Lectures on Grothendieck Duality

III: Adjoint monoidal pseudofunctors between closed categories.

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Contents

1 Derived direct- and inverse-image pseudofunctors.

$\mathcal{H}om^\bullet_X$ denotes the sheaf-Hom functor of $\mathcal{O}_X$-complexes:

$$\mathcal{H}om^\bullet_X(E,F)(U) := \mathrm{Hom}^\bullet_U(E|_U, F|_U) \qquad (U \subset X \text{ open}),$$

the restriction map for $U' \subset U$ being the obvious one.
This "dynamic" sheafified version of $\mathrm{Hom}^\bullet$ has a derived functor $\mathbf{R}\mathcal{H}om^\bullet$, defined as usual via q-injective resolutions (which always exist!).
Similarly, we have a sheaf-theoretic version of $\otimes$, and its left-derived functor $\underline{\underline{\otimes}}$, defined via q-flat resolutions.

First, two more examples of derived functors: right-derived direct image and left-derived inverse image.
Most of the lecture is about the formalism of relations between these two and $\mathbf{R}\mathcal{H}om^\bullet$ and $\underline{\underline{\otimes}}$—
all four of which are central characters in Grothendieck duality theory.

Direct image

A ringed space is a pair $(X, \mathcal{O}_X)$ with X a topological space and $\mathcal{O}_X$ a sheaf of commutative rings.
For a continuous map of topological spaces $f\colon X \to Y$, the direct image functor f_* from (the category of) sheaves on X to sheaves on Y is such that for any sheaf E on X,

$$f_*E(U) = E(f^{-1}U) \text{ for all open } U \subset Y.$$

A map of ringed spaces $f\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a pair (f, φ) with $f\colon X \to Y$ a continuous map and $\varphi\colon \mathcal{O}_Y \to f_*\mathcal{O}_X$ a homomorphism of sheaves of rings. (Often, $\mathcal{O}_X$, $\mathcal{O}_Y$ and φ are omitted in the notation.)
Given such an (f, φ), and an $\mathcal{O}_X$-module (sheaf) E, one uses φ to make the $f_*\mathcal{O}_X$-module f_*E into an $\mathcal{O}_Y$-module. There results a left-exact additive functor from the abelian category of $\mathcal{O}_X$-modules to the abelian category of $\mathcal{O}_Y$-modules, that extends in the obvious way to the respective homotopy categories $\mathbf{K}(X)$ and $\mathbf{K}(Y)$: f_* produces a left-exact additive functor from $\mathbf{K}(X)$ to $\mathbf{K}(Y)$.

Example 1. Let $\mathcal{O}_X$ (resp. $\mathcal{O}_Y$) be the sheaf of continuous functions on X (resp. Y). For any continuous map $f\colon X \to Y$, and open $U \subset Y$, define

$$\varphi_f(U)\colon \mathcal{O}_Y(U) \to \mathcal{O}_X(f^{-1}U)$$

to be composition with f. Varying U, one gets a homomorphism $\varphi\colon \mathcal{O}_Y \to f_*\mathcal{O}_X$ of sheaves of rings; so (f, φ_f) is a map of ringed spaces.

Similar examples can be constructed by assuming X and Y to be, say, differentiable (resp. complex, resp. algebraic) manifolds, and all maps and functions to be differentiable (resp. holomorphic, resp. rational).
Such geometric situations originally motivated the general definition of f_*.

Inverse image

f_* has a right-exact left adjoint $f^*\colon \mathcal{O}_Y$-modules $\to \mathcal{O}_X$-modules.
The stalk of f^*E at $x \in X$ is

$$(f^*E)_x = E_{Y,fx} \otimes_{\mathcal{O}_{Y,fx}} \mathcal{O}_{X,x}.$$

There results a functor $f^*\colon \mathbf{K}(Y) \to \mathbf{K}(X)$ and an isomorphism

$$\mathrm{Hom}_{\mathbf{K}(X)}(f^*E, F) \cong \mathrm{Hom}_{\mathbf{K}(Y)}(E, f_*F) \qquad (E \in \mathbf{K}(Y),\ F \in \mathbf{K}(X)).$$

Example 2. When $f\colon X \to Y$ is the map of affine schemes corresponding to a ring homomorphism $R \to S$, then the adjoint pair (f^*, f_*) of functors of quasi-coherent sheaves corresponds to the adjoint pair of module-functors (extension of scalars, restriction of scalars).

