REGULAR DIFFERENTIALS AND EQUIDIMENSIONAL SCHEME-MAPS

JOSEPH LIPMAN† AND PRAMATHANATH SASTRY‡

Introduction

In [Km, p. 43, Thm. 4], Kleiman proves (more than) the following version of *relative duality* for quasi-coherent sheaves in algebraic geometry:

For any scheme Z, let Z_{qc} denote the category of quasi-coherent $\mathcal{O}_Z$-Modules. Let $f\colon X \to Y$ be a finitely presentable proper map of schemes, and let d be an integer such that all the fibres of f have dimension $\leq d$. Then there exists a functor $f^!\colon Y_{\mathrm{qc}} \to X_{\mathrm{qc}}$ and a bifunctorial isomorphism

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F},\, f^!\mathcal{G}) \xrightarrow{\ \sim\ } \mathrm{Hom}_{\mathcal{O}_Y}(R^d f_* \mathcal{F},\, \mathcal{G}) \qquad (\mathcal{F} \in X_{\mathrm{qc}},\ \mathcal{G} \in Y_{\mathrm{qc}}).$$

In other words, $f^!$ is right-adjoint to the higher direct image functor $R^d f_*\colon X_{\mathrm{qc}} \to Y_{\mathrm{qc}}$; as such, it is unique up to isomorphism.

Deeply intriguing and enlivening features of the subject of Duality emerge when one seeks to render concrete realizations—often involving differential forms—of the abstract theory. The passage between abstract and concrete is unexpectedly demanding; the payoff is vital illumination in both directions. It is in this light that our results should be viewed.

The importance of differential forms in this area grows out of the following result: though the *relative dualizing sheaf* $f^!\mathcal{O}_Y$ is determined only up to isomorphism, under suitable additional hypotheses (X and Y noetherian excellent schemes without embedded associated points, $f\colon X \to Y$ generically smooth and equidimensional of relative dimension d, cf. §1), there is a *canonical choice* for it, namely the sheaf of *regular d-forms* defined by Kunz and Waldi in [KW, §3], a certain coherent $\mathcal{O}_X$-submodule of the sheaf of meromorphic relative differential d-forms, restricting over the (open, dense) smooth locus $U \subset X$ of f to the usual sheaf $\Omega^d_{U/Y}$ of holomorphic relative d-forms. This result has a long history, beginning with Roch's contribution to the Riemann-Roch theorem, if not earlier, cf. [L1, pp. 5–6]. For $Y = \mathrm{Spec}(k)$, k a perfect field, it is part of the main theorem in [L1] (*ibid.* p. 26, (e))—the projective case had been done before in [Kz]. (In this situation, and no doubt more generally, it can also be deduced from Grothendieck Duality [RD].) Recently it was proved by Hübl and Kunz for any projective map f as above [HK2, Duality Theorem], and then for any proper f by Hübl and Sastry [HS, main Theorem].

† Department of Mathematics, Purdue University, W. Lafayette IN 47907

‡ Department of Mathematics, University of Missouri-St. Louis, St. Louis, MO 63121

Now fix a base scheme S, and restrict attention to those S-schemes Z whose structure map $g \colon Z \to S$ is proper and satisfies the "additional hypotheses" in the preceding paragraph. Let $\omega_Z \cong g^! \mathcal{O}_S$ be the canonical dualizing sheaf of regular differential forms. A motivating problem is: *given a suitably restricted map $f \colon X \to Y$ of such S-schemes, relate ω_X to ω_Y by invariants of f.* There is a fascinating interplay between this global question and local functorial properties of *residues,* such as are given by (R3), (R4), and (R10) in [RD, pp. 197–199]. The case where f is finite and flat leads to generalizations of (R10) like the "trace formula" of [L2, p. 92, Thm. 4.7.1]. The case of a closed immersion, treated in [L1, §13], leads to generalizations of (R3), cf. [*ibid.,* p. 117, Thm. 13.12]. Here we consider the case where f satisfies the "additional hypotheses." This case will involve a local version of (R4). The territory is not undiscovered, but still rather unexplored; our emphasis will be on bringing out the inherent local-global relationships.

Suppose then that f has relative dimension d and that $Y \to S$ has relative dimension n, so that $X \to S$ has relative dimension $n+d$. Let ω_f be the canonical dualizing sheaf for f. The right-adjointness property of $f^!$ leads directly to a natural isomorphism $\omega_X \xrightarrow{\sim} f^! \omega_Y$, and hence (cf. §2) to a natural map

$$\eta \colon f^* \omega_Y \otimes \omega_f \to \omega_X = f^! \omega_Y \,.$$

This η is an *isomorphism* if, for example, $Y \to S$ is flat, with Gorenstein fibres (so that ω_Y is invertible); or if f is flat and locally projective, with Cohen-Macaulay fibres (cf. Remark (2.3)). In particular, the case where f is flat and finite contains the classical "Hurwitz formula." So we have in these circumstances some kind of solution to our motivating problem (given implicitly in [V, p. 396, Cor. 2], and more explicitly in [Km, p. 57, Remark (vii)]). But what does the abstractly-defined map η look like in concrete terms?

Assume for simplicity that X, Y, and S are reduced and irreducible, with function fields $k(X)$, $k(Y)$, and $k(S)$. Then ω_f is a subsheaf of the constant sheaf $\Omega^d_{k(X)/k(Y)}$, and similarly $\omega_X \subset \Omega^{n+d}_{k(X)/k(S)}$ and $\omega_Y \subset \Omega^n_{k(Y)/k(S)}$. Using only local methods (i.e., commutative algebra), Hübl shows in [H2, p.216, Thm. 1] that *the image of the natural composed map*

$$f^* \omega_Y \otimes \omega_f \to f^* \Omega^n_{k(Y)/k(S)} \otimes \Omega^d_{k(X)/k(Y)} \to \Omega^{n+d}_{k(X)/k(S)}$$

lies in ω_X. Thus there is a down-to-earth map

$$\varphi \colon f^* \omega_Y \otimes \omega_f \to \omega_X \,.$$

Hübl shows further [*ibid.,* p. 221, Cor. 2] that φ is an *isomorphism* at any point $x \in X$ such that f is Cohen-Macaulay at x and g is Cohen-Macaulay at $f(x)$,[1] or such that g is Gorenstein at $f(x)$. So here again is a solution—local, and concrete—to our problem.

---

[1] This restriction on g is superfluous.

Our **main result** Theorem (4.1) asserts, in essence, that $\eta = \varphi$. Thus the global, functorial perspective and the local, algebraic perspective complement each other.

We close this Introduction with a few remarks about the proof of Theorem (4.1). It suffices to compare the stalks η_x and φ_x at a closed point x of X at which f is smooth and such that g is smooth at $f(x)$; and since all the relevant data are compatible with flat base change, cf. (2.2.6), we may therefore assume that $S = \operatorname{Spec}(k)$, where $k = k(S)$, and that furthermore the maps $X \to Y$ and $Y \to S$ are smooth. In this case one could, presumably, find a *global* proof of the Theorem by working with the definition of the isomorphisms $f^! \mathcal{O}_Y \xrightarrow{\sim} \omega_f = \Omega^d_{X/Y}$ etc. described in [Km, p. 55, Prop. (22)] (based on [V, p. 397, Thm. 3])—and verifying that those isomorphisms coincide with the ones from [HS] which we have been using up to now. However, in the above-mentioned spirit, we will base our proof on the interconnection between global duality and local residues, as expressed fundamentally by the Residue Theorem, which in various degrees of generality is a principal result in [Kz], [L1], [HK2], and [HS]. (We need a still more general version, given in (4.2.2).)

By *local duality,* we can determine that $\eta_x = \varphi_x$ after applying local cohomology H^{n+d}_x and the residue map $\operatorname{res}_x \colon H^{n+d}_x(\Omega^{n+d}_{X/k}) \to k$, cf. e.g., [L1, §7]. Set $R := \mathcal{O}_{X,x}$ and $A := \mathcal{O}_{Y,f(x)}$; let $\mathbf{s} = (s_1, \ldots, s_n)$ be a system of parameters in A, and extend $\mathbf{s}$ to a system of parameters $(\mathbf{s}, \mathbf{t}) = (s_1, \ldots, s_n, t_1 \ldots, t_d)$ in R. The key point is to show that for $\xi_1 \in \Omega^n_{A/k}$ and $\xi_2 \in \Omega^d_{R/A}$, the image of the generalized fraction (cf. §3)

$$\begin{bmatrix} \xi_1 \otimes \xi_2 \\ \mathbf{s}, \mathbf{t} \end{bmatrix} \in H^{n+d}_x(f^* \omega_Y \otimes \omega_f)$$

under the composition

$$H^{n+d}_x(f^* \omega_Y \otimes \omega_f) \xrightarrow{\eta_x} H^{n+d}_x(\omega_X) \xrightarrow{\operatorname{res}_x} k$$

is

$$\operatorname{res}_{f(x)} \begin{bmatrix} \left(\operatorname{Res}_{R/A} \begin{bmatrix} \xi_2 \\ \mathbf{t} \end{bmatrix} \right) \xi_1 \\ \mathbf{s} \end{bmatrix}$$

where the residue $\operatorname{Res}_{R/A}[\ ] \in A$ is as in [L2, p. 19, (1.9)]. (N.B. This is a *local* characterization of the *globally defined* map η.) This requires, among other things, a transitivity relation for cohomology with supports, given in §3.

