Lectures on Grothendieck Duality

IV: A basic setup for duality.

Joseph Lipman

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1 Formal duality setup.

Let there be given, on a category $\mathbf{C}$, a pair $({}^*, {}_*)$ of adjoint monoidal closed-category-valued pseudofunctors.

Thus, to each object $X \in \mathbf{C}$ is associated a closed category $\mathbf{D}_X$, with unit object $\mathcal{O}_X$;

and to each $\mathbf{C}$-map $\psi\colon X \to Y$, adjoint monoidal functors $\mathbf{D}_X \underset{\psi_*}{\overset{\psi^*}{\rightleftarrows}} \mathbf{D}_Y$.

There are also, as before, compatibilities—expressed by commutative diagrams—among adjunction, pseudofunctoriality, and monoidality.

The maps giving the monoidal structure on ψ_* are denoted

$$\begin{gathered}\mathbf{e}_\psi(E,E')\colon \psi_*E\otimes\psi_*E' \to \psi_*(E\otimes E') \qquad \big(E,E'\in\mathbf{D}_X\big),\\ \nu_\psi\colon \mathcal{O}_Y \to \psi_*\mathcal{O}_X\,.\end{gathered}$$

Adjoint to the natural composition

$$F\otimes F' \to \psi_*\psi^*F\otimes\psi_*\psi^*F' \xrightarrow{\mathbf{e}} \psi_*(\psi^*F\otimes\psi^*F') \qquad (F,F'\in\mathbf{D}_Y)$$

(resp. to $\nu_\psi\colon \mathcal{O}_Y \to \psi_*\mathcal{O}_X$) we have maps

$$\begin{gathered}\mathbf{d}_\psi(F,F')\colon \psi^*(F\otimes F') \to \psi^*F\otimes\psi^*F',\\ \mu_\psi\colon \psi^*\mathcal{O}_Y \to \mathcal{O}_X\,.\end{gathered}$$

For $E \in \mathbf{D}_X$ and $F \in \mathbf{D}_Y$ the composite map

$$\mathbf{p}_1(E,F)\colon \psi_*E \otimes F \xrightarrow{\text{natural}} \psi_*E \otimes \psi_*\psi^*F \xrightarrow{\mathbf{e}} \psi_*(E \otimes \psi^*F)$$

and the map deduced from it by application of the symmetry isomorphism

$$\mathbf{p}_2(F,E)\colon F \otimes \psi_*E \xrightarrow{\text{natural}} \psi_*\psi^*F \otimes \psi_*E \xrightarrow{\mathbf{e}} \psi_*(\psi^*F \otimes E)$$

are called projection maps.

Axioms

- For $X = Y$ and $\psi = \mathbf{1}_X$ the identity map of X, $(\mathbf{1}_X)_*$ is the identity functor of $\mathbf{D}_X$.
- The map μ_ψ is an isomorphism $\psi^*\mathcal{O}_Y \xrightarrow{\sim} \mathcal{O}_X$.
- For all $F, G \in \mathbf{D}_Y$, the map $\mathbf{d}_\psi$ is an isomorphism $\psi^*(F \otimes G) \xrightarrow{\sim} \psi^*F \otimes \psi^*G$.
- For all $E \in \mathbf{D}_X$ and $F \in \mathbf{D}_Y$ the projection maps are isomorphisms
$$\mathbf{p}_1\colon \psi_*E \otimes F \xrightarrow{\sim} \psi_*(E \otimes \psi^*F), \qquad \mathbf{p}_2\colon F \otimes \psi_*E \xrightarrow{\sim} \psi_*(\psi^*F \otimes E).$$
- The functor $\psi_*\colon \mathbf{D}_X \to \mathbf{D}_Y$ has a right adjoint $\psi^\#$.

So there is a **duality isomorphism**

$$\mathrm{Hom}_{\mathbf{D}_Y}(\psi_*E, F) \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{D}_X}(E, \psi^\# F) \qquad (E \in \mathbf{D}_X,\ F \in \mathbf{D}_Y).$$

Example: Commutative algebra

$\mathbf{C} :=$ opposite of the category of commutative rings.

For $R \in \mathbf{C}$, $\mathbf{D}_R := \{R\text{-modules}\}$, with the obvious closed structure: $\otimes$ is the usual tensor product, and $[E,F] := \mathrm{Hom}_R(E,F)$.

