Bigg|\\
&\lesssim \left\|\omega^{-1}\rho\otimes\rho\frac{\beta_1(x-\cdot)\beta_3(\cdot-x)}{|x-\cdot|^{p_1}|\cdot-x|^{p_3}}\boldsymbol{M}_1(x-\cdot)\boldsymbol{M}_3(\cdot-x)\boldsymbol{a}\right\|_{\boldsymbol{W}_0^{-2s,q}(D\times D)}\\
&\qquad \times \left\|\omega^{-\frac{1}{2}}\frac{\beta_2(\cdot-\cdot)}{|\cdot-\cdot|^{p_2}}\boldsymbol{M}_2(\cdot-\cdot)\right\|_{(W^{2s,p}(D\times D))^{2\times 2}}\\
&\lesssim \omega^{-\frac{3}{2}+3s}
\end{aligned}
$$

based on Lemma 5.6 and a similar argument used in (5.25). As a result,

$$
\begin{aligned}
&\lim_{Q\to\infty}\frac{1}{Q-1}\sum_{k=2}^{4}\int_1^Q \omega^{m-(2k-1)}|\boldsymbol{v}_k(x,\omega,\boldsymbol{a})|^2 d\omega\\
&\lesssim \lim_{Q\to\infty}\frac{1}{Q-1}\sum_{k=2}^{4}\int_1^Q \omega^{m-(2k-1)+6s}d\omega\\
\text{(A.2)}\qquad &\lesssim \lim_{Q\to\infty}\frac{Q^{m-2+6s}-1}{Q-1} = 0
\end{aligned}
$$

almost surely by choosing $s\in(\frac{2-m}{2},\frac{1}{2})$ with $m\in(\frac{5}{3},2]$ such that $m-2+6s<1$. Hence, to prove (5.27), according to (A.1) and (A.2), we only need to show for $x\in U$ that

$$\text{(A.3)}\qquad \lim_{Q\to\infty}\frac{1}{Q-1}\int_1^Q \omega^{m-1}|\boldsymbol{v}_1(x,\omega,\boldsymbol{a})|^2 d\omega = 0$$

in the almost surely sense. Note that

$$\frac{1}{Q-1}\int_1^Q \omega^{m-1}|\boldsymbol{v}_1(x,\omega,\boldsymbol{a})|^2 d\omega \le \int_1^\infty \frac{\omega \mathbf{1}_{[1,Q]}(\omega)}{Q-1}\omega^{m-2}|\boldsymbol{v}_1(x,\omega,\boldsymbol{a})|^2 d\omega$$

with $\mathbf{1}_{[1,Q]}$ being the characteristic function on the interval $[1,Q]$, and it holds pointwise that

$$\lim_{Q\to\infty}\frac{\omega \mathbf{1}_{[1,Q]}(\omega)}{Q-1} = 0 \quad \text{and} \quad \left|\frac{\omega \mathbf{1}_{[1,Q]}(\omega)}{Q-1}\omega^{m-2}|\boldsymbol{v}_1(x,\omega,\boldsymbol{a})|^2\right| \lesssim \omega^{m-2}|\boldsymbol{v}_1(x,\omega,\boldsymbol{a})|^2.$$

By the dominated convergence theorem, to show (A.3), it suffices to prove

$$\int_1^\infty \omega^{m-2}\mathbb{E}|\boldsymbol{v}_1(x,\omega,\boldsymbol{a})|^2 d\omega < \infty. \tag{A.4}$$