The desired conclusion then follows, because in [H1, p. 102, Cor. (7.9)] Hübl has proved a similar transitivity formula for residues, with φ in place of η.

What we just described is our original argument. R. Hübl has made us aware that the residue formalism recently developed by him and E. Kunz in [HK1] is very well suited to the present context. For example, it makes the reduction to smoooth points unnecessary. Therefore the argument given in §4, though basically the same as the foregoing one, will make use of the Hübl-Kunz approach to residues.

§1. Preliminaries

For any scheme Z, Z_{qc} denotes the category of quasi-coherent (sheaves of) $\mathcal{O}_Z$-Modules.

A locally finitely presentable (lfp) scheme-map $f\colon X \to Y$ is said to be *generically smooth* if f is smooth at each maximal point x of X, i.e., at the generic point of each irreducible component of X. (This means that f is flat at x and that, with $y = f(x)$, the local ring of x on the fibre $f^{-1}(y)$ is a field, separable over the residue field at y.) Equivalently, f is generically smooth if the (open) subset of X where f is smooth is dense in X [EGA IV, (12.1.7)].

An lfp scheme-map $f\colon X \to Y$ is said to be *equidimensional* if f takes each maximal point of X to a maximal point of Y, and if there exists an integer d such that every component of every non-empty fibre of f has dimension d. Such a d is called the *relative dimension* of f.

Let $\mathfrak{C}$ be the category whose objects are noetherian excellent schemes without embedded associated points (i.e., every associated point is maximal), and whose morphisms are finite-type (hence lfp) generically smooth equidimensional scheme-maps. Let $\mathfrak{C}^d$ consist of all morphisms in $\mathfrak{C}$ of relative dimension d. If $f\colon X \to Y$ is in $\mathfrak{C}^d$ and $g\colon Y \to S$ is in $\mathfrak{C}^n$, then one checks, using e.g. [EGA IV, (13.3.1), b)], that $gf\colon X \to S$ is in $\mathfrak{C}^{n+d}$.

To any $f\colon X \to Y$ in $\mathfrak{C}^d$, Kunz and Waldi have associated a coherent $\mathcal{O}_X$-Module ω_f^d—the sheaf of *regular differential forms of f* (with respect to the trivial differential algebra $\Omega = \mathcal{O}_Y$) *of degree d*. (Cf. [KW, p. 45, 3.2 b), and p. 51]; or, for a quick survey, [H2, pp. 214–215].) This is a subsheaf of the sheaf of meromorphic differentials $\mathcal{M}_X(\Omega_{X/Y}^d)$ defined in [EGA IV, (20.1.3)].

Let $f\colon X \to Y$ be a proper map in $\mathfrak{C}^d$, and let $(f^!, t_f)$ be a d-dualizing pair ([Km, p. 41, Definition 1]), i.e., $f^!\colon Y_{\mathrm{qc}} \to X_{\mathrm{qc}}$ is a functor right-adjoint to $R^d f_*$, and $t_f\colon R^d f_* f^! \to \mathrm{id}$ is the corresponding functorial map. The main results in [HS] yield:

(1.1) THEOREM. *There is a canonical isomorphism of $\mathcal{O}_X$-Modules*

$$\gamma_f : \omega_f^d \xrightarrow{\sim} f^! \mathcal{O}_Y .$$

(1.2) *Example.* If $d = 0$, $X = \mathrm{Spec}(B)$, $Y = \mathrm{Spec}(A)$, then the total ring of fractions L of B is étale over the total ring of fractions K of A, so we have the usual *trace map* $\mathrm{tr}\colon L \to K$; and γ_f is the sheafification of the B-module isomorphism

$$\{\, x \in L \mid \mathrm{tr}(xB) \subset A \,\} \xrightarrow{\sim} \mathrm{Hom}_A(B, A)$$

which takes x to the map $y \mapsto \mathrm{tr}(xy)$.

§2. An abstract transitivity relation

Fix a proper finitely presentable map $f\colon X \to Y$ of schemes (not necessarily noetherian). Let d be an integer such that the fibres of f are all of dimension $\leq d$, let $(f^!, t_f)$ be a d-dualizing pair [Km, p. 43, Thm. 4], and set

$$\omega_f = f^! \mathcal{O}_Y \, .$$

In this section we define a canonical functorial map

$$\eta_f(\mathcal{F}) \colon f^* \mathcal{F} \otimes_{\mathcal{O}_X} \omega_f \to f^! \mathcal{F} \qquad (\mathcal{F} \in Y_{\mathrm{qc}}),$$

and prove its compatibility with flat base change.[2] (The section heading refers to the special case where $\mathcal{F} = \omega_g$ for suitable $g\colon Y \to Z$, so that $f^! \mathcal{F} \cong \omega_{gf}$ [Km, p. 57, Remark (vii)].)

(2.1) Here is the definition of η_f. In the following diagram, with $\int_f = t_f(\mathcal{O}_Y)$, the map $F_f(\mathcal{F})$ is a natural *isomorphism,* to be described in a moment. The defining right-adjointness property of $(f^!, t_f)$ guarantees then that there is a unique map $\eta_f(\mathcal{F})$—clearly functorial in $\mathcal{F} \in Y_{\mathrm{qc}}$—making the diagram commute.

(2.1.1)
$$
\begin{array}{ccc}
R^d f_*(f^* \mathcal{F} \otimes_{\mathcal{O}_X} \omega_f) & \xrightarrow{\ R^d f_*(\eta_f(\mathcal{F}))\ } & R^d f_*(f^! \mathcal{F}) \\[4pt]
{\scriptstyle F_f(\mathcal{F})} \uparrow {\scriptstyle \simeq} & & \downarrow {\scriptstyle t_f(\mathcal{F})} \\[4pt]
\mathcal{F} \otimes_{\mathcal{O}_Y} R^d f_* \omega_f & \xrightarrow[\ \mathrm{id} \otimes \int_f\ ]{} & \mathcal{F} \otimes_{\mathcal{O}_Y} \mathcal{O}_Y
\end{array}
$$

The functorial map

$$F_f(\mathcal{F}) \colon \mathcal{F} \otimes_{\mathcal{O}_Y} R^d f_* \omega_f \to R^d f_*(f^* \mathcal{F} \otimes_{\mathcal{O}_X} \omega_f) \qquad (\mathcal{F} \in Y_{\mathrm{qc}})$$

is defined by setting $\mathcal{E} = \omega_f$ in the natural composition

$$G_f(\mathcal{F}, \mathcal{E}) \colon \mathcal{F} \otimes_{\mathcal{O}_Y} R^d f_* \mathcal{E} \to f_* f^* \mathcal{F} \otimes_{\mathcal{O}_Y} R^d f_* \mathcal{E} \to R^d f_*(f^* \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{E})$$

where $\mathcal{E}$ is any $\mathcal{O}_X$-Module, the first map arises from the canonical map $\mathcal{F} \to f_* f^* \mathcal{F}$, and the second one ("cup product") from [EGA III, p. 58, (12.2.2.1)]. This $G_f(\mathcal{F}, \mathcal{E})$ is an *isomorphism,* to verify which observe that:

(a) The question is local (check, or refer to Lemma (2.2.3) below), so we may assume that Y is affine, say $Y = \mathrm{Spec}(A)$, and that $\mathcal{F} = \tilde{M}$, the $\mathcal{O}_Y$-Module corresponding to an A-module M.

---

[2] This map is implicit in [V, p. 396, Proof of Cor. 2].

(b) The map f, being proper, is quasi-compact and separated, and so for any affine open subset U of Y, cohomology on $f^{-1}U$ commutes with (filtered) direct limits [Kf, p. 641, Thm. 8]; it follows directly—or by [*ibid.,* p. 643, Cor. 11]—that all the higher direct images $R^i f_*$ commute with direct limits. Hence, since M is a direct limit of finitely presentable A-modules [GD, p. 133, (6.3.1.4)], we can reduce to where M itself is finitely presentable, i.e., there exists an exact sequence

$$\mathcal{O}_Y^m \to \mathcal{O}_Y^n \to \mathcal{F} \to 0 \qquad (m, n \in \mathbb{N}).$$

(c) Both the functors $\mathcal{F} \otimes_{\mathcal{O}_Y} R^d f_* \mathcal{E}$ and $R^d f_* (f^* \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{E})$ are right exact in $\mathcal{F} \in Y_{\mathrm{qc}}$, because $R^i f_* = 0$ for $i > d$ [Km, p. 43, Lemma 3].

