INVERSE SOURCE PROBLEMS IN ELECTRODYNAMICS

YUE ZHAO, GUANGHUI HU, PEIJUN LI, AND XIAODONG LIU

ABSTRACT. This paper concerns inverse source problems for the time-dependent Maxwell equations. The electric current density is assumed to be a separable function, which is the product of a spatial function and a temporal function. We prove uniqueness and stability in determining the spatial or temporal function from the electric field, which is measured on a sphere or at a point over a finite time interval.

1. INTRODUCTION

Consider the time-dependent Maxwell equations in a homogeneous medium:

\[ \partial_t \mathbf{H}(x, t) + \nabla \times \mathbf{E}(x, t) = 0, \quad \partial_t \mathbf{E}(x, t) - \nabla \times \mathbf{H}(x, t) = \mathbf{F}(x, t), \quad x \in \mathbb{R}^3, \quad t > 0, \tag{1.1} \]

where \( \mathbf{E} \) and \( \mathbf{H} \) are the electric and magnetic fields, respectively, and the source \( \mathbf{F} \) is known as the electric current density. Eliminating the magnetic field \( \mathbf{H} \) from (1.1), we obtain the Maxwell system for the electric field \( \mathbf{E} \):

\[ \partial_t^2 \mathbf{E}(x, t) + \nabla \times (\nabla \times \mathbf{E}(x, t)) = \partial_t \mathbf{F}(x, t), \quad x \in \mathbb{R}^3, \quad t > 0, \tag{1.2} \]

which is supplemented by the homogeneous initial conditions:

\[ \mathbf{E}(x, 0) = \partial_t \mathbf{E}(x, 0) = 0, \quad x \in \mathbb{R}^3. \tag{1.3} \]

This paper concerns the inverse source problem of determining the electric current density. We assume that the source is a separable function, and consider the following two inverse problems:

(i) Inverse problem one (IP1). The electric current density is assumed to take the form

\[ \mathbf{F}(x, t) = J(x)g(t), \]

where \( g \) is a known scalar function satisfying \( g(t) = 0, t \in (-\infty, 0] \cup [T_0, \infty) \), where \( T_0 > 0 \), and \( g \in C([-0, T_0]), g' \in L^2([0, T_0]) \); \( J \) is an unknown vector function satisfying \( J \in H^1(\mathbb{R}^3)^3 \), \( \text{supp}(J) \subset B_{\hat{R}} \), where \( B_{\hat{R}} \) is the ball with a radius \( \hat{R} > 0 \). In addition, we also assume that \( \nabla \cdot J = 0 \) in \( \mathbb{R}^3 \). Let \( R > \hat{R} \) and \( \Gamma_R = \{ x \in \mathbb{R}^3 : |x| = R \} \). Denote \( T = T_0 + \hat{R} + R \). The inverse problem is to determine \( J \) from the measurement \( \mathbf{E}(x, t) \times \nu, x \in \Gamma_R, t \in (0, T) \), where \( \nu \) is the unit normal vector on \( \Gamma_R \).

(ii) Inverse problem two (IP2). The electric current density has the form

\[ \mathbf{F}(x, t) = J(x)g(t), \]

where \( J \) is a given scalar function satisfying \( \text{supp}(J) \subset B_{\hat{R}} \) and \( g \) is an unknown vector function which is assumed to satisfy \( g \in H^1([-0, T])^3 \). In this paper we are interested in the inverse problem of determining \( g \) from the interior electric field \( \mathbf{E}(x_0, t), t \in (0, T) \) for some \( x_0 \in \text{supp}(J) \) or from the boundary tangential data \( \mathbf{E}(x, t) \times \nu, x \in \Gamma_R, t \in (0, T) \).

Inverse source problems have many significant applications in antenna synthesis, biomedical engineering, medical imaging, and optical tomography [3, 6, 15]. In general, there is no uniqueness for the inverse source problems with a single frequency data, due to the existence of non-radiating sources, e.g., see [12, 14] for acoustic problems. A non-uniqueness example was discussed in [1] to determine a volume current source.
in an inhomogeneous medium. But, the surface currents or dipole sources can be uniquely determined by
surface measurements at a fixed frequency. In [4], the authors showed uniqueness and stability, and presented
an inversion scheme to reconstruct dipole sources based on the low-frequency asymptotic analysis of the time-
harmonic Maxwell equations. A monograph for a formulation with impulsive inputs can be found in [24].
We refer to [16, 19, 20, 25] for the method of applying Carleman estimate to inverse source problems for
hyperbolic systems, and to [22, 23] for other formulation of inverse problems which are related to Maxwell’s
equations.

Recently, many efforts have been made on inverse source scattering problems by using multi-frequency
data to overcome nonuniqueness and to achieve increasing stability estimate for the Helmholtz, Navier, and
Maxwell equations in the frequency domain [2, 8–10, 13, 17, 18, 26]. In [11], the first attempt was made to
bridge the connection of stability estimate between the multi-frequency Helmholtz equation and the wave
equation. The spatial source function was transformed to be the inhomogeneous initial conditions for the
wave equation. In this work, we consider directly how to determine the temporal and spatial source func-
tions for the time-dependent Maxwell equations. A related work can be found in [7] for the inverse source
problems in elastodynamics. In section 2, we study IP1 and present the uniqueness and stability results for
recovering the spatial function. In particular, we show unique determination of the maximum and minimum
distance between one observation point and the support of the spatial function, and provide novel mathemat-
ical techniques for deriving the stability estimate with boundary observations. In section 3, we discuss IP2
and show the unique determination of the temporal function by using boundary measurement and the stability
estimate with an interior measurement at one point only.

2. IP1: DETERMINATION OF THE SPATIAL FUNCTION

2.1. Preliminaries. We first introduce the electrodynamic Green tensor $G$ to the system (1.1) and then
present an estimate for the electric field in terms of the regularity of $J$.

Consider the Maxwell system

$$\begin{aligned}
\partial_t H(x, t) + \nabla \times G(x, t) &= 0, \\
-\partial_t G(x, t) + \nabla \times H(x, t) &= \delta(t)\delta(x) I,
\end{aligned}$$

where $G$ and $H$ are the electric and magnetic Green tensors, respectively, $I$ is the $3 \times 3$ identity matrix, and $\delta$ is the Dirac distribution. It is easy to verify that $G$ satisfies

$$\partial_t^2 G(x, t) + \nabla \times (\nabla \times G(x, t)) = -\delta'(t)\delta(x) I. \quad (2.1)$$

The homogeneous initial conditions are imposed:

$$G(x, 0) = \partial_t G(x, 0) = 0, \quad |x| \neq 0.$$

To express $G$, we introduce a vector potential $\Phi$ and a matrix potential $A$ such that

$$\begin{aligned}
H &= \nabla \times A, \\
G &= -\partial_t A - \nabla \Phi, \\
\nabla \cdot A + \partial_t \Phi &= 0.
\end{aligned}$$

The last equation is known as the Lorentz gauge condition.

