INVERSE SCATTERING FOR THE BIHARMONIC WAVE EQUATION WITH A RANDOM POTENTIAL*

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Abstract. We consider the inverse random potential scattering problem for the two- and three-dimensional biharmonic wave equation in lossy media. The potential is assumed to be a microlocally isotropic Gaussian rough field. The main contributions of the work are twofold. First, the unique continuation principle is proved for the fourth order biharmonic wave equation with rough potentials and the well-posedness of the direct scattering problem is established in the distribution sense. Second, the correlation strength of the random potential is shown to be uniquely determined by the high frequency limit of the second moment of the backscattering data averaged over the frequency band. Moreover, we demonstrate that the expectation in the data can be removed and the data of a single realization is sufficient for the uniqueness of the inverse problem with probability one when the medium is lossless.

Key words. Inverse scattering, random potential, biharmonic operator, pseudo-differential operator, principal symbol, uniqueness

AMS subject classifications. 35R30, 35R60, 60H15

1. Introduction.

Scattering problems arise from the interaction between waves and media. They play a fundamental role in many scientific areas such as medical imaging, exploration geophysics, and remote sensing. Driven by significant applications, scattering problems have been extensively studied by many researchers, especially for acoustic and electromagnetic waves [8,24]. Recently, scattering problems for biharmonic waves have attracted much attention due to their important applications in thin plate elasticity, which include offshore runway design [31], seismic cloaks [9,28], and platonic crystals [23]. Compared with the second order acoustic and electromagnetic wave equations, many direct and inverse scattering problems remain unsolved for the fourth order biharmonic wave equation [10,27].

In this paper, we consider the biharmonic wave equation with a random potential

$$\Delta^2 u - (k^2 + \mathrm{i}\sigma k)u + \rho u = -\delta_y \quad \text{in } \mathbb{R}^d, \tag{1.1}$$

where $d = 2$ or 3, $k > 0$ is the wavenumber, $\sigma \geq 0$ is the damping coefficient, and $\delta_y(x) := \delta(x - y)$ denotes the point source located at $y \in \mathbb{R}^d$ with δ being the Dirac delta distribution. The term ρu describes physically an external linear load added to the system and represents a multiplicative noise from the point of view of stochastic partial differential equations. Denote by $\kappa = \kappa(k)$ the complex-valued wavenumber which is given by

$$\kappa^4 = k^2 + \mathrm{i}\sigma k.$$

Let $\kappa_{\rm r} := \Re(\kappa) > 0$ and $\kappa_{\rm i} := \Im(\kappa) \geq 0$, where $\Re(\cdot)$ and $\Im(\cdot)$ denote the real and imaginary parts of a complex number, respectively. As an outgoing wave condition

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for the fourth order equation, the Sommerfeld radiation condition is imposed to both the wave field u and its Laplacian Δu:

$$(1.2)\qquad \lim_{r\to\infty} r^{\frac{d-1}{2}}\left(\partial_r u - \mathrm{i}\kappa u\right) = 0, \quad \lim_{r\to\infty} r^{\frac{d-1}{2}}\left(\partial_r \Delta u - \mathrm{i}\kappa \Delta u\right) = 0, \quad r = |x|.$$

We refer to [30] for the radiation condition in the lossless case with $\sigma = 0$. In the case where $\sigma > 0$, the radiation condition can be derived using the classical procedure (cf. [7, Theorem 3.2]) by utilizing the exponential decay property of the fundamental solution described in (2.2).

The potential ρ is assumed to be a Gaussian random field defined in a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where $\mathbb{P}$ is the probability measure. More precisely, ρ is required to satisfy the following assumption (cf. [16]).

Assumption 1.1. *Let the potential ρ be a real-valued centered microlocally isotropic Gaussian random field of order $m \in (d-1, d]$ in a bounded domain $D \subset \mathbb{R}^d$, i.e., the covariance operator Q_ρ of ρ is a classical pseudo-differential operator with the principal symbol $\mu(x)|\xi|^{-m}$, where μ is the correlation strength of ρ and is a function that is compactly supported in D satisfying $\mu \in C_0^\infty(D)$ and $\mu \geq 0$.*

Apparently, the regularity of the microlocally isotropic Gaussian random potential depends on the order m. It has been proved in [21, Lemma 2.6] that the potential is relatively regular and satisfies $\rho \in C^{0,\alpha}(D)$ with $\alpha \in (0, \frac{m-d}{2})$ if $m \in (d, d+2)$; the potential is rough and satisfies $\rho \in W^{\frac{m-d}{2}-\epsilon,p}(D)$ with $\epsilon > 0$ and $p > 1$ if $m \leq d$. This work focuses on the rough case, i.e., $m \leq d$.

Given the rough potential ρ, the direct scattering problem is to study the well-posedness and examine the regularity of the solution to (1.1)–(1.2); the inverse scattering problem is to determine the correlation strength μ of the random potential ρ from some statistics of the wave field u satisfying (1.1)–(1.2). Both the direct and inverse scattering problems pose challenges due to the rough nature of the random potential ρ. Specifically, the equation (1.1) should be studied in the distribution sense, treating ρ as a distribution. In this context, it is more reasonable to focus on the statistics of ρ, such as its covariance or correlation strength, rather than attempting to directly reconstruct ρ itself. The unique continuation principle is crucial for the well-posedenss of the direct scattering problem, which is nontrivial for the biharmonic wave equation with a rough potential. Moreover, the inverse scattering problem is nonlinear.

The inverse scattering problems for random potentials with potential ρ that satisfy Assumption 1.1 were investigated in [5, 16–19] for second-order wave equations. The approach for two-dimensional problems involves utilizing point source illumination and near-field data, while the three-dimensional problems require plane wave incidence and far-field pattern analysis due to the distinct configurations in each dimension. For the Schrödinger equation, the unique continuation principle was extended in [16] from the integrable potential $\rho \in L^p(D)$ with $p \in (1, \infty]$ (cf. [12, 13, 25]) to the rough potential $\rho \in W^{-\epsilon,p}(D)$, i.e., $m = d$. The uniqueness was also established for the two-dimensional inverse problem with $m \in [d, d+1)$. It was shown that the strength μ of the random potential ρ can be uniquely determined by a single realization of the near-field data almost surely. The corresponding three-dimensional inverse problem with $m = d$ was studied in [5] by using the far-field pattern of the scattered field. In [19], the authors considered a generalized setting for the three-dimensional Schrödinger equation, where both the potential and source are random. The uniqueness was obtained to determine the strength of the potential and source simultaneously based on far-field patterns. Recently, the unique continuation principle was proved in [20] for the second order elliptic operators with rougher potentials or medium parameters of order $m \in (d-1, d]$. In [17], the rough model was taken to study the inverse random potential problem for the two-dimensional elastic wave equation. It was shown that the correlation strength of the random potential is uniquely determined by the near-field data under the assumption $m \in (d - \frac{1}{3}, d]$. For the three-dimensional elastic wave equation, due to the lack of decay property of the fundamental solution with respect to the frequency, the far-field data was utilized in [18] to uniquely determine the strength of the random potential under the condition $m \in (d - \frac{1}{5}, d]$.

In the deterministic setting, the unique continuation principle was investigated in [4] and [26] for the general higher order linear elliptic operators with a weak vanishing assumption and for the biharmonic operator with a nonlinear coefficient satisfying a Lipschitz-type condition, respectively. In [15], the authors studied the inverse boundary value problem of determining a first order perturbation for the polyharmonic operator $(-\Delta)^n, n \geq 2$ by using the Cauchy data. It was shown in [14] that the first order perturbation of the biharmonic operator in a bounded domain can be uniquely determined from the knowledge of the Dirichlet-to-Neumann map given on a part of the boundary. We refer to [11, 29, 30, 32] and references therein for related direct and inverse scattering problems of the biharmonic operators with regular potentials. To the best of our knowledge, the unique continuation principle is not available for the biharmonic wave equation with rough potentials.

This paper is concerned with the direct and inverse random potential scattering problems for the two- and three-dimensional biharmonic wave equation. As previously mentioned, the configurations for the inverse scattering problems involving second-order wave equations differ in two and three dimensions. Nevertheless, due to the high regularity of the fundamental solution, a unified approach can be employed to tackle the inverse scattering problems associated with the biharmonic wave equation in both two and three dimensions. This can be achieved by utilizing the point source illumination and near-field data. The work contains two main contributions. First, the unique continuation principle is proved for the biharmonic wave equation with a rough potential and the well-posedness is established in the distribution sense for the direct scattering problem. Second, the uniqueness is established for the inverse scattering problem. Denote by $u(x, y, k)$ the solution of (1.1). The scattered wave, denoted by u^s, satisfies $u^s(x, y, k) = u(x, y, k) - \Phi(x, y, k)$, where Φ is the fundamental solution given in (2.2). We show that the correlation strength of the random potential can be uniquely determined by the high frequency limit of the second moment of the backscattering data, denoted as $u^s(x, k) := u^s(x, x, k)$, which is averaged over the frequency band $(K, 2K)$ as $K \to \infty$. It is noteworthy that the scattered wave $u^s(x, y, k)$ does not exhibit any singularity when $y = x$, and the backscattering data $u^s(x, x, k)$ holds significant importance in practical measurement scenarios. In the case of a lossless medium, where the damping coefficient $\sigma = 0$, we establish that the expectation in the data can be eliminated. Moreover, we show that the uniqueness of the inverse problem can be guaranteed with a probability of one by utilizing the data from a single realization. Our main result for the inverse scattering problem is summarized as follows.

THEOREM 1.2. *Let ρ be a random potential satisfying Assumption 1.1 and $U \subset \mathbb{R}^d$ be a bounded and convex domain having a positive distance to the support D of the strength μ. Assume in addition that $m > \frac{6}{5}d - 1$ if $\sigma > 0$. For any $x \in U$, the scattered* *field u^s satisfies*

$$\lim_{K\to\infty}\frac{1}{K}\int_K^{2K}\kappa_{\rm r}^{m+14-2d}\mathbb{E}|u^s(x,k)|^2d\kappa_{\rm r}=T_d(x), \tag{1.3}$$

where

$$T_d(x):=\frac{1}{8^4\pi^{4(d-2)}}\int_D\frac{1}{|x-z|^{2(d-1)}}\mu(z)dz$$

and $\mathbb{E}|u^s(x,k)|^2$ is known as the second moment of $u^s(x,k)$. In addition, in the case of a lossless medium where $\sigma=0$, it holds that

$$\lim_{K\to\infty}\frac{1}{2K}\int_{K^2}^{4K^2}k^{\frac{m+13}{2}-d}|u^s(x,k)|^2dk=T_d(x)\quad \mathbb{P}\text{-}a.s. \tag{1.4}$$

Moreover, the strength μ of the random potential ρ can be uniquely determined by $\{T_d(x)\}_{x\in U}$.

Hereafter, we use the notation "$\mathbb{P}$-$a.s.$" to indicate that the formula holds with probability one. The notation $a\lesssim b$ stands for $a\le Cb$, where C is a positive constant and may change from line to line in the proofs.

Note that the additional restrictions of $m>\frac{5}{3}$ for $d=2$ and $m>\frac{14}{5}$ for $d=3$ in the case of a lossless medium (i.e., $\sigma=0$), as stated in our previous works [17, Theorem 1.2] and [18, Theorem 1.2] respectively, can be removed for the biharmonic wave equation. It is important to mention that the range of the order $m\in(d-1,d]$ specified in our current result for the inverse scattering problem with $\sigma=0$ is optimal. This means that it coincides with the range of m required in the unique continuation principle to ensure the well-posedness of the direct scattering problem.

The rest of the paper is organized as follows. Section 2 introduces the fundamental solution to the biharmonic wave equation. Section 3 presents the unique continuation principle for the biharmonic wave equation with rough potentials. Based on the Lippmann–Schwinger integral equation, the well-posedness for the direct scattering problem is addressed in section 4. Section 5 is dedicated to the uniqueness of the inverse scattering problem. The paper is concluded with some general remarks in section 6.

2. Preliminaries.

In this section, we introduce the fundamental solution to the two- and three-dimensional biharmonic wave equation and examine some important properties of the integral operators defined by the fundamental solution.

2.1. The fundamental solution.

Recalling $\kappa^4=k^2+{\rm i}\sigma k$, we have from a straightforward calculation that

$$\kappa_{\rm r}=\Re(\kappa)=\left[\left(\frac{k^4+\sigma^2k^2}{16}\right)^{\frac14}+\left(\frac{\sqrt{k^4+\sigma^2k^2}+k^2}{8}\right)^{\frac12}\right]^{\frac12},$$

$$\kappa_{\rm i}=\Im(\kappa)=\left[\left(\frac{k^4+\sigma^2k^2}{16}\right)^{\frac14}-\left(\frac{\sqrt{k^4+\sigma^2k^2}+k^2}{8}\right)^{\frac12}\right]^{\frac12}.$$

It is clear to note that

$$k^{\frac12}\kappa_{\rm i}=\left[\frac{\sqrt{k^4+\sigma^2k^2}-k^2}{8\left(\frac{k^4+\sigma^2k^2}{16k^4}\right)^{\frac14}+8\left(\frac{\sqrt{k^4+\sigma^2k^2}+k^2}{8k^2}\right)^{\frac12}}\right]^{\frac12},$$

where

$$\lim_{k\to\infty}\left(\sqrt{k^4+\sigma^2k^2}-k^2\right)=\lim_{k\to\infty}\frac{\sigma^2k^2}{\sqrt{k^4+\sigma^2k^2}+k^2}=\frac{\sigma^2}{2}.$$

Hence we get

$$\lim_{k\to\infty}\frac{\kappa_{\rm r}}{k^{\frac12}}=1,\quad \lim_{k\to\infty}k^{\frac12}\kappa_{\rm i}=\frac{\sigma}{4}, \tag{2.1}$$

which implies for sufficiently large k that the following quantities are equivalent:

$$|\kappa|\sim\kappa_{\rm r}\sim k^{\frac12}.$$

Let $\Phi(x,y,k)$ be the fundamental solution to the biharmonic wave equation, i.e., it satisfies

$$\Delta^2\Phi(x,y,k)-\kappa^4\Phi(x,y,k)=-\delta(x-y).$$

It follows from the identity $\Delta^2-\kappa^4=(\Delta+\kappa^2)(\Delta-\kappa^2)$ that Φ is a linear combination of the fundamental solutions to the Helmholtz operator $\Delta+\kappa^2$ and the modified Helmholtz operator $\Delta-\kappa^2$ (cf. [29,30]):

$$\Phi(x,y,k)=-\frac{\rm i}{8\kappa^2}\left(\frac{\kappa}{2\pi|x-y|}\right)^{\frac{d-2}{2}}\left(H^{(1)}_{\frac{d-2}{2}}(\kappa|x-y|)+\frac{2{\rm i}}{\pi}K_{\frac{d-2}{2}}(\kappa|x-y|)\right),$$

where $H^{(1)}_\nu$ and K_ν are the Hankel function of the first kind and the Macdonald function with order $\nu\in\mathbb{R}$, respectively. Noting

$$K_\nu(z)=\frac{\pi}{2}{\rm i}^{\nu+1}H^{(1)}_\nu({\rm i}z),\quad -\pi<\arg z\le\frac{\pi}{2}$$

and

$$H^{(1)}_{\frac12}(z)=\sqrt{\frac{2}{\pi z}}\frac{e^{{\rm i}z}}{\rm i},$$

we have

$$\Phi(x,y,k)=\begin{cases}-\dfrac{\rm i}{8\kappa^2}\left(H^{(1)}_0(\kappa|x-y|)-H^{(1)}_0({\rm i}\kappa|x-y|)\right), & d=2,\\ -\dfrac{1}{8\pi\kappa^2|x-y|}\left(e^{{\rm i}\kappa|x-y|}-e^{-\kappa|x-y|}\right), & d=3.\end{cases} \tag{2.2}$$

The following lemma gives the regularity of Φ and its dependence on the wavenumber k.

LEMMA 2.1. *Let $G\subset\mathbb{R}^d$ be any bounded domain with a strong local Lipschitz boundary. For any fixed $y\in\mathbb{R}^d$, it holds $\Phi(\cdot,y,k)\in W^{\gamma,q}(G)$ for any $\gamma\in[0,1]$ and $q\in(1,\frac{2}{\gamma})$. In particular, for any fixed $y\in D$ and G having a positive distance from D, it holds for sufficiently large k that*

$$\|\Phi(\cdot,y,k)\|_{W^{\gamma,q}(G)}\lesssim k^{\frac{d-7}{4}+\frac{\gamma}{2}}$$

for any $\gamma\in[0,1]$ and $q>1$.

Proof. Let $r^* := \sup_{x\in G}|x-y|$ for any fixed $y\in\mathbb{R}^d$ and $r_0 := \inf_{x\in G}|x-y|>0$ if $y\in D$. We discuss the two- and three-dimensional problems separately.

First we consider the two-dimensional case, where the fundamental solution takes the form $\Phi(x,y,k) = -\frac{\mathrm{i}}{8\kappa^2}(H_0^{(1)}(\kappa|x-y|)+\frac{2\mathrm{i}}{\pi}K_0(\kappa|x-y|))$ for any fixed $y\in\mathbb{R}^2$.

By [6, Lemmas 2.1 and 2.2], it holds for any $z\in\mathbb{C}$ that

$$\left|H_\nu^{(1)}(z)\right| \le e^{-\Im(z)\left(1-\frac{\Theta^2}{|z|^2}\right)^{\frac12}}\left|H_\nu^{(1)}(\Theta)\right|, \tag{2.3}$$

$$|K_\nu(z)| \le \frac{\pi}{2}e^{-\Re(z)\left(1-\frac{\Theta^2}{|z|^2}\right)^{\frac12}}\left|H_\nu^{(1)}(\Theta)\right|, \tag{2.4}$$

where $\nu\in\mathbb{R}$ and Θ is any real number satisfying $0<\Theta\le|z|$. Choosing $z=\kappa|x-y|$ and $\Theta=\Re(z)=\kappa_{\mathrm{r}}|x-y|$, we get

$$\begin{aligned}\int_G|\Phi(x,y,k)|^p\,dx &\lesssim |\kappa|^{-2p}\int_G\left|H_0^{(1)}(\kappa_{\mathrm{r}}|x-y|)\right|^p dx \lesssim \kappa_{\mathrm{r}}^{-2p}\int_0^{r^*}\left|H_0^{(1)}(\kappa_{\mathrm{r}}r)\right|^p r dr\\ &= \kappa_{\mathrm{r}}^{-2p}\int_0^{\kappa_{\mathrm{r}}^{-1}}\left|H_0^{(1)}(\kappa_{\mathrm{r}}r)\right|^p r dr + \kappa_{\mathrm{r}}^{-2p}\int_{\kappa_{\mathrm{r}}^{-1}}^{r^*}\left|H_0^{(1)}(\kappa_{\mathrm{r}}r)\right|^p r dr,\end{aligned}$$

where the second term is bounded due to the regularity of $H_0^{(1)}(\kappa_{\mathrm{r}}r)$ for $r\in(\kappa_{\mathrm{r}}^{-1},r^*)$. For the first term, according to the fact $H_0^{(1)}(\kappa_{\mathrm{r}}r)\sim\frac{2\mathrm{i}}{\pi}\ln(\kappa_{\mathrm{r}}r)$ as $r\to0$ (cf. [2, Section 9.1.8]), it holds

$$\int_0^{\kappa_{\mathrm{r}}^{-1}}\left|H_0^{(1)}(\kappa_{\mathrm{r}}r)\right|^p r dr \lesssim \kappa_{\mathrm{r}}^{-2}\int_0^1|\ln(r)|^p r dr \lesssim \kappa_{\mathrm{r}}^{-2} \quad \forall\, p>1, \epsilon>0.$$

We then get

$$\|\Phi(\cdot,y,k)\|_{L^p(G)}<\infty \quad \forall\, p>1,\epsilon>0.$$

Moreover, noting

$$\begin{aligned}\partial_{x_i}H_0^{(1)}(\kappa|x-y|) &= \kappa H_0^{(1)'}(\kappa|x-y|)\frac{x_i-y_i}{|x-y|} = -\kappa H_1^{(1)}(\kappa|x-y|)\frac{x_i-y_i}{|x-y|},\\ \partial_{x_i}K_0(\kappa|x-y|) &= \frac{\mathrm{i}\pi}{2}\partial_{x_i}H_0^{(1)}(\mathrm{i}\kappa|x-y|) = -\mathrm{i}\kappa K_1(\kappa|x-y|)\frac{x_i-y_i}{|x-y|}\end{aligned}$$

for $i=1,2$ and using $H_1^{(1)}(\kappa_{\mathrm{r}}r)\sim\frac{2\mathrm{i}}{\pi}\frac{1}{\kappa_{\mathrm{r}}r}$ as $r\to0$ (cf. [2, Section 9.1.9]), following the same procedure, we obtain for any $p'\in(1,2)$ that

$$\begin{aligned}\int_G|\partial_{x_i}\Phi(x,y,k)|^{p'}dx &\lesssim |\kappa|^{-p'}\int_G\left|H_1^{(1)}(\kappa_{\mathrm{r}}|x-y|)\right|^{p'}dx \lesssim \kappa_{\mathrm{r}}^{-p'}\int_0^{r^*}\left|H_1^{(1)}(\kappa_{\mathrm{r}}r)\right|^{p'} r dr\\ &\lesssim \kappa_{\mathrm{r}}^{-p'}\int_0^{\kappa_{\mathrm{r}}^{-1}}\frac{1}{(\kappa_{\mathrm{r}}r)^{p'}} r dr + \kappa_{\mathrm{r}}^{-p'}\int_{\kappa_{\mathrm{r}}^{-1}}^{r^*}\left|H_1^{(1)}(\kappa_{\mathrm{r}}r)\right|^{p'} r dr<\infty,\end{aligned}$$

which shows

$$\|\Phi(\cdot,y,k)\|_{W^{1,p'}(G)}<\infty \quad \forall\, p'\in(1,2)$$

and hence $\Phi(\cdot,y,k)\in W^{1,p'}(G)$.