Recall: right- and left-derived functors

$Q_{\mathcal{A}}\colon \mathcal{A} \to \mathbf{D}(\mathcal{A})$ denotes the canonical functor from an abelian category $\mathcal{A}$ to its derived category.
Let $\mathcal{A}_1$, $\mathcal{A}_2$ be abelian categories, and set $Q_i := Q_{\mathcal{A}_i}$. Let $\gamma\colon \mathbf{K}(\mathcal{A}_1) \to \mathbf{K}(\mathcal{A}_2)$ be a Δ-functor.

A right-derived functor $(\mathbf{R}\gamma, \zeta)$ of γ consists of a Δ-functor $\mathbf{R}\gamma\colon \mathbf{D}(\mathcal{A}_1) \to \mathbf{D}(\mathcal{A}_2)$ and a Δ-functorial map $\zeta\colon Q_2\gamma \to \mathbf{R}\gamma Q_1$ such that
every Δ-functorial map $Q_2\gamma \to \Gamma$, where $\Gamma\colon \mathbf{K}(\mathcal{A}_1) \to \mathbf{D}(\mathcal{A}_2)$ takes quasi-isomorphisms to isomorphisms, factors uniquely as

$$Q_2\gamma \xrightarrow{\zeta} \mathbf{R}\gamma Q_1 \to \Gamma.$$

A left-derived functor $(\mathbf{R}\gamma, \xi)$ of γ consists of a Δ-functor $\mathbf{L}\gamma\colon \mathbf{D}(\mathcal{A}_1) \to \mathbf{D}(\mathcal{A}_2)$ and a Δ-functorial map $\xi\colon \mathbf{L}\gamma Q_1 \to Q_2\gamma$ such that
every Δ-functorial map $\Gamma \to Q_2\gamma$, where $\Gamma\colon \mathbf{K}(\mathcal{A}_1) \to \mathbf{D}(\mathcal{A}_2)$ takes quasi-isomorphisms to isomorphisms, factors uniquely as

$$\Gamma \to \mathbf{L}\gamma Q_1 \xrightarrow{\xi} Q_2\gamma.$$

Relations between direct- and inverse-image functors

1. Derived adjointness of direct- and inverse-image functors.

The left-exact functor f_* has a right-derived functor $\mathbf{R}f_*$, the canonical map $\zeta\colon f_*C \to \mathbf{R}f_*C$ (more precisely, $Q_Y f_*C \to \mathbf{R}f_*Q_XC$) being given for any complex $C \in \mathbf{D}(X)$ by applying f_* to a q-injective resolution $C \to E_C$.

The right-exact functor f^* has a left-derived functor $\mathbf{L}f^*$, the canonical map $\xi\colon \mathbf{L}f^*D \to f^*D$ (more precisely, $\mathbf{L}f^*Q_YD \to Q_Xf^*D$) being given for any complex $D \in \mathbf{D}(Y)$ by applying f^* to a q-flat resolution $F_D \to D$.

With a little care, one "derives" $(f^*\text{-}f_*)$-adjointness:

For any ringed-space map $f\colon X \to Y$,

$\mathbf{L}f^*\colon \mathbf{D}(Y) \to \mathbf{D}(X)$ is left-adjoint to $\mathbf{R}f_*$, i.e., for $E \in \mathbf{D}(Y)$, $F \in \mathbf{D}(X)$,

$$\mathrm{Hom}_{\mathbf{D}(X)}(\mathbf{L}f^*E, F) \cong \mathrm{Hom}_{\mathbf{D}(Y)}(E, \mathbf{R}f_*F). \tag{$*$}$$

Elaboration

Proposition 3. *Let $f\colon X \to Y$ be a ringed-space map, $A \in \mathbf{D}(Y)$, $B \in \mathbf{D}(X)$. Write $\mathrm{Hom}^\bullet_X$ for $\mathrm{Hom}^\bullet_{\mathbf{D}(X)}$. There is a unique Δ-functorial isomorphism*

$$\alpha\colon \mathbf{R}\mathrm{Hom}^\bullet_X(\mathbf{L}f^*A, B) \xrightarrow{\ \sim\ } \mathbf{R}\mathrm{Hom}^\bullet_Y(A, \mathbf{R}f_*B)$$

such that the following natural diagram in $\mathbf{D}(X)$ commutes:

$$\begin{tikzcd}
\mathrm{Hom}^\bullet_X(f^*A,\, B) \arrow[r] \arrow[d, "\simeq"] & \mathbf{R}\mathrm{Hom}^\bullet_X(f^*A,\, B) \arrow[r] & \mathbf{R}\mathrm{Hom}^\bullet_X(\mathbf{L}f^*A,\, B) \arrow[d, "\alpha", "\simeq"'] \\
\mathrm{Hom}^\bullet_Y(A,\, f_*B) \arrow[r] & \mathbf{R}\mathrm{Hom}^\bullet_Y(A,\, f_*B) \arrow[r] & \mathbf{R}\mathrm{Hom}^\bullet_Y(A,\, \mathbf{R}f_*B).
\end{tikzcd}$$