Applying $G_f(-, \mathcal{E})$ to the exact sequence in (b), we can now reduce further to the case $\mathcal{F} = \mathcal{O}_Y$, which is covered by [EGA III, p. 58, (12.2.3)]. □

(2.2) (Base change) With f and d as before, consider a fibre square

(2.2.1)
$$
\begin{array}{ccc}
X' & \xrightarrow{\ g'\ } & X \\
{\scriptstyle f'}\big\downarrow & & \big\downarrow{\scriptstyle f} \\
Y' & \xrightarrow[\ g\ ]{} & Y
\end{array}
$$

i.e., a commutative square such that the associated map $X \times_Y Y' \to X'$ is an isomorphism. We have functorial maps

(2.2.2) $$\qquad\qquad g^* R^i f_* \to (R^i f'_*) g'^* \qquad (i \geq 0)$$

corresponding via adjointness of g^* and g_* to the natural compositions

$$R^i f_* \to (R^i f_*) g'_* g'^* \to g_* (R^i f'_*) g'^*,$$

cf. [EGA III, p. 58, (12.2.5)].

(2.2.3) LEMMA. *With preceding notation, the following diagram commutes:*

$$
\begin{array}{ccc}
g^*(\mathcal{F} \otimes_{\mathcal{O}_Y} R^d f_* \mathcal{E}) & \xrightarrow{\ g^* G_f(\mathcal{F}, \mathcal{E})\ } & g^* R^d f_*(f^* \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{E}) \\
{\scriptstyle (2.2.2)}\big\downarrow & & \big\downarrow{\scriptstyle (2.2.2)} \\
g^* \mathcal{F} \otimes_{\mathcal{O}_{Y'}} R^d f'_* g'^* \mathcal{E} & \xrightarrow[\ G_{f'}(g^* \mathcal{F},\, g'^* \mathcal{E})\ ]{} & R^d f'_*(f'^* g^* \mathcal{F} \otimes_{\mathcal{O}_{X'}} g'^* \mathcal{E})
\end{array}
$$

Proof. The functorial map G_f is defined above as a composition of two others. It will suffice to show that each of these two is "compatible with g^*." For the first, this compatibility amounts to commutativity of the diagram of natural functorial maps

$$
\begin{array}{ccc}
g^* & \xrightarrow{\quad\quad} & g^* f_* f^* \\
\big\downarrow & & \big\downarrow{\scriptstyle (2.2.2)} \\
f'_* f'^* g^* & \xrightarrow{\ \sim\ } & f'_* g'^* f^*
\end{array}
$$

6

which commutativity becomes apparent when the diagram is expanded naturally as follows:

$$
\begin{array}{ccccccc}
g^* & \!\!=\!\!=\!\! & g^* & \longrightarrow & g^*f_*f^* & \longrightarrow & g^*f_*g'_*g'^*f^* \\
\| & & \downarrow & & & & \downarrow{\scriptstyle\simeq} \\
g^* & \longleftarrow & g^*g_*g^* & & & & g^*(fg')_*(fg')^* \\
\| & & \downarrow & & & & \| \\
 & & g^*g_*f'_*f'^*g^* & \rightsquigarrow & g^*(gf')_*(gf')^* & \!\!=\!\!=\!\! & g^*(gf')_*(fg')^* \\
\| & & \downarrow & & & & \downarrow{\scriptstyle\simeq} \\
g^* & \longrightarrow & f'_*f'^*g^* & \rightsquigarrow & f'_*g'^*f^* & \longleftarrow & g^*g_*f'_*g'^*f^*
\end{array}
$$

For the second map, compatibility amounts to commutativity of the natural diagram

$$
\begin{array}{ccc}
g^*(f_*f^*\mathcal{F} \otimes R^d f_*\mathcal{E}) & \longrightarrow & g^*R^d f_*(f^*\mathcal{F} \otimes \mathcal{E}) \\
{\scriptstyle\simeq}\downarrow & & \downarrow{\scriptstyle(2.2.2)} \\
g^*f_*f^*\mathcal{F} \otimes g^*R^d f_*\mathcal{E} & & R^d f'_* g'^*(f^*\mathcal{F} \otimes \mathcal{E}) \\
{\scriptstyle(2.2.2)}\downarrow & & \downarrow{\scriptstyle\simeq} \\
f'_*g'^*f^*\mathcal{F} \otimes R^d f'_* g'^*\mathcal{E} & \longrightarrow & R^d f'_*(g'^*f^*\mathcal{F} \otimes g'^*\mathcal{E})
\end{array}
$$

i.e., to commutativity of the adjoint diagram (where $\mathcal{D} = f^*\mathcal{F}$)

$$
\begin{array}{ccc}
f_*\mathcal{D} \otimes R^d f_*\mathcal{E} & \longrightarrow & R^d f_*(\mathcal{D} \otimes \mathcal{E}) \\
\downarrow & & \downarrow \\
g_*(g^*f_*\mathcal{D} \otimes g^*R^d f_*\mathcal{E}) & & g_*R^d f'_* g'^*(\mathcal{D} \otimes \mathcal{E}) \\
\downarrow & & \downarrow \\
g_*(f'_*g'^*\mathcal{D} \otimes R^d f'_* g'^*\mathcal{E}) & \longrightarrow & g_*R^d f'_*(g'^*\mathcal{D} \otimes g'^*\mathcal{E})
\end{array}
$$

But this last diagram—without its middle row—is just the sheafification of the following natural diagram of *presheaves* on Y (U being any open subset of Y, and $V = g^{-1}U$), a diagram whose commutativity results from [EGA III, p. 53, (12.1.5)]:[3]

$$
\begin{array}{ccc}
H^0(f^{-1}U, \mathcal{D}) \otimes_{\Gamma(U,\mathcal{O}_Y)} H^d(f^{-1}U, \mathcal{E}) & \longrightarrow & H^d(f^{-1}U, \mathcal{D} \otimes_{\mathcal{O}_X} \mathcal{E}) \\
\downarrow & & \downarrow \\
H^0(f'^{-1}V, g'^*\mathcal{D}) \otimes_{\Gamma(V,\mathcal{O}_{Y'})} H^d(f'^{-1}V, g'^*\mathcal{E}) & \longrightarrow & H^d(f'^{-1}V, g'^*\mathcal{D} \otimes_{\mathcal{O}_{X'}} g'^*\mathcal{E})
\end{array}
$$

Filling in of details is left to the reader. □

---

[3] See also (III) in (4.5) below for an explicit description of the cup product.

Assume now that g is *flat*. Then the maps in (2.2.2) are all *isomorphisms* [EGA III, (1.4.15)]. And from [Km, p. 44, Thm. 5] and its proof we get a functorial isomorphism

$$(2.2.4) \qquad\qquad g'^* f^! \xrightarrow{\ \sim\ } f'^! g^*$$

which at any $\mathcal{F} \in Y_{\mathrm{qc}}$ is the unique map such that the following diagram commutes:

$$(2.2.5)$$
$$
\begin{array}{ccc}
R^d f'_*(g'^* f^! \mathcal{F}) & \xrightarrow{\ R^d f'_*(2.2.4)\ } & R^d f'_*(f'^! g^* \mathcal{F}) \\
{\scriptstyle (2.2.2)}\Big\downarrow{\scriptstyle \simeq} & & \Big\downarrow{\scriptstyle t_{f'}(g^*\mathcal{F})} \\
g^* R^d f_* f^! \mathcal{F} & \xrightarrow[\ g^*(t_f(\mathcal{F}))\ ]{} & g^* \mathcal{F}
\end{array}
$$

In particular, we have (taking $\mathcal{F} = \mathcal{O}_Y$) a canonical isomorphism

$$(2.2.4)_\omega \qquad\qquad g'^* \omega_f \xrightarrow{\ \sim\ } \omega_{f'} .$$

(2.2.6) PROPOSITION (Compatibility of η_f with flat base change). *Let f and d be as before, let*

$$
\begin{array}{ccc}
X' & \xrightarrow{\ g'\ } & X \\
{\scriptstyle f'}\Big\downarrow & & \Big\downarrow{\scriptstyle f} \\
Y' & \xrightarrow[\ g\ ]{} & Y
\end{array}
$$

be a fibre square with g flat, and let $(f^!, t_f)$ and $(f'^!, t_{f'})$ be d-dualizing pairs for f and f' respectively. Then for every $\mathcal{F} \in Y_{\mathrm{qc}}$, the following diagram, with vertical isomorphisms arising from (2.2.4), commutes:

$$
\begin{array}{ccc}
g'^*(f^*\mathcal{F} \otimes_{\mathcal{O}_X} \omega_f) & \xrightarrow{\ g'^*(\eta_f(\mathcal{F}))\ } & g'^* f^! \mathcal{F} \\
{\scriptstyle \simeq}\Big\downarrow & & \Big\downarrow{\scriptstyle \simeq} \\
f'^* g^* \mathcal{F} \otimes_{\mathcal{O}_X} \omega_{f'} & \xrightarrow[\ \eta_{f'}(g^*\mathcal{F})\ ]{} & f'^! g^* \mathcal{F}
\end{array}
$$