For $\psi\colon S \to R$ (a ring-homomorphism $R \to S$), $\psi_*\colon \mathbf{D}_S \to \mathbf{D}_R$ is restriction of scalars: for any S-module E, ψ_*E is the naturally resulting R-module E; and $\mathbf{e}_\psi\colon E \otimes_R E' \to E \otimes_S E'$ the natural map.

$\psi^*\colon \mathbf{D}_S \to \mathbf{D}_R$ is extension of scalars: for any R-module F, ψ^*F is the S-module $S \otimes_R F$.

One verifies that

$$\mu_\psi\colon S \otimes_R R \xrightarrow{\sim} S,$$
$$\mathbf{d}_\psi\colon S \otimes_R (F \otimes_R G) \xrightarrow{\sim} (S \otimes_R F) \otimes_S (S \otimes_R G)$$

are the usual isomorphisms; and that $\mathbf{p}_1$ is the natural R-isomorphism

$$E \otimes_R F \xrightarrow{\sim} E \otimes_S (S \otimes_R F) \qquad (E \in \mathbf{D}_S,\ F \in \mathbf{D}_R).$$

Finally, a right adjoint $\psi^\#$ of ψ_* is given by $\psi^\# F := \mathrm{Hom}_R(S, F)$.

Remarks

In the preceding example, one can substitute derived categories and functors for ordinary ones. Then, at least in the noetherian case, the existence of the right adjoint $\psi^\#$ is a consequence of the local duality isomorphism from Lecture 2, with J the unit ideal:

$$\mathrm{Hom}_{\mathbf{D}(R)}(\psi_*\mathbf{R}\Gamma_J E, G) \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{D}(S)}(E, \psi_J^\# G).$$

One can deal with arbitrary J in a similar way, but at the cost of further elaborating the basic setup.
Globalizing (as we are about to do in the unit-ideal case) then leads to *duality over formal schemes.*
Thus we have a common framework for local and global duality.

Such "topological" generalizations are beyond the scope of the present lectures. But several papers dealing with formal schemes are available at **www.math.purdue.edu/~lipman**

Globalization: Noetherian schemes

$\mathbf{C}:=$ category of noetherian schemes.[1]

For any $X \in \mathbf{C}$, $\mathbf{D}_X := \mathbf{D}_{\mathsf{qc}}(X)$, the full subcategory of $\mathbf{D}(X)$ whose objects are complexes with quasi-coherent homology. Together with the derived tensor product, this is a monoidal category.

To make it closed, set $[E,F]:= \mathsf{Q}_X\mathbf{R}\mathrm{Hom}(E,F)$, where

Q_X is a right adjoint to the inclusion functor $\mathbf{D}_X \hookrightarrow \mathbf{D}(X)$.

(*Existence* of such a right adjoint—a *derived quasi-coherator*—is a very special case of the duality theorem to be discussed later.)

Indeed, using derived adjoint associativity, one has, for all $E, F, G \in \mathbf{D}_X$,

$$\begin{aligned}\mathrm{Hom}_{\mathbf{D}_X}\big(E \underset{=}{\otimes} F, G\big) &= \mathrm{Hom}_{\mathbf{D}(X)}\big(E \underset{=}{\otimes} F, G\big)\\ &\cong \mathrm{Hom}_{\mathbf{D}(X)}\big(E, \mathbf{R}\mathcal{H}om(F,G)\big)\\ &\cong \mathrm{Hom}_{\mathbf{D}_X}\big(E, \mathsf{Q}_X\mathbf{R}\mathcal{H}om(F,G)\big).\end{aligned}$$

Noetherian schemes (continued)

For each $f\colon X \to Y$ in $\mathbf{C}$, one shows that $\mathbf{R}f_*\mathbf{D}_{\mathsf{qc}}(X) \subset \mathbf{D}_{\mathsf{qc}}(Y)$;[2] we denote this functor simply by $f_*\colon \mathbf{D}_X \to \mathbf{D}_Y$.

Also, one shows (easily) that $\mathbf{L}f^*\mathbf{D}_{\mathsf{qc}}(Y) \subset \mathbf{D}_{\mathsf{qc}}(X)$; we denote this functor simply by $f^*\colon \mathbf{D}_Y \to \mathbf{D}_X$.

As in the main example of the preceding lecture, this gives us
an adjoint pair of closed-category-valued pseudofunctors.