Moreover, the induced homology map

$$H^0(\alpha)\colon \mathrm{Hom}_{\mathbf{D}(X)}(\mathbf{L}f^*A,\, B) \xrightarrow{\ \sim\ } \mathrm{Hom}_{\mathbf{D}(Y)}(A,\, \mathbf{R}f_*B)$$

is just the adjunction isomorphism $(*)$.

Further elaboration (sheafification)

Proposition 4. *There is a unique Δ-functorial isomorphism*

$$\tilde\alpha\colon \mathbf{R}f_*\mathbf{R}\mathcal{H}om^\bullet_X(\mathbf{L}f^*A,\, B) \xrightarrow{\ \sim\ } \mathbf{R}\mathcal{H}om^\bullet_Y(A,\, \mathbf{R}f_*B)$$
$$\big(A \in \mathbf{D}(Y),\ B \in \mathbf{D}(X)\big)$$

such that the following natural diagram commutes:

$$\begin{tikzcd}
f_*\mathcal{H}om^\bullet_X(f^*A, B) \arrow[r] \arrow[d, "\simeq"] & \mathbf{R}f_*\mathbf{R}\mathcal{H}om^\bullet_X(f^*A, B) \arrow[r] & \mathbf{R}f_*\mathbf{R}\mathcal{H}om^\bullet_X(\mathbf{L}f^*A, B) \arrow[d, "\tilde\alpha", "\simeq"'] \\
\mathcal{H}om^\bullet_Y(A,\, f_*B) \arrow[r] & \mathbf{R}\mathcal{H}om^\bullet_Y(A,\, f_*B) \arrow[r] & \mathbf{R}\mathcal{H}om^\bullet_Y(A,\, \mathbf{R}f_*B)
\end{tikzcd}$$

The previous elaboration is obtained from this one by application of the derived functor $\mathbf{R}\Gamma(Y, -) = \mathbf{R}\mathrm{Hom}^\bullet_Y(\mathcal{O}_Y, -)$.

2. Base change I. To a commutative square of ringed-space maps

$$\begin{tikzcd} X' \arrow[r, "v"] \arrow[d, "g"'] \arrow[dr, phantom, "\sigma"] & X \arrow[d, "f"] \\ Y' \arrow[r, "u"'] & Y \end{tikzcd}$$

associate the map $\theta = \theta_\sigma \colon \mathbf{L}u^*\mathbf{R}f_* \to \mathbf{R}g_*\mathbf{L}v^*$, adjoint to the natural composition

$$\mathbf{R}f_* \to \mathbf{R}f_*\mathbf{R}v_*\mathbf{L}v^* \xrightarrow{\sim} \mathbf{R}u_*\mathbf{R}g_*\mathbf{L}v^*.$$

For affine schemes, σ corresponds to a commutative square of ring-maps

$$\begin{tikzcd} S' & S \arrow[l, "\bar v"'] \\ R' \arrow[u, "\bar g"] \arrow[ur, phantom, "\bar\sigma"] & R \arrow[l, "\bar u"] \arrow[u, "\bar f"'] \end{tikzcd}$$

and θ_σ is the derived upgrade of the usual functorial map $R' \otimes_R C \to S' \otimes_S C \quad (C \in \mathbf{M}(S))$.

Tor-independence

If

$$\begin{tikzcd} X' \arrow[r, "v"] \arrow[d, "g"'] \arrow[dr, phantom, "\sigma"] & X \arrow[d, "f"] \\ Y' \arrow[r, "u"'] & Y \end{tikzcd}$$

is a fiber square of concentrated (= quasi-compact, quasi-separated) schemes then, with $\mathbf{D}_{\mathsf{qc}}$ the full subcategory of $\mathbf{D}$ whose objects are the complexes with quasi-coherent homology,

$$\theta_\sigma \text{ is an isomorphism of functors on } \mathbf{D}_{\mathsf{qc}} \iff \sigma \text{ is tor-independent},$$

i.e., for all $x \in X$ and $y' \in Y'$ such that $f(x) = u(y') =$ (say) y,

$$\mathrm{Tor}_i^{\mathcal{O}_{Y,y}}\big(\mathcal{O}_{X,x}, \mathcal{O}_{Y',y'}\big) = 0 \quad \text{for all } i \neq 0.$$

Pseudofunctor (a special case of 2-functor)

Formalize behavior of inverse and direct-image functors vis-à-vis composition.