Substituting ψ_1, ψ_2, ψ_3, and $\boldsymbol{J}$ into $\boldsymbol{v}_1$ gives that $\boldsymbol{v}_1$ is a linear combination of the integral

$$\mathbb{B}(x,\omega) := \int_D\int_D e^{\mathrm{i}\omega(c_1|x-z|+c_2|z-z'|+c_3|z'-x|)}\mathbb{K}(x,z,z')\rho(z)\rho(z')dzdz', \tag{A.5}$$

where $c_1, c_2, c_3 \in \{c_{\mathrm{s}}, c_{\mathrm{p}}\}$ and

$$\mathbb{K}(x,z,z') = \frac{(x_1-z_1)^{p_1}(x_2-z_2)^{p_2}(z_1-z_1')^{p_3}(z_2-z_2')^{p_4}(z_1'-x_1)^{p_5}(z_2'-x_2)^{p_6}}{|x-z|^{p_7}|z-z'|^{p_8}|z'-x|^{p_9}} \tag{A.6}$$

with

$$\begin{aligned}(p_1,\dots,p_9) \in \Big\{(p_1,\dots,p_9)\Big| p_i \in \{0,1,2\}, 1\le i\le 6, p_j \in \Big\{\frac12,\frac52\Big\}, 7\le j\le 9,\\ p_7-p_1-p_2=\frac12, p_8-p_3-p_4=\frac12, p_9-p_5-p_6=\frac12\Big\}.\end{aligned}$$

It follows from the Cauchy–Schwarz inequality that a sufficient condition for (A.4) is

$$\int_1^\infty \omega^{m-2}\mathbb{E}|\mathbb{B}(x,\omega)|^2 d\omega < \infty. \tag{A.7}$$

To deal with the roughness of the random potential ρ, similar to the technique used in [22], we introduce a modification $\rho_\varepsilon := \rho * \varphi_\varepsilon$ with $\varphi_\varepsilon(x) := \varepsilon^{-2}\varphi(x/\varepsilon)$ for $\varepsilon > 0$, where $\varphi \in C_0^\infty(\mathbb{R}^2)$ is a radially symmetric function satisfying $\int_{\mathbb{R}^2}\varphi(x)dx = 1$, and define

$$\mathbb{B}_\varepsilon(x,\omega) := \int_D\int_D e^{\mathrm{i}\omega(c_1|x-z|+c_2|z-z'|+c_3|z'-x|)}\mathbb{K}(x,z,z')\rho_\varepsilon(z)\rho_\varepsilon(z')dzdz' \tag{A.8}$$

by replacing ρ in (A.5) by ρ_ε. It is easy to show that $\lim_{\varepsilon\to\infty}\mathbb{E}|\mathbb{B}_\varepsilon|^2 = \mathbb{E}|\mathbb{B}|^2$ (cf. [22]), which, together with Fatou's lemma and the fact $m \in (\frac53, 2]$, leads to

$$\int_1^\infty \omega^{m-2}\mathbb{E}|\mathbb{B}(x,\omega)|^2 d\omega \le \varlimsup_{\varepsilon\to 0}\int_1^\infty \mathbb{E}|\mathbb{B}_\varepsilon(x,\omega)|^2 d\omega.$$

Hence, it suffices to show

$$\overline{\lim_{\varepsilon\to 0}} \int_1^\infty \mathbb{E}|\mathbb{B}_\varepsilon(x,\omega)|^2 d\omega < \infty \quad \forall\, x \in U. \tag{A.9}$$

The procedure to show (A.9) is similar to the proof for the acoustic wave case with $m = d = 2$ given in [19]. To be self-contained and for completeness, we present the details below.

The basic idea is to express $\mathbb{B}_\varepsilon$ in terms of a one-dimensional Fourier transform and then get the estimate with respect to ω by utilizing the Parseval formula. To this end, we first consider the phase function $L(z,z') := c_1|x-z| + c_2|z-z'| + c_3|z'-x|$, which is smooth in the domain $\Theta := \{(z,z') \in D \times D | z \neq z'\}$ and

$$\nabla_z L(z,z') = c_1 \frac{z-x}{|z-x|} + c_2 \frac{z-z'}{|z-z'|}, \quad \nabla_{z'} L(z,z') = c_2 \frac{z'-z}{|z'-z|} + c_3 \frac{z'-x}{|z'-x|}$$