The interpolation $[L^p(G), W^{1,p'}(G)]_\gamma = W^{\gamma,q}(G)$ with $\gamma \in [0,1]$ and q satisfying $\frac{1}{q} = \frac{1-\gamma}{p} + \frac{\gamma}{p'}$ (cf. [3, Theorem 6.4.5]) yields $\Phi(\cdot, y, k) \in W^{\gamma,q}(G)$ for any $\gamma \in [0,1]$ and $q \in (1, \frac{2}{\gamma})$.

In particular, if $y \in D$ and k is sufficiently large, then $r_0 := \inf_{x\in G} |x-y| > 0$ and the Hankel function has the following asymptotic expansion (cf. [2, Section 9.2.3]):

$$H_\nu^{(1)}(\kappa_{\rm r}|x-y|) \sim \left(\frac{2}{\pi\kappa_{\rm r}|x-y|}\right)^{\frac{1}{2}} e^{{\rm i}(\kappa_{\rm r}|x-y|-\frac{1}{2}\nu\pi-\frac{1}{4}\pi)}$$

for $\nu \in \mathbb{R}$. Following from the interpolation between $L^q(G)$ and $W^{1,q}(G)$ provided that G is bounded with a strong local Lipschitz boundary (cf. [1, Section 7.69]), we have

$$\int_G |\Phi(x,y,k)|^q\,dx \lesssim |\kappa|^{-2q} \int_G \left|H_0^{(1)}(\kappa_{\rm r}|x-y|)\right|^q dy \lesssim \kappa_{\rm r}^{-2q} \int_{r_0}^{r^*} \frac{1}{(\kappa_r r)^{\frac{q}{2}}} r dr \lesssim \kappa_{\rm r}^{-\frac{5}{2}q},$$

$$\int_G |\partial_{x_i}\Phi(x,y,k)|^q\,dx \lesssim |\kappa|^{-q} \int_G \left|H_1^{(1)}(\kappa_{\rm r}|x-y|)\right|^q dx \lesssim \kappa_{\rm r}^{-q} \int_{r_0}^{r^*} \frac{1}{(\kappa_{\rm r} r)^{\frac{q}{2}}} r dr \lesssim \kappa_{\rm r}^{-\frac{3}{2}q},$$

which leads to

$$\|\Phi(\cdot,y,k)\|_{W^{\gamma,q}(G)} \lesssim \kappa_{\rm r}^{-\frac{5}{2}+\gamma} \lesssim k^{-\frac{5}{4}+\frac{\gamma}{2}} \quad (2.5)$$

for any $\gamma \in [0,1]$ and $q > 1$.

Next we examine the three-dimensional problem, where

$$\Phi(x,y,k) = -\frac{1}{8\pi\kappa^2|x-y|}\left(e^{{\rm i}\kappa|x-y|} - e^{-\kappa|x-y|}\right).$$

The estimates are similar to the two-dimensional case.

For any $y \in \mathbb{R}^3$, it holds

$$\|\Phi(\cdot,y,k)\|_{L^q(G)} \lesssim |\kappa|^{-2}\left(\int_0^{r^*} \frac{|e^{{\rm i}\kappa r} - e^{-\kappa r}|^q}{r^q} r^2 dr\right)^{\frac{1}{q}} < \infty \quad \forall\ q > 1$$

by utilizing the fact that $|e^{{\rm i}\kappa r} - e^{-\kappa r}| \lesssim \kappa r$ for sufficiently small r. The derivatives of Φ satisfy

$$\begin{aligned}
&\int_G |\partial_{x_i}\Phi(x,y,k)|^q dx \\
&= \int_G \left|\frac{x_i-y_i}{8\pi\kappa^2|x-y|^3}\left[e^{{\rm i}\kappa|x-y|}({\rm i}\kappa|x-y|-1) + e^{-\kappa|x-y|}(\kappa|x-y|+1)\right]\right|^q dx \\
&\lesssim |\kappa|^{-2q} \int_0^{r^*} \frac{|e^{{\rm i}\kappa r}({\rm i}\kappa r-1) + e^{-\kappa r}(\kappa r+1)|^q}{r^{2q}} r^2 dr < \infty \quad \forall\ q > 1,
\end{aligned}$$

which implies $\Phi(\cdot,y,k) \in W^{\gamma,q}(G)$ for any $\gamma \in [0,1]$ and $q > 1$.

In particular, for $y \in D$, a straightforward calculation gives

$$\|\Phi(\cdot,y,k)\|_{L^q(G)} \lesssim |\kappa|^{-2}\left(\int_{r_0}^{r^*} \frac{|e^{{\rm i}\kappa r} - e^{-\kappa r}|^q}{r^q} r^2 dr\right)^{\frac{1}{q}}$$

$$\lesssim |\kappa|^{-2}\left(\int_{r_0}^{r^*} r^{2-q}dr\right)^{\frac{1}{q}} \lesssim |\kappa|^{-2}$$

and

$$\int_G |\partial_{x_i}\Phi(x,y,k)|^q dx = |\kappa|^{-2q}\int_{r_0}^{r^*}\frac{|\kappa r|^q+1}{r^{2q}}r^2dr \lesssim |\kappa|^{-q}.$$

Hence, for sufficiently large k, it holds

$$\|\Phi(\cdot,y,k)\|_{W^{\gamma,q}(G)} \lesssim |\kappa|^{-2+\gamma} \lesssim k^{-1+\frac{\gamma}{2}}$$

for any $\gamma \in [0,1]$ and $q > 1$. ∎

2.2. Integral operators. Define the integral operators

$$\mathcal{H}_k(\phi)(\cdot) := \int_{\mathbb{R}^d}\Phi(\cdot,z,k)\phi(z)dz,$$

$$\mathcal{K}_k(\phi)(\cdot) := \mathcal{H}_k(\rho\phi)(\cdot) = \int_{\mathbb{R}^d}\Phi(\cdot,z,k)\rho(z)\phi(z)dz,$$

where Φ is the fundamental solution given in (2.2) and ρ is the random potential satisfying Assumption 1.1.

LEMMA 2.2. *Let B and G be two bounded domains in $\mathbb{R}^d$, and G has a strong local Lipschitz boundary. Assume that the wave number k is sufficiently large.*

(i) The operator $\mathcal{H}_k : H^{-s_1}(B) \to H^{s_2}(G)$ is bounded and satisfies

$$\|\mathcal{H}_k\|_{\mathcal{L}(H^{-s_1}(B),H^{s_2}(G))} \lesssim k^{\frac{s-(3-\chi_\sigma)}{2}}$$

for $s := s_1 + s_2 \in (0, 3-\chi_\sigma)$ with $s_1, s_2 \geq 0$ and

$$\chi_\sigma := \begin{cases} 0, & \sigma = 0,\\ 1, & \sigma > 0.\end{cases}$$

(ii) The operator $\mathcal{H}_k : H^{-s}(B) \to L^\infty(G)$ is bounded and satisfies

$$\|\mathcal{H}_k\|_{\mathcal{L}(H^{-s}(B),L^\infty(G))} \lesssim k^{\frac{2s+d-2(3-\chi_\sigma)+\epsilon}{4}}$$

for any $s \in (0, 3-\chi_\sigma)$ and $\epsilon > 0$.

(iii) The operator $\mathcal{H}_k : W^{-\gamma,p}(B) \to W^{\gamma,q}(G)$ is compact for any $1 < p < 2 < q$ satisfying $\frac{1}{p}+\frac{1}{q} = 1$ and $0 < \gamma < \min\{\frac{3-\chi_\sigma}{2}, \frac{3-\chi_\sigma}{2} + (\frac{1}{q}-\frac{1}{2})d\}$.

Proof. (i) Since the case $\sigma = 0$ is discussed in [22, Lemma 3.1], we only show the proof for the case $\sigma > 0$ where $\kappa_{\rm i} > 0$. For any two smooth test functions $\phi \in C_0^\infty(B)$ and $\psi \in C_0^\infty(G)$, we consider

$$\begin{aligned}\langle\mathcal{H}_k(\phi),\psi\rangle &= \int_{\mathbb{R}^d}\frac{1}{|\xi|^4-\kappa^4}\hat{\phi}(\xi)\hat{\psi}(\xi)d\xi\\ &= \int_{\mathbb{R}^d}\frac{(1+|\xi|^2)^{\frac{s}{2}}}{(|\xi|^2+\kappa^2)(|\xi|+\kappa)(|\xi|-\kappa)}\widehat{\mathcal{J}^{-s_1}\phi}(\xi)\widehat{\mathcal{J}^{-s_2}\psi}(\xi)d\xi,\end{aligned} \tag{2.6}$$

where $\hat{\phi}$ and $\hat{\psi}$ are the Fourier transform of ϕ and ψ, respectively, and $\mathcal{J}^{-s}$ stands for the Bessel potential of order $-s$ and is defined by (cf. [20])

$$\mathcal{J}^{-s}f := \mathcal{F}^{-1}\big((1+|\cdot|^2)^{-\frac{s}{2}}\hat{f}\big)$$

with $\mathcal{F}^{-1}$ denoting the inverse Fourier transform.

The integral domain $\mathbb{R}^d$ of (2.6) can be split into two parts

$$\Omega_1 := \left\{\xi \in \mathbb{R}^d : ||\xi| - \kappa_{\rm r}| > \frac{\kappa_{\rm r}}{2}\right\}, \quad \Omega_2 := \left\{\xi \in \mathbb{R}^d : ||\xi| - \kappa_{\rm r}| < \frac{\kappa_{\rm r}}{2}\right\}$$

such that (2.6) turns to be

$$\begin{aligned}\langle \mathcal{H}_k(\varphi), \psi\rangle =& \int_{\Omega_1} \frac{(1+|\xi|^2)^{\frac{s}{2}}}{(|\xi|^2+\kappa^2)(|\xi|+\kappa)(|\xi|-\kappa)} \widehat{\mathcal{J}^{-s_1}\phi}(\xi)\widehat{\mathcal{J}^{-s_2}\psi}(\xi)d\xi \\ &+ \int_{\Omega_2} \frac{(1+|\xi|^2)^{\frac{s}{2}}}{(|\xi|^2+\kappa^2)(|\xi|+\kappa)(|\xi|-\kappa)} \widehat{\mathcal{J}^{-s_1}\phi}(\xi)\widehat{\mathcal{J}^{-s_2}\psi}(\xi)d\xi \\ =&: \Lambda_1 + \Lambda_2.\end{aligned}$$

The term Λ_1 can be estimated following a similar procedure as in [22, Lemma 3.1]. In fact, we get for $s < 3$ that

$$\begin{aligned}|\Lambda_1| \leq& \int_{\Omega_1} \frac{(1+|\xi|^2)^{\frac{s}{2}}}{||\xi|^2+\kappa_{\rm r}^2-\kappa_{\rm i}^2|(|\xi|+\kappa_{\rm r})||\xi|-\kappa_{\rm r}|} \left|\widehat{\mathcal{J}^{-s_1}\phi}(\xi)\widehat{\mathcal{J}^{-s_2}\psi}(\xi)\right| d\xi \\ \leq& \frac{2}{\kappa_{\rm r}} \int_{\{|\xi|>\frac{3\kappa_{\rm r}}{2}\}\cup\{|\xi|<\frac{\kappa_{\rm r}}{2}\}} \frac{(1+|\xi|^2)^{\frac{s}{2}}}{||\xi|^2+\kappa_{\rm r}^2-\kappa_{\rm i}^2|(|\xi|+\kappa_{\rm r})} \left|\widehat{\mathcal{J}^{-s_1}\phi}(\xi)\widehat{\mathcal{J}^{-s_2}\psi}(\xi)\right| d\xi \\ \lesssim& \frac{1}{\kappa_{\rm r}^{4-s}} \|\varphi\|_{H^{-s_1}(B)}\|\psi\|_{H^{-s_2}(G)}\end{aligned}$$

using the fact that $\kappa_{\rm i} \ll 1 \ll \kappa_{\rm r}$ for sufficiently large k according to (2.1). For Λ_2, since the term $\frac{1}{|\xi|^4-\kappa^4}$ is not singular for $\kappa_{\rm i} > 0$, one can easily get

$$\begin{aligned}|\Lambda_2| \leq& \int_{\Omega_2} \frac{(1+|\xi|^2)^{\frac{s}{2}}}{||\xi|^2+\kappa_{\rm r}^2-\kappa_{\rm i}^2|(|\xi|+\kappa_{\rm r})\kappa_{\rm i}} \left|\widehat{\mathcal{J}^{-s_1}\phi}(\xi)\widehat{\mathcal{J}^{-s_2}\psi}(\xi)\right| d\xi \\ \lesssim& \frac{1}{\kappa_{\rm r}^{3-s}\kappa_{\rm i}} |\varphi\|_{H^{-s_1}(B)}\|\psi\|_{H^{-s_2}(G)}.\end{aligned}$$

As a result, using (2.1), we get

$$|\langle \mathcal{H}_k(\phi), \psi\rangle| \lesssim \kappa_{\rm r}^{s-3}\kappa_{\rm i}^{-1}\|\phi\|_{H^{-s_1}(B)}\|\psi\|_{H^{-s_2}(G)} \lesssim k^{\frac{s-2}{2}}\|\phi\|_{H^{-s_1}(B)}\|\psi\|_{H^{-s_2}(G)}$$

with $s < 2$, which completes the proof by extending the above result to $\phi \in H^{-s_1}(B)$ and $\psi \in H^{-s_2}(G)$.

(ii) For any $\phi \in C_0^\infty(B)$, we still denote by ϕ its zero extension outside of B. It follows from the Plancherel theorem that

$$\begin{aligned}\mathcal{H}_k(\phi)(x) =& \int_{\mathbb{R}^d} \Phi(x,z,k)\phi(z)dz \\ =& \int_{\mathbb{R}^d} (1+|\xi|^2)^{\frac{s}{2}}\widehat{\Phi}(x,\xi,k)\widehat{\mathcal{J}^{-s}\phi}(\xi)d\xi, \\ =& -\int_{\mathbb{R}^d} \frac{(1+|\xi|^2)^{\frac{2s+d+\epsilon}{4}}}{|\xi|^4-\kappa^4}\widehat{\mathcal{J}^{-s}\phi}(\xi)\big(e^{-{\rm i}x\cdot\xi}(1+|\xi|^2)^{-\frac{d+\epsilon}{4}}\big)d\xi,\end{aligned}$$

where

$$\widehat{\Phi}(x,\xi,k) := \mathcal{F}[\Phi(x,\cdot,k)](\xi) = \frac{-e^{-{\rm i}x\cdot\xi}}{|\xi|^4-\kappa^4}$$

is the Fourier transform of $\Phi(x,y,k)$ with respect to y. Comparing the above integral with (2.6) and replacing $\widehat{\mathcal{J}^{-s_2}\psi}(\xi)$ by $g(\xi) := e^{-\mathrm{i}x\cdot\xi}(1+|\xi|^2)^{-\frac{d+\epsilon}{4}}$, we obtain

$$|\mathcal{H}_k(\phi)(x)| \lesssim k^{\frac{\frac{2s+d+\epsilon}{2}-(3-\chi_\sigma)}{2}}\|\phi\|_{H^{-s}(B)} \lesssim k^{\frac{2s+d-2(3-\chi_\sigma)+\epsilon}{4}}\|\phi\|_{H^{-s}(B)},$$

which can also be extended to $\phi \in H^{-s}(B)$. We mention that $g \in H^1(\mathbb{R}^d)$ is utilized in the above estimate, which is required in the estimate of (2.6) (see e.g., [17, 20]).

(iii) The compactness of $\mathcal{H}_k$ can be obtained from the boundedness shown in (i) and the Sobolev embedding theorem. In fact, according to the Kondrachov embedding theorem, the embeddings

$$W^{-\gamma,p}(B) \hookrightarrow H^{-s_1}(B),$$
$$H^{s_2}(G) \hookrightarrow W^{\gamma,q}(G)$$

are continuous under conditions $1 < p < 2 < q$,

$$\gamma < s_1, \quad \frac{1}{2} > \frac{1}{p} - \frac{s_1-\gamma}{d},$$
$$\gamma < s_2, \quad \frac{1}{q} > \frac{1}{2} - \frac{s_2-\gamma}{d},$$

and $s_1+s_2 \in (0, 3-\chi_\sigma)$. It is easy to check that the above conditions are satisfied if $\frac{1}{p}+\frac{1}{q}=1$ and

$$0 < \gamma < \min\left\{\frac{s_1+s_2}{2}, \frac{s_1+s_2}{2} + d\left(\frac{1}{q}-\frac{1}{2}\right)\right\},$$

which completes the proof of (iii) due to $s_1+s_2 < 3-\chi_\sigma$. □

The estimates for the operator $\mathcal{K}_k$ can be obtained from the estimates of $\mathcal{H}_k$ given in Lemma 2.2 and the relation $\mathcal{K}_k(\phi) = \mathcal{H}_k(\rho\phi)$.

LEMMA 2.3. *Let $G \subset \mathbb{R}^d$ be a bounded domain with a strong local Lipschitz boundary and the random potential ρ satisfy Assumption 1.1. Assume that the wave number k is sufficiently large.*

(i) The operator $\mathcal{K}_k : W^{\gamma,q}(G) \to W^{\gamma,q}(G)$ is compact for any $q \in (2, A)$ and $\gamma \in (\frac{d-m}{2}, \frac{3-\chi_\sigma}{2} + (\frac{1}{q}-\frac{1}{2})d)$ with

$$A := \begin{cases} \infty & \text{if} \quad 2d-m-(3-\chi_\sigma) \le 0, \\ \dfrac{2d}{2d-m-(3-\chi_\sigma)} & \text{if} \quad 2d-m-(3-\chi_\sigma) > 0, \end{cases}$$

and satisfies

$$\|\mathcal{K}_k\|_{\mathcal{L}(W^{\gamma,q}(G))} \lesssim k^{\gamma+(\frac{1}{2}-\frac{1}{q})d-\frac{3-\chi_\sigma}{2}} \quad \mathbb{P}\text{-a.s.}$$

(ii) The following estimates hold:

$$\|\mathcal{K}_k\|_{\mathcal{L}(H^s(G))} \lesssim k^{s-\frac{3-\chi_\sigma}{2}} \quad \mathbb{P}\text{-a.s.}$$

for any $s \in (\frac{d-m}{2}, \frac{3-\chi_\sigma}{2})$ and

$$\|\mathcal{K}_k\|_{\mathcal{L}(H^s(G), L^\infty(G))} \lesssim k^{\frac{2s+d-2(3-\chi_\sigma)+\epsilon}{4}} \quad \mathbb{P}\text{-a.s.}$$

for any $s \in (\frac{d-m}{2}, 3-\chi_\sigma)$ and $\epsilon > 0$.