A contravariant pseudofunctor on a category $\mathbf{S}$ assigns:

- to each $X \in \mathbf{S}$ a category $\mathbf{X}^{\#}$,
- to each map $f \colon X \to Y$ a functor $f^{\#} \colon \mathbf{Y}^{\#} \to \mathbf{X}^{\#}$ (with $\mathbf{1}^{\#} = \mathbf{1}$), and
- to each map-pair $X \xrightarrow{f} Y \xrightarrow{g} Z$ a functorial "transitivity" isomorphism

$$d_{f,g} \colon f^{\#}g^{\#} \xrightarrow{\sim} (gf)^{\#}$$

satisfying $d_{\mathbf{1},g} = d_{g,\mathbf{1}} =$ identity, and a kind of associativity: for each triple of maps $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W$ the following diagram commutes:

$$\begin{tikzcd} (hgf)^{\#} & f^{\#}(hg)^{\#} \arrow[l, "d_{f,hg}"'] \\ (gf)^{\#}h^{\#} \arrow[u, "d_{gf,h}"] & f^{\#}g^{\#}h^{\#} \arrow[l, "d_{f,g}"] \arrow[u, "d_{g,h}"'] \end{tikzcd}$$

Covariant pseudofunctor is similarly defined, with arrows reversed. It means contravariant functor on $\mathbf{S}^{\mathsf{op}}$.

Examples: **Derived inverse-image (contravariant).** **Derived direct-image (covariant).**

$$\begin{aligned} &\mathbf{S} := \text{category of ringed spaces}\\ &\mathbf{X}^{\#} := \mathbf{D}(X) \quad \big(\text{derived category of } \{\mathcal{O}_X\text{-modules}\}\big)\\ &f^{\#} := \mathbf{L}f^{*} \qquad \text{resp.} \qquad f_{\#} := \mathbf{R}f_{*} \end{aligned}$$

Compatibility of pseudofunctoriality and adjointness

For ringed-space maps $\bullet \xrightarrow{f} \bullet \xrightarrow{g} \bullet$, the following natural diagram of functors, involving three different counit maps and two transitivity isomorphisms, commutes:

$$\begin{tikzcd} 1 \arrow[r] \arrow[d] & f_*f^* \arrow[r] & f_*(g_*g^*f^*) \arrow[d, equal] \\ (gf)_*(gf)^* \arrow[r, "\sim"] & f_*g_*(gf)^* \arrow[r, "\sim"] & f_*g_*g^*f^* \end{tikzcd}$$

Equivalently (categorically), the "dual" diagram commutes:

$$\begin{tikzcd} 1 & g^*g_* \arrow[l] & g^*(f^*f_*g_*) \arrow[l] \arrow[d, equal] \\ (gf)^*(gf)_* \arrow[u] & g^*f^*(gf)_* \arrow[l, "\sim"'] & g^*f^*f_*g_* \arrow[l, "\sim"'] \end{tikzcd}$$

Similar commutativity relations hold with f_* (resp. f^*) replaced by $\mathbf{R}f_*$ (resp. $\mathbf{L}f^*$).

2 Interaction with tensor product: symmetric monoidal categories.

Continue to formalize the behavior of the four basic operations.
Axiomatic properties of $\underline{\underline{\otimes}}$ are summarized in the following definition:

A *symmetric monoidal category* $\overline{\mathbf{C}} = (\mathbf{C}, \otimes, \mathcal{O}, \alpha, \lambda, \rho, \gamma)$ consists of:
a category $\mathbf{C}$, a "product" functor $\otimes\colon \mathbf{C}\times\mathbf{C} \to \mathbf{C}$, an object $\mathcal{O}$ of $\mathbf{C}$, and functorial isomorphisms,

$$\alpha\colon (A\otimes B)\otimes C \xrightarrow{\sim} A\otimes(B\otimes C) \qquad \text{(associativity)}$$

$$\lambda\colon \mathcal{O}\otimes A \xrightarrow{\sim} A \qquad \rho\colon A\otimes\mathcal{O} \xrightarrow{\sim} A \qquad \text{(left and right units)}$$

$$\gamma\colon A\otimes B \xrightarrow{\sim} B\otimes A \qquad \text{(symmetry)}$$