Proof. Let $\eta : f'^* g^* \mathcal{F} \otimes_{\mathcal{O}_X} \omega_{f'} \to f'^! g^* \mathcal{F}$ be the unique map such that the diagram

$$
\begin{array}{ccc}
g'^*(f^*\mathcal{F} \otimes_{\mathcal{O}_X} \omega_f) & \xrightarrow{\ g'^*(\eta_f(\mathcal{F}))\ } & g'^* f^! \mathcal{F} \\
{\scriptstyle \simeq}\Big\downarrow & & \Big\downarrow{\scriptstyle \simeq} \\
f'^* g^* \mathcal{F} \otimes_{\mathcal{O}_X} \omega_{f'} & \xrightarrow[\ \eta\ ]{} & f'^! g^* \mathcal{F}
\end{array}
$$

commutes. Our task is to show that $\eta = \eta_{f'}(g^*\mathcal{F})$, i.e., that the following diagram commutes (cf. (2.1.1)):

$$
\begin{array}{ccc}
R^d f'_*(f'^*g^*\mathcal{F} \otimes_{\mathcal{O}_{X'}} \omega_{f'}) & \xrightarrow{\ R^d f'_*(\eta)\ } & R^d f'_*(f'^! g^*\mathcal{F}) \\[2mm]
{\scriptstyle F_{f'}(g^*\mathcal{F})} \Big\uparrow {\scriptstyle \simeq} & & \Big\downarrow {\scriptstyle t_{f'}(g^*\mathcal{F})} \\[2mm]
g^*\mathcal{F} \otimes_{\mathcal{O}_{Y'}} R^d f'_* \omega_{f'} & \xrightarrow[\ \mathrm{id}\otimes \int_{f'}\ ]{} & g^*\mathcal{F} \otimes_{\mathcal{O}_{Y'}} \mathcal{O}_{Y'}
\end{array}
$$

For this purpose, expand the diagram as follows, with all unlabeled arrows representing *isomorphisms* arising from (2.2.2) or (2.2.4):

Commutativity of subdiagram ① follows from the definition of η; of ② from functoriality of (2.2.2); of ③ from Lemma 2.2.3, with $\mathcal{E} = \omega_f$ and $g'^*\mathcal{E} = \omega_{f'}$, cf. $(2.2.4)_\omega$; of ④ and ⑤ from the commutativity of (2.2.5); and of ⑥ from the commutativity of (2.1.1). Since the southwest-pointing arrows—which represent isomorphisms—can be reversed, commutativity of the outer border results. □

(2.3) *Remarks.* 1. One checks that $\eta_f(\mathcal{O}_Y) = $ identity; and so Proposition (2.2.6) implies that $\eta_f(\mathcal{F})$ is an *isomorphism* whenever $\mathcal{F}$ is locally free of finite rank. The same

holds in fact for any *flat* $\mathcal{F} \in Y_{\mathrm{qc}}$, since any such $\mathcal{F}$ is locally a direct limit of finite-rank free ones (by Lazard's theorem [GD, p. 163, (6.6.24)]), and since $f^!$ commutes with direct limits [Km, p.42, Prop. 2, (iv)].

2. If η_f is a functorial isomorphism, and if Y is quasi-separated, then $\omega_f = f^! \mathcal{O}_Y$ is a *dualizing sheaf* in the sense of [Km, p. 46, Definition 6]. Indeed, since η_f is, as in (2.2.6), compatible with open immersions, and since for every affine open $U \subset Y$, quasi-coherent $\mathcal{O}_U$-Modules extend to quasi-coherent $\mathcal{O}_Y$-Modules [GD, p. 317, (6.9.2)], therefore $\eta_{f|U}$ is an isomorphism, and we conclude via [Km, p. 47, Prop. 8, (ii) $\Rightarrow$ (i)].

Conversely—and with no assumption on Y—if a dualizing sheaf exists, then for all open $U \subset Y$, $\eta_{f|U}$ is the isomorphism in [Km, p. 46, Definition 6].[4] This need only be verified (trivially) at $\mathcal{O}_Y$, because all the functors in sight preserve epimorphisms and arbitrary direct sums, and all the maps involved are compatible with open immersions.

Dualizing sheaves exist, for example, whenever Y is the spectrum of a field [Km, p.46, Example 7 (i)]; or whenever f is flat, equidimensional, locally projective, and has Cohen-Macaulay fibres [Km, p. 48, Definition 10, and p. 55, Thm. 21].

§3 Transitivity for cohomology with supports

(3.1) Let I be an ideal in a noetherian commutative ring A. The left-exact additive functor Γ_I of A-modules M is given by

$$\Gamma_I(M) := \{\, m \in M \mid I^n m = 0 \text{ for some } n > 0 \,\}.$$

The right-derived functors of Γ_I—cohomology with supports in I (or, more accurately, in $\mathrm{Spec}(A/I)$)—are denoted by H_I^i ($i \geq 0$).

If J is a second A-ideal, then

$$(3.1.1) \qquad\qquad\qquad \Gamma_{J+I} = \Gamma_J \circ \Gamma_I \,.$$

As is well known (or follows from (3.2.3) below), if $I = (t_1, \ldots, t_d)A$ and $J = (s_1, \ldots, s_n)A$, then

$$H_I^p(M) = 0 \text{ for } p > d \quad \text{and} \quad H_J^q(M) = 0 \text{ for } q > n;$$

hence the spectral sequence associated to the decomposition (3.1.1) of Γ_{J+I} gives rise to a canonical isomorphism

$$(3.1.2) \qquad\qquad\qquad \nu \colon H_J^n\big(H_I^d(M)\big) \overset{\sim}{\longrightarrow} H_{J+I}^{n+d}(M) \,.$$

The main result in this section is Proposition (3.3.1), which describes ν in terms of "generalized fraction" representations of elements in modules of the form $H_I^d(M)$.

---

[4] which is presumably meant to be the identity when applied to $\mathcal{O}_Y$.

10

(3.2) To prepare the ground, we review a few facts about cohomology with supports.

Let $X^\bullet$ be a bounded-below complex of A-modules. Recall from [RD, p. 56, Cor. 5.3 α] the definition of $\mathbb{R}\Gamma_I(X^\bullet)$, a complex whose homology is the hyperhomology of $X^\bullet$ with supports in I.

Let $t \in A$. For any A-module M, $\Gamma_{tA}(M)$ is just the kernel of the natural map λ_M from M to the localization M_t. Moreover, if M is an injective A-module, then λ_M is surjective [Ha, p. 214, Lemma 3.3].

Denote by $\mathbf{K}^\bullet(t)$ the complex which looks like $A \xrightarrow{\lambda_A} A_t$ in degrees 0 and 1, and which vanishes elsewhere.

(3.2.1) LEMMA. *For any $X^\bullet$ in the derived category of bounded-below complexes of A-modules, and $t \in A$, there is a natural isomorphism*

$$\mathbb{R}\Gamma_{tA}(X^\bullet) \xrightarrow{\sim} \mathbf{K}^\bullet(t) \otimes X^\bullet.$$

Proof. Let $\lambda := \lambda_{X^\bullet} \colon X^\bullet \to X_t^\bullet$ be the natural map. There is an obvious map of complexes

$$\Gamma_{tA}(X^\bullet) = \ker(\lambda) \xhookrightarrow{\iota} \mathbf{K}^\bullet(t) \otimes X^\bullet .$$

One checks directly that if λ is surjective then ι induces homology isomorphisms (i.e., ι is a derived-category isomorphism); so in case $X^\bullet$ is *injective,* then we are done, since then the canonical map $\Gamma_{tA}(X^\bullet) \to \mathbb{R}\Gamma_{tA}(X^\bullet)$ is an isomorphism. Reduce the general case to this one by choosing an isomorphism of $X^\bullet$ into an injective bounded-below complex $E^\bullet$, and noting that the resulting maps $\mathbb{R}\Gamma_{tA}(X^\bullet) \to \mathbb{R}\Gamma_{tA}(E^\bullet)$ and $\mathbf{K}^\bullet(t) \otimes X^\bullet \to \mathbf{K}^\bullet(t) \otimes E^\bullet$ are both isomorphisms (the latter, since $\mathbf{K}^\bullet(t)$ is flat and bounded, via [RD, p. 93, Lemma 4.1, part b2]). □

From (3.1.1) we get an isomorphism

(3.2.2) $$\mathbb{R}\Gamma_{J+I} \xrightarrow{\sim} \mathbb{R}\Gamma_J \circ \mathbb{R}\Gamma_I$$

[RD, p. 60, b)]. Hence, by induction on d, the case $d = 0$ being trivial and $d = 1$ given by (3.2.1):