Proof. (i) Under Assumption 1.1, it holds that $\rho \in W^{\frac{m-d}{2}-\epsilon,p'}(D)$ for any $\epsilon > 0$ and $p' > 1$ based on [22, Lemma 2.2]. Then for any $m \in (d-1, d]$, $q \in (2, A) \neq \emptyset$ and $\gamma \in (\frac{d-m}{2}, \frac{3-\chi_\sigma}{2} + (\frac{1}{q} - \frac{1}{2})d) \neq \emptyset$, there exists some $p' > 1$ such that the embedding

$$W^{\frac{m-d}{2}-\epsilon,p'}(D) \hookrightarrow W^{-\gamma,\tilde{p}}(D)$$

is continuous with $\tilde{p} := \frac{q}{q-2} > 1$. Moreover, for any $\phi \in W^{\gamma,q}(G)$, we have from [16, Lemma 2] that $\rho\phi \in W^{-\gamma,p}(D)$ with $\frac{1}{p} + \frac{1}{q} = 1$ and

$$\|\rho\phi\|_{W^{-\gamma,p}(D)} \lesssim \|\rho\|_{W^{-\gamma,\tilde{p}}(D)} \|\phi\|_{W^{\gamma,q}(G)}. \tag{2.7}$$

Hence

$$\|\mathcal{K}_k(\phi)\|_{W^{\gamma,q}(G)} \lesssim \|\mathcal{H}_k\|_{\mathcal{L}(W^{-\gamma,p}(D), W^{\gamma,q}(G))} \|\rho\phi\|_{W^{-\gamma,p}(D)} \quad \mathbb{P}\text{-}a.s.,$$

which implies the compactness of $\mathcal{K}_k$ due to the compactness of $\mathcal{H}_k$ proved in Lemma 2.2.

To estimate the operator norm, we choose $s = \gamma + (\frac{1}{2} - \frac{1}{q})d$ such that the embeddings

$$\begin{aligned} H^s(G) &\hookrightarrow W^{\gamma,q}(G), \\ W^{-\gamma,p}(D) &\hookrightarrow H^{-s}(D) \end{aligned} \tag{2.8}$$

hold with $p < 2$ and $q > 2$ satisfying $\frac{1}{p} + \frac{1}{q} = 1$. The result is obtained by noting

$$\begin{aligned} \|\mathcal{K}_k(\phi)\|_{W^{\gamma,q}(G)} \lesssim \|\mathcal{K}_k(\phi)\|_{H^s(G)} &\lesssim \|\mathcal{H}_k\|_{\mathcal{L}(H^{-s}(D), H^s(G))} \|\rho\phi\|_{H^{-s}(D)} \\ &\lesssim \|\mathcal{H}_k\|_{\mathcal{L}(H^{-s}(D), H^s(G))} \|\rho\phi\|_{W^{-\gamma,p}(D)} \\ &\lesssim k^{\gamma + (\frac{1}{2} - \frac{1}{q})d - \frac{3-\chi_\sigma}{2}} \|\phi\|_{W^{\gamma,q}(G)}. \end{aligned}$$

(ii) For any $\phi \in H^s(G)$ with $s > \frac{d-m}{2}$, there exist $\gamma \in (\frac{d-m}{2}, s)$ and $q \in (2, A)$ satisfying $\frac{1}{q} > \frac{1}{2} - \frac{s-\gamma}{d}$ such that the embeddings (2.8) hold. It follows from Lemma 2.2 and (2.7) that we have

$$\begin{aligned} \|\mathcal{K}_k(\phi)\|_{H^s(G)} &\lesssim \|\mathcal{H}_k\|_{\mathcal{L}(H^{-s}(D), H^s(G))} \|\rho\phi\|_{H^{-s}(D)} \\ &\lesssim \|\mathcal{H}_k\|_{\mathcal{L}(H^{-s}(D), H^s(G))} \|\rho\phi\|_{W^{-\gamma,p}(D)} \\ &\lesssim k^{\frac{2s-(3-\chi_\sigma)}{2}} \|\rho\|_{W^{-\gamma,\tilde{p}}(D)} \|\phi\|_{W^{\gamma,q}(G)} \lesssim k^{s - \frac{3-\chi_\sigma}{2}} \|\phi\|_{H^s(G)} \quad \mathbb{P}\text{-}a.s. \end{aligned} \tag{2.9}$$

with $s \in (\frac{d-m}{2}, \frac{3-\chi_\sigma}{2})$, and

$$\|\mathcal{K}_k(\phi)\|_{L^\infty(G)} \lesssim \|\mathcal{H}_k\|_{\mathcal{L}(H^{-s}(D), L^\infty(G))} \|\rho\phi\|_{H^{-s}(D)} \lesssim k^{\frac{2s+d-2(3-\chi_\sigma)+\epsilon}{4}} \|\phi\|_{H^s(G)} \quad \mathbb{P}\text{-}a.s.$$

with $s \in (\frac{d-m}{2}, 3 - \chi_\sigma)$ and $\epsilon > 0$. ∎

3. The unique continuation.

This section is to investigate the unique continuation principle, which is essential for the uniqueness of the solution to the biharmonic wave scattering problem with a random potential. We refer to [16, 20] for the unique continuation of the solutions to the stochastic acoustic and elastic wave equations.

THEOREM 3.1. *Let ρ satisfy Assumption 1.1, $q \in (2, \frac{2d}{3d-2m-2})$ and $\gamma \in (\frac{d-m}{2}, \frac{1}{2} + (\frac{1}{q} - \frac{1}{2})\frac{d}{2})$. If $u \in W^{\gamma,q}(\mathbb{R}^d)$ is compactly supported in $\mathbb{R}^d$ and is a distributional solution to the homogeneous biharmonic wave equation*

$$\Delta^2 u - \kappa^4 u + \rho u = 0,$$

then $u \equiv 0$ in $\mathbb{R}^d$.

Proof. We consider an auxiliary function $v(x) := e^{-\mathrm{i}\eta\cdot x}u(x)$, where the complex vector η is defined by

$$\eta := \begin{cases} (\omega t, \eta_d)^\top, & d = 2, \\ (\omega t, 0, \eta_d)^\top, & d = 3, \end{cases}$$

where $t \gg 1$,

$$\omega := \left(\frac{\sqrt{k^4 + \sigma^2 k^2} + k^2}{2}\right)^{\frac{1}{4}},$$

and $\eta_d = \eta_d^{\mathrm{r}} + \mathrm{i}\eta_d^{\mathrm{i}}$ with the real and imaginary parts being given by

$$\eta_d^{\mathrm{r}} = \left(\frac{\sqrt{\omega^4(t^2-1)^2 + \omega^4 - k^2} - \omega^2(t^2-1)}{2}\right)^{\frac{1}{2}},$$

$$\eta_d^{\mathrm{i}} = \left(\frac{\sqrt{\omega^4(t^2-1)^2 + \omega^4 - k^2} + \omega^2(t^2-1)}{2}\right)^{\frac{1}{2}},$$

respectively. It is clear to note $\eta\cdot\eta = \kappa^2 = \omega^2 + \mathrm{i}(\omega^4 - k^2)^{\frac{1}{2}}$. Moreover, a simple calculation shows that

$$\lim_{t\to\infty} \eta_d^{\mathrm{r}} = 0, \quad \lim_{t\to\infty} \frac{\eta_d^{\mathrm{i}}}{t} = \omega. \tag{3.1}$$

Then v is also compactly supported in $\mathbb{R}^d$ and satisfies

$$\Delta^2 v + 4\mathrm{i}\eta\cdot\nabla\Delta v - 4\eta^\top(\nabla^2 v)\eta - 2(\eta\cdot\eta)\Delta v - 4\mathrm{i}(\eta\cdot\eta)(\eta\cdot\nabla v) = -\rho v.$$

Taking the Fourier transform of the above equation yields

$$v = -\mathcal{G}_\eta(\rho v), \tag{3.2}$$

where $\mathcal{G}_\eta$ is defined by

$$\mathcal{G}_\eta(f)(x) := \mathcal{F}^{-1}\left[\frac{\hat{f}(\xi)}{|\xi|^4 + 4|\xi|^2(\eta\cdot\xi) + 4(\eta\cdot\xi)^2 + 2(\eta\cdot\eta)|\xi|^2 + 4(\eta\cdot\eta)(\eta\cdot\xi)}\right](x).$$

Using the Plancherel theorem, we have from a straightforward calculation that

$$\begin{aligned} \langle \mathcal{G}_\eta f, g\rangle = \langle \widehat{\mathcal{G}_\eta f}, \hat{g}\rangle &= \int_{\mathbb{R}^d} \frac{\hat{f}(\xi)\overline{\hat{g}(\xi)}}{|\xi|^4 + 4|\xi|^2(\eta\cdot\xi) + 4(\eta\cdot\xi)^2 + 2\kappa^2|\xi|^2 + 4\kappa^2(\eta\cdot\xi)} d\xi \\ &= \int_{\mathbb{R}^d} \frac{\hat{f}(\xi)\overline{\hat{g}(\xi)}}{(|\xi|^2 + 2\eta\cdot\xi + 2\kappa^2)(|\xi|^2 + 2\eta\cdot\xi)} d\xi \\ &= \frac{1}{2\kappa^2}\left[\int_{\mathbb{R}^d} \frac{\hat{f}(\xi)\overline{\hat{g}(\xi)}}{|\xi|^2 + 2\eta\cdot\xi} d\xi - \int_{\mathbb{R}^d} \frac{\hat{f}(\xi)\overline{\hat{g}(\xi)}}{|\xi|^2 + 2\eta\cdot\xi + 2\kappa^2} d\xi\right]. \end{aligned} \tag{3.3}$$

It suffices to show $v \equiv 0$ in order to show $u \equiv 0$. The proof consists of two steps.

The first step is to estimate the operator $\mathcal{G}_\eta$ in Hilbert spaces. Let $G \subset \mathbb{R}^d$ be a bounded domain with a strong local Lipschitz boundary containing the compact supports of both ρ and u. For $s \in (0, \frac{1}{2})$, we have the following estimate:

$$\|\mathcal{G}_\eta\|_{\mathcal{L}(H^{-s}(G),H^s(G))} \lesssim \frac{1}{\omega^{3-2s}t^{1-2s}}. \tag{3.4}$$

The proof of this inequality is postponed to the subsequent lemma for the sake of brevity.

The second step is to estimate the operator $\mathcal{G}_\eta$ in Sobolev spaces and show $v \equiv 0$ in $\mathbb{R}^d$. To extend the estimate of $\mathcal{G}_\eta$ from Hilbert spaces to Sobolev spaces, we claim that $\mathcal{G}_\eta : L^r(G) \to L^{r'}(G)$ is bounded and satisfies

$$\|\mathcal{G}_\eta\|_{\mathcal{L}(L^r(G),L^{r'}(G))} \lesssim 1 \tag{3.5}$$

for some proper r and r'. In fact, it follows from the decomposition of the operator $\mathcal{G}_\eta$ given in (3.3) that we may rewrite it as

$$\mathcal{G}_\eta = \frac{1}{2\kappa^2}\left(\mathcal{G}_{\eta,1} - \mathcal{G}_{\eta,2}\right),$$

where

$$\mathcal{G}_{\eta,1}(f)(x) := \mathcal{F}^{-1}\left[\frac{\hat{f}}{|\xi|^2 + 2\eta\cdot\xi}\right](x), \quad \mathcal{G}_{\eta,2}(f)(x) := \mathcal{F}^{-1}\left[\frac{\hat{f}}{|\xi|^2 + 2\eta\cdot\xi + 2\kappa^2}\right](x).$$

Next we consider the cases $d = 3$ and $d = 2$, separately.

For $d = 3$, the claim (3.5) holds under the conditions

$$\frac{1}{r} - \frac{1}{r'} = \frac{2}{d}, \quad \min\left\{\left|\frac{1}{r} - \frac{1}{2}\right|, \left|\frac{1}{r'} - \frac{1}{2}\right|\right\} > \frac{1}{2d},$$

since operators $\mathcal{G}_{\eta,i}$, $i = 1, 2$, are both bounded from $L^r(G)$ to $L^{r'}(G)$ according to [13, Theorem 2.2] and [16, Proposition 2]. To deduce the estimate for $\mathcal{G}_\eta$ between the dual Sobolev spaces $W^{-\gamma,p}(G)$ and $W^{\gamma,q}(G)$ with $\frac{1}{p} + \frac{1}{q} = 1$, we consider the interpolation of (3.4) and (3.5). Noting

$$[L^r(G), H^{-s}(G)]_\theta = W^{-\gamma,p}(G),$$
$$[L^{r'}(G), H^s(G)]_\theta = W^{\gamma,q}(G)$$

and choosing $\theta = 1 + (\frac{1}{q} - \frac{1}{2})d \in (0, 1)$ and $r = \frac{2d}{d+2}$ such that $\gamma = \theta s < \frac{1}{2} + (\frac{1}{q} - \frac{1}{2})\frac{d}{2}$, $\frac{1}{p} = \frac{1-\theta}{r} + \frac{\theta}{2}$ and $\frac{1}{q} = \frac{1-\theta}{r'} + \frac{\theta}{2}$, we obtain

$$\|\mathcal{G}_\eta\|_{\mathcal{L}(W^{-\gamma,p}(G),W^{\gamma,q}(G))} \lesssim \frac{1}{\omega^{(3-2s)\theta}t^{(1-2s)\theta}}. \tag{3.6}$$

As is proved in [16, Lemma 2], $\rho v \in W^{-\gamma,p}(G)$ for any $v \in W^{\gamma,q}(G)$, where γ is required to satisfy $\gamma < \frac{1}{2} + (\frac{1}{q} - \frac{1}{2})\frac{d}{2}$. Hence an additional restriction on q is also required due to $\gamma > \frac{d-m}{2}$, i.e., $q < \frac{2d}{3d-2m-2}$. Consequently, (3.2) leads to

$$\|v\|_{W^{\gamma,q}(G)} \le \|\mathcal{G}_\eta\|_{\mathcal{L}(W^{-\gamma,p}(G),W^{\gamma,q}(G))}\|\rho v\|_{W^{-\gamma,p}(G)} \lesssim \frac{1}{\omega^{(3-2s)\theta}t^{(1-2s)\theta}}\|v\|_{W^{\gamma,q}(G)}$$

with $s \in (0, \frac{1}{2})$, which implies $v \equiv 0$ by choosing $t \gg 1$.

For $d = 2$, it is shown in [16, Proposition 2] that (3.5) holds for any $r > 1$. Similarly, (3.6) can be deduced from the interpolation between (3.4) and (3.5) by choosing $r = 1 + \epsilon$ with an arbitrary small parameter $\epsilon > 0$ and $\theta = \frac{2(1+\epsilon)-2\epsilon q}{q(1-\epsilon)}$ such that $\gamma = \theta s < \frac{(1+\epsilon)-\epsilon q}{q(1-\epsilon)}$. Following the same procedure as the three-dimensional case and letting $\epsilon \to 0$, we get $v \equiv 0$ under the restrictions $\gamma < \frac{1}{q} = \frac{1}{2} + (\frac{1}{q} - \frac{1}{2})\frac{d}{2}$ and $q < \frac{2}{2-m} = \frac{2d}{3d-2m-2}$. □

LEMMA 3.2. *Let the assumptions given in Theorem 3.1 hold and $G \subset \mathbb{R}^d$ be a bounded domain with a strong local Lipschitz boundary containing the compact supports of both ρ and u. Then for $s \in (0, \frac{1}{2})$, the operator $\mathcal{G}_\eta$ defined in Theorem 3.1 satisfies*

$$\|\mathcal{G}_\eta\|_{\mathcal{L}(H^{-s}(G),H^s(G))} \lesssim \frac{1}{\omega^{3-2s}t^{1-2s}}.$$

Proof. We denote (3.3) by

$$\langle \mathcal{G}_\eta f, g\rangle =: \frac{1}{2\kappa^2}\big[\mathcal{A} - \mathcal{B}\big].$$

For any $f, g \in C_0^\infty(G)$, we denote their zero extensions outside of G still by f, g for simplicity. Denote $\xi^- := (\xi_1, \cdots, \xi_{d-1})^\top \in \mathbb{R}^{d-1}$ and $\xi^{--} := (\xi_2, \cdots, \xi_{d-1})^\top \in \mathbb{R}^{d-2}$ with $\xi^{--} = 0$ if $d = 2$. Then $\mathcal{A}$ can be rewritten as

$$\begin{aligned}
\mathcal{A} &= \int_{\mathbb{R}^d} \frac{\hat{f}(\xi)\overline{\hat{g}(\xi)}}{|\xi|^2 + 2\omega t\xi_1 + 2\eta_d\xi_d} d\xi \\
&= \int_{\mathbb{R}^d} \frac{\hat{f}(\xi)\overline{\hat{g}(\xi)}}{(\xi_1+\omega t)^2 + |\xi^{--}|^2 - \omega^2t^2 + (\xi_d + \eta_d^{\rm r})^2 - (\eta_d^{\rm r})^2 + 2{\rm i}\eta_d^{\rm i}\xi_d} d\xi \\
&= \int_{\mathbb{R}^d} \frac{\hat{f}(\xi)\overline{\hat{g}(\xi)}}{|\xi|^2 - \omega^2t^2 - (\eta_d^{\rm r})^2 + 2{\rm i}\eta_d^{\rm i}(\xi_d - \eta_d^{\rm r})} d\xi,
\end{aligned}$$

where in the last step we used the transformation of variables $(\xi_1 + \omega t, \xi_2, \cdots, \xi_d + \eta_d^{\rm r})^\top \mapsto (\xi_1, \cdots, \xi_d)^\top$ and $\hat{f}(\xi_1, \cdots, \xi_j - a, \cdots, \xi_d) = e^{-{\rm i}a\xi_j}\hat{f}(\xi)$. Using $\kappa^2 = \eta \cdot \eta = \omega^2t^2 + \eta_d^2$ and the transformation $(\xi_1 + \omega t, \xi_2, \cdots, \xi_d + \eta_d^{\rm r})^\top \mapsto (\xi_1, \cdots, \xi_d)^\top$, we have

$$\begin{aligned}
\mathcal{B} &= \int_{\mathbb{R}^d} \frac{\hat{f}(\xi)\overline{\hat{g}(\xi)}}{(\xi_1+\omega t)^2 + |\xi^{--}|^2 + (\xi_d+\eta_d^{\rm r})^2 + \omega^2t^2 + (\eta_d^{\rm r})^2 - 2(\eta_d^{\rm i})^2 + 2{\rm i}\eta_d^{\rm i}(\xi_d + 2\eta_d^{\rm r})} d\xi \\
&= \int_{\mathbb{R}^d} \frac{\hat{f}(\xi)\overline{\hat{g}(\xi)}}{|\xi|^2 + \omega^2t^2 + (\eta_d^{\rm r})^2 - 2(\eta_d^{\rm i})^2 + 2{\rm i}\eta_d^{\rm i}(\xi_d + \eta_d^{\rm r})} d\xi.
\end{aligned}$$

It is easy to see that the function

$$\begin{aligned}
&\frac{1}{|\xi|^2 - \omega^2t^2 - (\eta_d^{\rm r})^2 + 2{\rm i}\eta_d^{\rm i}(\xi_d - \eta_d^{\rm r})} \\
&= \frac{1}{|\xi^-|^2 - \omega^2t^2 + (\xi_d - \eta_d^{\rm r})(\xi_d + \eta_d^{\rm r}) + 2{\rm i}\eta_d^{\rm i}(\xi_d - \eta_d^{\rm r})}
\end{aligned}$$

involved in $\mathcal{A}$ is singular on the manifold $\{\xi \in \mathbb{R}^d : |\xi^-| = \omega t,\ \xi_d = \eta_d^{\rm r}\}$, and the function

$$\frac{1}{|\xi|^2 + \omega^2t^2 + (\eta_d^{\rm r})^2 - 2(\eta_d^{\rm i})^2 + 2{\rm i}\eta_d^{\rm i}(\xi_d + \eta_d^{\rm r})}$$

$$= \frac{1}{|\xi^-|^2 + \omega^2 t^2 + 2(\eta_d^{\rm r})^2 - 2(\eta_d^{\rm i})^2 + (\xi_d - \eta_d^{\rm r})(\xi_d + \eta_d^{\rm r}) + 2{\rm i}\eta_d^{\rm i}(\xi_d + \eta_d^{\rm r})}$$

involved in $\mathcal{B}$ is singular on the manifold

$$\left\{\xi \in \mathbb{R}^d : |\xi^-| = \sqrt{2(\eta_d^{\rm i})^2 - 2(\eta_d^{\rm r})^2 - \omega^2 t^2},\ \xi_d = -\eta_d^{\rm r}\right\},$$

where $2(\eta_d^{\rm i})^2 - 2(\eta_d^{\rm r})^2 - \omega^2 t^2$ is equivalent to ωt as $t \gg 1$ according to (3.1).