The first three of the above axioms are easy to check.

The fourth, that the projection maps are isomorphisms, will be discussed below.

The fifth, *one of the basic facts of duality theory*, is the existence of a right adjoint for $\mathbf{R}f_*$,
to be discussed in a subsequent lecture.

2 Projection isomorphisms.

Theorem 1. *Let $f\colon X \to Y$ be a map of noetherian schemes, $F \in \mathbf{D}_{\mathsf{qc}}(X)$, $G \in \mathbf{D}_{\mathsf{qc}}(Y)$. Then the projection maps are isomorphisms*

$$\mathbf{p}_1\colon (\mathbf{R}f_*F) \underset{=}{\otimes} G \xrightarrow{\ \sim\ } \mathbf{R}f_*(F \underset{=}{\otimes} \mathbf{L}f^*G), \quad \mathbf{p}_2\colon G \underset{=}{\otimes} \mathbf{R}f_*F \xrightarrow{\ \sim\ } \mathbf{R}f_*(\mathbf{L}f^*G \underset{=}{\otimes} F).$$

Sketch of proof
A *key fact* is that $\mathbf{R}f_*\colon \mathbf{D}_{\mathsf{qc}}(X) \to \mathbf{D}_{\mathsf{qc}}(Y)$ is a bounded-above functor:
there is an integer d such that for all $E \in \mathbf{D}_{\mathsf{qc}}(X)$ and all $n \in \mathbb{Z}$,

$$H^i(E) = 0 \text{ for all } i \geq n \implies H^i(\mathbf{R}f_*E) = 0 \text{ for all } i \geq n+d.$$

This is shown by induction on the least number of affines covering X and Y.

Turning to the theorem, we treat only $\mathbf{p}_1$. ($\mathbf{p}_2$ can be handled similarly, or by symmetry.)
The question is local on Y, so we may assume Y affine.

[1] Much of what follows applies, with some elaborations, to arbitrary quasi-compact quasi-separated schemes.

[2] Showing that $E \in \mathbf{D}_{\mathsf{qc}}(X) \implies \mathbf{R}f_*E \in \mathbf{D}_{\mathsf{qc}}(Y)$ involves only standard arguments when $H^iE = 0$ for all $i \ll 0$, but is somewhat trickier otherwise.

Sketch of proof (continued)

Suppose first that both F and G are bounded-above complexes. Then boundedness of $\mathbf{R}f_*$ implies that the source and target of

$$\mathbf{p}_1\colon (\mathbf{R}f_*F) \underset{=}{\otimes} G \xrightarrow{\;\sim\;} \mathbf{R}f_*(F \underset{=}{\otimes} \mathbf{L}f^*G)\,,$$

are, for fixed F, bounded-above functors of G.
This allows us to use inductive "way-out" methods to reduce the question to where G is a single *free* $\mathcal{O}_Y$-module G^0, whence $\mathbf{L}f^*G$ is isomorphic to the free $\mathcal{O}_X$-module f^*G^0.
One verifies that everything in sight commutes with direct sums, so we have a further reduction to the case $G = \mathcal{O}_Y$.
In that case, $\mathbf{p}_1$ is isomorphic to the identity map of $\mathbf{R}f_*F$.

The unbounded case requires additional considerations, omitted here. (Full details in the reference notes.)

Remark: An example in the reference notes shows that quasi-coherence of homology is necessary for the theorem to hold.

3 Independent squares.

We describe a certain class of commutative squares which will play an important role later on, in connection with a fundamental base-change theorem for the right adjoint of $\mathbf{R}f_*$.

Recall that to a commutative $\mathbf{C}$-square

$$\begin{tikzcd} X' \arrow[r, "v"] \arrow[d, "g"'] \arrow[dr, phantom, "\sigma"] & X \arrow[d, "f"] \\ Y' \arrow[r, "u"'] & Y \end{tikzcd}$$

one associates the map

$$\theta = \theta_\sigma\colon u^*f_* \to g_*v^*,$$

adjoint to the natural composition $f_* \to f_*v_*v^* \xrightarrow{\;\sim\;} u_*g_*v^*$.