Substituting the above decomposition and gauge condition to (2.1), we may verify that $A$ satisfies the
following wave equation:

$$\partial_t^2 A - \Delta A = \delta(t)\delta(x) I.$$

It is well-known that

$$G(x, t) = \frac{1}{4\pi |x|} \delta(|x| - t)$$

is the fundamental solution of the wave equation in $\mathbb{R}^3 \times [0, \infty)$, i.e., it satisfies

$$\partial_t^2 G(x, t) - \Delta G(x, t) = \delta(t)\delta(x).$$
and the homogeneous initial conditions. Therefore, we have
\[ \mathbb{A} = G(x, t) \mathbb{I}. \]

On the other hand, it follows from the Lorentz gauge condition that we get
\[ \partial_t \Phi(x, t) = -\nabla \cdot \mathbb{A}(x, t) = -\nabla G(x, t). \]

Recall that the derivative of the Heaviside step function coincides with the Dirac distribution, i.e., \( H'(t) = \delta(t) \). Then we obtain
\[ \Phi(x, t) = \nabla \left( \frac{1}{4\pi |x|} H(|x| - t) \right), \]

Consequently, it follows from the relation \( G = -\partial_t \mathbb{A} - \nabla \Phi \) that the electrodynamic Green tensor can be expressed as
\[ G(x, t) = \frac{1}{4\pi |x|} \delta'(|x| - t) \mathbb{I} - \nabla \nabla^\top \left( \frac{1}{4\pi |x|} H(|x| - t) \right). \tag{2.2} \]

Denote by \( \hat{G}(x, \kappa) \) the Fourier transform of \( G(x, \cdot) \) with respect to the time variable, i.e.,
\[ \hat{G}(x, \kappa) = \int_{\mathbb{R}} G(x, t) e^{-i \kappa t} dt. \]

Then it follows from (2.1) and (2.2) that
\[ \nabla \times (\nabla \times \hat{G}) - \kappa^2 \hat{G} = -i \kappa \delta(x) \mathbb{I}, \quad x \in \mathbb{R}^3, \]
which gives
\[ \hat{G}(x, \kappa) = -i \kappa \left( g(x, \kappa) \mathbb{I} + \frac{1}{\kappa^2} \nabla \nabla^\top g(x, \kappa) \right). \tag{2.3} \]

Here \( g \) is the fundamental solution of the three-dimensional Helmholtz equation. Explicitly, we have
\[ g(x, \kappa) = \frac{1}{4\pi} e^{i\kappa |x|}. \]

Clearly, \( \hat{G}(x, \kappa) \) satisfies the Silver–Müller radiation condition in the frequency domain.

The following lemma states that the electric fields \( \mathbf{E} \) over \( B_R \) must vanish after a finite time. Physically, this phenomenon can be interpreted by Huygens’ principle.

**Lemma 2.1.** We have \( \mathbf{E}(x, t) = 0 \) for all \( x \in B_R, t > T \).

**Proof.** Using Green’s tensor (2.2), we have
\[
\mathbf{E}(x, t) = \int_0^\infty \int_{\mathbb{R}^3} G(x - y, t - s) \mathbf{J}(y) g(s) dy ds \\
= \int_0^\infty \int_{\mathbb{R}^3} \frac{1}{4\pi |x - y|} \delta'(|x - y| - (t - s)) \mathbf{J}(y) g(s) dy ds \\
- \int_0^\infty \int_{\mathbb{R}^3} \nabla_x \nabla_x^\top \left( \frac{1}{4\pi |x - y|} H(|x - y| + s - t) \right) \mathbf{J}(y) g(s) dy ds \\
= - \int_{B_R} \frac{1}{4\pi |x - y|} g'(t - |x - y|) \mathbf{J}(y) dy \\
- \int_0^{T_0} \int_{B_R} \left( \frac{1}{4\pi |x - y|} H(|x - y| + s - t) \right) \nabla_y \nabla_y^\top \cdot \mathbf{J}(y) g(s) dy ds.
\]

For \( t > T = T_0 + R + \hat{R}, \) one can easily observe that
\[ g'(t - |x - y|) = 0, \quad H(|x - y| + s - t) = 0 \]
hold uniformly for all \( x \in B_R, \ y \in B_{\hat{R}}, s \in (0, T_0), \) which imply the result. \( \square \)
Recalling $\nabla \cdot J = 0$, taking the divergence on both sides of (1.2), and using the initial conditions (1.3), we have

$$
\frac{\partial^2}{\partial t^2} \left( \nabla \cdot E(x, t) \right) = 0, \quad x \in \mathbb{R}^3, \ t > 0
$$

and

$$
\nabla \cdot E(x, 0) = \frac{\partial}{\partial t} \left( \nabla \cdot E(x, 0) \right) = 0.
$$

Therefore, $\nabla \cdot E(x, t) = 0$ for all $x \in \mathbb{R}^3$ and $t > 0$. In view of the identify $\nabla \times \nabla \times = -\Delta + \nabla \nabla \cdot$, we obtain from (1.2) that

$$
\begin{cases}
\frac{\partial^2}{\partial t^2} E(x, t) - \Delta E(x, t) = J(x) g'(t), \quad x \in \mathbb{R}^3, \ t > 0, \\
E(x, 0) = \frac{\partial}{\partial t} E(x, 0) = 0, \quad x \in \mathbb{R}^3.
\end{cases}
$$

(2.4)

To state the regularity of the solution for the initial value problem (2.4), we recall definition of spaces involving time variables. Given the Banach space $X$ with norm $\| \cdot \|_X$, the space $C([0, T]; X)$ consists of all continuous functions $f : [0, T] \to X$ with the norm

$$
\| f \|_{C([0, T]; X)} := \max_{t \in [0, T]} \| f(t, \cdot) \|_X.
$$

The Sobolev space $H^m(0, T; X)$ comprises all functions $f : L^2(0, T; X)$ such that $\partial^k_t f$ exists in the weak sense and belongs to $L^2(0, T; X)$ for $k = 0, 1, 2, \ldots, m$.