The estimates for $\mathcal{A}$ and $\mathcal{B}$ follow a similar procedure, requiring the decomposition of the integral domain $\mathbb{R}^d$ into several subdomains based on the singularity of the integrands. In the following, we present a detailed analysis of the estimate for $\mathcal{A}$. The analysis of $\mathcal{B}$ can be carried out in a similar manner and is omitted here for brevity.

To estimate $\mathcal{A}$, we define two domains

$$\Omega_1 := \left\{\xi : ||\xi^-| - \omega t| > \frac{\omega t}{2}\right\} = \left\{\xi : |\xi^-| > \frac{3\omega t}{2}\right\} \cup \left\{\xi : |\xi^-| < \frac{\omega t}{2}\right\},$$

$$\Omega_2 := \left\{\xi : ||\xi^-| - \omega t| < \frac{\omega t}{2}\right\} = \left\{\xi : \frac{\omega t}{2} < |\xi^-| < \frac{3\omega t}{2}\right\}.$$

Based on Ω_1 and Ω_2, $\mathcal{A}$ can be split into the following two terms:

$$\begin{aligned}\mathcal{A} &= \int_{\Omega_1} \frac{(1+|\xi|^2)^s}{|\xi|^2 - \omega^2 t^2 - (\eta_d^{\rm r})^2 + 2{\rm i}\eta_d^{\rm i}(\xi_d - \eta_d^{\rm r})} \widehat{\mathcal{J}^{-s} f}(\xi)\overline{\widehat{\mathcal{J}^{-s} g}(\xi)} d\xi \\ &\quad + \int_{\Omega_2} \frac{(1+|\xi|^2)^s}{|\xi|^2 - \omega^2 t^2 - (\eta_d^{\rm r})^2 + 2{\rm i}\eta_d^{\rm i}(\xi_d - \eta_d^{\rm r})} \widehat{\mathcal{J}^{-s} f}(\xi)\overline{\widehat{\mathcal{J}^{-s} g}(\xi)} d\xi \\ &=: {\rm I} + {\rm II},\end{aligned}$$

where $s \in (0, \frac{1}{2})$. Next is to estimate I and II, respectively.

Term I satisfies

$$\begin{aligned}|{\rm I}| &\le \int_{\Omega_1} \frac{(1+|\xi|^2)^s}{\left[(|\xi|^2 - \omega^2 t^2 - (\eta_d^{\rm r})^2)^2 + 4(\eta_d^{\rm i})^2(\xi_d - \eta_d^{\rm r})^2\right]^{\frac{1}{2}}} |\widehat{\mathcal{J}^{-s} f}||\widehat{\mathcal{J}^{-s} g}| d\xi \\ &= \int_{\{\xi:|\xi^-|>\frac{3\omega t}{2}\}} \frac{(1+|\xi|^2)^s |\widehat{\mathcal{J}^{-s} f}||\widehat{\mathcal{J}^{-s} g}|}{\left[(|\xi|^2 - \omega^2 t^2 - (\eta_d^{\rm r})^2)^2 + 4(\eta_d^{\rm i})^2(\xi_d - \eta_d^{\rm r})^2\right]^{\frac{1}{2}}} d\xi \\ &\quad + \int_{\{\xi:|\xi^-|<\frac{\omega t}{2}, |\xi_d - \eta_d^{\rm r}|>\frac{\omega t}{2}\}} \frac{(1+|\xi|^2)^s |\widehat{\mathcal{J}^{-s} f}||\widehat{\mathcal{J}^{-s} g}|}{\left[(|\xi|^2 - \omega^2 t^2 - (\eta_d^{\rm r})^2)^2 + 4(\eta_d^{\rm i})^2(\xi_d - \eta_d^{\rm r})^2\right]^{\frac{1}{2}}} d\xi \\ &\quad + \int_{\{\xi:|\xi^-|<\frac{\omega t}{2}, |\xi_d - \eta_d^{\rm r}|<\frac{\omega t}{2}\}} \frac{(1+|\xi|^2)^s |\widehat{\mathcal{J}^{-s} f}||\widehat{\mathcal{J}^{-s} g}|}{\left[(|\xi|^2 - \omega^2 t^2 - (\eta_d^{\rm r})^2)^2 + 4(\eta_d^{\rm i})^2(\xi_d - \eta_d^{\rm r})^2\right]^{\frac{1}{2}}} d\xi \\ &=: {\rm I}_1 + {\rm I}_2 + {\rm I}_3.\end{aligned}$$

By (3.1), we may choose a sufficiently large t^* such that $\eta_d^{\rm r} < \frac{\omega t}{4}$ for all $t > t^*$, which leads to

$$\frac{3\omega t}{2} - \sqrt{\omega^2 t^2 + (\eta_d^{\rm r})^2} > \frac{\omega t}{4}, \quad t > t^*.$$

We then get

$${\rm I}_1 \le \int_{\{\xi:|\xi|>\frac{3\omega t}{2}\}} \frac{(1+|\xi|^2)^s}{(|\xi| - \sqrt{\omega^2 t^2 + (\eta_d^{\rm r})^2})(|\xi| + \sqrt{\omega^2 t^2 + (\eta_d^{\rm r})^2})} |\widehat{\mathcal{J}^{-s} f}||\widehat{\mathcal{J}^{-s} g}| d\xi$$

$$\lesssim \frac{1}{\omega t}\int_{\{\xi:|\xi|>\frac{3\omega t}{2}\}} \frac{1}{|\xi|^{1-2s}}|\widehat{\mathcal{J}^{-s}f}||\widehat{\mathcal{J}^{-s}g}|d\xi$$
$$\lesssim \frac{1}{(\omega t)^{2-2s}}\|f\|_{H^{-s}(G)}\|g\|_{H^{-s}(G)}.$$

Note also that $\eta_d^{\rm i}$ is equivalent to ωt as $t\to\infty$, which yields

$$\mathrm{I}_2 \le \int_{\{\xi:|\xi^-|<\frac{\omega t}{2},|\xi_d-\eta_d^{\rm r}|>\frac{\omega t}{2}\}} \frac{(1+\frac{\omega^2t^2}{4}+\xi_d^2)^s}{2\eta_d^{\rm i}|\xi_d-\eta_d^{\rm r}|}|\widehat{\mathcal{J}^{-s}f}||\widehat{\mathcal{J}^{-s}g}|d\xi$$
$$\lesssim \int_{\{\xi:|\xi^-|<\frac{\omega t}{2},|\xi_d-\eta_d^{\rm r}|>\frac{\omega t}{2}\}} \frac{(\omega t)^{2s}+|\xi_d-\eta_d^{\rm r}|^{2s}+(\eta_d^{\rm r})^{2s}}{2\eta_d^{\rm i}|\xi_d-\eta_d^{\rm r}|}|\widehat{\mathcal{J}^{-s}f}||\widehat{\mathcal{J}^{-s}g}|d\xi$$
$$\lesssim \int_{\{\xi:|\xi^-|<\frac{\omega t}{2},|\xi_d-\eta_d^{\rm r}|>\frac{\omega t}{2}\}} \left(\frac{1}{(\omega t)^{2-2s}}+\frac{1}{\omega t|\xi_d-\eta_d^{\rm r}|^{1-2s}}\right)|\widehat{\mathcal{J}^{-s}f}||\widehat{\mathcal{J}^{-s}g}|d\xi$$
$$\lesssim \frac{1}{(\omega t)^{2-2s}}\|f\|_{H^{-s}(G)}\|g\|_{H^{-s}(G)}.$$

Moreover, for any $\xi\in\{\xi:|\xi^-|<\frac{\omega t}{2},|\xi_d-\eta_d^{\rm r}|<\frac{\omega t}{2}\}$, it holds

$$|\xi|^2=|\xi^-|^2+|\xi_d|^2<\left(\frac{\omega t}{2}\right)^2+\left(\frac{\omega t}{2}+\eta_d^{\rm r}\right)^2=\frac{\omega^2t^2}{2}+\omega t\eta_d^{\rm r}+(\eta_d^{\rm r})^2.$$

Hence, for $t>t^*$,

$$\omega^2t^2+(\eta_d^{\rm r})^2-|\xi|^2>\frac{\omega^2t^2}{2}-\omega t\eta_d^{\rm r}>\frac{\omega^2t^2}{4},$$

which gives

$$\mathrm{I}_3 \le \int_{\{\xi:|\xi^-|<\frac{\omega t}{2},|\xi_d-\eta_d^{\rm r}|<\frac{\omega t}{2}\}} \frac{(1+|\xi|^2)^s}{||\xi|^2-\omega^2t^2-(\eta_d^{\rm r})^2|}|\widehat{\mathcal{J}^{-s}f}||\widehat{\mathcal{J}^{-s}g}|d\xi$$
$$\lesssim \frac{1}{(\omega t)^{2-2s}}\|f\|_{H^{-s}(G)}\|g\|_{H^{-s}(G)}.$$

We then conclude

$$|\mathrm{I}|\lesssim \frac{1}{(\omega t)^{2-2s}}\|f\|_{H^{-s}(G)}\|g\|_{H^{-s}(G)}. \tag{3.7}$$

To estimate II, we divide it into two parts

$$\mathrm{II}=\int_{\Omega_2\cap\{\xi:|\xi_d-\eta_d^{\rm r}|>\frac{\omega t}{2}\}} \frac{(1+|\xi|^2)^s}{|\xi|^2-\omega^2t^2-(\eta_d^{\rm r})^2+2{\rm i}\eta_d^{\rm i}(\xi_d-\eta_d^{\rm r})}\widehat{\mathcal{J}^{-s}f}(\xi)\overline{\widehat{\mathcal{J}^{-s}g}(\xi)}d\xi$$
$$+\int_{\Omega_2\cap\{\xi:|\xi_d-\eta_d^{\rm r}|<\frac{\omega t}{2}\}} \frac{(1+|\xi|^2)^s}{|\xi|^2-\omega^2t^2-(\eta_d^{\rm r})^2+2{\rm i}\eta_d^{\rm i}(\xi_d-\eta_d^{\rm r})}\widehat{\mathcal{J}^{-s}f}(\xi)\overline{\widehat{\mathcal{J}^{-s}g}(\xi)}d\xi$$
$$=:\mathrm{II}_1+\mathrm{II}_2,$$

where II_1 can be estimated similarly as I_2 by utilizing the boundedness of $|\xi^-|$:

$$|\mathrm{II}_1|\lesssim \frac{1}{(\omega t)^{2-2s}}\|f\|_{H^{-s}(G)}\|g\|_{H^{-s}(G)}.$$

It suffices to estimate II_2 where the integrand is singular. To deal with the singularity, we denote

$$n_t(\xi) := \frac{1}{|\xi|^2 - \omega^2 t^2 - (\eta_d^{\mathrm{r}})^2 + 2\mathrm{i}\eta_d^{\mathrm{i}}(\xi_d - \eta_d^{\mathrm{r}})}$$

and define the transformation

$$\tau : \xi \mapsto \xi^* = (\xi', -\xi_d + 2\eta_d^{\mathrm{r}}), \quad \xi \in \Omega_2,$$

where

$$\xi' := \left(\frac{2\omega t}{|\xi^-|} - 1\right)\xi^-.$$

A simple calculation yields that $|\xi'| = 2\omega t - |\xi^-|$ and the Jacobian of the transformation is

$$J_{d,t}(\xi) = \left|\det \frac{\partial \xi^*}{\partial \xi}\right| = \left(\frac{2\omega t}{|\xi^-|} - 1\right)^{d-2}.$$

Moreover, it can be verified that the transformation maps the subdomain

$$\Omega_{21} := \left\{\xi : \frac{\omega t}{2} < |\xi^-| < \omega t, |\xi_d - \eta_d^{\mathrm{r}}| < \frac{\omega t}{2}\right\}$$

to the subdomain

$$\Omega_{22} := \left\{\xi : \omega t < |\xi^-| < \frac{3\omega t}{2}, |\xi_d - \eta_d^{\mathrm{r}}| < \frac{\omega t}{2}\right\},$$

and vice versa.

Based on Ω_{21} and Ω_{22}, II_2 can be subdivided into several parts:

$$\begin{aligned}
\mathrm{II}_2 &= \int_{\Omega_2 \cap \{\xi : |\xi_d - \eta_d^{\mathrm{r}}| < \frac{\omega t}{2}\}} \frac{(1+|\xi|^2)^s}{|\xi|^2 - \omega^2 t^2 - (\eta_d^{\mathrm{r}})^2 + 2\mathrm{i}\eta_d^{\mathrm{i}}(\xi_d - \eta_d^{\mathrm{r}})} \widehat{\mathcal{J}^{-s} f}(\xi) \overline{\widehat{\mathcal{J}^{-s} g}(\xi)} d\xi \\
&= \int_{\Omega_{21} \cup \Omega_{22}} n_t(\xi)(1+|\xi|^2)^s \widehat{\mathcal{J}^{-s} f}(\xi) \overline{\widehat{\mathcal{J}^{-s} g}(\xi)} d\xi \\
&= \int_{\Omega_{22}} \Big[n_t(\xi)(1+|\xi|^2)^s \widehat{\mathcal{J}^{-s} f}(\xi) \overline{\widehat{\mathcal{J}^{-s} g}(\xi)} \\
&\quad + n_t(\xi^*) J_{d,t}(\xi)(1+|\xi^*|^2)^s \widehat{\mathcal{J}^{-s} f}(\xi^*) \overline{\widehat{\mathcal{J}^{-s} g}(\xi^*)} \Big] d\xi \\
&= \int_{\Omega_{22}} \left[n_t(\xi) + n_t(\xi^*) J_{d,t}(\xi)\right] (1+|\xi|^2)^s \widehat{\mathcal{J}^{-s} f}(\xi) \overline{\widehat{\mathcal{J}^{-s} g}(\xi)} d\xi \\
&\quad + \int_{\Omega_{22}} n_t(\xi^*) J_{d,t}(\xi) \left[(1+|\xi^*|^2)^s - (1+|\xi|^2)^s\right] \widehat{\mathcal{J}^{-s} f}(\xi) \overline{\widehat{\mathcal{J}^{-s} g}(\xi)} d\xi \\
&\quad + \int_{\Omega_{22}} n_t(\xi^*) J_{d,t}(\xi)(1+|\xi^*|^2)^s \left[\widehat{\mathcal{J}^{-s} f}(\xi^*) - \widehat{\mathcal{J}^{-s} f}(\xi)\right] \overline{\widehat{\mathcal{J}^{-s} g}(\xi)} d\xi \\
&\quad + \int_{\Omega_{22}} n_t(\xi^*) J_{d,t}(\xi)(1+|\xi^*|^2)^s \widehat{\mathcal{J}^{-s} f}(\xi^*) \overline{\left[\widehat{\mathcal{J}^{-s} g}(\xi^*) - \widehat{\mathcal{J}^{-s} g}(\xi)\right]} d\xi \\
&=: \mathrm{II}_{21} + \mathrm{II}_{22} + \mathrm{II}_{23} + \mathrm{II}_{24},
\end{aligned}$$

where we used the fact

$$\begin{aligned}
&\int_{\Omega_{21}} n_t(\xi)(1+|\xi|^2)^s \widehat{\mathcal{J}^{-s}f}(\xi)\overline{\widehat{\mathcal{J}^{-s}g}(\xi)}d\xi \\
&= \int_{\Omega_{21}} n_t(\xi^*)(1+|\xi^*|^2)^s \widehat{\mathcal{J}^{-s}f}(\xi^*)\overline{\widehat{\mathcal{J}^{-s}g}(\xi^*)}d\xi^* \\
&= \int_{\Omega_{22}} n_t(\xi^*)(1+|\xi^*|^2)^s \widehat{\mathcal{J}^{-s}f}(\xi^*)\overline{\widehat{\mathcal{J}^{-s}g}(\xi^*)}J_{d,t}(\xi)d\xi.
\end{aligned}$$

Noting

$$\begin{aligned}
n_t(\xi^*) &= \frac{1}{|\xi^*|^2-\omega^2t^2-(\eta_d^{\rm r})^2+2{\rm i}\eta_d^{\rm i}(\xi_d^*-\eta_d^{\rm r})} \\
&= \frac{1}{|\xi'|^2-\omega^2t^2+(\xi_d^*-\eta_d^{\rm r})(\xi_d^*+\eta_d^{\rm r})+2{\rm i}\eta_d^{\rm i}(\xi_d^*-\eta_d^{\rm r})} \\
&= \frac{1}{|\xi'|^2-\omega^2t^2+(\xi_d-\eta_d^{\rm r})(\xi_d-3\eta_d^{\rm r})-2{\rm i}\eta_d^{\rm i}(\xi_d-\eta_d^{\rm r})},
\end{aligned}$$

we get for $d=2$ that

$$\begin{aligned}
h_2(\xi) :&= |n_t(\xi)+n_t(\xi^*)J_{2,t}(\xi)| \\
&= \Bigg|\frac{1}{|\xi^-|^2-\omega^2t^2+(\xi_d-\eta_d^{\rm r})(\xi_d+\eta_d^{\rm r})+2{\rm i}\eta_d^{\rm i}(\xi_d-\eta_d^{\rm r})} \\
&\quad + \frac{1}{|\xi'|^2-\omega^2t^2+(\xi_d-\eta_d^{\rm r})(\xi_d-3\eta_d^{\rm r})-2{\rm i}\eta_d^{\rm i}(\xi_d-\eta_d^{\rm r})}\Bigg| \\
&= \frac{2(|\xi^-|-\omega t)^2+2(\xi_d-\eta_d^{\rm r})^2}{\left[((|\xi^-|-\omega t)(|\xi^-|+\omega t)+(\xi_d-\eta_d^{\rm r})(\xi_d+\eta_d^{\rm r}))^2+4(\eta_d^{\rm i})^2(\xi_d-\eta_d^{\rm r})^2\right]^{\frac{1}{2}}} \\
&\quad \times \frac{1}{\left[((|\xi^-|-\omega t)(|\xi^-|-3\omega t)+(\xi_d-\eta_d^{\rm r})(\xi_d-3\eta_d^{\rm r}))^2+4(\eta_d^{\rm i})^2(\xi_d-\eta_d^{\rm r})^2\right]^{\frac{1}{2}}},
\end{aligned}$$

which is bounded

$$h_2(\xi) \lesssim \frac{1}{\omega^2t^2}, \quad \xi\in\Omega_{22}$$

as $t \gg 1$ according to the boundedness of $\xi\in\Omega_{22}$. Similarly, it holds for $d=3$ and $t\gg 1$ that

$$\begin{aligned}
h_3(\xi) :&= |n_t(\xi)+n_t(\xi^*)J_{3,t}(\xi)| \\
&= \Bigg|\frac{1}{|\xi^-|^2-\omega^2t^2+(\xi_d-\eta_d^{\rm r})(\xi_d+\eta_d^{\rm r})+2{\rm i}\eta_d^{\rm i}(\xi_d-\eta_d^{\rm r})} \\
&\quad + \frac{\frac{2\omega t}{|\xi^-|}-1}{|\xi'|^2-\omega^2t^2+(\xi_d-\eta_d^{\rm r})(\xi_d-3\eta_d^{\rm r})-2{\rm i}\eta_d^{\rm i}(\xi_d-\eta_d^{\rm r})}\Bigg| \\
&\lesssim \frac{1}{\omega^2t^2}.
\end{aligned}$$