(with A, B, C in $\mathbf{C}$), such that $\gamma^2 = 1$ and the following diagrams commute:

$$\begin{tikzcd} (A\otimes\mathcal{O})\otimes B \arrow[r, "\alpha"] \arrow[d, "\rho\otimes 1"'] & A\otimes(\mathcal{O}\otimes B) \arrow[d, "1\otimes\lambda"] \\ A\otimes B \arrow[r, equal] & A\otimes B \end{tikzcd} \qquad \begin{tikzcd} \mathcal{O}\otimes A \arrow[r, "\gamma"] \arrow[d, "\lambda"'] & A\otimes\mathcal{O} \arrow[d, "\rho"] \\ A \arrow[r, equal] & A \end{tikzcd}$$

$$\begin{tikzcd} ((A\otimes B)\otimes C)\otimes D \arrow[r, "\alpha"] \arrow[d, "\alpha\otimes 1"'] & (A\otimes B)\otimes(C\otimes D) \arrow[r, "\alpha"] & A\otimes(B\otimes(C\otimes D)) \arrow[d, "1\otimes\alpha"] \\ (A\otimes(B\otimes C))\otimes D \arrow[rr, "\alpha"] & & A\otimes((B\otimes C)\otimes D) \end{tikzcd}$$

$$\begin{tikzcd} (A\otimes B)\otimes C \arrow[r, "\alpha"] \arrow[d, "\gamma\otimes 1"'] & A\otimes(B\otimes C) \arrow[r, "\gamma"] & (B\otimes C)\otimes A \arrow[d, "\alpha"] \\ (B\otimes A)\otimes C \arrow[r, "\alpha"] & B\otimes(A\otimes C) \arrow[r, "1\otimes\gamma"] & B\otimes(C\otimes A) \end{tikzcd}$$

Examples

Example 5. Let S be a commutative ring, let $\mathbf{C}$ be the category of S-modules, let $\otimes$ be the usual tensor product over S, and let $\mathcal{O}_S := S$.

Example 6. Let $(X, \mathcal{O}_X)$ be a ringed space, $\mathbf{C} := \mathbf{D}(X)$ (derived category of $\{\mathcal{O}_X\text{-modules}\}$), $\otimes := \underline{\underline{\otimes}}_X$ (derived tensor product over X), and $\mathcal{O} := \mathcal{O}_X$.

The first example requires a few straightforward verifications.

For the second example, verifications are slightly less straightforward—one has to work with q-flat resolutions of $\mathcal{O}_X$-complexes.

3 Adjoint monoidal pseudofunctors.

Symmetric monoidal functors

Let $\overline{\mathbf{X}}$, $\overline{\mathbf{Y}}$ be symmetric monoidal categories.

A symmetric monoidal functor $\overline{f}_*\colon \overline{\mathbf{X}} \to \overline{\mathbf{Y}}$ is a functor $f_*\colon \mathbf{X} \to \mathbf{Y}$ together with two functorial maps

$$f_*A \otimes f_*B \longrightarrow f_*(A \otimes B)$$
$$\mathcal{O}_Y \longrightarrow f_*\mathcal{O}_X$$

(where $\otimes$ is in $\mathbf{X}$ or in $\mathbf{Y}$, as appropriate), compatible with α, λ, γ (and hence ρ), in that the following natural diagrams commute:

$$\begin{tikzcd}
(f_*A \otimes f_*B) \otimes f_*C \arrow[r] \arrow[d, "\alpha"'] & f_*(A \otimes B) \otimes f_*C \arrow[r] & f_*((A \otimes B) \otimes C) \arrow[d, "f_*(\alpha)"] \\
f_*A \otimes (f_*B \otimes f_*C) \arrow[r] & f_*A \otimes f_*(B \otimes C) \arrow[r] & f_*(A \otimes (B \otimes C))
\end{tikzcd}$$

$$\begin{tikzcd}
f_*\mathcal{O}_X \otimes f_*A \arrow[r] & f_*(\mathcal{O}_X \otimes A) \arrow[d, "f_*(\lambda_X)"] \\
\mathcal{O}_Y \otimes f_*A \arrow[u] \arrow[r, "\lambda_Y"'] & f_*A
\end{tikzcd}
\qquad
\begin{tikzcd}
f_*A \otimes f_*B \arrow[r] \arrow[d, "\gamma_Y"'] & f_*(A \otimes B) \arrow[d, "f_*(\gamma_X)"] \\
f_*B \otimes f_*A \arrow[r] & f_*(B \otimes A)
\end{tikzcd}$$

Examples

1. Let $f\colon R \to S$ be a ring homomorphism, and let f_* be the "restriction of scalars" functor from $\{S\text{-modules}\}$ to $\{R\text{-modules}\}$ (see above), monoidalized by the obvious maps.