(3.2.3) COROLLARY. *For any sequence $\mathbf{t} = (t_1, \ldots, t_d)$ in A, there is a natural functorial isomorphism*

$$\alpha_\mathbf{t} \colon \mathbb{R}\Gamma_{\mathbf{t}A}(X^\bullet) \xrightarrow{\sim} \mathbf{K}^\bullet(\mathbf{t}) \otimes X^\bullet \overset{\text{def}}{=} \mathbf{K}^\bullet(t_1) \otimes \cdots \otimes \mathbf{K}^\bullet(t_d) \otimes X^\bullet;$$

and if $\mathbf{s} = (s_1, \ldots, s_n)$ is a second such sequence, then the following natural diagram commutes:

$$
\begin{array}{ccc}
\mathbb{R}\Gamma_{\mathbf{s}A}\mathbb{R}\Gamma_{\mathbf{t}A}(X^\bullet) & \xrightarrow{\;\sim\;} & \mathbb{R}\Gamma_{(\mathbf{s},\mathbf{t})A}(X^\bullet) \\
{\scriptstyle \alpha_\mathbf{s}(\mathbb{R}\Gamma_{\mathbf{t}A}(X^\bullet))}\Big\downarrow & & \Big\downarrow{\scriptstyle \alpha_{\mathbf{s},\mathbf{t}}(X^\bullet)} \\
\mathbf{K}^\bullet(\mathbf{s}) \otimes \mathbf{K}^\bullet(\mathbf{t}) \otimes X^\bullet & \xrightarrow{\;\sim\;} & \mathbf{K}^\bullet(\mathbf{s},\mathbf{t}) \otimes X^\bullet
\end{array}
$$

Now for any A-module M, the complex $\mathbf{K}^\bullet(\mathbf{t}) \otimes M$ vanishes in all degrees $> d$, and in degree d it is the localization $M_{t_1 t_2 \ldots t_d}$. So $\alpha_{\mathbf{t}}^{-1}$ induces a surjection

$$\pi = \pi(\mathbf{t}, M) \colon M_{t_1 t_2 \ldots t_d} \twoheadrightarrow H^d\big(\mathbb{R}\Gamma_{\mathbf{t}A}(M)\big) = H^d_{\mathbf{t}A}(M) \,.$$

Thus we can represent any element of $H^d_{\mathbf{t}A}(M)$ as a "generalized fraction":

(3.2.4) Definition. With preceding notation, and $m \in M$, and $a_1, a_2, \ldots, a_d$ any positive integers,

$$\left[\frac{m}{t_1^{a_1}, t_2^{a_2}, \ldots, t_d^{a_d}} \right] := \pi\left(\frac{m}{t_1^{a_1} t_2^{a_2} \ldots t_d^{a_d}} \right) \in H^d_{\mathbf{t}A}(M) \,.$$

For $d = 0$ and the empty sequence ϕ, we set

$$\left[\begin{matrix} m \\ \phi \end{matrix} \right] := m \in M = H^0_{(0)}(M) \,.$$

Remark. There is a possible ambiguity here in that we can have equalities of the form $t_1^{a_1} = s_1^{b_1}$, etc. But in fact there is no problem: denoting $(t_1^{a_1}, \ldots t_d^{a_d})$ by $\mathbf{t^a}$, one verifies easily that $\mathbf{K}^\bullet(\mathbf{t^a}) = \mathbf{K}^\bullet(\mathbf{t})$ and that $\alpha_{\mathbf{t^a}} = \alpha_{\mathbf{t}}$, whence $\pi(\mathbf{t}, M) = \pi(\mathbf{t^a}, M)$, so that the map π depends only on the sequence $\mathbf{t^a}$.

(3.3) From (3.2.3) we deduce, for A-modules L, M, a natural functorial isomorphism

$$\mu = \mu_{\mathbf{s},\mathbf{t}}(L, M) \colon H^n_{\mathbf{s}A}(L) \otimes H^d_{\mathbf{t}A}(M) \xrightarrow{\sim} H^{n+d}_{(\mathbf{s},\mathbf{t})A}(L \otimes M)$$

such that

$$\mu\left(\left[\begin{matrix} a \\ \mathbf{s} \end{matrix} \right] \otimes \left[\begin{matrix} b \\ \mathbf{t} \end{matrix} \right] \right) = \left[\begin{matrix} a \otimes b \\ \mathbf{s}, \mathbf{t} \end{matrix} \right] \qquad (a \in L, \ b \in M).$$

In particular, we have the isomorphism

$$\mu_{\mathbf{s},\phi}\big(A, H^d_{\mathbf{t}}(M)\big) \colon H^n_{\mathbf{s}A}(A) \otimes H^d_{\mathbf{t}A}(M) \xrightarrow{\sim} H^n_{\mathbf{s}A}\big(H^d_{\mathbf{t}A}(M)\big) \,.$$

So we have the isomorphism

$$\mu_{\mathbf{s},\mathbf{t}}(A, M) \circ \mu_{\mathbf{s},\phi}\big(A, H^d_{\mathbf{t}}(M)\big)^{-1} \colon H^n_{\mathbf{s}A}\big(H^d_{\mathbf{t}A}(M)\big) \xrightarrow{\sim} H^{n+d}_{(\mathbf{s},\mathbf{t})A}(M) \,.$$

The following Proposition identifies this isomorphism with the isomorphism ν of (3.1.2).

(3.3.1) PROPOSITION. *Let I and J be ideals in a noetherian commutative ring A, generated respectively by sequences $(t_1, \ldots, t_d)$ and $(s_1, \ldots, s_n)$. Then for any A-module M, and with notation as in (3.2.4), the isomorphism ν of (3.1.2) satisfies*

$$\nu \begin{bmatrix} \begin{bmatrix} m \\ \mathbf{t} \end{bmatrix} \\ \mathbf{s} \end{bmatrix} = \begin{bmatrix} m \\ \mathbf{s}, \mathbf{t} \end{bmatrix} \qquad (m \in M).$$

Proof. For any complex $Z^\bullet$ of A-modules, and any integer r, the truncation $\tau_{\geq r} Z^\bullet$ is defined to be the complex

$$\cdots \to 0 \to 0 \to \operatorname{coker}(Z^{r-1} \to Z^r) \to Z^{r+1} \to Z^{r+2} \to \cdots$$

This $\tau_{\geq r}$ can be viewed, in the obvious way, as a functor from the category of complexes into itself. If f and g are homotopic maps of complexes, then so are $\tau_{\geq r} f$ and $\tau_{\geq r} g$; and if f induces homology isomorphisms, then so does $\tau_{\geq r} f$. Hence $\tau_{\geq r}$ can be made into a *derived-category functor.*

From (3.2.3) we deduce, for A-modules M, a functorial derived-category isomorphism

$$\tau_{\geq d} \mathbb{R}\Gamma_I(M) \xrightarrow{\sim} H_I^d(M)[-d],$$

and similarly for the pair (J, n).[5] By (3.2.3) again, the desired conclusion results quickly from commutativity of the following natural diagram:

$$
\begin{array}{ccc}
\tau_{\geq n+d} \mathbb{R}\Gamma_{J+I}(M) & \xrightarrow[(3.2.2)]{\sim} \tau_{\geq n+d} \mathbb{R}\Gamma_J\big(\mathbb{R}\Gamma_I(M)\big) \longrightarrow & \tau_{\geq n+d} \mathbb{R}\Gamma_J\big(\tau_{\geq d} \mathbb{R}\Gamma_I(M)\big) \\
\simeq \downarrow & & \downarrow \simeq \\
H_{J+I}^{n+d}(M)[-n-d] & \xrightarrow[\nu^{-1}]{} & H_J^n\big(H_I^d(M)\big)[-n-d]
\end{array}
$$

The proof of commutativity is an exercise on spectral sequences. (One may, for example, begin by identifying ν, via the natural map $\mathbb{R}\Gamma_I \to \tau_{\geq d}\mathbb{R}\Gamma_I$, with an edge homomorphism in the (degenerate) spectral sequence of Γ_J-hyperhomology of the complex $\tau_{\geq d}\mathbb{R}\Gamma_I(M)$.) □

(3.4) *Remarks.* (i) We will be making use of the residue maps defined in [HK]. Since their notation for generalized fractions is, on the surface, different than ours, we need to establish the equivalence of the two.

For any $t \in A$, $\mathcal{K}^\bullet(t)$ will denote the complex which is "multiplication by t": $A \to A$ in degrees 0 and 1, and which vanishes elsewhere. For any two positive integers $r \leq s$, there is a map of complexes $\mathcal{K}^\bullet(t^r) \to \mathcal{K}^\bullet(t^s)$ which is the identity in degree 0 and multiplication

---

[5] Recall that for an A-module N, $N[-d]$ is the complex which is N in degree d and 0 everywhere else.

by t^{s-r} in degree 1. The resulting direct system of complexes has as its $\varinjlim$ the complex $\mathbf{K}^\bullet(t)$ defined above.