The above estimates lead to

$$|{\rm II}_{21}| \lesssim \frac{1}{\omega^2t^2}\int_{\Omega_{22}}(1+|\xi|^2)^s|\widehat{\mathcal{J}^{-s}f}(\xi)||\widehat{\mathcal{J}^{-s}g}(\xi)|d\xi \lesssim \frac{1}{(\omega t)^{2-2s}}\|f\|_{H^{-s}(G)}\|g\|_{H^{-s}(G)}.$$

For II_{22}, we apply the mean value theorem and get for some $\theta \in (0,1)$ that

$$
\begin{aligned}
&\left|n_t(\xi^*)J_{d,t}(\xi)\left[(1+|\xi^*|^2)^s-(1+|\xi|^2)^s\right]\right|\\
&=\left|n_t(\xi^*)J_{d,t}(\xi)s\left(1+\theta|\xi^*|^2+(1-\theta)|\xi|^2\right)^{s-1}(|\xi^*|^2-|\xi|^2)\right|\\
&\lesssim\left|n_t(\xi^*)J_{d,t}(\xi)(|\xi^*|^2-|\xi|^2)\right|\left(1+\theta|\xi^*|^2+(1-\theta)|\xi|^2\right)^{s-1}\\
&\lesssim\left(1+\theta|\xi^*|^2+(1-\theta)|\xi|^2\right)^{s-1}\lesssim\frac{1}{(\omega t)^{2-2s}},
\end{aligned}
$$

where in the third step we used the following estimate similar to $h_2(\xi)$:

$$
\begin{aligned}
&\left|n_t(\xi^*)J_{d,t}(\xi)(|\xi^*|^2-|\xi|^2)\right|\\
&=\left|\frac{\left(\frac{2\omega t}{|\xi^-|}-1\right)^{d-2}(|\xi^*|^2-|\xi|^2)}{|\xi'|^2-\omega^2t^2+(\xi_d-\eta_d^{\mathrm{r}})(\xi_d-3\eta_d^{\mathrm{r}})-2\mathrm{i}\eta_d^{\mathrm{i}}(\xi_d-\eta_d^{\mathrm{r}})}\right|
\end{aligned}
\tag{3.8}
$$

$$
=\frac{\left(\frac{2\omega t}{|\xi^-|}-1\right)^{d-2}\left|4\omega t(|\xi^-|-\omega t)+4\eta_d^{\mathrm{r}}(\xi_d-\eta_d^{\mathrm{r}})\right|}{\left[((|\xi^-|-\omega t)(|\xi^-|-3\omega t)+(\xi_d-\eta_d^{\mathrm{r}})(\xi_d-3\eta_d^{\mathrm{r}}))^2+4(\eta_d^{\mathrm{i}})^2(\xi_d-\eta_d^{\mathrm{r}})^2\right]^{\frac12}}\lesssim 1.
$$

Therefore

$$
|\mathrm{II}_{22}|\lesssim\frac{1}{(\omega t)^{2-2s}}\int_{\Omega_{22}}|\widehat{\mathcal{J}^{-s}f}(\xi)||\widehat{\mathcal{J}^{-s}g}(\xi)|d\xi\lesssim\frac{1}{(\omega t)^{2-2s}}\|f\|_{H^{-s}(G)}\|g\|_{H^{-s}(G)}.
$$

Terms II_{23} and II_{24} can be estimated similarly by following the procedure used in [20, Theorem 3.2]. In fact, it can be shown that the Bessel potential satisfies

$$
\left|\widehat{\mathcal{J}^{-s}f}(\xi^*)-\widehat{\mathcal{J}^{-s}f}(\xi)\right|\lesssim\left||\xi^*|-|\xi|\right|\left[M(|\nabla\widehat{\mathcal{J}^{-s}f}|)(\xi^*)+M(|\nabla\widehat{\mathcal{J}^{-s}f}|)(\xi^*)\right],
$$

where M is the Hardy–Littlewood maximal function defined by

$$
M(f)(x)=\sup_{r>0}\frac{1}{|B(x,r)|}\int_{B(x,r)}|f(y)|dy
$$

with $B(x,r)$ being the ball of center x and radius r, and satisfies (cf. [20, Theorem 3.2])

$$
\|M(|\nabla\widehat{\mathcal{J}^{-s}f}|)\|_{L^2(\mathbb{R}^d)}\lesssim\|f\|_{H^{-s}(G)}.
$$

The above estimates, together with (3.8), yield

$$
\begin{aligned}
|\mathrm{II}_{23}|\lesssim&\int_{\Omega_{22}}\frac{|n_t(\xi^*)J_{d,t}(|\xi^*|^2-|\xi|^2)|}{|\xi^*|+|\xi|}(1+|\xi^*|^2)^s\\
&\times\left|M(|\nabla\widehat{\mathcal{J}^{-s}f}|)(\xi^*)+M(|\nabla\widehat{\mathcal{J}^{-s}f}|)(\xi^*)\right||\widehat{\mathcal{J}^{-s}g}(\xi)|d\xi\\
\lesssim&\frac{1}{(\omega t)^{1-2s}}\|f\|_{H^{-s}(G)}\|g\|_{H^{-s}(G)}
\end{aligned}
$$

and

$$
|\mathrm{II}_{24}|\lesssim\int_{\Omega_{22}}\frac{|n_t(\xi^*)J_{d,t}(|\xi^*|^2-|\xi|^2)|}{|\xi^*|+|\xi|}(1+|\xi^*|^2)^s
$$

$$\times |\widehat{\mathcal{J}^{-s}f}(\xi^*)|\Big|M(|\nabla\widehat{\mathcal{J}^{-s}g}|)(\xi^*)+M(|\nabla\widehat{\mathcal{J}^{-s}g}|)(\xi^*)\Big|d\xi$$

$$\lesssim \frac{1}{(\omega t)^{1-2s}}\|f\|_{H^{-s}(G)}\|g\|_{H^{-s}(G)}.$$

Hence, II satisfies

$$|\mathrm{II}| \lesssim \frac{1}{(\omega t)^{1-2s}}\|f\|_{H^{-s}(G)}\|g\|_{H^{-s}(G)}. \tag{3.9}$$

Combining (3.7) and (3.9), we obtain the estimate of $\mathcal{A}$ and get

$$|\langle \mathcal{G}_\eta f, g\rangle| \lesssim \frac{1}{\omega^{3-2s}t^{1-2s}}\|f\|_{H^{-s}(G)}\|g\|_{H^{-s}(G)}$$

for any $f, g \in C_0^\infty(G)$. Since $C_0^\infty(G)$ is dense in $L^2(G)$ and $H^{-s}(G) \subset H^{-1}(G) = \overline{L^2(G)}^{\|\cdot\|_{H^{-1}(G)}}$ (cf. [1, Sections 2.30, 3.13]), the above result can be extended to $f, g \in H^{-s}(G)$ with $s \in (0, \frac{1}{2})$, which completes the proof. ∎

Remark 3.3. The unique continuation principle established in Theorem 3.1 holds for any damping coefficient $\sigma \geq 0$. If the medium is lossless with $\sigma = 0$, the proof can be simplified by letting $\omega = k^{\frac{1}{2}}$ and

$$\eta = \begin{cases} \left(k^{\frac{1}{2}}t, \mathrm{i}k^{\frac{1}{2}}\sqrt{t^2-1}\right)^\top, & d = 2, \\ \left(k^{\frac{1}{2}}t, 0, \mathrm{i}k^{\frac{1}{2}}\sqrt{t^2-1}\right)^\top, & d = 3. \end{cases}$$

We refer to [25] for the unique continuation principle of the Schrödinger equation without damping. The unique continuation principle will be utilized to show the uniqueness of the solution to the direct scattering problem when $\sigma = 0$.

4. The Lippmann–Schwinger equation. In this section, we examine the well-posedness of the scattering problem (1.1)–(1.2) by studying the equivalent Lippmann–Schwinger integral equation.

4.1. Well-posedness. Based on the integral operators, the scattering problem (1.1)–(1.2) can be written formally as the Lippmann–Schwinger equation

$$u = \mathcal{K}_k u + \mathcal{H}_k \delta_y = \mathcal{K}_k u + \Phi, \tag{4.1}$$

where the fundamental solution Φ is given in (2.2).

THEOREM 4.1. *Let ρ satisfy Assumption 1.1. The Lippmann–Schwinger equation (4.1) has a unique solution in $W^{\gamma,q}_{loc}(\mathbb{R}^d)$ with $q \in (2, \frac{2d}{3d-2m-2})$ and $\gamma \in (\frac{d-m}{2}, \frac{1}{2} + (\frac{1}{q} - \frac{1}{2})\frac{d}{2})$.*

Proof. According to the compactness of the operator $\mathcal{K}_k$ proved in Lemma 2.3 and the Fredholm alternative theorem, it suffices to show that the homogeneous equation

$$u = \mathcal{K}_k u \tag{4.2}$$

has only the trivial solution $u \equiv 0$.

Assume that u^* is a solution to the homogeneous equation (4.2). Then it satisfies the following equation in the distribution sense:

$$\Delta^2 u^* - \kappa^4 u^* + \rho u^* = 0 \quad \text{in } \mathbb{R}^d. \tag{4.3}$$

Let us consider two auxiliary functions

$$(4.4)\qquad u_H := -\frac{1}{2\kappa^2}(\Delta u^* - \kappa^2 u^*), \quad u_M := \frac{1}{2\kappa^2}(\Delta u^* + \kappa^2 u^*).$$

It is clear to note that $u^* = u_H + u_M$ and $\Delta u^* = \kappa^2(u_M - u_H)$.

Since ρ is compactly supported in D, there exists a constant $R > 0$ such that $D \subset B_R$ with B_R being the open ball of radius R centered at zero. It can be verified that u_H and u_M satisfy the homogeneous Helmholtz and modified Helmholtz equation with the wavenumber κ, respectively, in $\mathbb{R}^d \setminus \overline{B_R}$:

$$\Delta u_H + \kappa^2 u_H = 0, \quad \Delta u_M - \kappa^2 u_M = 0.$$

Hence, u_H and u_M admit the following Fourier series expansions for any $r = |x| > R$:

$$(4.5)\qquad \begin{aligned} u_H(r,\theta) &= \sum_{n=-\infty}^{\infty} \frac{H_n^{(1)}(\kappa r)}{H_n^{(1)}(\kappa R)} \hat{u}_H^{(n)}(R) e^{\mathrm{i}n\theta}, \\ u_M(r,\theta) &= \sum_{n=-\infty}^{\infty} \frac{K_n(\kappa r)}{K_n(\kappa R)} \hat{u}_M^{(n)}(R) e^{\mathrm{i}n\theta}, \end{aligned} \qquad \text{if} \quad d = 2,$$

where

$$\hat{u}_J^{(n)}(R) = \frac{1}{2\pi} \int_0^{2\pi} u_J(R,\theta) e^{-\mathrm{i}n\theta} d\theta, \quad J \in \{H, M\}$$

are the Fourier coefficients, and

$$(4.6)\qquad \begin{aligned} u_H(r,\theta,\varphi) &= \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \frac{h_n^{(1)}(\kappa r)}{h_n^{(1)}(\kappa R)} \hat{u}_H^{(m,n)}(R) Y_n^m(\theta,\varphi), \\ u_M(r,\theta,\varphi) &= \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \frac{k_n(\kappa r)}{k_n(\kappa R)} \hat{u}_M^{(m,n)}(R) Y_n^m(\theta,\varphi), \end{aligned} \qquad \text{if} \quad d = 3,$$

where $h_n^{(1)}$ and k_n are the spherical and modified spherical Hankel functions, respectively, satisfying

$$h_n^{(1)}(z) = \sqrt{\frac{\pi}{2z}} H_{n+\frac{1}{2}}^{(1)}(z), \quad k_n(z) = \sqrt{\frac{\pi}{2z}} K_{n+\frac{1}{2}}(z), \quad z \in \mathbb{C},$$

Y_n^m are the spherical harmonics of order n, and the Fourier coefficients $\hat{u}_J^{(m,n)}(R)$ are given by

$$\hat{u}_J^{(m,n)}(R) = \int_{\mathbb{S}^2} u_J(R,\theta,\varphi) \overline{Y_n^m(\theta,\varphi)} ds.$$

If $\sigma > 0$, then we have $\kappa_{\rm r} = \Re(\kappa) > 0, \kappa_{\rm i} = \Im(\kappa) > 0$. It follows from (2.3)–(2.4) and (4.5)–(4.6) that u_H, u_M and thus $u^*, \Delta u^*$ decay exponentially as $r \to \infty$. Multiplying (4.3) by the complex conjugate of u^*, integrating over B_r, and applying Green's formula, we obtain

$$\int_{B_r} \left(|\Delta u^*|^2 - \kappa^4 |u^*|^2 + \rho |u^*|^2\right) dx = \int_{\partial B_r} \left(\Delta u^* \overline{\partial_\nu u^*} - \overline{u^*} \partial_\nu \Delta u^*\right) ds,$$

where ν is the unit outward normal vector to ∂B_r. Taking the imaginary part of the above equation yields

$$-\Im(\kappa^4)\|u^*\|^2_{L^2(B_r)} = \Im\left[\int_{\partial B_r} \left(\Delta u^* \overline{\partial_\nu u^*} - \overline{u^*}\partial_\nu \Delta u^*\right) ds\right] \to 0$$

as $r \to \infty$ and hence $u^* \equiv 0$ in $\mathbb{R}^d$.

If $\sigma = 0$, then $\kappa = k^{\frac{1}{2}}$ is real. By (4.5)–(4.6), only $u_M|_{\partial B_r}$ and $\partial_\nu u_M|_{\partial B_r}$ decay exponentially as $r \to \infty$. It is easy to verify from (4.3) that u_H and u_M satisfy the following equations in $\mathbb{R}^d$:

$$\Delta u_H + k u_H - \frac{1}{2k}\rho u^* = 0, \quad \Delta u_M - k u_M + \frac{1}{2k}\rho u^* = 0.$$

Indeed, based on the definition of u_H given in (4.4) with $\kappa^2 = k$ and (4.3), we have the following relationship:

$$\begin{aligned}\Delta u_H + k u_H - \frac{1}{2k}\rho u^* &= -\frac{1}{2k}(\Delta + k)(\Delta u^* - k u^*) - \frac{1}{2k}\rho u^* \\ &= -\frac{1}{2k}(\Delta^2 u^* - k^2 u^* + \rho u^*) = 0.\end{aligned}$$

Similarly, the equation for u_M can be obtained. Using the integration by parts and the fact $u^* = u_H + u_M$, we have from Green's formula that

$$\begin{aligned}\int_{\partial B_r} u_M \overline{\partial_\nu u_M} ds &= \int_{B_r} \left(|\nabla u_M|^2 + k|u_M|^2 - \frac{1}{2k}\rho |u_M|^2 - \frac{1}{2k}\rho u_M \overline{u_H}\right) dx, \\ \int_{\partial B_r} u_H \overline{\partial_\nu u_H} ds &= \int_{B_r} \left(|\nabla u_H|^2 - k|u_H|^2 + \frac{1}{2k}\rho |u_H|^2 + \frac{1}{2k}\rho \overline{u_M} u_H\right) dx,\end{aligned}$$

which are well-defined since $\nabla \Delta u^* \in L^2_{loc}(\mathbb{R}^d)$ due to $\Delta^2 u^* = k^2 u^* - \rho u^*$ with $u^* \in W^{\gamma,q}_{loc}(\mathbb{R}^d)$ and $\rho u^* \in W^{-\gamma,p}(D)$ (cf. (2.7)). Taking the imaginary parts of the above two equations yields

$$\Im\left[\int_{\partial B_r} u_M \overline{\partial_\nu u_M} ds\right] = \Im\left[\int_{\partial B_r} u_H \overline{\partial_\nu u_H} ds\right],$$

which leads to

$$\int_{\partial B_r} \left(|\partial_\nu u_H|^2 + k|u_H|^2\right) ds = \int_{\partial B_r} \left|\partial_\nu u_H - \mathrm{i} k^{\frac{1}{2}} u_H\right|^2 ds - 2k^{\frac{1}{2}} \Im\left[\int_{\partial B_r} u_M \overline{\partial_\nu u_M} ds\right].$$

By the Sommerfeld radiation condition (1.2), the first integral on the right-hand side of the above equation tends to zero as $r \to \infty$. The second integral also tends to zero due to the exponential decay of u_M. Therefore,

$$\lim_{r\to\infty} \int_{\partial B_r} \left(|\partial_\nu u_H|^2 + k|u_H|^2\right) ds = \lim_{r\to\infty} \int_{\partial B_r} \left(|\partial_\nu u_M|^2 + k|u_M|^2\right) ds = 0.$$

It follows from Rellich's lemma that $u_H = u_M = 0$ in $\mathbb{R}^d \setminus \overline{B_R}$ and thus $u^* \equiv 0$ in $\mathbb{R}^d \setminus \overline{B_R}$. The proof is completed by applying the unique continuation in Theorem 3.1. ∎

The well-posedness of the scattering problem (1.1)–(1.2) can be obtained by showing the equivalence to the Lippmann–Schwinger equation. The proof is similar to that of [20, Theorem 3.5] and is omitted here for brevity.

Corollary 4.2. *Under Assumption 1.1, the scattering problem* (1.1)–(1.2) *is well-posed in the distribution sense and has a unique solution $u \in W^{\gamma,q}_{loc}(\mathbb{R}^d)$, where q and γ are given in Theorem 4.1.*

4.2. Born series. Based on the Lippmann–Schwinger equation (4.1), we formally define the Born series

$$\sum_{n=0}^{\infty} u_n(x,y,k),$$

where

(4.7)
$$u_n(x,y,k) := \mathcal{K}_k\left(u_{n-1}(\cdot,y,k)\right)(x) = \int_{\mathbb{R}^d} \Phi(x,z,k)\rho(z)u_{n-1}(z,y,k)dz, \quad n \geq 1$$

and $u_0(x,y,k) := \mathcal{H}_k(\delta_y)(x) = \Phi(x,y,k)$.

The Born series is crucial in our arguments for the inverse scattering problem. It helps to establish the recovery formula for the strength μ of the random potential ρ. Before addressing the inverse problem, we study the convergence of the Born series.

LEMMA 4.3. *There exists $k_0 > 0$ such that for any wavenumber $k \geq k_0$ and any fixed $x, y \in U$ with U having a positive distance to the support D, the Born series converges to the solution of* (1.1)–(1.2)*, i.e.,*

$$u(x,y,k) = \sum_{n=0}^{\infty} u_n(x,y,k).$$

Proof. The convergence of the Born series to the solution of (1.1)–(1.2) can be obtained by employing the same procedure as that in [17, Section 4.2] and the estimate of $u_0(x,y,k) = \Phi(x,y,k)$ given in Lemma 2.1.