2. Let $f\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a map of ringed spaces, and f_* the direct-image functor from $\{\mathcal{O}_X\text{-modules}\}$ to $\{\mathcal{O}_Y\text{-modules}\}$, monoidalized by the map $\mathbf{e}\colon f_*A \otimes f_*B \longrightarrow f_*(A \otimes B)$ obtained by passage to associated sheaves from the obvious composed *presheaf* map (where $U \subset X$ is open)

$$(f_*A)(U) \otimes (f_*B)(U) = A(f^{-1}U) \otimes B(f^{-1}U) \to (A \otimes B)(f^{-1}U) = f_*(A \otimes B)(U).$$

One shows that $\mathbf{e}$ is adjoint to the natural composition

$$f^*(f_*A \otimes f_*B) \xrightarrow{\ \sim\ } f^*f_*A \otimes f^*f_*B \to A \otimes B.$$

3. For f as in 2, one has the right-derived direct image functor $\mathbf{R}f_*\colon D(X)\to D(Y)$, monoidalized by the map

$$\mathbf{R}f_*A \underline{\underline{\otimes}} \mathbf{R}f_*B \to \mathbf{R}f_*(A \underline{\underline{\otimes}} B)$$

adjoint to the natural composition

$$\mathbf{L}f^*(\mathbf{R}f_*A \underline{\underline{\otimes}} \mathbf{R}f_*B) \xrightarrow{\sim} \mathbf{L}f^*\mathbf{R}f_*A \underline{\underline{\otimes}} \mathbf{L}f^*\mathbf{R}f_*B \to A \underline{\underline{\otimes}} B$$

with $\mathbf{L}f^*(C \underline{\underline{\otimes}} D) \xrightarrow{\sim} \mathbf{L}f^*C \underline{\underline{\otimes}} \mathbf{L}f^*D$ defined via q-flat resolutions of C, D.

Monoidality and Pseudofunctoriality

Now suppose we have a *covariant pseudofunctor* on a category $\mathbf{C}$,
assigning a monoidal category $\mathbf{X}_*$ to each object X in $\mathbf{C}$ and a monoidal functor $f_*\colon \mathbf{X}_*\to\mathbf{Y}_*$ to each map $f\colon X\to Y$ in $\mathbf{C}$.
We say this is a monoidal pseudofunctor if for every composition $X\xrightarrow{f}Y\xrightarrow{g}Z$, the following natural diagrams commute, for all $A,B\in\mathbf{X}_*$:

$$\begin{tikzcd}
\mathcal{O}_Z \arrow[r] \arrow[d] & (gf)_*\mathcal{O}_X \arrow[d] \\
g_*\mathcal{O}_Y \arrow[r] & g_*f_*\mathcal{O}_X
\end{tikzcd}$$

$$\begin{tikzcd}
(gf)_*A\otimes(gf)_*B \arrow[rr] \arrow[d, "\simeq"'] & & (gf)_*(A\otimes B) \arrow[d, "\simeq"] \\
g_*f_*A\otimes g_*f_*B \arrow[r] & g_*(f_*A\otimes f_*B) \arrow[r] & g_*f_*(A\otimes B)
\end{tikzcd}$$

3. Adjoint monoidal pseudofunctors

Assume further that each of the symmetric monoidal functors f_* has a *left adjoint* $f^*\colon \mathbf{Y}_*\to\mathbf{X}_*$.
There are then natural maps, for all A, $B\in\mathbf{Y}^*$,

$$f^*(A\otimes B)\to f^*A\otimes f^*B, \qquad f^*\mathcal{O}_Y\to\mathcal{O}_X$$

adjoint respectively to the two maps

$$A\otimes B\to f_*f^*A\otimes f_*f^*B\to f_*(f^*A\otimes f^*B), \qquad \mathcal{O}_Y\to f_*\mathcal{O}_X,$$

that, one shows, make the opposite functor

$$(f^*)^{\mathsf{op}}\colon(\mathbf{Y}^*)^{\mathsf{op}}\to(\mathbf{X}^*)^{\mathsf{op}}$$

monoidal.
There is, furthermore, a unique way to make these f^* into a
contravariant pseudofunctor with $\mathbf{X}^*=\mathbf{X}_*$ for all $X\in\mathbf{C}$ and such that for all $\bullet\xrightarrow{f}\bullet\xrightarrow{g}\bullet$ in $\mathbf{C}$, the following diagram commutes.

$$\begin{tikzcd}
1 \arrow[r] \arrow[d] & f_*f^* \arrow[r] & f_*(g_*g^*f^*) \arrow[d, equal] \\
(gf)_*(gf)^* \arrow[r, "\sim"] & f_*g_*(gf)^* \arrow[r, "\sim"] & f_*g_*g^*f^*
\end{tikzcd}$$

(We saw this diagram before, as a natural expression of the relation between
adjunction and pseudofunctoriality.)