**Lemma 2.2.** Let $J \in H^p(\mathbb{R}^3)^3$ with $p > 0$ be supported on $B_R$. The initial value problem (2.4) admits a unique solution $E \in C([0, T]; H^{p+1}(\mathbb{R}^3)^3) \cap H^p(0, T; H^{p+1}(\mathbb{R}^3)^3)$ $(\tau = 1, 2)$, which satisfies

$$
\| E \|_{C([0, T]; H^{p+1}(\mathbb{R}^3)^3)} + \| E \|_{H^\tau(0, T; H^{p+1}(\mathbb{R}^3)^3)} \leq C \| g' \|_{L^2(0, T)} \| J \|_{H^p(\mathbb{R}^3)^3},
$$

(2.5)

where $C$ is a positive constant depending on $R$.

**Proof.** Taking the Fourier transform of (2.4) with respect to the spatial variable $x$, we obtain

$$
\begin{cases}
\frac{\partial^2}{\partial t^2} \hat{E}(\xi, t) + A(\xi) \hat{E}(\xi, t) = g'(t) \hat{J}(\xi), \\
\hat{E}(\xi, 0) = \frac{\partial}{\partial t} \hat{E}(\xi, t) = 0,
\end{cases}
$$

(2.6)

where $\xi \in \mathbb{R}^3$, $A(\xi) = |\xi|^2 I$. By Duhamel’s principle, it is clear to note that the unique solution of (2.6) is

$$
\hat{E}(\xi, t) = \int_0^t g'(s) |\xi|^{-1} \sin((t - s)|\xi|) \hat{J}(\xi) ds.
$$

For all $t \in [0, T]$ and $s \in [0, t]$, define

$$
K(t - s, \cdot) := \xi \mapsto |\xi|^{-1} \sin((t - s)|\xi|) \hat{J}(\xi).
$$

Then we have

$$
\hat{E}(\xi, t) = \int_0^t g'(s) K(t - s, \xi) \ ds
$$

and

$$
\begin{aligned}
\| K(t - s, \cdot) \|^2_{L^2(\mathbb{R}^3)^3} &\leq \| \hat{J} \|^2_{L^\infty(\mathbb{R}^3)^3} \int_{B_R} |\xi|^{-2} d\xi + \int_{\mathbb{R}^3 \setminus B_R} |\xi|^{-2} |\hat{J}|^2 d\xi \\
&\leq C_1 \| \hat{J} \|^2_{L^\infty(\mathbb{R}^3)^3} + C_2 \| \hat{J} \|^2_{L^2(\mathbb{R}^3)^3} \leq C \| \hat{J} \|^2_{L^2(\mathbb{R}^3)^3},
\end{aligned}
$$

(2.7)

where $C_1, C_2, C$ are positive constants depending on $R$. For some positive constant $C$. Similarly, we obtain for $p > 0$ that

$$
\| (1 + |\xi|^2)^{p/2} K(t - s, \cdot) \|^2_{L^2(\mathbb{R}^3)^3} \leq C \int_{\mathbb{R}^3} (1 + |\xi|^2)^p |\hat{J}(\xi)|^2 d\xi \leq C \| \hat{J} \|^2_{H^p(\mathbb{R}^3)^3},
$$

(2.8)
which implies that the equation (1.2) becomes

\[ \partial_t E(x,t) = \partial_t^2 \hat{K}(t-s,\cdot) \]

imposed on condition (2.12) can be equivalently written as

\[ T: H^p(\mathbb{R}^3) \rightarrow H^{p+1}(\mathbb{R}^3) \]

Moreover, for almost every \( \xi \in \mathbb{R}^3 \), we have

\[ \partial^2_\xi E(x,t) = g(s)\hat{J}(\xi) + \int_0^t g(s)\partial^2_\xi K(t-s,\xi)ds. \]

It is easy to prove that

\[ \hat{E}(\xi,\cdot): t \mapsto \hat{E}(\xi,t) \in H^2(0,T). \]

Combing the above estimate with (2.8)–(2.9), we deduce that \( E \in H^\tau(0,T; H^{p-\tau+1}(\mathbb{R}^3)^3) \) (\( \tau = 1, 2 \)), which completes the proof.

Assuming that the temporal function \( g \) is given, we present a Fourier approach to determine the unknown spatial function \( J \) in the subsequent two subsections. Our arguments rely on the Fourier transform and are motivated by recent studies on inverse source problems for the time-harmonic elastic and electromagnetic wave equations [9].

### 2.2. Uniqueness

First we consider the uniqueness for IP1.

**Theorem 2.3.** The spatial source function \( J \) can be uniquely determined by the data set \( \{ E(x,t) \times \nu : x \in \Gamma_R, t \in (0,T) \} \).

**Proof.** It suffices to show \( J = 0 \) in \( B_R \) if \( E(x,t) \times \nu(x) = 0 \) for all \( x \in \Gamma_R, t \in (0,T) \). Recalling Lemma 2.1, we have \( E(x,t) \times \nu(x) = 0 \) for all \( x \in \Gamma_R, t \in \mathbb{R}^+ \). Combining this with the fact that \( E(x,t) \) for \( t \leq 0 \), we deduce that \( E(x,t) \times \nu(x) = 0 \) for all \( x \in \Gamma_R, t \in \mathbb{R} \). Define by \( \hat{E}(x,\nu) \) the Fourier transform of \( E(x,t) \) with respect to the time \( t \), i.e.,

\[ \hat{E}(x,\nu) = \int_\mathbb{R} E(x,t)e^{-i\kappa t}dt, \quad \forall x \in \Gamma_R, \nu \in \mathbb{R}^+. \]

We have

\[ \hat{E}(x,\nu) \times \nu(x) = 0, \quad \forall x \in \Gamma_R, \nu \in \mathbb{R}^+. \]

Then the equation (1.2) becomes

\[ \nabla \times (\nabla \times \hat{E}) - \kappa^2 \hat{E} = i\kappa\hat{g} \]

in \( \mathbb{R}^3 \).