Similarly, one has the map

$$\theta'_\sigma\colon f^*u_* \to v_*g^*$$

Example 2. In the commutative algebra situation, σ corresponds to a commutative square of ring-maps

$$\begin{tikzcd} S' \arrow[dr, phantom, "\bar\sigma"] & S \arrow[l, "\bar v"'] \\ R' \arrow[u, "\bar g"] & R \arrow[l, "\bar u"] \arrow[u, "\bar f"'] \end{tikzcd}$$

and θ_σ is the usual functorial map, for S-modules M,

$$R' \otimes_R M \to S' \otimes_S M,$$

while θ'_σ is the usual functorial map, for R'-modules N,

$$S \otimes_R N \to S' \otimes_{R'} N.$$

In the more significant scheme-theoretic context, with u^* standing for $\mathbf{L}u^*$, f_* for $\mathbf{R}f_*\ldots$, one replaces M and N by q-flat quasi-coherent complexes.

Künneth map

For a commutative $\mathbf{C}$-square

$$\begin{tikzcd} X' \arrow[r, "v"] \arrow[d, "g"'] \arrow[dr, phantom, "\sigma"] & X \arrow[d, "f"] \\ Y' \arrow[r, "u"'] & Y \end{tikzcd}$$

setting $h := fv = ug$, define the functorial Künneth map

$$\eta_\sigma(E,F)\colon u_*E\otimes f_*F \to h_*(g^*E\otimes v^*F) \qquad (E\in \mathbf{D}_{Y'}, F\in\mathbf{D}_X)$$

to be the natural composition

$$u_*E\otimes f_*F \to h_*h^*(u_*E\otimes f_*F) \to h_*(g^*u^*u_*E\otimes v^*f^*f_*F) \to h_*(g^*E\otimes v^*F).$$

Example 3. 1. When $X = X' = Y$, and v, g are identity maps (so that $u = f$), then

$$\eta = \mathbf{e}_f\colon f_*E\otimes f_*F \to f_*(E\otimes F).$$

2. When $X = Y$, $X' = Y'$, and f, g, are identity maps (so that $u = v$),

$$\eta = \mathbf{p}_1\colon u_*E\otimes F \to u_*(E\otimes u^*F).$$

Example 4. In the commutative algebra situation, σ corresponds to a commutative square of ring-maps

$$\begin{tikzcd} S' \arrow[dr, phantom, "\bar\sigma"] & S \arrow[l, "\bar v"'] \\ R' \arrow[u, "\bar g"] & R \arrow[l, "\bar u"] \arrow[u, "\bar f"'] \end{tikzcd}$$

and η_σ is the usual functorial map, for R'-modules M, and S-modules N,

$$M\otimes_R N \to (M\otimes_{R'} S')\otimes_{S'}(S'\otimes_S N).$$

In the corresponding scheme-theoretic context, with u^* standing for $\mathbf{L}u^*$, f_* for $\mathbf{R}f_*\dots$, one replaces M and N by q-flat quasi-coherent complexes.

Equivalent definitions of independence

Theorem 5. *Let*

$$\begin{tikzcd} X' \arrow[r, "v"] \arrow[d, "g"'] \arrow[dr, phantom, "\sigma"] & X \arrow[d, "f"] \\ Y' \arrow[r, "u"'] & Y \end{tikzcd}$$

be a fiber square of quasi-compact quasi-separated schemes (i.e., σ commutes and the associated map is an isomorphism $X' \xrightarrow{\sim} Y'\times_Y X$). Set $h := fv = gu$.
The following conditions are equivalent—and when they hold we say that σ is an independent square:

(i) *For all $E\in\mathbf{D}_{\mathsf{qc}}(X)$, θ_σ is an isomorphism*

$$Lu^*\mathbf{R}f_*E \xrightarrow{\sim} \mathbf{R}g_*Lv^*E.$$

(i)$'$ *For all $F\in\mathbf{D}_{\mathsf{qc}}(Y')$, θ'_σ is an isomorphism*

$$\mathbf{L}f^*\mathbf{R}u_*E \xrightarrow{\sim} \mathbf{R}v_*Lg^*E.$$

(ii) *For all $E\in\mathbf{D}_{\mathsf{qc}}(X)$ and $F\in\mathbf{D}_{\mathsf{qc}}(Y')$, η_σ is an isomorphism*

$$\mathbf{R}u_*E\otimes\mathbf{R}f_*F \xrightarrow{\sim} \mathbf{R}h_*(Lg^*E\otimes Lv^*F).$$

Theorem-definition (continued)