Moreover, the Born series admits the pointwise convergence. Using the estimates of $\mathcal{H}_k$ and $\mathcal{K}_k$ given in Lemmas 2.2 and 2.3, we get for any $s \in (\frac{d-m}{2}, \frac{3-\chi_\sigma}{2})$ that

$$\left\| u(\cdot,y,k) - \sum_{n=0}^{N} u_n(\cdot,y,k) \right\|_{L^\infty(U)} \lesssim \sum_{n=N+1}^{\infty} \|\mathcal{K}_k^n\left(u_0(\cdot,y,k)\right)\|_{L^\infty(U)}$$

$$\lesssim \sum_{n=N+1}^{\infty} \|\mathcal{K}_k\|_{\mathcal{L}(H^s(U),L^\infty(U))} \|\mathcal{K}_k\|_{\mathcal{L}(H^s(U))}^{n-2} \|\mathcal{H}_k\|_{\mathcal{L}(H^{-s}(D),H^s(U))} \|\rho\Phi(\cdot,y,k)\|_{H^{-s}(D)}$$

$$\lesssim \sum_{n=N+1}^{\infty} k^{\frac{2s+d-2(3-\chi_\sigma)+\epsilon}{4}} k^{\left(s-\frac{3-\chi_\sigma}{2}\right)(n-2)} k^{s-\frac{3-\chi_\sigma}{2}} \|\Phi(\cdot,y,k)\|_{H^s(D)}$$

(4.8)
$$\lesssim k^{\frac{2s+d-2(3-\chi_\sigma)+\epsilon}{4} + \left(s-\frac{3-\chi_\sigma}{2}\right)N + \frac{d-7}{4} + \frac{s}{2}} \to 0$$

as $N \to \infty$ for any $k \geq k_0$ and $\epsilon > 0$, where we used (2.9) and Lemma 2.1. ∎

5. The inverse scattering problem. This section is devoted to the inverse scattering problem, which is to determine the strength μ of the random potential ρ. More specifically, the point source is assumed to be located at $y = x$, where $x \in U$ is the observation point and U is the measurement domain having a positive distance to the support D of the random potential. Therefore, only the backscattering data is used for the inverse problem, as also discussed in [16, 17] for the cases of the Schrödinger equation and elastic wave equation. For simplicity, we use the notation $u_n(x,k) := u_n(x,x,k)$ for $n \geq 1$. Then the scattered field u^s has the form

$$u^s(x,k) = \sum_{n=1}^{\infty} u_n(x,k)$$

for $k \geq k_0$ with k_0 being given in Lemma 4.3.

Next we analyze the contribution of each term in the Born series in order to deduce the reconstruction formula and achieve the uniqueness of the inverse problem.

5.1. The analysis of u_1. Based on the definitions of the Born sequence (4.7) and the incident field u_0, the leading term u_1 can be expressed as

$$u_1(x,k) = \mathcal{K}_k(u_0(\cdot,x,k))(x) = \int_{\mathbb{R}^d} \Phi(x,z,k)^2 \rho(z)dz. \tag{5.1}$$

Since the fundamental solutions take different forms, the contribution of u_1 is discussed for the three- and two-dimensional cases, separately.

5.1.1. The three-dimensional case. By Assumption 1.1, we have $m \in (2,3]$ for $d = 3$. Substituting the fundamental solution

$$\Phi(x,z,k) = -\frac{1}{8\pi\kappa^2|x-z|}\left(e^{\mathrm{i}\kappa|x-z|} - e^{-\kappa|x-z|}\right)$$

into (5.1) gives

$$\begin{aligned}
\mathbb{E}|u_1(x,k)|^2 = &\frac{1}{(8\pi|\kappa|^2)^4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\left(\frac{e^{\mathrm{i}\kappa|x-z|}-e^{-\kappa|x-z|}}{|x-z|}\right)^2\left(\overline{\frac{e^{\mathrm{i}\kappa|x-z'|}-e^{-\kappa|x-z'|}}{|x-z'|}}\right)^2\\
&\times \mathbb{E}[\rho(z)\rho(z')]dzdz'\\
=&\frac{1}{(8\pi|\kappa|^2)^4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{e^{2\mathrm{i}(\kappa|x-z|-\overline{\kappa}|x-z'|)}}{|x-z|^2|x-z'|^2}\mathbb{E}[\rho(z)\rho(z')]dzdz'\\
&-\frac{2}{(8\pi|\kappa|^2)^4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{e^{2\mathrm{i}\kappa|x-z|-(\mathrm{i}+1)\overline{\kappa}|x-z'|}}{|x-z|^2|x-z'|^2}\mathbb{E}[\rho(z)\rho(z')]dzdz'\\
&+\frac{1}{(8\pi|\kappa|^2)^4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{e^{2\mathrm{i}\kappa|x-z|-2\overline{\kappa}|x-z'|}}{|x-z|^2|x-z'|^2}\mathbb{E}[\rho(z)\rho(z')]dzdz'\\
&-\frac{2}{(8\pi|\kappa|^2)^4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{e^{(\mathrm{i}-1)\kappa|x-z|-2\mathrm{i}\overline{\kappa}|x-z'|}}{|x-z|^2|x-z'|^2}\mathbb{E}[\rho(z)\rho(z')]dzdz'\\
&+\frac{4}{(8\pi|\kappa|^2)^4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{e^{(\mathrm{i}-1)\kappa|x-z|-(\mathrm{i}+1)\overline{\kappa}|x-z'|}}{|x-z|^2|x-z'|^2}\mathbb{E}[\rho(z)\rho(z')]dzdz'\\
&-\frac{2}{(8\pi|\kappa|^2)^4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{e^{(\mathrm{i}-1)\kappa|x-z|-2\overline{\kappa}|x-z'|}}{|x-z|^2|x-z'|^2}\mathbb{E}[\rho(z)\rho(z')]dzdz'\\
&+\frac{1}{(8\pi|\kappa|^2)^4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{e^{-2\kappa|x-z|-2\mathrm{i}\overline{\kappa}|x-z'|}}{|x-z|^2|x-z'|^2}\mathbb{E}[\rho(z)\rho(z')]dzdz'\\
&-\frac{2}{(8\pi|\kappa|^2)^4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{e^{-2\kappa|x-z|-(\mathrm{i}+1)\overline{\kappa}|x-z'|}}{|x-z|^2|x-z'|^2}\mathbb{E}[\rho(z)\rho(z')]dzdz'\\
&+\frac{1}{(8\pi|\kappa|^2)^4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{e^{-2\kappa|x-z|-2\overline{\kappa}|x-z'|}}{|x-z|^2|x-z'|^2}\mathbb{E}[\rho(z)\rho(z')]dzdz'\\
=&\frac{1}{(8\pi|\kappa|^2)^4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{e^{2\mathrm{i}(\kappa|x-z|-\overline{\kappa}|x-z'|)}}{|x-z|^2|x-z'|^2}\mathbb{E}[\rho(z)\rho(z')]dzdz'\\
&-\frac{4}{(8\pi|\kappa|^2)^4}\Re\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{e^{2\mathrm{i}\kappa|x-z|-(\mathrm{i}+1)\overline{\kappa}|x-z'|}}{|x-z|^2|x-z'|^2}\mathbb{E}[\rho(z)\rho(z')]dzdz'
\end{aligned}$$

$$
\begin{aligned}
&+\frac{2}{(8\pi|\kappa|^2)^4}\Re\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{e^{2\mathrm{i}\kappa|x-z|-2\overline{\kappa}|x-z'|}}{|x-z|^2|x-z'|^2}\mathbb{E}[\rho(z)\rho(z')]dzdz'\\
&+\frac{4}{(8\pi|\kappa|^2)^4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{e^{(\mathrm{i}-1)\kappa|x-z|-(\mathrm{i}+1)\overline{\kappa}|x-z'|}}{|x-z|^2|x-z'|^2}\mathbb{E}[\rho(z)\rho(z')]dzdz'\\
&-\frac{4}{(8\pi|\kappa|^2)^4}\Re\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{e^{(\mathrm{i}-1)\kappa|x-z|-2\overline{\kappa}|x-z'|}}{|x-z|^2|x-z'|^2}\mathbb{E}[\rho(z)\rho(z')]dzdz'\\
&+\frac{1}{(8\pi|\kappa|^2)^4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{e^{-2\kappa|x-z|-2\overline{\kappa}|x-z'|}}{|x-z|^2|x-z'|^2}\mathbb{E}[\rho(z)\rho(z')]dzdz'\\
=:&\ \mathrm{I}_1+\mathrm{I}_2+\mathrm{I}_3+\mathrm{I}_4+\mathrm{I}_5.
\end{aligned}
$$

For I_1, following the procedure used in [21, Theorem 4.5], we get

$$
\begin{aligned}
|\mathrm{I}_1| &= \frac{1}{(8\pi|\kappa|^2)^4}\left[\int_D \frac{e^{-4\kappa_{\mathrm{i}}|x-z|}}{|x-z|^4}\mu(z)dz\kappa_{\mathrm{r}}^{-m}+O\left(\kappa_{\mathrm{r}}^{-m-1}\right)\right]\\
&= \frac{\kappa_{\mathrm{r}}^{-m}}{(8\pi|\kappa|^2)^4}\int_D \frac{e^{-4\kappa_{\mathrm{i}}|x-z|}}{|x-z|^4}\mu(z)dz+O\left(\kappa_{\mathrm{r}}^{-m-9}\right).
\end{aligned}
$$

The other terms can be estimated by utilizing the exponential decay of the integrants with respect to κ_{r}. Since the estimates are analogous, we only show the detail for I_2. Note that $|x-z|$ is bounded below and above for any $x\in U$ and $z\in D$. A simple calculation yields

$$
\begin{aligned}
\mathrm{I}_2 = \frac{4}{(8\pi|\kappa|^2)^4}\Re\int_D\int_D &\frac{e^{\mathrm{i}(2\kappa_{\mathrm{r}}|x-z|+(\kappa_{\mathrm{i}}-\kappa_{\mathrm{r}})|x-z'|)}e^{-2\kappa_{\mathrm{i}}|x-z|-(\kappa_{\mathrm{r}}+\kappa_{\mathrm{i}})|x-z'|}}{|x-z|^2|x-z'|^2}\\
&\times\mathbb{E}[\rho(z)\rho(z')]dzdz',
\end{aligned}
$$

where

$$
e^{-2\kappa_{\mathrm{i}}|x-z|-(\kappa_{\mathrm{r}}+\kappa_{\mathrm{i}})|x-z'|}\lesssim\kappa_{\mathrm{r}}^{-M}
$$

for any $M>0$ as $\kappa_{\mathrm{r}}\to\infty$. Choosing $M=m+1$ gives

$$
|\mathrm{I}_2|\lesssim|\kappa|^{-8}\kappa_{\mathrm{r}}^{-m-1}\int_D\int_D|\mathbb{E}[\rho(z)\rho(z')]|dzdz'\lesssim\kappa_{\mathrm{r}}^{-m-9}\quad\forall\ x\in U,
$$

where we used the equivalence between $|\kappa|$ and κ_{r} as $\kappa_{\mathrm{r}}\to\infty$ and the following expression (up to a constant) of the leading term for the kernel $\mathbb{E}[\rho(z)\rho(z')]$ (cf. [22, Lemma 2.4]) with $d=2,3$:

$$
\mathbb{E}[\rho(z)\rho(z')]\sim\begin{cases}\mu(z)\ln|z-z'|, & m=d,\\ \mu(z)|z-z'|^{m-d}, & m\in(d-1,d).\end{cases}\tag{5.2}
$$

Terms $\mathrm{I}_3,\mathrm{I}_4$ and I_5 can be estimated similarly. Hence we obtain

$$
\mathbb{E}|u_1(x,k)|^2=\frac{\kappa_{\mathrm{r}}^{-m}}{(8\pi|\kappa|^2)^4}\int_D\frac{e^{-4\kappa_{\mathrm{i}}|x-z|}}{|x-z|^4}\mu(z)dz+O\left(\kappa_{\mathrm{r}}^{-m-9}\right)\quad\forall\ x\in U.\tag{5.3}
$$

5.1.2. The two-dimensional case. Now let us consider the two-dimensional problem where $d=2$ and $m\in(1,2]$. The fundamental solution Φ has the asymptotic expansion (cf. [2, 22])

$$
\Phi(x,z,k)=-\sum_{j=0}^{\infty}\frac{C_j}{8\kappa^2(\kappa|x-z|)^{j+\frac12}}\left(\mathrm{i}e^{\mathrm{i}\kappa|x-z|}-\mathrm{i}^{-j+\frac12}e^{-\kappa|x-z|}\right),
$$

where $C_0 = 1$ and

$$C_j = \sqrt{\frac{2}{\pi}} \frac{8^{-j}}{j!} \prod_{l=1}^{j} (2l-1)^2 e^{-\frac{\mathrm{i}\pi}{4}}, \quad j \geq 1.$$

Let the truncations of Φ and u_1 be defined as follows:

$$\Phi_N(x,z,k) := -\sum_{j=0}^{N} \frac{C_j}{8\kappa^2(\kappa|x-z|)^{j+\frac{1}{2}}} \left(\mathrm{i}e^{\mathrm{i}\kappa|x-z|} - \mathrm{i}^{-j+\frac{1}{2}} e^{-\kappa|x-z|}\right),$$

$$u_1^{(N)}(x,k) := \int_{\mathbb{R}^2} \Phi_N(x,z,k)^2 \rho(z) dz,$$

where

$$|\Phi(x,z,k)| \lesssim |\kappa|^{-\frac{5}{2}} |x-z|^{-\frac{1}{2}}, \quad |\Phi_N(x,z,k)| \lesssim |\kappa|^{-\frac{5}{2}} |x-z|^{-\frac{1}{2}}$$

and

$$\Phi(x,z,k) - \Phi_N(x,z,k) = O\left(|\kappa|^{-N-\frac{7}{2}} |x-z|^{-N-\frac{3}{2}}\right) \tag{5.4}$$

for any $N \in \mathbb{N}$ as $|\kappa||x-z| \to \infty$. The following lemma gives the truncation error of the fundamental solution.

LEMMA 5.1. *For any fixed $x \in U$, $N \in \mathbb{N}$, $\gamma \in [0,1]$ and $q > 1$, it holds*

$$\|\Phi(x,\cdot,k) - \Phi_N(x,\cdot,k)\|_{W^{\gamma,q}(D)} \lesssim |\kappa|^{-N-\frac{7}{2}+\gamma}. \tag{5.5}$$

In particular, for $N = 0$ and $\tilde{q} \in (1, \frac{4}{3})$, it holds

$$\|\Phi(\cdot,\cdot,k) - \Phi_0(\cdot,\cdot,k)\|_{W^{\gamma,\tilde{q}}(D\times D)} \lesssim |\kappa|^{-\frac{7}{2}+\gamma}. \tag{5.6}$$

Proof. Using (5.4) and

$$|\nabla_z \left(\Phi(x,z,k) - \Phi_N(x,z,k)\right)| = O\left(|\kappa|^{-N-\frac{5}{2}} |x-z|^{-N-\frac{3}{2}}\right),$$

we get

$$\|\Phi(x,\cdot,k) - \Phi_N(x,\cdot,k)\|_{L^q(D)} \lesssim |\kappa|^{-N-\frac{7}{2}},$$
$$\|\Phi(x,\cdot,k) - \Phi_N(x,\cdot,k)\|_{W^{1,q}(D)} \lesssim |\kappa|^{-N-\frac{5}{2}}.$$

Then (5.5) follows from the space interpolation $[L^q(D), W^{1,q}(D)]_\gamma = W^{\gamma,q}(D)$.

Similarly, (5.6) can be obtained by noting that

$$\|\Phi(\cdot,\cdot,k) - \Phi_0(\cdot,\cdot,k)\|_{L^{\tilde{q}}(D\times D)} \lesssim |\kappa|^{-\frac{7}{2}} \left(\int_D \int_D |z-z'|^{-\frac{3}{2}\tilde{q}} dz dz'\right)^{\frac{1}{\tilde{q}}} \lesssim |\kappa|^{-\frac{7}{2}}$$

and

$$\|\Phi(\cdot,\cdot,k) - \Phi_0(\cdot,\cdot,k)\|_{W^{1,\tilde{q}}(D\times D)} \lesssim |\kappa|^{-\frac{5}{2}}$$

for any $\tilde{q} \in (1, \frac{4}{3})$. ∎

Choosing $N = 1$ and using (2.5), (5.2), and (5.4), we get for any $x \in U$ that

$$\mathbb{E}\left|u_1(x,k) - u_1^{(1)}(x,k)\right|^2$$

$$
\begin{aligned}
&= \int_D\int_D (\Phi^2-\Phi_1^2)(x,z,k)\overline{(\Phi^2-\Phi_1^2)(x,z',k)}\mathbb{E}[\rho(z)\rho(z')]dzdz'\\
&\lesssim \sup_{(x,z)\in U\times D}\left[|(\Phi+\Phi_1)(x,z,k)|^2|(\Phi-\Phi_1)(x,z,k)|^2\right]\int_D\int_D|\mathbb{E}[\rho(z)\rho(z')]|dzdz'\\
&\lesssim |\kappa|^{-14}.
\end{aligned}
$$

The second moment of $u_1^{(1)}$ satisfies

$$
\begin{aligned}
\mathbb{E}|u_1^{(1)}(x,k)|^2 &= \frac{1}{(8|\kappa|^2)^4}\sum_{j,l=0}^{1}\frac{C_j^2\overline{C_l^2}}{\kappa^{2j+1}\overline{\kappa}^{2l+1}}\int_D\int_D\left(\frac{\mathrm{i}e^{\mathrm{i}\kappa|x-z|}-\mathrm{i}^{-j+\frac12}e^{-\kappa|x-z|}}{|x-z|^{j+\frac12}}\right)^2\\
&\quad\times\left(\overline{\frac{\mathrm{i}e^{\mathrm{i}\kappa|x-z'|}-\mathrm{i}^{-l+\frac12}e^{-\kappa|x-z'|}}{|x-z'|^{l+\frac12}}}\right)^2\mathbb{E}[\rho(z)\rho(z')]dzdz'\\
&= \frac{\kappa_{\mathrm{r}}^{-m}}{8^4|\kappa|^{10}}\int_D\frac{e^{-4\kappa_{\mathrm{i}}|x-z|}}{|x-z|^2}\mu(z)dz+O\left(\kappa_{\mathrm{r}}^{-m-11}\right)
\end{aligned}
$$

for any $x\in U$ and $\kappa_{\mathrm{r}}\to\infty$.

Combining the above estimates leads to

$$
\begin{aligned}
\mathbb{E}|u_1(x,k)|^2 &= \mathbb{E}|u_1^{(1)}(x,k)|^2+2\Re\mathbb{E}\big[\overline{u_1^{(1)}(x,k)}(u_1(x,k)-u_1^{(1)}(x,k))\big]\\
&\quad+\mathbb{E}\big|u_1(x,k)-u_1^{(1)}(x,k)\big|^2\\
&= \frac{\kappa_{\mathrm{r}}^{-m}}{8^4|\kappa|^{10}}\int_D\frac{e^{-4\kappa_{\mathrm{i}}|x-z|}}{|x-z|^2}\mu(z)dz+O\big(\kappa_{\mathrm{r}}^{-m-11}\big)\\
&\quad+O\big((\kappa_{\mathrm{r}}^{-m}|\kappa|^{-10})^{\frac12}\kappa_{\mathrm{r}}^{-7}\big)+O\big(\kappa_{\mathrm{r}}^{-14}\big)\\
&= \frac{\kappa_{\mathrm{r}}^{-m}}{8^4|\kappa|^{10}}\int_D\frac{e^{-4\kappa_{\mathrm{i}}|x-z|}}{|x-z|^2}\mu(z)dz+O\big(\kappa_{\mathrm{r}}^{-m-11}\big)\quad\forall\, x\in U. \qquad (5.7)
\end{aligned}
$$

The following theorem is concerned with the contribution of u_1 to the reconstruction formula for both the two- and three-dimensional problems.