Next, let $\mathbf{t} := (t_1, \ldots, t_d)$ be as in (3.2.3). For any A-module M, we set

$$\mathcal{K}^\bullet(\mathbf{t}, M) := \mathcal{K}^\bullet(t_1) \otimes \cdots \otimes \mathcal{K}^\bullet(t_d) \otimes M \,.$$

There is a natural isomorphism between this $\mathcal{K}^\bullet(\mathbf{t}, M)$ and the Koszul cochain complex $K^\bullet(\mathbf{t}, M)$ defined in [EGA III, p. 82, (1.1.2.2)], inducing in degree d the identity map of $M = \mathcal{K}^d(\mathbf{t}, M) = K^d(\mathbf{t}, M)$. The specification of this isomorphism (by induction on d, or directly) is left to the reader. Setting $\mathbf{t}^r := (t_1^r, \ldots, t_d^r)$, we get from the direct systems $\mathcal{K}^\bullet(t_i^r)$ $(1 \le i \le d)$ a direct system $\mathcal{K}^\bullet(\mathbf{t}^r, M)$ whose $\varinjlim$ is $\mathbf{K}^\bullet(\mathbf{t}) \otimes M$. Hence we have a map

$$\beta_{\mathbf{t}} \colon M/\mathbf{t}M = H^d\big(\mathcal{K}^\bullet(\mathbf{t}, M)\big) \to H^d\big(\mathbf{K}^\bullet(\mathbf{t}) \otimes M\big) \xrightarrow{\sim} H_I^d(M)$$

such that

$$\beta_{\mathbf{t}}(m + \mathbf{t}M) = \begin{bmatrix} m \\ \mathbf{t} \end{bmatrix}.$$

With these remarks, the equivalence of the generalized fraction notation here and in [HK] becomes straightforward to check.

(ii) Let I be an ideal in a noetherian ring A, and let $\mathcal{P} = \mathcal{P}(I)$ be the set of all sequences $\mathbf{t}$ of length d such that the ideals I and $\mathbf{t}A$ have the same radical. Assume that $\mathcal{P}$ is non-empty. Let $\mathbf{t}$ and $\mathbf{t}'$ be sequences in $\mathcal{P}$, and think of them as $1 \times d$ column vectors. For any $d \times d$ matrix Λ such that $\Lambda \mathbf{t} = \mathbf{t}'$, Cramer's rule shows that multiplication by the determinant $|\Lambda|$ induces a map $M/\mathbf{t}M \to M/\mathbf{t}'M$. The family $(M/\mathbf{t}M)_{\mathbf{t} \in \mathcal{P}}$ together with all such determinantal maps forms a directed inductive system. It follows easily from [L1, p. 60, Lemma 7.2] (whose proof in the present more general context is the same) that $H_I^d(M)$ *together with the preceding maps $\beta_{\mathbf{t}}$ is a $\varinjlim$ of this system.* Thus to define a map of groups $\rho \colon H_I^d(M) \to S$ is the same as to define a family of maps $\rho_{\mathbf{t}} \colon M/\mathbf{t}M \to S$ ($\mathbf{t} \in \mathcal{P}$) commuting with the determinantal maps, the correspondence being such that

$$\rho \begin{bmatrix} m \\ \mathbf{t} \end{bmatrix} = \rho_{\mathbf{t}}(m + \mathbf{t}M) \,.$$

§4 Transitivity for regular differentials

(4.0) We need some preliminary definitions before stating the main Theorem (4.1).

Let X be a noetherian scheme without embedded associated points. Let X_0 be the artinian scheme

$$X_0 := \coprod_s \operatorname{Spec}(\mathcal{O}_{X,s})$$

where s runs through the set of associated (= maximal) points of X, let $i_X\colon X_0 \to X$ be the canonical map, and set

$$k(X) := i_{X*}(\mathcal{O}_{X_0}),$$

the sheaf of germs of meromorphic functions on X. The map $i := i_X$ being affine, we have for any $\mathcal{E} \in X_{\mathrm{qc}}$ and $\mathcal{F} \in (X_0)_{\mathrm{qc}}$, a natural isomorphism

$$(4.0.1) \qquad \mathcal{E} \otimes_{\mathcal{O}_X} i_* \mathcal{F} \xrightarrow{\sim} i_*(i^* \mathcal{E} \otimes_{\mathcal{O}_{X_0}} \mathcal{F});$$

and in particular there is a natural isomorphism

$$\mathcal{E} \otimes_{\mathcal{O}_X} k(X) \xrightarrow{\sim} i_* i^* \mathcal{E}.$$

For any map $f\colon X \to Y$ of such schemes, $\Omega^j_{X/Y}$ denotes, as usual, the sheaf of relative differential j-forms. The map f induces a map $f_0\colon X_0 \to Y_0$, and we have $\Omega^j_{X_0/Y_0} = i_X^* \Omega^j_{X/Y}$. We define the sheaf of *meromorphic relative j-forms* to be

$$\Omega^j_{k(X)/k(Y)} := i_{X*}(\Omega^j_{X_0/Y_0}) = i_{X*} i_X^* \Omega^j_{X/Y} = \Omega^j_{X/Y} \otimes_{\mathcal{O}_X} k(X).$$

Now let $f\colon X \to Y$ be in $\mathfrak{C}^d$ and let $g\colon Y \to S$ be in $\mathfrak{C}^n$, so that $gf\colon X \to S$ is in $\mathfrak{C}^{n+d}$ (cf. §1). There results a commutative diagram

$$
\begin{array}{ccc}
X_0 & \xrightarrow{\ i_X\ } & X \\
{\scriptstyle f_0}\downarrow & & \downarrow{\scriptstyle f} \\
Y_0 & \xrightarrow[\ i_Y\ ]{} & Y \\
{\scriptstyle g_0}\downarrow & & \downarrow{\scriptstyle g} \\
S_0 & \xrightarrow[\ i_S\ ]{} & S
\end{array}
$$

The generic smoothness of f and g implies that f_0 and g_0 are smooth. So we have an exact sequence of locally free $\mathcal{O}_{X_0}$-Modules

$$0 \to f_0^* \Omega^1_{Y_0/S_0} \to \Omega^1_{X_0/S_0} \to \Omega^1_{X_0/Y_0} \to 0$$

whence (cf. e.g., [RD, p. 139]) an isomorphism

$$(4.0.2) \qquad f_0^* \Omega^n_{Y_0/S_0} \otimes_{\mathcal{O}_{X_0}} \Omega^d_{X_0/Y_0} \xrightarrow{\sim} \Omega^{n+d}_{X_0/S_0}.$$

Since $i_Y^* k(Y) = \mathcal{O}_{Y_0}$, there are natural identifications

$$f_0^* \Omega^n_{Y_0/S_0} = f_0^*\left(i_Y^* \Omega^n_{Y/S} \otimes_{\mathcal{O}_{Y_0}} i_Y^* k(Y)\right) = f_0^* i_Y^*\left(\Omega^n_{Y/S} \otimes_{\mathcal{O}_Y} k(Y)\right) = i_X^* f^*\left(\Omega^n_{k(Y)/k(S)}\right).$$

Applying i_{X*} to the isomorphism (4.0.2), and keeping in mind (4.0.1), we get a canonical isomorphism (which, for convenience, we will treat as an identity)

$$f^* \Omega^n_{k(Y)/k(S)} \otimes_{\mathcal{O}_X} \Omega^d_{k(X)/k(Y)} \xrightarrow{\sim} \Omega^{n+d}_{k(X)/k(S)}.$$

Locally, this isomorphism is the same as the isomorphism Ψ in [H2, p. 214].

15

Suppose now that the maps f and g are proper. Then there is a functorial isomorphism $R^n g_* R^d f_* \xrightarrow{\sim} R^{n+d}(gf)_*$ (cf. e.g., [Km, p. 57, Remark (vii)]), and hence by the defining right-adjointness property of dualizing pairs (§1) there is a unique functorial isomorphism $f^! g^! \xrightarrow{\sim} (gf)^!$ such that the following resulting diagram commutes for any $\mathcal{D} \in S_{\mathrm{qc}}$:

(4.0.3)

$$
\begin{array}{ccc}
R^{n+d}(gf)_* f^! g^! \mathcal{D} & \xrightarrow{\hspace{1cm}\sim\hspace{1cm}} & R^{n+d}(gf)_*(gf)^! \mathcal{D} \\[2mm]
{\scriptstyle\simeq}\downarrow & & \Big\downarrow {\scriptstyle t_{gf}(\mathcal{D})} \\[2mm]
R^n g_* R^d f_* f^! g^! \mathcal{D} & & \\[2mm]
{\scriptstyle R^n g_*(t_f(g^! \mathcal{D}))}\downarrow & & \\[2mm]
R^n g_* g^! \mathcal{D} & \xrightarrow[\;t_g(\mathcal{D})\;]{} & \mathcal{D}
\end{array}
$$

From Theorem (1.1), it follows now that we can identify $f^! \omega_g^n$ with ω_{gf}^{n+d}; and then as in §2 we have the map

$$
\eta_f(\omega_g^n) : f^* \omega_g^n \otimes \omega_f^d \to \omega_{gf}^{n+d} \,.
$$

Our principal result is:

(4.1) THEOREM. *Let $f : X \to Y$, $g : Y \to S$ be proper maps, with $f \in \mathfrak{C}^d$ and $g \in \mathfrak{C}^n$. Then the following diagram, in which the vertical maps arise from the embedding of regular differentials into meromorphic differentials, commutes:*

$$
\begin{array}{ccc}
f^* \omega_g^n \otimes \omega_f^d & \xrightarrow{\;\eta_f(\omega_g^n)\;} & \omega_{gf}^{n+d} \\[2mm]
\downarrow & & \downarrow \\[2mm]
f^* \Omega_{k(Y)/k(S)}^n \otimes \Omega_{k(X)/k(Y)}^d & =\!=\!=\!= & \Omega_{k(X)/k(S)}^{n+d}
\end{array}
$$