for any $(z,z') \in \Theta$. Without loss of generality, we assume that $0 \in U$ such that $|z|$ and $|z'|$ are bounded from below and above for $z, z' \in D$ since U has a positive distance to D. Hence, it holds for $(z,z') \in \Theta$ that

$$0 < C_1 \le |\nabla L(z,z')| \le C_2 < \infty \tag{A.10}$$

for some constants C_1 and C_2, where we use the facts that U is bounded and convex, and that

$$\begin{aligned}(z,z') \cdot \nabla L(z,z') &= c_1 z \cdot \frac{z-x}{|z-x|} + c_2|z-z'| + c_3 z' \cdot \frac{z'-x}{|z'-x|} \\ &= c_1|z|\cos\theta_1 + c_2|z-z'| + c_3|z'|\cos\theta_2 \ge C_3 > 0\end{aligned}$$

for some constant C_3 with θ_1 and θ_2 being the angle between z and $z-x$ and the angle between z' and $z'-x$, respectively. Due to the boundedness of D and U and the fact that they have a positive distance, the surface

$$\Gamma'_t := \{(z,z') \in D \times D | L(z,z') = t\}, \quad t > 0,$$

is nonempty only for $t \in [T_0, T_1]$ with some positive values $T_0 = T_0(x)$ and $T_1 = T_1(x)$.

For a fixed $\tilde{t} \in [T_0, T_1]$, there exists a $\tilde{\eta} = \tilde{\eta}(\tilde{t})$ and an open cone $E = E(\tilde{t}) \subset \mathbb{R}^4$ centered at the origin such that it holds for $t_0 = t_0(\tilde{t}) := \tilde{t} - \tilde{\eta}$ and $t_1 = t_1(\tilde{t}) := \tilde{t} + \tilde{\eta}$ that

$$D \times D \cap \{t_0 < L(z,z') < t_1\} \subset E \cap \{t_0 < L(z,z') < t_1\} := \Gamma \tag{A.11}$$

and

$$\Gamma = \bigcup_{t\in[t_0,t_1]} \Gamma_t \quad \text{with} \quad \Gamma_t := \Gamma \cap \{(z,z') | L(z,z') = t\}.$$

According to (A.8) and (A.11), we obtain

$$\begin{aligned}\mathbb{B}_\varepsilon(x,\omega) &= \int_\Gamma e^{\mathrm{i}\omega L(z,z')} \mathbb{K}(x,z,z') \rho_\varepsilon(z)\rho_\varepsilon(z') dz dz' \\ &= \int_{t_0}^{t_1} e^{\mathrm{i}\omega t} \int_{\Gamma_t} \mathbb{K}(x,z,z') |\nabla L(z,z')|^{-1} \rho_\varepsilon(z)\rho_\varepsilon(z') d\mathcal{H}^3(z,z') dt \\ &=: \int_{t_0}^{t_1} e^{\mathrm{i}\omega t} S_\varepsilon(t) dt = (\mathcal{F} S_\varepsilon)(-\omega),\end{aligned}$$

where

$$S_\varepsilon(t) := \int_{\Gamma_t} \mathbb{K}(x,z,z')|\nabla L(z,z')|^{-1}\rho_\varepsilon(z)\rho_\varepsilon(z')d\mathcal{H}^3(z,z') \tag{A.12}$$

is compactly supported in $[T_0, T_1]$ and the integral in (A.12) is with respect to the three-dimensional Hausdorff measure $\mathcal{H}^3$ on Γ_t. Note that

$$\begin{aligned}\mathbb{E}|S_\varepsilon(t)|^2 = \int_{\Gamma_t\times\Gamma_t} & \mathbb{K}(x,z,z')\mathbb{K}(x,\tilde z,\tilde z')|\nabla L(z,z')|^{-1}|\nabla L(\tilde z,\tilde z')|^{-1} \\ & \times\mathbb{E}[\rho_\varepsilon(z)\rho_\epsilon(z')\rho_\varepsilon(\tilde z)\rho_\varepsilon(\tilde z')]d\mathcal{H}^3(z,z')d\mathcal{H}^3(\tilde z,\tilde z'),\end{aligned} \tag{A.13}$$