Such a family of pairs (f^*, f_*) is called an adjoint pair of monoidal pseudofunctors.
One could (but we won't) add further conditions relating to Δ-structures.

4 Further interaction, with $\mathcal{H}om$: closed categories.

A *closed category* is a symmetric monoidal category $\overline{\mathbf{R}}$ as above,
together with a functor, called internal hom:

$$[-,-]\colon \mathbf{R}^{\mathsf{op}} \times \mathbf{R} \to \mathbf{R}$$

(where $\mathbf{R}^{\mathsf{op}}$ is the dual category of $\mathbf{R}$) and a functorial isomorphism

$$\pi\colon \mathrm{Hom}_{\mathbf{R}}(A \otimes B,\, C) \xrightarrow{\;\sim\;} \mathrm{Hom}_{\mathbf{R}}(A,\, [B, C])\,.$$

Example 7. X a ringed space, $\mathbf{R} := \mathbf{D}(X)$, $\otimes := \underset{=}{\otimes}$, $[-,-] := \mathbf{R}\mathcal{H}om^{\bullet}(-,-)$.
For π use the sheafified version of $\mathbf{R}\mathcal{H}om^{\bullet}$–$\underset{=}{\otimes}$ adjunction.

We assume further, axiomatically, that

$$\nu\colon f^*(E \otimes_R F) \to f^*E \otimes_S f^*F \text{ is an isomorphism.}$$

This holds in the example, and also in most other cases of interest.

Example: commutative algebra

Let S be a ring and $\mathbf{R} := \mathbf{M}(S)$.
For S-modules E and F, let $\otimes$ be the usual tensor product,
and $[E, F] := Hom_S(E, F)$.
This makes $\mathbf{M}(S)$ into a closed category.
For a ring homomorphism $f\colon R \to S$, let f_* and f^* be the module-functors restriction and extension of scalars.
There results an adjoint pair of monoidal closed-category-valued pseudofunctors

For a ring homomorpism $f\colon R \to S$, one has, as before, $\mathbf{S}$-module maps

$$S \leftarrow f^*R, \qquad f^*E \otimes_S f^*F \xleftarrow{\nu} f^*(E \otimes_R F) \qquad (E, F \in \mathbf{R}).$$

with ν adjoint to the natural composite map

$$E \otimes_R F \to f_*f^*E \otimes_R f_*f^*F \xrightarrow{\mu} f_*(f^*E \otimes_S f^*F).$$

In fact (check!) , ν is just the standard isomorphism

$$S \otimes_R (E \otimes_R F) \xrightarrow{\;\sim\;} (E \otimes_R S) \otimes_S (F \otimes_R S).$$

Ditto (without explicit mention) for what follows.

Example: Ringed spaces

Here is the **focal point of this lecture**, summarizing the basic relations among four "Grothendieck operations."
It underlies all of the sequel.

Let $\mathbf{C}$ be the category of ringed spaces. For each object $X \in \mathbf{C}$, set $\mathbf{X}^* = \mathbf{X}_* := \mathbf{D}(X)$ (the derived category of the category of $\mathcal{O}_X$-modules),
a closed category with $\otimes := \underset{=}{\otimes}$ and $[-,-] := \mathbf{R}\mathcal{H}om(-,-)$.

For $X \xrightarrow{f} Y$ in $\mathbf{C}$, write

$$\begin{aligned} f^* &\;\text{ for }\; \mathbf{L}f^*\colon \mathbf{Y}^* \to \mathbf{X}^*,\\ f_* &\;\text{ for }\; \mathbf{R}f_*\colon \mathbf{X}_* \to \mathbf{Y}_*\,. \end{aligned}$$

There results an adjoint pair of monoidal closed-category-valued pseudofunctors on $\mathbf{C}$.

5 Let the games begin.

The formalism of adjoint monoidal closed-category-valued pseudofunctors which we have introduced to study relations among the four basic operations is very rich. Relations among operations are expressed by commutativity of diagrams of maps constructed from the axioms.
Many such diagrams force themselves on you when, for example, you delve into Grothendieck duality theory. Here we just scratch the surface.