(4.2) The proof of Theorem (4.1) will be based on Proposition (4.2.2) below, which generalizes the Residue Theorem [HS, Main Theorem, (iii); or Remark 2.7].[6]

We first fix some notation. Let $f : X \to Y$ be a proper map in $\mathfrak{C}^d$, and let $(f^!, t_f)$ be a d-dualizing pair (§1). Let x be a closed point of X, so that $y := f(x)$ is a closed point of Y. Set $R := \mathcal{O}_{X,x}$ and $A := \mathcal{O}_{Y,y}$. Let $n = \dim A$, and assume that $\dim R = n+d$ (i.e., at least one component of X through x maps densely to a component Y' of Y through y such that $\dim \mathcal{O}_{Y',y} = n$, cf. [EGA IV, 13.3.4]). Let $\mathbf{s} := (s_1, \ldots, s_n)$ be a system of parameters in A, and extend the image $\mathbf{s}'$ of $\mathbf{s}$ in the A-algebra R to a system of parameters $(\mathbf{s}', \mathbf{t})$. For any $\mathcal{F} \in Y_{\mathrm{qc}}$, the local cohomology $H_y^n(\mathcal{F}) = H_{\mathbf{s}A}^n(\mathcal{F}_y)$ is isomorphic to $\varinjlim \mathcal{F}_y / (s_1^r, \ldots, s_n^r)\mathcal{F}_y$, cf. (3.4); and hence, with $\hat{A}$ the completion of A, there is a canonical isomorphism

(4.2.1)
$$
H_{\mathbf{s}A}^n(\mathcal{F}_y) \xrightarrow{\sim} H_{\mathbf{s}\hat{A}}^n(\mathcal{F}_y \otimes_A \hat{A})
$$

---

[6] Prop. (4.2.2) is also related to [L1, p. 87, Thm. 10.2], and to [HS, Thm. 4.2].

16

(so that $H_y^n(\mathcal{F})$ is an $\hat{A}$-module). As in (3.3) there is a functorial map (with $\mathcal{E} \in X_{\mathrm{qc}}$)

$$\mu = \mu(\mathcal{F}, \mathcal{E})\colon H_y^n(\mathcal{F}) \otimes_A H_{\mathbf{t}R}^d(\mathcal{E}_x) \to H_x^{n+d}(f^*\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{E})$$

such that for $a \in \mathcal{F}_y$ and $b \in \mathcal{E}_x$,

$$\mu\left(\begin{bmatrix} a \\ \mathbf{s} \end{bmatrix} \otimes \begin{bmatrix} b \\ \mathbf{t} \end{bmatrix}\right) = \begin{bmatrix} a \otimes b \\ \mathbf{s}', \mathbf{t} \end{bmatrix}.$$

With $\mathfrak{m}$ the maximal ideal of R, we set

$$\operatorname{Res}_{R/A}\begin{bmatrix} \nu \\ \mathbf{t} \end{bmatrix} := \operatorname{Res}_{\mathfrak{m}}\begin{bmatrix} \nu \\ \mathbf{t} \end{bmatrix} \in \hat{A} \qquad (\nu \in \omega_{R/A}^d = \omega_{f,x}^d)$$

where $\operatorname{Res}_{\mathfrak{m}}$ is as in [HK1, Definition 2.1]. By the *transition formula* for the residue symbol [*ibid.*, Thm. 2.4], and the remarks at the end of (3.4) above, we can think of $\operatorname{Res}_{R/A}$ as arising from a map

$$\operatorname{Res}_{R/A}^J\colon H_J^d(\omega_{R/A}^d) \to \hat{A} \qquad (J = \sqrt{\mathbf{t}R}).$$

Let $E := f^{-1}\{y\}$. Since $H_y^p(R^q f_* f^!\mathcal{F}) = 0$ for $p > n$ or $q > d$, the spectral sequence associated to the functorial decomposition $\Gamma_E = \Gamma_y \circ f_*$ gives an isomorphism

$$\delta(\mathcal{F})\colon H_y^n(R^d f_* f^!\mathcal{F}) \xrightarrow{\sim} H_E^{n+d}(f^!\mathcal{F}) \quad (\mathcal{F} \in Y_{\mathrm{qc}}).$$

We define the *local trace*

$$t_f^x(\mathcal{F})\colon H_x^{n+d}(f^!\mathcal{F}) \to H_y^n(\mathcal{F})$$

to be the functorial composition

$$H_x^{n+d}(f^!\mathcal{F}) \xrightarrow{\text{natural}} H_E^{n+d}(f^!\mathcal{F}) \xrightarrow{\delta^{-1}} H_y^n(R^d f_* f^!\mathcal{F}) \xrightarrow{H_y^n(t_f)} H_y^n(\mathcal{F}).$$

In particular, if $Y = \operatorname{Spec}(k)$ for some local artin ring k, then $E = X$, $n = 0$, $H_y^n = \Gamma_Y$, and δ is the canonical isomorphism $\Gamma_Y(R^d f_* f^!\mathcal{F}) \xrightarrow{\sim} H^d(X, f^!\mathcal{F})$. In this case we let res_x be the composition

$$\operatorname{res}_x\colon H_x^d(\omega_f^d) \xrightarrow{\sim} H_x^d(f^!\mathcal{O}_Y) \xrightarrow{t_f^x(\mathcal{O}_S)} k.$$

This is the same as the map $\operatorname{Res}_{R/k}^{\mathfrak{m}}$, which, in view of [HK1, 1.7 and 2.6], is the map

$$H_x^d(\omega_f^d) = \Gamma_Y(R_Z^d f_* \omega_f^d) \xrightarrow{\Gamma_Y\left(\int_{X/Y,Z}\right)} k$$

with $Z = \{x\}$ and $\int_{X/Y,Z}$ as in 4.4 of *loc. cit.* Indeed, the above-mentioned Residue Theorem of [HS] tells us that $\int_{X/Y,Z}$ factors as

$$R_Z^d f_* \omega_f^d \xrightarrow{\text{natural}} R^d f_* \omega_f^d \xrightarrow{\sim} R^d f_* f^!\mathcal{O}_Y \xrightarrow{t_f} \mathcal{O}_Y,$$

and the equality $\operatorname{Res}_{R/k}^{\mathfrak{m}} = \operatorname{res}_x$ results. Thus res_x depends only on the k-algebra $\hat{R}$, and not on f.

(4.2.2) PROPOSITION. (i) (Residue Theorem) *Under the preceding circumstances, and with $\eta = \eta_f(\mathcal{F})$, cf. (2.1), the following diagram commutes:*

$$
\begin{array}{ccc}
H^n_y(\mathcal{F}) \otimes_A H^d_{\mathbf{t}R}((f^!\mathcal{O}_Y)_x) & \xrightarrow{\ \mu\ } & H^{n+d}_x(f^*\mathcal{F} \otimes_{\mathcal{O}_X} f^!\mathcal{O}_Y) \\
\simeq \downarrow & & \downarrow \eta \\
H^n_y(\mathcal{F}) \otimes_A H^d_{\mathbf{t}R}(\omega^d_{R/A}) & & H^{n+d}_x(f^!\mathcal{F}) \\
1 \otimes \mathrm{Res}^J_{R/A} \downarrow & & \downarrow t^x_f \\
H^n_y(\mathcal{F}) \otimes_A \hat{A} & \xrightarrow[\text{natural}]{} & H^n_y(\mathcal{F})
\end{array}
$$

(ii) (Transitivity for local trace) *Let $X \xrightarrow{f} Y \xrightarrow{g} S$ be proper maps in $\mathfrak{C}^d$, $\mathfrak{C}^n$ respectively. Let x be a closed point of X, and set $y := f(x)$, $z := g(y)$. Let $m := \dim \mathcal{O}_{S,z}$, and assume that $\dim \mathcal{O}_{Y,y} = m + n$ and $\dim \mathcal{O}_{X,x} = m + n + d$. Then for any $\mathcal{D} \in S_{\mathrm{qc}}$, the following diagram commutes:*

$$
\begin{array}{ccc}
H^{m+n+d}_x(f^!g^!\mathcal{D}) & \xrightarrow{\ t^x_f(g^!\mathcal{D})\ } & H^{m+n}_y(g^!\mathcal{D}) \\
\simeq \uparrow & & \downarrow t^y_g(\mathcal{D}) \\
H^{m+n+d}_x((gf)^!\mathcal{D}) & \xrightarrow[\ t^x_{gf}(\mathcal{D})\ ]{} & H^m_z(\mathcal{D})
\end{array}
$$

In particular, if $S = \mathrm{Spec}(k)$ for some local artin ring k (so that $m = 0$), then the following diagram commutes:

$$
\begin{array}{ccc}
H^{n+d}_x(f^!\omega^n_g) & \xrightarrow{\ t^x_f(\omega^n_g)\ } & H^n_y(\omega^n_g) \\
\simeq \uparrow & & \downarrow \mathrm{res}_y \\
H^{n+d}_x(\omega^{n+d}_{gf}) & \xrightarrow[\ \mathrm{res}_x\ ]{} & k
\end{array}
$$

(4.3) Before proving (4.2.2), we deduce Theorem (4.1) from it.