THEOREM 5.2. *Let the random potential ρ satisfy Assumption 1.1 and $U\subset\mathbb{R}^d$ be a bounded domain having a positive distance to the support D of the strength μ. For any $x\in U$, it holds*

$$
\lim_{K\to\infty}\frac1K\int_K^{2K}\kappa_{\mathrm{r}}^{m+14-2d}\mathbb{E}|u_1(x,k)|^2d\kappa_{\mathrm{r}}=T_d(x), \qquad (5.8)
$$

where $T_d(x)$ is given in Theorem 1.2. Moreover, if $\sigma=0$, then it holds

$$
\lim_{K\to\infty}\frac1K\int_K^{2K}\kappa^{m+14-2d}|u_1(x,k)|^2d\kappa=T_d(x)\quad\mathbb{P}\text{-}a.s. \qquad (5.9)
$$

Proof. To prove (5.8), we consider the imaginary part of κ as a function of κ_{r}, i.e., $\kappa_{\mathrm{i}}=\kappa_{\mathrm{i}}(\kappa_{\mathrm{r}})$, which satisfies $\lim_{\kappa_{\mathrm{r}}\to\infty}\kappa_{\mathrm{i}}(\kappa_{\mathrm{r}})=0$. From (5.3) and (5.7), we get

$$
\lim_{\kappa_{\mathrm{r}}\to\infty}\kappa_{\mathrm{r}}^{m+14-2d}\mathbb{E}|u_1(x,k)|^2=T_d(x). \qquad (5.10)
$$

Based on the mean value theorem, (5.8) follows from the identity

$$
\lim_{\kappa_{\mathrm{r}}\to\infty}\kappa_{\mathrm{r}}^{m+14-2d}\mathbb{E}|u_1(x,k)|^2=\lim_{K\to\infty}\frac1K\int_K^{2K}\kappa_{\mathrm{r}}^{m+14-2d}\mathbb{E}|u_1(x,k)|^2d\kappa_{\mathrm{r}}.
$$

It then suffices to show (5.9) for the case $\sigma = 0$, i.e., $\kappa = \kappa_{\rm r} = k^{\frac{1}{2}} \in \mathbb{R}_+$. Noting

$$\lim_{k\to\infty} e^{-4\kappa_{\rm i}|x-z|} = 1,$$

and combining (2.1) and (5.8), we have

$$\lim_{k\to\infty} \kappa^{m+14-2d}\mathbb{E}|u_1(x,k)|^2 = T_d(x).$$

To replace the expectation in the above formula by the frequency average, an asymptotic version of the law of large numbers is required. Such a replacement is an analogue of ergodicity in the frequency domain, and has been adopted in the analysis of stochastic inverse problems (cf. [16, 17, 22]).

For $d = 3$, consider the correlations $\mathbb{E}[u_1(x,k_1)\overline{u_1(x,k_2)}]$ and $\mathbb{E}[u_1(x,k_1)u_1(x,k_2)]$ with $k_i = \kappa_i^2, i = 1,2$ at different wavenumbers κ_1 and κ_2. Following the same procedure as that used in [22, Lemma 4.1], we may show that

$$\left|\mathbb{E}[u_1(x,k_1)\overline{u_1(x,k_2)}]\right| \lesssim \kappa_1^{-4}\kappa_2^{-4}\left[(\kappa_1+\kappa_2)^{-m}(1+|\kappa_1-\kappa_2|)^{-M_1} + \kappa_1^{-M_2} + \kappa_2^{-M_2}\right],$$

$$\left|\mathbb{E}[u_1(x,k_1)u_1(x,k_2)]\right| \lesssim \kappa_1^{-4}\kappa_2^{-4}\left[(\kappa_1+\kappa_2)^{-M_1}(1+|\kappa_1-\kappa_2|)^{-m} + \kappa_1^{-M_2} + \kappa_2^{-M_2}\right],$$

where $M_1, M_2 > 0$ are arbitrary integers. The above estimates indicate the asymptotic independence of $u_1(x,k_1)$ and $u_1(x,k_2)$ for $|\kappa_1 - \kappa_2| \gg 1$. Then, according to [22, Theorem 4.2], the expectation in (5.8) can be replaced by the frequency average with respect to κ:

$$\lim_{K\to\infty}\frac{1}{K}\int_K^{2K} \kappa^{m+8}|u_1(x,k)|^2 d\kappa = T_3(x) \quad \mathbb{P}\text{-}a.s.$$

For $d = 2$, we need to consider $u_1^{(3)}$, which is the truncated u_1 with $N = 3$. Its correlations at different wavenumbers can be carried out similarly as those for the three-dimensional case (cf. [22, Lemma 4.4]). Hence

$$\lim_{K\to\infty}\frac{1}{K}\int_K^{2K} \kappa^{m+10}|u_1^{(3)}(x,k)|^2 d\kappa = T_2(x) \quad \mathbb{P}\text{-}a.s. \tag{5.11}$$

The residual $u_1 - u_1^{(3)}$ satisfies

$$\begin{aligned}
&|u_1(x,k) - u_1^{(3)}(x,k)| \\
&= \left|\int_D (\Phi^2 - \Phi_3^2)(x,z,k)\rho(z)dz\right| \\
&\lesssim \|\Phi^2(x,\cdot,k) - \Phi_3^2(x,\cdot,k)\|_{W^{1,q}(D)}\|\rho\|_{W^{-1,p}(D)} \\
&\lesssim \|\Phi(x,\cdot,k) + \Phi_3(x,\cdot,k)\|_{W^{1,2q}(D)}\|\Phi(x,\cdot,k) - \Phi_3(x,\cdot,k)\|_{W^{1,2q}(D)}\|\rho\|_{W^{-1,p}(D)} \\
&\lesssim k^{-\frac{3}{4}}\kappa^{-\frac{11}{2}} \lesssim \kappa^{-7} \quad \mathbb{P}\text{-}a.s.
\end{aligned}$$

for any $p > 1$ and q satisfying $\frac{1}{p} + \frac{1}{q} = 1$, where we used Lemmas 2.1 and 5.1, and $\rho \in W^{\frac{m-2}{2}-\epsilon,p}(D) \subset W^{-1,p}(D)$ for $m \in (1,2]$ and any sufficiently small $\epsilon \in (0, \frac{m}{2})$. We have from a simple calculation that

$$\lim_{K\to\infty}\frac{1}{K}\int_K^{2K} \kappa^{m+10}|u_1(x,k) - u_1^{(3)}(x,k)|^2 d\kappa \lesssim \lim_{K\to\infty}\frac{1}{K}\int_K^{2K} \kappa^{m-4} d\kappa = 0 \quad \mathbb{P}\text{-}a.s.$$

Combining the above estimate with (5.11) leads to

$$\lim_{K\to\infty}\frac{1}{K}\int_K^{2K}\kappa^{m+10}|u_1(x,k)|^2d\kappa=T_2(x)\quad \mathbb{P}\text{-}a.s.,$$

which completes the proof of (5.9). ∎

5.2. The analysis of u_2. It follows from (4.7) and (5.1) that

$$\begin{aligned}u_2(x,k)&=\int_{\mathbb{R}^d}\Phi(x,z,k)\rho(z)u_1(z,x,k)dz\\&=\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\Phi(x,z,k)\rho(z)\Phi(z,z',k)\rho(z')\Phi(z',x,k)dzdz',\end{aligned}$$

which does not contribute to the inversion formula as stated in the following theorem.

THEOREM 5.3. *Let the random potential ρ satisfy Assumption 1.1 and $U\subset\mathbb{R}^d$ be a bounded and convex domain having a positive distance to the support D of the strength μ. For any $x\in U$, it holds*

$$\lim_{K\to\infty}\frac{1}{K}\int_K^{2K}\kappa_{\rm r}^{m+14-2d}|u_2(x,k)|^2d\kappa_{\rm r}=0\quad \mathbb{P}\text{-}a.s.$$

Proof. The proof is motivated by [16], where the inverse random potential scattering problem is studied for the two-dimensional Schrödinger equation with $m\ge d$. In what follows, we provide some details to demonstrate the differences for the biharmonic wave equation of rougher potentials with $m\in(d-1,d]$.

(i) First we consider the case $d=3$. As a function of x and $\kappa_{\rm r}$, $u_2(x,k)$ satisfies

$$\begin{aligned}\frac{1}{K}\int_K^{2K}\kappa_{\rm r}^{m+8}|u_2(x,k)|^2d\kappa_{\rm r}&\le\int_K^{2K}\frac{\kappa_{\rm r}}{K}\kappa_{\rm r}^{m+7}|u_2(x,k)|^2d\kappa_{\rm r}\\&\le\int_1^{\infty}\min\Big\{2,\frac{\kappa_{\rm r}}{K}\Big\}\kappa_{\rm r}^{m+7}|u_2(x,k)|^2d\kappa_{\rm r}\quad \mathbb{P}\text{-}a.s.\end{aligned}$$

Then the required result is obtained by taking $K\to\infty$ if the following estimate holds:

$$(5.12)\qquad \int_1^{\infty}\kappa_{\rm r}^{m+7}\mathbb{E}|u_2(x,k)|^2d\kappa_{\rm r}<\infty\quad\forall\,x\in U.$$

To deal with the product of the rough potentials in $\mathbb{E}|u_2(x,k)|^2$, we consider the smooth modification $\rho_\varepsilon:=\rho*\varphi_\varepsilon$ with $\varphi_\varepsilon(x)=\varepsilon^{-2}\varphi(x/\varepsilon)$ for $\varepsilon>0$ and $\varphi\in C_0^\infty(\mathbb{R}^3)$. Define

$$\begin{aligned}u_{2,\varepsilon}(x,k):&=\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\Phi(x,z,k)\rho_\varepsilon(z)\Phi(z,z',k)\rho_\varepsilon(z')\Phi(z',x,k)dzdz'\\&=-\frac{1}{(8\pi\kappa^2)^3}\int_D\int_D\frac{(e^{{\rm i}\kappa|x-z|}-e^{-\kappa|x-z|})e^{{\rm i}\kappa|z-z'|}(e^{{\rm i}\kappa|x-z'|}-e^{-\kappa|x-z'|})}{|x-z||z-z'||x-z'|}\\&\qquad\times\rho_\varepsilon(z)\rho_\varepsilon(z')dzdz'\\&\quad+\frac{1}{(8\pi\kappa^2)^3}\int_D\int_D\frac{(e^{{\rm i}\kappa|x-z|}-e^{-\kappa|x-z|})e^{-\kappa|z-z'|}(e^{{\rm i}\kappa|x-z'|}-e^{-\kappa|x-z'|})}{|x-z||z-z'||x-z'|}\end{aligned}$$

$$\times \rho_\varepsilon(z)\rho_\varepsilon(z')dzdz'$$
$$=: -\frac{1}{(8\pi\kappa^2)^3}\mathrm{II}_1(x,k,\varepsilon) + \frac{1}{(8\pi\kappa^2)^3}\mathrm{II}_2(x,k,\varepsilon).$$

Note that

$$\int_1^\infty \kappa_{\mathrm{r}}^{m+7}\mathbb{E}|u_{2,\varepsilon}(x,k)|^2 d\kappa_{\mathrm{r}} \lesssim \sum_{i=1}^2 \int_1^\infty |\kappa|^{-12}\kappa_{\mathrm{r}}^{m+7}\mathbb{E}|\mathrm{II}_i(x,k,\varepsilon)|^2 d\kappa_{\mathrm{r}}$$
$$\lesssim \sum_{i=1}^2 \int_1^\infty \mathbb{E}|\mathrm{II}_i(x,k,\varepsilon)|^2 d\kappa_{\mathrm{r}},$$

where in the last inequality we used

$$|\kappa|^{-12}\kappa_{\mathrm{r}}^{m+7} \le \kappa_{\mathrm{r}}^{m-5} \le 1 \quad \forall\, m \in (2,3].$$

Based on the Fubini theorem and Fatou's lemma, to show (5.12), it suffices to prove

$$\sup_{\varepsilon\in(0,1)} \int_1^\infty \mathbb{E}|\mathrm{II}_i(x,k,\varepsilon)|^2 d\kappa_{\mathrm{r}} < \infty \quad \forall\, x \in U,\ i=1,2.$$

The estimates for II_1 and II_2 are parallel, and they are similar to the procedure used in [16, 17] for the inverse potential scattering problems of the two-dimensional acoustic and elastic wave equations without attenuation. The basic idea is to rewrite each term II_i, $i=1,2$, as the Fourier or inverse Fourier transform of some well-defined function. In the following, we only give the estimate for II_1 to show the differences in handling the attenuation.

Denote

$$\mathbb{K}(x,z,z') := \frac{(e^{\mathrm{i}\kappa|x-z|} - e^{-\kappa|x-z|})e^{-\mathrm{i}\kappa_{\mathrm{r}}|x-z|}e^{-\kappa_{\mathrm{i}}|z-z'|}e^{-\mathrm{i}\kappa_{\mathrm{r}}|z'-x|}(e^{\mathrm{i}\kappa|x-z'|} - e^{-\kappa|x-z'|})}{|x-z||z-z'||x-z'|},$$

then II_1 can be rewritten as

$$\mathrm{II}_1(x,k,\varepsilon) = \int_D\int_D e^{\mathrm{i}\kappa_{\mathrm{r}}(|x-z|+|z-z'|+|z'-x|)}\mathbb{K}(x,z,z')\rho_\varepsilon(z)\rho_\varepsilon(z')dzdz'.$$

Define a phase function

$$L(z,z') = |x-z| + |z-z'| + |z'-x|,$$

which is uniformly bounded below and above for any $(z,z') \in D\times D$ and $x \in U$. Hence the set

$$\{(z,z') \in D\times D : L(z,z') = t\}, \quad t>0$$

is non-empty only for t lying in a finite interval $[T_0,T_1]$ with $0<T_0<T_1$.

For any fixed $\tilde{t} \in [T_0,T_1]$, there exist $\eta = \eta(\tilde{t})$ and an open cone $K = K(\tilde{t}) \subset \mathbb{R}^6$ such that

$$D\times D\ \cap \{(z,z') : t_0 < L(z,z') < t_1\} \subset K \cap \{(z,z') : t_0 < L(z,z') < t_1\} =: \Gamma,$$

where $t_0 = \tilde{t} - \eta$ and $t_1 = \tilde{t}+\eta$. Letting $\Gamma_t := \Gamma \cap \{(z,z') : L(z,z') = t\}$, we have

$$\int_\Gamma e^{\mathrm{i}\kappa_{\mathrm{r}}L(z,z')}\mathbb{K}(x,z,z')\rho_\varepsilon(z)\rho_\varepsilon(z')dzdz'$$

$$
\begin{aligned}
&= \int_{t_0}^{t_1} e^{\mathrm{i}\kappa_{\mathrm{r}} t} \left[\int_{\Gamma_t} \mathbb{K}(x,z,z') |\nabla L(z,z')|^{-1} \rho_\varepsilon(z)\rho_\varepsilon(z') d\mathcal{H}^5(z,z') \right] dt \\
&=: \int_{t_0}^{t_1} e^{\mathrm{i}\kappa_{\mathrm{r}} t} S_\varepsilon(t) dt = \mathcal{F}[S_\varepsilon](-\kappa_{\mathrm{r}}),
\end{aligned}
$$

where $\mathcal{H}^5$ is the Hausdorff measure on Γ_t and S_ε is compactly supported in $[T_0, T_1]$. Applying Parseval's identity yields

$$
\int_1^\infty \mathbb{E}|\mathrm{II}_1(x,k,\varepsilon)|^2 d\kappa_{\mathrm{r}} \lesssim \mathbb{E}\|S_\varepsilon\|^2_{L^2(T_0,T_1)}.
$$

Using Isserlis' theorem, we obtain

$$
\begin{aligned}
\mathbb{E}|S_\varepsilon(t)|^2 &= \int_{\Gamma_t}\int_{\Gamma_t} \mathbb{K}(x,z_1,z_1')\overline{\mathbb{K}(x,z_2,z_2')} |\nabla L(z_1,z_1')|^{-1} |\nabla L(z_2,z_2')|^{-1} \\
&\quad \times \mathbb{E}\left[\rho_\varepsilon(z_1)\rho_\varepsilon(z_1')\rho_\varepsilon(z_2)\rho_\varepsilon(z_2')\right] d\mathcal{H}^5(z_1,z_1') d\mathcal{H}^5(z_2,z_2') \\
&= \int_{\Gamma_t}\int_{\Gamma_t} \mathbb{K}(x,z_1,z_1')\overline{\mathbb{K}(x,z_2,z_2')} |\nabla L(z_1,z_1')|^{-1} |\nabla L(z_2,z_2')|^{-1} \\
&\quad \times \Big(\mathbb{E}[\rho_\varepsilon(z_1)\rho_\varepsilon(z_1')]\mathbb{E}[\rho_\varepsilon(z_2)\rho_\varepsilon(z_2')] + \mathbb{E}[\rho_\varepsilon(z_1)\rho_\varepsilon(z_2)]\mathbb{E}[\rho_\varepsilon(z_1')\rho_\varepsilon(z_2')] \\
&\quad + \mathbb{E}[\rho_\varepsilon(z_1)\rho_\varepsilon(z_2')]\mathbb{E}[\rho_\varepsilon(z_1')\rho_\varepsilon(z_2)] \Big) d\mathcal{H}^5(z_1,z_1') d\mathcal{H}^5(z_2,z_2'),
\end{aligned}
$$

where $\mathbb{K}$ and ∇L satisfy $|\mathbb{K}(x,z,z')| \lesssim |z-z'|^{-1}$ and $0 < C_1 \le |\nabla L(z,z')| \le C_2$, respectively, for any $(z,z') \in D \times D$ with $z \neq z'$ (cf. [16]), and $|\mathbb{E}[\rho_\varepsilon(z)\rho_\varepsilon(z')]| \lesssim |z-z'|^{m-3-\epsilon}$ for any $\epsilon > 0$ and $m \in (2,3]$ according to (5.2). It follows from the Hölder inequality and the symmetry of the integral that

$$
\begin{aligned}
\mathbb{E}|S_\varepsilon(t)|^2 &\lesssim \int_{\Gamma_t}\int_{\Gamma_t} |z_1-z_1'|^{-1}|z_2-z_2'|^{-1}|z_1-z_1'|^{m-3-\epsilon} \\
&\qquad\qquad \times |z_2-z_2'|^{m-3-\epsilon} d\mathcal{H}^5(z_1,z_1') d\mathcal{H}^5(z_2,z_2') \\
&\quad + \int_{\Gamma_t}\int_{\Gamma_t} |z_1-z_1'|^{-1}|z_2-z_2'|^{-1}|z_1-z_2|^{m-3-\epsilon} \\
&\qquad\qquad \times |z_1'-z_2'|^{m-3-\epsilon} d\mathcal{H}^5(z_1,z_1') d\mathcal{H}^5(z_2,z_2') \\
&\quad + \int_{\Gamma_t}\int_{\Gamma_t} |z_1-z_1'|^{-1}|z_2-z_2'|^{-1}|z_1-z_2'|^{m-3-\epsilon} \\
&\qquad\qquad \times |z_1'-z_2|^{m-3-\epsilon} d\mathcal{H}^5(z_1,z_1') d\mathcal{H}^5(z_2,z_2') \\
&= \left(\int_{\Gamma_t} |z_1-z_1'|^{m-4-\epsilon} d\mathcal{H}^5(z_1,z_1') \right)^2 + 2\int_{\Gamma_t}\int_{\Gamma_t} |z_1-z_1'|^{-1}|z_2-z_2'|^{-1} \\
&\qquad\qquad \times |z_1-z_2|^{m-3-\epsilon}|z_1'-z_2'|^{m-3-\epsilon} d\mathcal{H}^5(z_1,z_1') d\mathcal{H}^5(z_2,z_2') \\
&\lesssim \left(\int_{\Gamma_t} |z_1-z_1'|^{m-4-\epsilon} d\mathcal{H}^5(z_1,z_1') \right)^2 \\
&\quad + \left[\int_{\Gamma_t}\int_{\Gamma_t} |z_1-z_1'|^{-3}|z_2-z_2'|^{-3} d\mathcal{H}^5(z_1,z_1') d\mathcal{H}^5(z_2,z_2') \right]^{\frac{1}{3}} \\
&\quad \times \left[\int_{\Gamma_t}\int_{\Gamma_t} |z_1-z_2|^{\frac{3}{2}(m-3-\epsilon)}|z_1'-z_2'|^{\frac{3}{2}(m-3-\epsilon)} d\mathcal{H}^5(z_1,z_1') d\mathcal{H}^5(z_2,z_2') \right]^{\frac{2}{3}}
\end{aligned}
$$

$$\lesssim \left(\int_{\Gamma_t} |z_1 - z_1'|^{m-4-\epsilon} d\mathcal{H}^5(z_1, z_1')\right)^2 + \left(\int_{\Gamma_t} |z_1 - z_1'|^{-3} d\mathcal{H}^5(z_1, z_1')\right)^{\frac{4}{3}}$$
$$+ \left(\int_{\Gamma_t}\int_{\Gamma_t} |z_1 - z_2|^{3(m-3-\epsilon)} d\mathcal{H}^5(z_1, z_1') d\mathcal{H}^5(z_2, z_2')\right)^{\frac{4}{3}},$$

where the boundedness of all the last three integrals can be obtained similarly to the two-dimensional problem shown in [16, Lemma 6].