From (2.3) we obtain

\[ \hat{E}(x,\nu) = -\hat{g}(\kappa) \int_{\mathbb{R}^3} \hat{G}(x-y,\kappa)J(y)dy. \]

Since \( \text{supp}(J) \subset B_R \), it is clear to note that \( \hat{E} \) satisfies the Silver–Müller radiation condition:

\[ \lim_{r \to \infty} ((\nabla \times \hat{E}) \times x - i\kappa r \hat{E}) = 0, \quad r = |x|, \]

for any fixed frequency \( \kappa > 0 \). Let \( \hat{E} \times \nu \) and \( \hat{H} \times \nu \) be the tangential trace of the electric and the magnetic fields in the frequency domain, respectively. In the Fourier domain, there exists a capacity operator \( T : H^{-1/2}(\text{curl}, \Gamma_R) \rightarrow H^{-1/2}(\text{div}, \Gamma_R) \) such that the following transparent boundary condition can be imposed on \( \Gamma_R \) (see e.g., [5, 21]):

\[ \hat{H} \times \nu = T(\hat{E} \times \nu) \] on \( \Gamma_R, \]

which implies that \( \hat{H} \times \nu \) can be computed once \( \hat{E} \times \nu \) is available on \( \Gamma_R \). The transparent boundary condition (2.12) can be equivalently written as

\[ (\nabla \times \hat{E}) \times \nu = i\kappa T(\hat{E} \times \nu) \] on \( \Gamma_R. \]
Next we introduce the functions $\mathbf{E}^{\text{inc}}$ and $\mathbf{H}^{\text{inc}}$ by
\[
\mathbf{E}^{\text{inc}}(x) = p e^{i\mathbf{x} \cdot \mathbf{d}} \quad \text{and} \quad \mathbf{H}^{\text{inc}}(x) = q e^{i\mathbf{x} \cdot \mathbf{d}},
\]
where $\mathbf{d} := d(\theta, \varphi) = (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta)^\top$ is the unit propagation vector, and $p, q$ are two unit polarization vectors satisfying $p(\theta, \varphi) \cdot \mathbf{d}(\theta, \varphi) = q(\theta, \varphi) = p(\theta, \varphi) \times \mathbf{d}(\theta, \varphi)$ for any fixed $\theta \in [0, \pi], \varphi \in [0, 2\pi]$. It is easy to verify that $\mathbf{E}^{\text{inc}}$ and $\mathbf{H}^{\text{inc}}$ satisfy the homogeneous time-harmonic Maxwell equations in $\mathbb{R}^3$:
\[
\nabla \times (\nabla \times \mathbf{E}^{\text{inc}}) - \kappa^2 \mathbf{E}^{\text{inc}} = 0 \quad \text{(2.15)}
\]
and
\[
\nabla \times (\nabla \times \mathbf{H}^{\text{inc}}) - \kappa^2 \mathbf{H}^{\text{inc}} = 0. \quad \text{(2.16)}
\]

Let $\xi = -\kappa \mathbf{d}$ with $|\xi| = \kappa \in (0, \infty)$. We have from (2.14) that $\mathbf{E}^{\text{inc}} = p e^{-i\xi \cdot \mathbf{x}}$ and $\mathbf{H}^{\text{inc}} = q e^{-i\xi \cdot \mathbf{x}}$.

Multiplying both sides of (2.10) by $\mathbf{E}^{\text{inc}}$ and using the integration by parts over $B_R$ and (2.15), we have from $\mathbf{E}(x, \kappa) \times \nu = 0$ and the transparent boundary condition (2.13) that
\[
\begin{align*}
\int_{B_R} p e^{-i\xi \cdot \mathbf{x}} \cdot \mathbf{J}(x) \, dx &= \i \kappa \hat{g}(\kappa) \int_{B_R} \mathbf{E}^{\text{inc}}(x, \kappa) \cdot \mathbf{J}(x) \, dx \\
&= \i \kappa \int_{\Gamma_R} \nu \times (\nabla \times \mathbf{E}(x, \kappa)) \cdot \mathbf{E}^{\text{inc}}(x, \kappa) \, dS \\
&= - \i \kappa \int_{\Gamma_R} \left( \i \kappa T(\mathbf{E}(x, \kappa) \times \nu) \cdot \mathbf{E}^{\text{inc}}(x, \kappa) + (\mathbf{E}(x, \kappa) \times \nu) \cdot (\nabla \times \mathbf{E}^{\text{inc}}(x, \kappa)) \right) \, dS \\
&= 0. \quad \text{(2.17)}
\end{align*}
\]

Hence we obtain
\[
\hat{g}(\kappa) p \cdot \mathbf{J}(\xi) = 0, \quad \forall \kappa \in \mathbb{R}^+. \quad \text{(2.18)}
\]

Similarly, we may deduce from (2.16) and the integration by parts that
\[
\hat{g}(\kappa) q \cdot \mathbf{J}(\xi) = 0, \quad \forall \kappa \in \mathbb{R}^+.
\]

On the other hand, since $\mathbf{J}$ is compactly supported in $B_R$ and $\nabla \cdot \mathbf{J} = 0$, we have
\[
\begin{align*}
\int_{\mathbb{R}^3} d e^{-i\mathbf{x} \cdot \mathbf{d}} \cdot \mathbf{J}(x) \, dx &= - \frac{1}{i \kappa} \int_{\mathbb{R}^3} \nabla e^{-i\mathbf{x} \cdot \mathbf{d}} \cdot \mathbf{J}(x) \, dx \\
&= \frac{1}{i \kappa} \int_{\mathbb{R}^3} e^{-i\mathbf{x} \cdot \mathbf{d}} \nabla \cdot \mathbf{J}(x) \, dx = 0,
\end{align*}
\]
which implies that $\mathbf{d} \cdot \mathbf{J}(\xi) = 0$. Since $p, q, d$ are orthonormal vectors, they form an orthonormal basis in $\mathbb{R}^3$. It follows from the previous identities that
\[
\hat{g}(\kappa) \mathbf{J}(\xi) = \hat{g}(\kappa) \left( p \cdot \mathbf{J}(\xi) p + q \cdot \mathbf{J}(\xi) q + d \cdot \mathbf{J}(\xi) d \right) = 0.
\]

Since $g \neq 0$, we can always find an interval $(a, b) \in \mathbb{R}^+$ such that $\hat{g}(\kappa) \neq 0$ for $\kappa \in (a, b)$. Hence, we have
\[
\mathbf{J}(\xi) = 0, \quad \xi = -\kappa \mathbf{d}, \quad \text{for all} \quad \mathbf{d} \in \mathbb{S}^2, \kappa \in \mathbb{R}^+.
\]

Knowing that $\mathbf{J}$ is an analytical function and $\mathbf{d}$ is an arbitrary unit vector, we obtain $\mathbf{J} = 0$, which completes the proof by taking the inverse Fourier transform. \(\square\)

If the electric field is measured at one point $z \in \Gamma_R$, it is impossible to determine the source function $\mathbf{J}$ in general. However, it is also interesting to know what kind of information can be extracted from a single receiver. To this end, following [2], we prove that the maximum and minimum distance between $z$ and the support of $\mathbf{J}$ can be uniquely determined using the electric data at $z$. 