where $|\nabla L(z,z')|^{-1}|\nabla L(\tilde z,\tilde z')|^{-1}$ is bounded according to (A.10) and

$$|\mathbb{K}(x,z,z')| \le |x-z|^{-\frac12}|z-z'|^{-\frac12}|z'-x|^{-\frac12} \lesssim |z-z'|^{-\frac12}$$

for $x\in U$ and $(z,z')\in\Theta$ according to (A.6). Moreover, the Wick formula leads to

$$\begin{aligned}\mathbb{E}[\rho_\varepsilon(z)\rho_\varepsilon(z')\rho_\varepsilon(\tilde z)\rho_\varepsilon(\tilde z')] = \mathcal{K}_{\rho_\varepsilon}(z,z')\mathcal{K}_{\rho_\varepsilon}(\tilde z,\tilde z') + \mathcal{K}_{\rho_\varepsilon}(z,\tilde z)\mathcal{K}_{\rho_\varepsilon}(z',\tilde z') \\ + \mathcal{K}_{\rho_\varepsilon}(z,\tilde z')\mathcal{K}_{\rho_\varepsilon}(z',\tilde z),\end{aligned}$$

where $\mathcal{K}_{\rho_\varepsilon}(z,z') := \mathbb{E}[\rho_\varepsilon(z)\rho_\varepsilon(z')]$ is the covariance function of ρ_ε. Thus, (A.13) turns out to be

$$\begin{aligned}\mathbb{E}|S_\varepsilon(t)|^2 &\lesssim \int_{\Gamma_t\times\Gamma_t} |z-z'|^{-\frac12}|\tilde z-\tilde z'|^{-\frac12}\left|\mathcal{K}_{\rho_\varepsilon}(z,z')\mathcal{K}_{\rho_\varepsilon}(\tilde z,\tilde z')\right| d\mathcal{H}^3(z,z')d\mathcal{H}^3(\tilde z,\tilde z') \\ &\quad + \int_{\Gamma_t\times\Gamma_t} |z-z'|^{-\frac12}|\tilde z-\tilde z'|^{-\frac12}\left|\mathcal{K}_{\rho_\varepsilon}(z,\tilde z)\mathcal{K}_{\rho_\varepsilon}(z',\tilde z')\right| d\mathcal{H}^3(z,z')d\mathcal{H}^3(\tilde z,\tilde z') \\ &\quad + \int_{\Gamma_t\times\Gamma_t} |z-z'|^{-\frac12}|\tilde z-\tilde z'|^{-\frac12}\left|\mathcal{K}_{\rho_\varepsilon}(z,\tilde z')\mathcal{K}_{\rho_\varepsilon}(z',\tilde z)\right| d\mathcal{H}^3(z,z')d\mathcal{H}^3(\tilde z,\tilde z') \\ &=: \mathrm{I}_1^* + \mathrm{I}_2^* + \mathrm{I}_3^*.\end{aligned}$$

For sufficiently small $\varepsilon > 0$, it follows from [22, Lemma 10] that

$$\begin{aligned}|\mathcal{K}_{\rho_\varepsilon}(z,z')| &\lesssim |\ln|z-z'|| + O(1) \quad \text{for } m = 2, \\ |\mathcal{K}_{\rho_\varepsilon}(z,z')| &\lesssim |z-z'|^{-(2-m)} + O(1) \quad \text{for } m\in(1,2).\end{aligned}$$

Hence, for any $m\in(\frac53, 2]$, there exists a sufficiently small $\epsilon > 0$ such that

$$|\mathcal{K}_{\rho_\varepsilon}(z,z')| \lesssim |z-z'|^{-(2-m+\epsilon)}$$

when $|z-z'| \ll 1$.