• There is a functorial map $f_*[A,B] \longrightarrow [f_*A,\, f_*B]$ corresponding under π to the composed map

$$f_*[A,B]\otimes f_*A \xrightarrow{\mu} f_*\big([A,B]\otimes A\big) \xrightarrow{f_*t_{AB}} f_*B\,.$$

where t_{AB} is the unit map associated to $\otimes$ - [,] adjunction.

• For fixed A the functorial isomorphism $\nu\colon f^*(C\otimes A) \to f^*C\otimes f^*A$ induces a *conjugate* "internal adjunction" isomorphism on right adjoints, namely $[A,\, f_*B] \underset{\xi}{\xleftarrow{\sim}} f_*[f^*A, B]$.

For ringed spaces one gets the "sheafified elaboration" of $\mathbf{R}f_*$-$\mathbf{L}f^*$ adjunction.

• There is a functorial map $f^*[A,B] \longrightarrow [f^*A,\, f^*B]$, adjoint to

$$[A,B] \longrightarrow [A,\, f_*f^*B] \xrightarrow{\xi^{-1}} f_*[f^*A,\, f^*B]\,.$$

A first step in getting familiar with such maps is to interpret them in the commutative model—modules over rings—where they all turn out to be standard maps. Of course we are interested also in other contexts...

Here's an instructive example, involving all four operations.
Working through it is a key to the motivation behind this lecture.

Example 8 (**Exercise**). Establish (from axioms) a natural commutative diagram

$$\begin{array}{ccccc} f^*\big(f_*[f^*F,G]\otimes F\big) & \longrightarrow & f^*\big([F,f_*G]\otimes F\big) & \longrightarrow & f^*f_*G \\ \big\downarrow & & & & \big\downarrow \\ f^*f_*[f^*F,G]\otimes f^*F & \longrightarrow & [f^*F,G]\otimes f^*F & \longrightarrow & G \end{array}$$

Interpret this in the context of rings.
For comparison, work directly in the ringed-space derived-category context, using the definitions of the operations via resolutions.

Projection map

Here is another ubiquitous character.
For any $f\colon X\to Y$ in $\mathbf{C}$, $E\in\mathbf{X}_*$, $F\in\mathbf{Y}^*$, one has the composite projection map

$$p(E,F)\colon f_*E\otimes F \longrightarrow f_*E\otimes f_*f^*F \longrightarrow f_*(E\otimes f^*F).$$

We'll assume, axiomatically, that p is an isomorphism.

This assumption holds in the most interesting models.

For example, in the commutative-algebra context, where $f\colon R\to S$ is a ring-homomorphism, E an S-module and F an R-module, $p(E,F)$ is just the usual isomorphism $E\otimes_R F \xrightarrow{\sim} E\otimes_S (S\otimes_R F)$. (Check!)

More generally, in the context of quasi-compact quasi-separated scheme-maps and quasi-coherent complexes, p turns out to be an isomorphism (next lecture).

Interaction of Projection and Base Change

Example 9 (Exercise). For a commutative $\mathbf{C}$-square, with $h = fv = ug$,

$$\begin{tikzcd} X' \arrow[r, "v"] \arrow[d, "g"'] \arrow[dr, phantom, "\sigma"] & X \arrow[d, "f"] \\ Y' \arrow[r, "u"'] & Y \end{tikzcd}$$

$\theta = \theta_\sigma \colon u^*f_* \to g_*v^*$ adjoint to $f_* \to f_*v_*v^* \xrightarrow{\sim} u_*g_*v^*$, and $C \in \mathbf{Y}^* = \mathbf{Y}_*$, $D \in \mathbf{X}^* = \mathbf{X}_*$, the following diagram commutes:

$$\begin{tikzcd} u^*C \otimes u^*f_*D \arrow[d, "1\otimes\theta"'] & u^*(C \otimes f_*D) \arrow[l, "\nu"'] \arrow[r, "u^*(p)"] & u^*f_*(f^*C \otimes D) \arrow[d, "\theta"] \\ u^*C \otimes g_*v^*D \arrow[d, "p"'] & & g_*v^*(f^*C \otimes D) \arrow[d, "\nu"] \\ g_*(g^*u^*C \otimes v^*D) \arrow[r, "\sim"] & g_*(h^*C \otimes v^*D) & g_*(v^*f^*C \otimes v^*D) \arrow[l, "\sim"'] \end{tikzcd}$$