Theorem 2.4. Let \( D = \text{supp}(J) \) be connected and \( \nabla \cdot J = 0 \). Let \( z \in \Gamma_R \) and define
\[
h_z := \inf_{y \in D} |z - y|, \quad H_z := \sup_{y \in D} |z - y|.
\]
Then the interval \((h_z, H_z)\) is uniquely determined by \( \{E(z, t) : t \in (0, T)\} \) at one receiver \( z \in \mathbb{R}^3, |z| = R \), provided that the zeros of the function
\[
F_z : r \mapsto \int_{|z-y|=r} J(y) \, ds(y), \quad r > 0
\]
form a discrete set in the interval \((h_z, H_z)\).

Proof. Taking the Fourier transform of (2.11) and following the same proof as that for Theorem 2.3, we obtain
\[
\hat{E}(z, \kappa) = -\hat{g}(\kappa) \int_{\mathbb{R}^3} \hat{\zeta}(z - y, \kappa) J(y) \, dy
= -i\kappa \hat{g}(\kappa) \int_{\mathbb{R}^3} \left(g(z - y, \kappa) I + \frac{1}{\kappa^2} \nabla \nabla^\top g(z - y, \kappa)\right) J(y) \, dy.
\]
Noting that \( \hat{E}(z, \kappa), \kappa \in \mathbb{R}^+ \) is uniquely determined by the data \( \{E(z, t) : t \in (0, T)\} \), \( \hat{E}(z, \kappa) = \overline{E}(z, -\kappa) \) for \( \kappa < 0 \), and \( \nabla \cdot J = 0 \), we have from the integration by parts that
\[
\hat{E}(z, \kappa) = -i\kappa \hat{g}(\kappa) \int_{\mathbb{R}^3} g(z - y, \kappa) J(y) \, dy
= -i\kappa \hat{g}(\kappa) \int_{0}^{\infty} \int_{|z-y|=r} \left(\frac{e^{i\kappa r}}{r}\right) J(y) \, ds(y) \, dr
= -i\kappa \hat{g}(\kappa) \int_{0}^{\infty} e^{i\kappa r} \left(\frac{F_z(r)}{r}\right) \, dr.
\]
We may extend \( F_z \) from \( \mathbb{R}^+ \) to \( \mathbb{R} \) by zero, since \( \text{supp}(F_z) \subseteq [h_z, H_z] \). This suggests that the Fourier transform of the one-dimensional function \( r \rightarrow F_z(r)/r \) with the Fourier variable \( \kappa > 0 \) is uniquely determined, provided \( \hat{g}(\kappa) \neq 0 \). By the analyticity of \( \hat{g} \), we may conclude that the function \( F_z(r) \) can be uniquely determined. Recalling our assumption that the zeros of \( F_z \) in \((h_z, H_z)\) are discrete, we get \( \text{supp}(F_z) = [h_z, H_z] \).

We end up the uniqueness results with the following remarks:

(i) It follows from the proof of Theorem 2.4 that the function \( F_z \) could be essentially identified. If the Lebesgue measure of the zeros of \( F_z \) in \((h_z, H_z)\) is not zero, we may construct examples to show that the distance \( h_z \) and \( H_z \) can not be uniquely determined. We refer to [2] for discussions in the acoustic case. When \( \text{supp}(J) \) consists of several disconnected components, one can prove the unique determination of the union of the subintervals formed by the maximum and minimum distance to each connected component of \( J \).

(ii) If one component of the electric data \( E = (E_1, E_2, E_3) \) is measured at \( z \), say e.g., \( E_j \), then the maximum and minimum distance between \( z \) and the support of the \( j \)-th component of \( J \) can be recovered. This follows directly from the proof of Theorem 2.4.

2.3. Stability estimate. In this section, we consider the stability estimate of the source term \( J \). Since the temporal function \( g \) is given, we assume that there exists a subset \( I \subset \mathbb{R} \) and constants \( M, K > 0 \) such that
\[
\sup\{|\kappa|, \kappa \in I\} < K, \quad |\hat{g}(\kappa)| \geq M, \quad \forall \kappa \in I.
\] (2.19)
In many practical applications, the Gaussian type excitation signals always appear. For instance, \( g(t) \) can be taken as a Gaussian-modulated sinusoidal pulse of the form

\[
g(t) = \chi(t; \omega, \sigma, \tau) := \begin{cases} 
\sin(\omega t) \exp(-\sigma(t-\tau)^2), & 0 \leq t \leq 2m\pi/\omega, \\
0, & t < 0 \text{ or } t > 2m\pi/\omega,
\end{cases}
\]

(2.20)

for some \( m \in \mathbb{N} \), where \( \omega > 0 \) is the center frequency, \( \sigma > 0 \) is the frequency bandwidth parameter, and \( \tau > 0 \) is a time-shift parameter related to the pulse peak time. In this case, the interval \( I \) can be chosen as

\[
I = (\omega - \sigma, \omega + \sigma);
\]

see e.g., Figure 1. In general, \( g \) can be a linear combination of such pulse functions, i.e.,

\[
g(t) = \sum_{j=1}^{N} \chi(t; \omega_j, \sigma_j, \tau_j).
\]

To get an stability estimate, we restrict our discussions to an admissible set of spatial functions as follows

\[
A_{\eta} := \left\{ J \in L^2(B_R)^3, \quad \text{supp}(J) \subset B_R, \quad \int_{\Sigma^2} |\xi|^2 \int_{S^2} |\hat{J}(\xi)|^2 \, d|\xi| \geq \eta \|\hat{J}\|_{L^2(R^3)}^2 \right\}, \quad \eta > 0.
\]

(2.21)

Note that if \( \|J\|_{L^2(B_R)^3} \neq 0 \), we can always claim that \( J \in A_\eta \) for some \( \eta > 0 \) depending on \( I \) and the regularity of \( J \).

**Lemma 2.5.** Let \( E(x,t) \) be the solution to the initial value problem (1.2)–(1.3) with \( J \in A_\eta \) for some \( \eta > 0 \). Then

\[
\|J\|_{L^2(B_R)^3}^2 \leq \frac{C}{M^2 \eta^2} \int_I (1 + \kappa^2) \|\hat{E}(x,\kappa) \times \nu\|_{L^2(\Gamma_R)^3}^2 \, d\kappa,
\]

where \( C \) is a constant depending on \( R \) and \( K \).