(ii) Next we consider the case $d = 2$. Define the following auxiliary functions (cf. [17, Section 5.2]) via the truncated fundamental solution Φ_0:

$$u_{2,l}(x,k) := \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \Phi_0(x,z,k)\rho(z)\Phi(z,z',k)\rho(z')\Phi(z',x,k)dzdz',$$
$$u_{2,r}(x,k) := \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \Phi_0(x,z,k)\rho(z)\Phi(z,z',k)\rho(z')\Phi_0(z',x,k)dzdz',$$
$$v(x,k) := \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \Phi_0(x,z,k)\rho(z)\Phi_0(z,z',k)\rho(z')\Phi_0(z',x,k)dzdz'.$$

By Lemmas 2.1, 2.3, and 5.1, we have

$$\begin{aligned} &|u_2(x,k) - u_{2,l}(x,k)| \\ &\lesssim \|\rho\|_{W^{-\gamma,p}(D)} \|[\Phi(x,\cdot,k) - \Phi_0(x,\cdot,k)]\,\mathcal{K}_k\Phi(\cdot,x,k)\|_{W^{\gamma,q}(D)} \\ &\lesssim \|\Phi(x,\cdot,k) - \Phi_0(x,\cdot,k)\|_{W^{\gamma,2q}(D)}\|\mathcal{K}_k\|_{\mathcal{L}(W^{\gamma,2q}(D))}\|\Phi(\cdot,x,k)\|_{W^{\gamma,2q}(D)} \\ &\lesssim |\kappa|^{-\frac{7}{2}+\gamma} k^{\gamma-\frac{1}{q}-\frac{1}{2}+\frac{\chi_\sigma}{2}} k^{-\frac{5}{4}+\frac{\gamma}{2}} \lesssim \kappa_{\rm r}^{-7-\frac{2}{q}+4\gamma+\chi_\sigma} \quad \mathbb{P}\text{-}a.s., \end{aligned}$$

$$\begin{aligned} &|u_{2,l}(x,k) - u_{2,r}(x,k)| \\ &\lesssim \|\rho\|_{W^{-\gamma,p}(D)} \|\Phi_0(x,\cdot,k)\mathcal{K}_k\,[\Phi(\cdot,x,k) - \Phi_0(\cdot,x,k)]\|_{W^{\gamma,q}(D)} \\ &\lesssim \|\Phi_0(x,\cdot,k)\|_{W^{\gamma,2q}(D)}\|\mathcal{K}_k\|_{\mathcal{L}(W^{\gamma,2q}(D))}\|\Phi(\cdot,x,k) - \Phi_0(\cdot,x,k)\|_{W^{\gamma,2q}(D)} \\ &\lesssim \kappa_{\rm r}^{-7-\frac{2}{q}+4\gamma+\chi_\sigma} \quad \mathbb{P}\text{-}a.s., \end{aligned}$$

$$\begin{aligned} &|u_{2,r}(x,k) - v(x,k)| \\ &\lesssim \|\Phi(\cdot,\cdot,k) - \Phi_0(\cdot,\cdot,k)\|_{W^{\gamma,\tilde{q}}(D\times D)}\|(\rho\otimes\rho)(\Phi_0\otimes\Phi_0(x,\cdot,k))\|_{W^{-2\gamma,\tilde{p}}(D\times D)} \\ &\lesssim |\kappa|^{-\frac{7}{2}+\gamma}\|\rho\|^2_{W^{-\gamma,\infty}(D)}\|\Phi_0(x,\cdot,k)\otimes\Phi_0(\cdot,x,k)\|_{W^{2\gamma,\infty}(D\times D)} \\ &\lesssim \kappa_{\rm r}^{-\frac{17}{2}+4\gamma} \quad \mathbb{P}\text{-}a.s., \end{aligned}$$

where (p,q) and $(\tilde{p},\tilde{q})$ are conjugate pairs with $q > 1$, $\gamma \in (\frac{2-m}{2}, \frac{1}{2}+\frac{1}{q})$, and $\tilde{q} \in (1, \frac{4}{3})$. Choosing $q = \frac{1}{1-\epsilon}$ and $\gamma = \frac{2-m}{2} + \epsilon$ with a sufficiently small $\epsilon > 0$ in above estimates, we get

$$\begin{aligned} &\lim_{K\to\infty}\frac{1}{K}\int_K^{2K} \kappa_{\rm r}^{m+10}|u_2(x,k) - v(x,k)|^2 d\kappa_{\rm r} \\ &\lesssim \lim_{K\to\infty}\frac{1}{K}\int_K^{2K} \kappa_{\rm r}^{m+10}\left(\kappa_{\rm r}^{-7-\frac{2}{q}+4\gamma+\chi_\sigma} + \kappa_{\rm r}^{-\frac{17}{2}+4\gamma}\right)^2 d\kappa_{\rm r} \\ &\lesssim \lim_{K\to\infty}\frac{1}{K}\int_K^{2K} \left(\kappa_{\rm r}^{-3m+12\epsilon+2\chi_\sigma} + \kappa_{\rm r}^{1-3m+8\epsilon}\right) d\kappa_{\rm r} = 0 \quad \mathbb{P}\text{-}a.s. \end{aligned}$$

Hence, to show the result in the theorem, it suffices to prove that the contribution of v is zero. Similar to the three-dimensional case, we consider the smooth modification

$$
\begin{aligned}
v_\varepsilon(x,k) :&= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \Phi_0(x,z,k)\rho_\varepsilon(z)\Phi_0(z,z',k)\rho_\varepsilon(z')\Phi_0(z',x,k)dzdz' \\
&= -\frac{\mathrm{i}}{8^3\kappa^{\frac{15}{2}}}\int_D\int_D \frac{(\mathrm{i}e^{\mathrm{i}\kappa|x-z|}-\mathrm{i}^{\frac12}e^{-\kappa|x-z|})e^{\mathrm{i}\kappa|z-z'|}(\mathrm{i}e^{\mathrm{i}\kappa|z'-x|}-\mathrm{i}^{\frac12}e^{-\kappa|z'-x|})}{|x-z|^{\frac12}|z-z'|^{\frac12}|z'-x|^{\frac12}} \\
&\qquad\qquad \times \rho_\varepsilon(z)\rho_\varepsilon(z')dzdz' \\
&\quad + \frac{\mathrm{i}^{\frac12}}{8^3\kappa^{\frac{15}{2}}}\int_D\int_D \frac{(\mathrm{i}e^{\mathrm{i}\kappa|x-z|}-\mathrm{i}^{\frac12}e^{-\kappa|x-z|})e^{-\kappa|z-z'|}(\mathrm{i}e^{\mathrm{i}\kappa|z'-x|}-\mathrm{i}^{\frac12}e^{-\kappa|z'-x|})}{|x-z|^{\frac12}|z-z'|^{\frac12}|z'-x|^{\frac12}} \\
&\qquad\qquad \times \rho_\varepsilon(z)\rho_\varepsilon(z')dzdz' \\
&=: -\frac{\mathrm{i}}{8^3\kappa^{\frac{15}{2}}}\tilde{\Pi}_1(x,k,\varepsilon) + \frac{\mathrm{i}^{\frac12}}{8^3\kappa^{\frac{15}{2}}}\tilde{\Pi}_2(x,k,\varepsilon).
\end{aligned}
$$

Following the same procedure as used in the three-dimensional case, we may show

$$
\int_1^\infty \kappa_{\mathrm{r}}^{m+9}\mathbb{E}|v_\varepsilon(x,k)|^2 d\kappa_{\mathrm{r}} \lesssim \sum_{i=1}^2 \int_1^\infty \mathbb{E}|\tilde{\Pi}_i(x,k,\varepsilon)|^2 d\kappa_{\mathrm{r}} < \infty \quad \forall\, x\in U,
$$

which completes the proof. ∎

5.3. The analysis of residual. Taking out u_1 and u_2, we define the residual in the Born series

$$
b(x,k) := \sum_{n=3}^\infty u_n(x,k),
$$

which has no contribution to the reconstruction formula as shown in the following theorem.

THEOREM 5.4. *Let assumptions in Theorem 5.3 hold and in addition $m > \frac65 d - 1$ if $\sigma > 0$. Then for any $x \in U$, it holds*

$$
\lim_{k\to\infty} \kappa_{\mathrm{r}}^{m+14-2d}|b(x,k)|^2 = 0 \quad \mathbb{P}\text{-}a.s.
$$

Proof. Following the similar estimate in (4.8) with $N = 2$, we have

$$
\begin{aligned}
\|b(\cdot,k)\|_{L^\infty(U)} &\le \sum_{n=3}^\infty \|\mathcal{K}_k^n u_0(\cdot,k)\|_{L^\infty(U)} \lesssim k^{3s+\frac{d}{2}-\frac{25-6\chi_\sigma}{4}+\frac{\epsilon}{4}} \\
&\lesssim \kappa_{\mathrm{r}}^{6s+d-\frac{25-6\chi_\sigma}{2}+\frac{\epsilon}{2}} \qquad \mathbb{P}\text{-}a.s.
\end{aligned}
$$

for any $s \in (\frac{d-m}{2}, \frac{3-\chi_\sigma}{2})$, $\kappa_{\mathrm{r}} \ge C_{k_0}$ and $\epsilon > 0$, where $C_{k_0} = \Re[\kappa(k_0)]$ is the a constant depending on k_0 given in Lemma 4.3. Hence, we obtain by choosing $s = \frac{d-m}{2} + \epsilon$ that

$$
\kappa_{\mathrm{r}}^{m+14-2d}|b(x,k)|^2 \lesssim \kappa_{\mathrm{r}}^{6d-5m-11+6\chi_\sigma+13\epsilon} \to 0 \quad \mathbb{P}\text{-}a.s. \tag{5.13}
$$

as $k \to \infty$ under the condition $m \in (d-1, d]$ for $\sigma = 0$ or $m \in (\frac65 d - 1, d]$ for $\sigma > 0$, which completes the proof. ∎

5.4. The proof of Theorem 1.2. Considering the Born series of the scattered field

$$u^s(x,k) = u_1(x,k) + u_2(x,k) + b(x,k)$$

for $k \geq k_0$ with k_0 being given in Lemma 4.3, we obtain

$$\begin{aligned}
&\frac{1}{K}\int_K^{2K} \kappa_{\rm r}^{m+14-2d}\mathbb{E}|u^s(x,k)|^2 d\kappa_{\rm r} \\
&= \frac{1}{K}\int_K^{2K} \kappa_{\rm r}^{m+14-2d}\mathbb{E}|u_1(x,k)|^2 d\kappa_{\rm r} + \frac{1}{K}\int_K^{2K} \kappa_{\rm r}^{m+14-2d}\mathbb{E}|u_2(x,k)|^2 d\kappa_{\rm r} \\
&\quad + \frac{1}{K}\int_K^{2K} \kappa_{\rm r}^{m+14-2d}\mathbb{E}|b(x,k)|^2 d\kappa_{\rm r} \\
&\quad + 2\Re\left[\frac{1}{K}\int_K^{2K} \kappa_{\rm r}^{m+14-2d}\mathbb{E}\big[u_1(x,k)\overline{u_2(x,k)}\big] d\kappa_{\rm r}\right] \\
&\quad + 2\Re\left[\frac{1}{K}\int_K^{2K} \kappa_{\rm r}^{m+14-2d}\mathbb{E}\big[u_1(x,k)\overline{b(x,k)}\big] d\kappa_{\rm r}\right] \\
&\quad + 2\Re\left[\frac{1}{K}\int_K^{2K} \kappa_{\rm r}^{m+14-2d}\mathbb{E}\big[u_2(x,k)\overline{b(x,k)}\big] d\kappa_{\rm r}\right] \\
&=: \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3 + \mathcal{I}_4 + \mathcal{I}_5 + \mathcal{I}_6,
\end{aligned}$$

where $\mathcal{I}_4 \lesssim \mathcal{I}_1^{\frac12}\mathcal{I}_2^{\frac12}$, $\mathcal{I}_5 \lesssim \mathcal{I}_1^{\frac12}\mathcal{I}_3^{\frac12}$, and $\mathcal{I}_6 \lesssim \mathcal{I}_2^{\frac12}\mathcal{I}_3^{\frac12}$.

According to Theorems 5.2, 5.3, and 5.4, it is clear to note

$$\lim_{K\to\infty}\mathcal{I}_1 = T_d(x), \quad \lim_{K\to\infty}\mathcal{I}_j = 0, \quad j = 2, 3,$$

which lead to

$$\lim_{K\to\infty}\frac{1}{K}\int_K^{2K} \kappa_{\rm r}^{m+14-2d}\mathbb{E}|u^s(x,k)|^2 d\kappa_{\rm r} = T_d(x)$$

and completes the proof of (1.3).

If $\sigma = 0$, then $\kappa = \kappa_{\rm r} = k^{\frac12}$. The expectation in the above estimates can be removed due to Theorem 5.2. We then get

$$\begin{aligned}
T_d(x) &= \lim_{K\to\infty}\frac{1}{K}\int_K^{2K} \kappa^{m+14-2d}|u^s(x,k)|^2 d\kappa \\
&= \lim_{K\to\infty}\frac{1}{K}\int_{K^2}^{4K^2} k^{\frac{m+14-2d}{2}}|u^s(x,k)|^2\frac{1}{2}k^{-\frac12}dk \\
&= \lim_{K\to\infty}\frac{1}{2K}\int_{K^2}^{4K^2} k^{\frac{m+13}{2}-d}|u^s(x,k)|^2 dk \quad \mathbb{P}\text{-}a.s.,
\end{aligned}$$

which completes the proof of (1.4).

The uniqueness for the recovery of the strength μ from $\{T_d(x)\}_{x\in U}$ can be proved by following the same argument in [16, Theorem 1] or [21, Theorem 4.4].

COROLLARY 5.5. *The expression in* (1.3) *can be interchangeably substituted with*

$$\lim_{K\to\infty}\frac{1}{K}\int_1^{K} \kappa_{\rm r}^{m+14-2d}\mathbb{E}|u^{\rm s}(x,k)|^2 d\kappa_{\rm r} = T_d(x), \quad x\in U. \tag{5.14}$$

In particular, for the lossless case where $\sigma = 0$, (1.4) *can also be replaced by*

$$\lim_{K\to\infty} \frac{1}{2K} \int_1^{K^2} k^{\frac{m+13}{2}-d} |u^s(x,k)|^2 dk = T_d(x) \quad \mathbb{P}\text{-}a.s. \tag{5.15}$$

Proof. Based on the notation $u^s = u_1 + u_2 + b$, we only need to study the limits for u_1, u_2, and b, respectively.

For u_1, we denote $f(x,\kappa_{\rm r}) := \kappa_{\rm r}^{m+14-2d}\mathbb{E}|u_1(x,k)|^2$ for simplicity. To demonstrate

$$\lim_{K\to\infty} \frac{1}{K} \int_1^K f(x,\kappa_{\rm r}) d\kappa_{\rm r} = T_d(x), \tag{5.16}$$

we equivalently need to prove that for any $x \in U$ and $\epsilon > 0$, there exists some $K_* = K_*(x,\epsilon) > 0$ such that for any $K > K_0$, it holds

$$\left| \frac{1}{K} \int_1^K f(x,\kappa_{\rm r}) d\kappa_{\rm r} - T_d(x) \right| < \epsilon.$$

Indeed, according to (5.10), there exists $K_0 = K_0(x,\epsilon) > 1$ such that for any $\kappa_{\rm r} > K_0$, it holds

$$|f(x,\kappa_{\rm r}) - T_d(x)| < \frac{\epsilon}{2}.$$

Moreover, for any fixed x, $f(x,\kappa_{\rm r})$ is uniformly bounded for $\kappa_{\rm r} \in [1,K_0]$ according to (5.3) and (5.7). Hence, denoting $C = C(x,K_0) := \sup_{\kappa_{\rm r}\in[1,K_0]} f(x,\kappa_{\rm r}) + T_d(x)$ such that

$$|f(x,\kappa_{\rm r}) - T_d(x)| \le C \quad \forall\, \kappa_{\rm r} \in [1,K_0]$$

and choosing $K_* = C(K_0-1)\frac{2}{\epsilon} > 0$, we deduce that for any $K > \max\{K_{,}K_0\}$:

$$\begin{aligned}
&\left| \frac{1}{K} \int_1^K f(x,\kappa_{\rm r}) d\kappa_{\rm r} - T_d(x) \right| \\
\le& \frac{1}{K} \int_1^{K_0} |f(x,\kappa_{\rm r}) - T_d(x)| d\kappa_{\rm r} + \frac{1}{K} \int_{K_0}^K |f(x,\kappa_{\rm r}) - T_d(x)| d\kappa_{\rm r} \\
\le& \frac{(K_0-1)C}{K} + \frac{K-K_0}{K}\frac{\epsilon}{2} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,
\end{aligned}$$

which completes the proof of (5.16).

For u_2, it is true that

$$\lim_{K\to\infty} \frac{1}{K} \int_1^K \kappa_{\rm r}^{m+14-2d} |u_2(x,k)|^2 d\kappa_{\rm r} = 0 \quad \mathbb{P}\text{-}a.s.,$$

and its proof is identical to that of Theorem 5.3. This can be seen by observing that

$$\frac{1}{K} \int_1^K \kappa_{\rm r}^{m+14-2d} |u_2(x,k)|^2 d\kappa_{\rm r} \le \int_1^\infty \min\left\{1, \frac{\kappa_{\rm r}}{K}\right\} \kappa_{\rm r}^{m+14-2d} |u_2(x,k)|^2 d\kappa_{\rm r} \quad \mathbb{P}\text{-}a.s.$$

For term b, its estimate (5.13) implies that

$$\lim_{K\to\infty} \frac{1}{K} \int_1^K \kappa_{\rm r}^{m+14-2d} |b(x,k)|^2 d\kappa_{\rm r} = 0 \quad \mathbb{P}\text{-}a.s.$$

We can then deduce (5.14). If, in particular, $\sigma = 0$, (5.15) can be obtained using the procedure employed in Theorem 5.2, along with the result (5.14). ∎

6. Conclusion. In this paper, we have studied the random potential scattering for biharmonic waves in lossy media. The unique continuation principle is proved for the biharmonic wave equation with rough potentials. Based on the equivalent Lippmann–Schwinger integral equation, the well-posedness is established for the direct scattering problem in the distribution sense. The uniqueness is attained for the inverse scattering problem. Particularly, we show that the correlation strength of the random potential is uniquely determined by the high frequency limit of the second moment of the scattered wave field averaged over the frequency band. Moreover, we demonstrate that the expectation can be removed and the data of only a single realization is needed almost surely to ensure the uniqueness of the inverse problem when the medium is lossless.

Finally, we point out some important future directions along the line of this research. In this work, the convergence of the Born series is crucial for the inverse problem. However, this approach is not applicable to the inverse random medium scattering problems, since the Born series for the medium scattering problem does not converge any more in the high frequency regime. It is unclear whether the correlation strength of the random medium can be uniquely determined by some statistics of the wave field. Other interesting problems include the inverse random source or potential problems for the wave equations with higher order differential operators, such as the stochastic polyharmonic wave equation.

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