For I_1^*, we have

$$\mathrm{I}_1^* \lesssim \int_{\Gamma_t} |z-z'|^{-\frac12}|\mathcal{K}_{\rho_\varepsilon}(z,z')|d\mathcal{H}^3(z,z')\int_{\Gamma_t} |\tilde z-\tilde z'|^{-\frac12}|\mathcal{K}_{\rho_\varepsilon}(\tilde z,\tilde z')|d\mathcal{H}^3(\tilde z,\tilde z') < \infty$$

according to [19, Lemma 6].

For I_2^*, it follows from the Hölder inequality and [19, Lemma 6] that

$$\begin{aligned}\mathrm{I}_2^* &\lesssim \int_{\Gamma_t\times\Gamma_t} |z-z'|^{-\frac12}|\tilde z-\tilde z'|^{-\frac12}|z-\tilde z|^{-(2-m+\epsilon)}|z'-\tilde z'|^{-(2-m+\epsilon)}d\mathcal{H}^3(z,z')d\mathcal{H}^3(\tilde z,\tilde z')\\
&\lesssim \left[\int_{\Gamma_t\times\Gamma_t} |z-z'|^{-\frac32}|\tilde z-\tilde z'|^{-\frac32}d\mathcal{H}^3(z,z')d\mathcal{H}^3(\tilde z,\tilde z')\right]^{\frac13}\\
&\quad\times\left[\int_{\Gamma_t\times\Gamma_t} |z-\tilde z|^{-\frac32(2-m+\epsilon)}|z'-\tilde z'|^{-\frac32(2-m+\epsilon)}d\mathcal{H}^3(z,z')d\mathcal{H}^3(\tilde z,\tilde z')\right]^{\frac23}\\
&\lesssim \left[\int_{\Gamma_t} |z-z'|^{-\frac32}d\mathcal{H}^3(z,z')\int_{\Gamma_t}|\tilde z-\tilde z'|^{-\frac32}d\mathcal{H}^3(\tilde z,\tilde z')\right]^{\frac13}\\
&\quad\times\left[\int_{\Gamma_t\times\Gamma_t} |z-\tilde z|^{-3(2-m+\epsilon)}d\mathcal{H}^3(z,z')d\mathcal{H}^3(\tilde z,\tilde z')\right]^{\frac13}\\
&\quad\times\left[\int_{\Gamma_t\times\Gamma_t} |z'-\tilde z'|^{-3(2-m+\epsilon)}d\mathcal{H}^3(z,z')d\mathcal{H}^3(\tilde z,\tilde z')\right]^{\frac13}<\infty\end{aligned}$$

for $m\in(\frac53,2]$. An argument similar to the one used in I_2^* shows that $\mathrm{I}_3^*<\infty$.

Hence, for any fixed $\tilde t\in[T_0,T_1]$, there exists a constant $C(\tilde t)$ such that

$$\mathbb{E}|S_\varepsilon(t)|^2\le C(\tilde t)\quad\forall\ t\in(t_0(\tilde t),t_1(\tilde t))$$

and sufficiently small $\varepsilon>0$. By compactness, there is a countable subset $\Lambda\subset[T_0,T_1]$ with finite elements such that

$$[T_0,T_1]\subset\bigcup_{\tilde t\in\Lambda}(t_0(\tilde t),t_1(\tilde t)).$$

By defining $C:=\sum_{\tilde t\in\Lambda}C(\tilde t)$, we get

$$\mathbb{E}|S_\varepsilon(t)|^2\le C\quad\forall\ t\in[T_0,T_1]$$

and sufficiently small $\varepsilon>0$. Then it follows from the Parseval formula that

$$\overline{\lim_{\varepsilon\to0}}\int_1^\infty\mathbb{E}|\mathbb{B}_\varepsilon(x,\omega)|^2d\omega=\overline{\lim_{\varepsilon\to0}}\int_{T_0}^{T_1}\mathbb{E}|S_\varepsilon(t)|^2dt\le C(T_1-T_0)<\infty,$$

which yields (A.9) and thus (A.7). Then (A.4) holds due to (A.7), which completes the proof together with (A.2).

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