**Proof.** It follows from (2.18) and the proof of Theorem 2.3 that we have

\[
\im \kappa \hat{g}(\kappa) \hat{J}(\xi) = -\int_{\Gamma_R} \left( (\nabla \times \hat{E}(x,\kappa)) \times \nu \cdot \hat{E}^\text{inc} + (\hat{E}(x,\kappa) \times \nu) \cdot (\nabla \times \hat{E}^\text{inc}) \right) \, dS,
\]

which implies that

\[
|\hat{J}(\xi)|^2 \leq \frac{4\pi R^2}{\kappa^2 |g(\kappa)|^2} \int_{\Gamma_R} \left( (\nabla \times \hat{E}(x,\kappa)) \times \nu \cdot \hat{E}^\text{inc} + (\hat{E}(x,\kappa) \times \nu) \cdot (\nabla \times \hat{E}^\text{inc}) \right)^2 \, dS, \quad \forall \kappa \in I.
\]
Integrating over $\Sigma := S^2 \times I$ by spherical coordinates and using the Cauchy–Schwarz inequality give
\[
\int_I \kappa^2 \int_{S^2} |\hat{J} (\xi)|^2 \, dS \, d\kappa \leq \int_I \frac{8\pi R^2}{M^2 \eta^2} \int_{\Gamma_R} |(\nabla \times \hat{E} (x, \kappa)) \times \nu|^2 + \kappa^2 |\nu \times \hat{E} (x, \kappa)|^2 \, dS \, d\kappa.
\]
In view of (2.19) and (2.21), we find
\[
\int_{\mathbb{R}^3} |\hat{J} (\xi)|^2 \, d\xi \leq \frac{8\pi R^2}{M^2 \eta^2} \int_I \int_{\Gamma_R} (\nabla \times \hat{E} (x, \kappa)) \times \nu|^2 + \kappa^2 |\nu \times \hat{E} (x, \kappa)|^2 \, dS \, d\kappa.
\]
Applying Plancherel theorem yields
\[
\parallel J \parallel_{L^2 (B_R)^3}^2 \leq \frac{8\pi R^2}{M^2 \eta^2} \int_I \int_{\Gamma_R} |(\nabla \times \hat{E} (x, \kappa)) \times \nu|^2 + \kappa^2 |\nu \times \hat{E} (x, \kappa)|^2 \, dS \, d\kappa. \tag{2.22}
\]
Recall that, for a tangential vector $f$ defined on $\Gamma_R$, it holds that (see e.g., [21])
\[
\parallel f \parallel_{L^2 (\Gamma_R)^3} \leq \parallel f \parallel_{H^{-1/2} (\text{div}, \Gamma_R)}, \quad \parallel f \parallel_{H^{-1/2} (\text{curl}, \Gamma_R)} \leq \parallel f \parallel_{H^1 (\Gamma_R)^3}.
\]
On the other hand, from the boundedness of the capacity operator (2.13) (see e.g., [21, Theorem 5.3.7]), we have
\[
\parallel (\nabla \times \hat{E} (x, \kappa)) \times \nu \parallel_{H^{-1/2} (\text{div}, \Gamma_R)} \leq C \parallel \hat{E} (x, \kappa) \times \nu \parallel_{H^{-1/2} (\text{curl}, \Gamma_R)},
\]
where the positive constant $C$ depends on $\kappa$ and $R$. Moreover, the constant $C = C (K, R)$ can be chosen to be uniform in all $\kappa \in I$. The previous two relations imply that
\[
\parallel (\nabla \times \hat{E} (x, \kappa)) \times \nu \parallel_{L^2 (\Gamma_R)^3} \leq C \parallel \hat{E} (x, \kappa) \times \nu \parallel_{H^1 (\Gamma_R)^3}^2.
\]
Combining this with (2.22), we obtain
\[
\parallel J \parallel_{L^2 (B_R)^3} \leq \frac{8\pi R^2 \max \{1, C\}}{M^2 \eta^2} \int_I (1 + \kappa^2) \parallel \hat{E} (x, \kappa) \times \nu \parallel_{H^1 (\Gamma_R)^3} \, d\kappa,
\]
which completes the proof. \[\Box\]

Below we state stability estimate of the source $J$ in terms of the tangential components of $E (x, t)$ on $\Gamma_R \times (0, T)$.

\textbf{Theorem 2.6.} Assume $J \in H^p (\mathbb{R}^3)^3 \cap A_\eta$ for some $p > 3/2$ and $\eta > 0$. Then there exists a constant $C = C (R, K) > 0$ such that
\[
\parallel J \parallel_{L^2 (B_R)^3} \leq \frac{C}{M^2 \eta^2} \parallel E \times \nu \parallel_{H^1 (0, T; H^1 (\Gamma_R))^3}.
\]

\textbf{Proof.} It follows from Lemma 2.2 and the Sobolev trace theorem that we have $E \in H^1 (0, T; H^1 (\Gamma_R))$ if $J \in H^p (\mathbb{R}^3)^3$, $p > 3/2$, which implies that $E \times \nu \in H^1 (\Gamma_R)^3$. Moreover, we obtain from the Plancherel theorem and Lemma 2.1 that
\[
\parallel E \times \nu \parallel_{H^1 (0, T; H^1 (\Gamma_R))}^2
= \int_0^T \parallel E (x, t) \times \nu \parallel_{H^1 (\Gamma_R)^3}^2 + \parallel E' (x, t) \times \nu \parallel_{H^1 (\Gamma_R)^3}^2 \, dt
= \int_R \parallel E (x, t) \times \nu \parallel_{H^1 (\Gamma_R)^3}^2 + \parallel E' (x, t) \times \nu \parallel_{H^1 (\Gamma_R)^3}^2 \, dt
= \int_R \parallel \hat{E} (x, \kappa) \times \nu \parallel_{H^1 (\Gamma_R)^3}^2 + \kappa^2 \parallel \hat{E} (x, \kappa) \times \nu \parallel_{H^1 (\Gamma_R)^3}^2 \, d\kappa
= \int_R (1 + \kappa^2) \parallel \hat{E} (x, \kappa) \times \nu \parallel_{H^1 (\Gamma_R)^3}^2 \, d\kappa,
\]
which completes the proof after combining the above identities with Lemma 2.5. \[\Box\]
3. IP2: DETERMINATION OF TEMPORAL FUNCTIONS

In this section, we consider IP2 and determine \( g \) from an observation of the solution for the initial value problem:

\[
\begin{cases}
\partial_t^2 E(x, t) - \Delta E(x, t) = J(x)g'(t), & x \in \mathbb{R}^3, \ t > 0, \\ E(x, 0) = \partial_t E(x, 0) = 0, & x \in \mathbb{R}^3,
\end{cases}
\] (3.1)

at one fixed point \( x_0 \in \text{supp}(J) \) (i.e., an interior observation) or at the boundary \( \Gamma_R \) (i.e., boundary observations).