(iii) The square σ is tor-independent, that is, for all pairs of points $y' \in Y'$, $x \in X$ such that $y := u(y') = f(x)$,

$$\mathrm{Tor}_i^{\mathcal{O}_{Y,y}}(\mathcal{O}_{Y',y'}, \mathcal{O}_{X,x}) = 0 \quad \text{for all } i > 0.$$

or, equivalently, for any affine open neighborhood $\mathrm{Spec}(A)$ of y and affine open sets $\mathrm{Spec}(A') \subset u^{-1}\mathrm{Spec}(A)$, $\mathrm{Spec}(B) \subset f^{-1}\mathrm{Spec}(A)$,

$$\mathrm{Tor}_i^A(A', B) = 0 \quad \text{for all } i > 0.$$

Remarks. (a) Condition (iii) holds if either f or u is flat.

(b) When f and g are identity maps, then of course (iii) holds, and so the implication (iii) $\Rightarrow$ (ii) amounts to saying that the projection map $\mathbf{p}_1$ is an isomorphism. But actually this latter fact is used in proving that (iii) $\Rightarrow$ (ii).

Outline of proof

That either (i) or (i)$'$ implies (ii) results from commutativity of the following natural diagram, for any $E \in \mathbf{D}_{\mathsf{qc}}(Y')$ and $F \in \mathbf{D}_{\mathsf{qc}}(X)$, the proof of which is a formal exercise on adjoint monoidal pseudofunctors:

$$\begin{tikzcd}
u_*(E \otimes u^*f_*F) \arrow[d, "u_*(1\otimes\theta)"'] & u_*E \otimes f_*F \arrow[l, "\sim"', "\mathbf{p}_1"] \arrow[r, "\sim", "\mathbf{p}_2"'] \arrow[dd, "\eta"] & f_*(f^*u_*E \otimes F) \arrow[d, "f_*(\theta'\otimes 1)"] \\
u_*(E \otimes g_*v^*F) \arrow[d, "u_*(\mathbf{p}_2)"', "\simeq"] & & f_*(v_*g^*E \otimes F) \arrow[d, "f_*(\mathbf{p}_1)", "\simeq"'] \\
u_*g_*(g^*E \otimes v^*F) \arrow[r, "\sim"] & h_*(g^*E \otimes v^*F) & f_*v_*(g^*E \otimes v^*F) \arrow[l, "\sim"']
\end{tikzcd}$$

For the rest, one first treats the case where all the schemes in σ are affine. To reduce to this case, by means of suitable affine covers, one needs to know that the conditions (i), (i)$'$, and (ii) are local. For this, one needs the behavior of independence under "concatenation of squares":

For each one of the following **C**-diagrams, assumed commutative,

$$\begin{tikzcd}
X'' \arrow[r, "v_1"] \arrow[d, "h"'] \arrow[dr, phantom, "\sigma_1"] & X' \arrow[r, "v"] \arrow[d, "g"'] \arrow[dr, phantom, "\sigma"] & X \arrow[d, "f"] \\
Y'' \arrow[r, "u_1"'] & Y' \arrow[r, "u"'] & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
Z' \arrow[r, "w"] \arrow[d, "g_1"'] \arrow[dr, phantom, "\sigma_1"] & Z \arrow[d, "f_1"] \\
X' \arrow[r, "v"] \arrow[d, "g"'] \arrow[dr, phantom, "\sigma"] & X \arrow[d, "f"] \\
Y' \arrow[r, "u"'] & Y
\end{tikzcd}$$

if σ and σ_1 satisfy (i) (resp. (i)$'$, resp. (ii)) then so does the rectangle σ_0 enclosed by the outer border.

This is shown via transitivity relations for θ, θ' and η. For instance, the θs for σ, σ_1 and σ_0 are related by commutativity, for any $G \in \mathbf{D}_X$, of the following **C**-diagram, a formal consequence of previously stated axioms:

$$\begin{tikzcd}
(uu_1)^*f_*G \arrow[rr, "\theta_{\sigma_0}(G)"] \arrow[d, "\simeq"'] & & h_*(vv_1)^*G \arrow[d, "\simeq"] \\
u_1^*u^*f_*G \arrow[r, "u_1^*\theta_\sigma(G)"'] & u_1^*g_*v^*G \arrow[r, "\theta_{\sigma_1}(v^*G)"'] & h_*v_1^*v^*G
\end{tikzcd}$$