3.1. Uniqueness and stability with interior data. Following similar arguments as those in Lemma 2.2, we have the regularity of the solution for the initial value problem (3.1).

**Lemma 3.1.** Let \( g' \in L^2(0,T)^3 \) and let \( J \in H^p(\mathbb{R}^3) \) (\( p > 0 \)) be supported in \( B_R^c \). Then the problem (3.1) admits a unique solution \( E \in C(0,T; H^{p+1}(\mathbb{R}^3))^3 \cap H^\tau(0,T; H^{p-\tau+1}(\mathbb{R}^3))^3 \) (\( \tau = 1, 2 \)) satisfying

\[
\|E\|_{C(0,T; H^{p+1}(\mathbb{R}^3))^3} + \|E\|_{H^\tau(0,T; H^{p-\tau+1}(\mathbb{R}^3))^3} \leq C\|g'\|_{L^2(0,T)^3}\|J\|_{H^p(\mathbb{R}^3)},
\] (3.2)

where the constant \( C > 0 \) depends on \( R \).

In the remaining part of this paper we assume that \( J \in H^p(\mathbb{R}^3) \) with \( p > 5/2 \). According to Lemma 3.1 and the Sobolev embedding theorem, we have \( E \in C(0,T; C^2(\mathbb{R}^3))^3 \cap H^2(0,T; C(\mathbb{R}^3))^3 \) and the trace \( t \mapsto E(x_0, t) \) for some point \( x_0 \in \mathbb{R}^3 \), is well-defined as an element of \( H^2([0,T]^3) \). Below we consider the inverse problem of determining the evolution function \( g(t) \) from the interior observation of the wave field \( E(x_0, t) \) for \( t \in (0,T) \) and some \( x_0 \in \text{supp}(J) \).

**Theorem 3.2.** Assume for some \( x_0 \in B_R^c \) that the set

\[
A_{x_0, p, \delta, M} := \{ h \in H^p(\mathbb{R}^3) : \|h\|_{H^p(\mathbb{R}^3)} \leq A, |h(x_0)| \geq \delta, \text{supp}(h) \subset B_R^c \}, \ M, \delta > 0,
\]

is not empty. Then, for \( J \in A_{x_0, p, \delta, M} \), it holds that

\[
\|g'\|_{L^2(0,T)^3} \leq C\|\partial_t^2 E(x_0, \cdot)\|_{L^2(0,T)^3},
\]

where \( C \) depends on \( p, x_0, A, R, \delta \) and \( T \). In particular, this estimate implies that the data \( \{E(x_0, t) : t \in (0,T)\} \) determines uniquely the temporal function \( g \).

**Proof.** Clearly, the solution \( E \) of (3.1) is given by

\[
E(x, t) = (2\pi)^{-3} \int_{\mathbb{R}^3} \left( \int_0^t \tilde{j}(\xi)|\xi|^{-1}\sin((t-s)|\xi|)g'(s)ds \right)e^{i\xi \cdot x}d\xi, \quad (x, t) \in \mathbb{R}^3 \times [0,T].
\]

Applying Fubini’s theorem yields

\[
E(x, t) = (2\pi)^{-3} \int_0^t \left( \int_{\mathbb{R}^3} \tilde{j}(\xi)|\xi|^{-1}\sin((t-s)|\xi|)e^{i\xi \cdot x}d\xi \right)g'(s)ds, \quad (x, t) \in \mathbb{R}^3 \times [0,T].
\]

In particular, in view of Lemma 3.1, \( E \in C(0,T; H^{p+1}(\mathbb{R}^3))^3 \cap H^2(0,T; H^{p-1}(\mathbb{R}^3))^3 \) satisfies (3.2). Further, direct calculations show that

\[
-\Delta E(x, t) = (2\pi)^{-3} \int_0^t \left( \int_{\mathbb{R}^3} \tilde{j}(\xi)|\sin((t-s)|\xi|)|e^{i\xi \cdot x}d\xi \right)g'(s)ds, \quad (x, t) \in \mathbb{R}^3 \times [0,T].
\]

Moreover, we have

\[
-\Delta E(x, t) \leq (2\pi)^{-3} \int_0^t |g'(s)|ds \int_{\mathbb{R}^3} |\tilde{j}(\xi)||\xi|d\xi \\
\leq (2\pi)^{-3} \int_0^t |g'(s)|ds \|\tilde{j}(\xi)(1 + |\xi|^2)^{p/2}\|_{L^2(\mathbb{R}^3)}\|1 + |\xi|^2\|_1^{1-p/2}\|_{L^2(\mathbb{R}^3)} \\
\leq A_0\|J\|_{H^p(\mathbb{R}^3)} \int_0^t |g'(s)|ds,
\] (3.3)
where \( A_0 = (2\pi)^{-3}\|(1 + |\xi|^2)^{(1-p)/2}\|_{L^2(\mathbb{R}^3)} < \infty \). Since \(|J(x_0)| \geq \delta\), we derive from (3.3) and the governing equation of \( E \) in (3.1) that

\[
|g'(t)| = \frac{1}{J(x_0)}|\partial_t^2 E(x_0, t) - \Delta E|
\]

\[
\leq A_1|\partial_t^2 E(x_0, t)| + A_2 \int_0^t |g'(s)|ds, \quad \forall t \in (0, T),
\]

where \( A_1 = 1/\delta, A_2 = A_0 A/\delta \). Applying the Gronwall inequality, we find

\[
|g'(t)| \leq A_1|\partial_t^2 E(x_0, t)| + A_1 A_2 \int_0^t \partial_t^2 E(x_0, s)|e^{A_2(t-s)}|ds
\]

\[
\leq A_1|\partial_t^2 E(x_0, t)| + A_1 A_2 T\frac{2}{3} |e^{A_2 T}|\|\partial_t^2 E(x_0, \cdot)\|_{L^2(0, T)^3}.
\]

Therefore, taking the norm \( L^2(0, T) \) on both sides of the above inequality, yields that

\[
\|g'\|_{L^2(0, T)^3} \leq (A_1 + A_1 A_2 T e^{A_2 T})\|\partial_t^2 E(x_0, \cdot)\|_{L^2(0, T)^3},
\]

which completes the proof. \( \square \)

3.2. **Uniqueness with boundary measurement data.** To state uniqueness with boundary measurement data, we need the concept of non-radiating source.

**Definition 3.3.** The compactly supported function \( J \) is called a non-radiating source at the frequency \( \kappa \in \mathbb{R}^+ \) to the Maxwell equations if there exists a vector \( P \in \mathbb{C}^3 \) such that the unique radiating solution to the inhomogeneous Maxwell system

\[
\nabla \times (\nabla \times E(x)) + \kappa^2 E(x) = J(x)P
\]

vanishes identically in \( \mathbb{R}^3 \setminus \text{supp}(J) \). The source \( J \) is not a non-radiating source at the frequency \( \kappa \in \mathbb{R}^+ \) if the unique solution to (3.4) does not vanish for all \( P \in \mathbb{C}^3 \).

**Theorem 3.4.** Suppose that \( J \in L^2(B_R) \) is a compacted supported function over \( B_R \) and that \( J \) is not a non-radiating source for all \( \kappa \in \mathbb{R}^+ \). Then the temporal function \( g \in C_0([0, T])^3 \) can be uniquely determined by the boundary measurement data \( \{E \times \nu : x \in \Gamma_R, t \in (0, T)\} \).

**Proof:** Denote by \( e_j \) (\( j = 1, 2, 3 \)) the unit vectors in Cartesian coordinate system. Let \( w_j = w_j(x, \kappa) \) be the unique radiating solution to the inhomogeneous equations

\[
\nabla \times (\nabla \times w_j(x, \kappa)) - \kappa^2 w_j(x, \kappa) = J(x)e_j, \quad j = 1, 2, 3,
\]

which does not vanish identically in \( |x| \geq R \) by our assumption. Let the matrix \( W = (w_1, w_2, w_3) \in \mathbb{C}^{3 \times 3} \) to be the unique radiating solution to the matrix equation

\[
\nabla \times \left( \nabla \times W(x, \kappa) \right) - \kappa^2 W(x, \kappa) = J(x)I, \quad x \in \mathbb{R}^3 \times (0, \infty),
\]

which gives that

\[
W(x, \kappa) = -\frac{1}{i\kappa} \int_{\mathbb{R}^3} \hat{G}(x - y, \kappa)J(y)dy, \quad x \in \mathbb{R}^3.
\]

Here \( \hat{G} \) is the Green tensor to the time-harmonic Maxwell equations; see (2.3). In view of (1.2), the Fourier transform \( \hat{E}(x; \kappa) \) of \( E(x, \kappa) \) can be written as

\[
\hat{E}(x; \kappa) = i\kappa W(x, \kappa)\hat{g}(\kappa), \quad \forall \kappa \in \mathbb{R}^+, \ |x| = R.
\]

We claim that for each \( \kappa_0 \in \mathbb{R}^+ \), there always exists \( x_0 \in \Gamma_R \) such that \( \det(W(x_0, \kappa_0)) \neq 0 \). Suppose on the contrary that \( \det(W(x, \kappa_0)) = 0 \) for all \( x \in \Gamma_R \). This implies that there exist \( c_j \in \mathbb{C}, \ j = 1, 2, 3 \) which are not all equal to zero such that

\[
V(x) := c_1 w_1(x, \kappa_0) + c_2 w_2(x, \kappa_0) + c_3 w_3(x, \kappa_0) = 0, \quad x \in \Gamma_R.
\]
By uniqueness of the exterior Dirichlet boundary value problem, we conclude that $V(x) = 0$ in $|x| > R$, and by unique continuation it holds that $V(x) = 0$ for all $x$ which are outside of the support of $J$. On the other hand, it is easy to observe that $V$ satisfies the inhomogeneous equation

$$\nabla \times \left( \nabla \times V(x) \right) - \kappa^2 V(x) = J(x)P,$$

where $P = c_1 e_1 + c_2 e_2 + c_3 e_3$, which contradicts the fact that $J$ is not a non-radiating source. This proves the existence of $x_0 \in \Gamma_R$ such that $\det(W(x_0, \kappa_0)) \neq 0$.

Therefore, we get by (3.5) that

$$i\kappa \hat{g}(\kappa_0) = \left[W(x_0, \kappa_0)\right]^{-1} \hat{E}(x_0, \kappa_0) \in \mathbb{C}^{3 \times 1} \quad \text{for some} \quad x_0 \in \Gamma_R. \quad (3.6)$$

Note that $\kappa_0$ is arbitrary and the point $x_0$ depends on $\kappa_0$. Hence, if $E \times \nu = 0$ for all $x \in \Gamma_R$ and $t \in (0, T)$, then $\hat{E}(x, \kappa) \times \nu = 0$ for all $x \in \Gamma_R$ and $\kappa \in \mathbb{R}^+$. From the uniqueness of the Dirichlet boundary value problem for the Maxwell equations, we get $\hat{E}(x, \kappa) = 0$ for all $x \in \Gamma_R$ and $\kappa \in \mathbb{R}^+$. This together with (3.6) implies that $\hat{g} = 0$ for all $\kappa \in \mathbb{R}$ and thus $g = 0$. \qed

**Remark 3.5.** Theorem 3.4 remains true if the measurement surface $\Gamma_R$ is replaced by an arbitrary open subset $\Lambda_R \subset \Gamma_R$ with positive Lebesgue measure, because in the frequency domain the knowledge of $\nu \times \hat{E}$ on $\Lambda_R$ uniquely determines $\nu \times \hat{E}|_{\Gamma_R}$ by analyticity.

**References**


SCHOOL OF MATHEMATICS AND STATISTICS, JIANGSU NORMAL UNIVERSITY XUZHOU, JIANGSU 221116, CHINA
E-mail address: zhaoy@jsnu.edu.cn

DEPARTMENT OF MATHEMATICS AND STATISTICS, YORK UNIVERSITY, 4700 KEELE ST., TORONTO, ONTARIO, M3J 1P3, CANADA
E-mail address: yzyork@yorku.ca

BEIJING COMPUTATIONAL SCIENCE RESEARCH CENTER, BEIJING 100193, CHINA
E-mail address: hu@csrc.ac.cn

DEPARTMENT OF MATHEMATICS, PURDUE UNIVERSITY, WEST LAFAYETTE, INDIANA 47907, USA.
E-mail address: lipeijun@math.purdue.edu

INSTITUTE OF APPLIED MATHEMATICS, ACADEMY OF MATHEMATICS AND SYSTEMS SCIENCE, CHINESE ACADEMY OF SCIENCES, BEIJING 100190, CHINA
E-mail address: xdliu@amt.ac.cn