First of all, (4.1) asserts the equality of two maps into $\Omega^{n+d}_{k(X)/k(S)}$, and it is clear that this equality can be checked "stalkwise" at the maximal points of X. Recall from (2.2.6) that η_f is compatible with flat base change, and by [HS, Prop. 3.1] it follows that the same is true of the isomorphism $\gamma_f \colon \omega^d_f \xrightarrow{\ \sim\ } f^!\mathcal{O}_Y$ of (1.1). Since the map $i_S \colon S_0 \to S$ in (4.0) is flat, it follows that we may replace S by a connected component of S_0, and then assume that $S = \mathrm{Spec}(k)$ where k is a local artin ring.

Next, let $U \subset X$ (resp. $V \subset Y$) be the (dense, open) smooth locus of f (resp. g), and let $W = U \cap f^{-1}V$. Then W contains all the maximal points of X, so that (4.1) need only be checked stalkwise at an arbitrary closed point $x \in W$ (since then it holds at any point

in W). Since f is smooth at x and g is smooth at $y := f(x)$ (so that gf is smooth at x), we have, by [KW, p. 52, Cor. 3.10], canonical identifications

$$(f^*\omega_g^n)_x \otimes (\omega_f^d)_x = (f^*\Omega_{Y/k}^n)_x \otimes (\Omega_{X/Y}^d)_x = (\Omega_{X/k}^{n+d})_x = \omega_{gf,x}^{n+d}.$$

Thus, with $\eta = \eta_f(\omega_g^n)$, our problem is reduced to showing that $\eta_x = $ identity.

By *local duality* [HK1, 3.4], for η_x to be the identity, it suffices, with $\mathbf{s}$, $\mathbf{s}'$, and $\mathbf{t}$ as in (4.2), that for any $\xi_1 \in \Omega_{A/k}^n$ and $\xi_2 \in \Omega_{R/A}^n$,

$$(4.3.1) \qquad \operatorname{res}_x \begin{bmatrix} \eta(\xi_1 \otimes \xi_2) \\ \mathbf{s}', \mathbf{t} \end{bmatrix} = \operatorname{res}_x \begin{bmatrix} \xi_1 \otimes \xi_2 \\ \mathbf{s}', \mathbf{t} \end{bmatrix}.$$

But (i) in (4.2.2), with $\mathcal{F} = \omega_g^n$, yields

$$t_f^x \begin{bmatrix} \eta(\xi_1 \otimes \xi_2) \\ \mathbf{s}', \mathbf{t} \end{bmatrix} = \left(\operatorname{Res}_{R/A}^J \begin{bmatrix} \xi_2 \\ \mathbf{t} \end{bmatrix} \right) \begin{bmatrix} \xi_1 \\ \mathbf{s} \end{bmatrix};$$

and then (ii) in (4.2.2) shows that applying res_y to this equality yields

$$(4.3.2) \qquad \operatorname{res}_x \begin{bmatrix} \eta(\xi_1 \otimes \xi_2) \\ \mathbf{s}', \mathbf{t} \end{bmatrix} = \operatorname{res}_y \left(\left(\operatorname{Res}_{R/A}^J \begin{bmatrix} \xi_2 \\ \mathbf{t} \end{bmatrix} \right) \begin{bmatrix} \xi_1 \\ \mathbf{s} \end{bmatrix} \right).$$

In the present smooth circumstances, all the residue maps appearing agree with those defined via Hochschild homology, cf. [HK1, 1.8]; and for the latter type of residues, Hübl has shown that the right hand sides of (4.3.1) and (4.3.2) agree [H1, p. 102, Cor. 7.9].[7]

Thus Theorem (4.1) holds. □

Exercise. What does all this mean when $d = 0$ or $n = 0$? (Cf. (1.2) and (3.2.4).)

(4.4) It remains to prove (4.2.2), (i) and (ii). We do the easier part (ii) first. Set $F := g^{-1}\{z\}$ and

$$E' := f^{-1}F = (gf)^{-1}\{z\} \supset E := f^{-1}\{y\}.$$

We have then the functorial decompositions

$$\Gamma_{E'} = \Gamma_z \circ (gf)_* = \Gamma_z \circ g_* \circ f_* = \Gamma_F \circ f_*.$$

---

[7] Note that this is a purely *local* result. Hübl has pointed out to us that similar arguments lead to a direct proof of this equality, with no reference to Hochschild homology. This is preferable not only because the result is then more generally applicable (smoothness is not needed), but also because the "trace property" employed is a rather difficult one to establish in the Hochschild context, but in contrast is built into the *definition* of residues in [HK1].

The assertion in (ii) is that *the outer border of the following natural functorial diagram commutes.*

$$
\begin{array}{ccccccc}
H_x^{m+n+d}(gf)^! & \longrightarrow & H_x^{m+n+d}f^!g^! & & & & \\
\downarrow & & \downarrow & & & & \\
H_E^{m+n+d}(gf)^! & \longrightarrow & H_E^{m+n+d}f^!g^! & \longrightarrow & H_y^{m+n}R^d f_* f^! g^! & \longrightarrow & H_y^{m+n}g^! \\
\downarrow & & \downarrow & \text{\textcircled{1}} & \downarrow & & \downarrow \\
H_{E'}^{m+n+d}(gf)^! & \longrightarrow & H_{E'}^{m+n+d}f^!g^! & \longrightarrow & H_F^{m+n}R^d f_* f^! g^! & \longrightarrow & H_F^{m+n}g^! \\
\simeq\downarrow & & \simeq\downarrow & \text{\textcircled{2}} & \downarrow\simeq & & \downarrow\simeq \\
H_z^m R^{n+d}(gf)_*(gf)^! & \longrightarrow & H_z^m R^{n+d}(gf)_* f^! g^! & \longrightarrow & H_z^m R^n g_* R^d f_* f^! g^! & \longrightarrow & H_z^m R^n g_* g^! \\
\downarrow & & & \text{\textcircled{3}} & & & \downarrow \\
H_z^m & = & = & = & = & = & H_z^m
\end{array}
$$

Subdiagram ① commutes because the natural functorial map $\Gamma_y \to \Gamma_F$ induces a morphism of the spectral sequences associated respectively to the decompositions $\Gamma_E = \Gamma_y \circ f_*$, $\Gamma_{E'} = \Gamma_F \circ f_*$. Commutativity of ② can be obtained as in the proof of (3.3.1) from commutativity of the following natural derived-category diagram:

$$
\begin{array}{ccc}
\tau_{\geq m+n+d}\mathbb{R}\Gamma_{E'} & \longrightarrow & (\tau_{\geq m+n+d}\mathbb{R}\Gamma_F)\circ(\tau_{\geq d}\mathbb{R}f_*) \\
\downarrow & & \downarrow \\
(\tau_{\geq m+n+d}\mathbb{R}\Gamma_z)\circ(\tau_{\geq n+d}\mathbb{R}(gf)_*) & \longrightarrow & (\tau_{\geq m+n+d}\mathbb{R}\Gamma_z)\circ(\tau_{\geq n+d}\mathbb{R}g_*)\circ(\tau_{\geq d}\mathbb{R}f_*)
\end{array}
$$

Commutativity of ③ follows from that of (4.0.3); and commutativity of the remaining subdiagrams is obvious. Thus (ii) holds.

(4.5) Now we prove (i) of (4.2.2).

(I) We first reduce to the case where A is complete and $Y = \mathrm{Spec}(A)$. For this purpose, let A' be the completion of A. Since A is excellent and has no embedded associated primes, the same holds for A' [EGA IV, (7.8.3)(v)]. Let $g \colon Y' := \mathrm{Spec}(A') \to Y$ be the canonical map, and consider a base change diagram (fibre square)

$$
\begin{array}{ccc}
X' & \xrightarrow{\;g'\;} & X \\
f'\downarrow & & \downarrow f \\
Y' & \xrightarrow[\;g\;]{} & Y
\end{array}
$$

Then $f' \in \mathfrak{C}^d$. (Equidimensionality of f' is given by [EGA IV, (13.3.8)]. Generic smoothness follows from the fact that the flat map g' takes each maximal point of X' to a maximal point of X. That X' has no embedded associated points is shown in [HK1, 1.9].) By [HS, Prop. 3.1] we have a commutative diagram of natural isomorphisms

$$
\begin{array}{ccc}
g'^{*}f^{!}\mathcal{O}_Y & \xrightarrow{\quad\sim\quad} & f'^{!}\mathcal{O}_{Y'} \\
\simeq\downarrow & & \downarrow\simeq \\
g'^{*}\omega_f^d & \xrightarrow{\quad\sim\quad} & \omega_{f'}^d
\end{array}
$$

(4.5.1)

Let y' be the closed point of Y', let $E' := f'^{-1}\{y'\} = g'^{-1}E$, and let x' correspond to x under the isomorphism $E' \xrightarrow{\sim} E$ induced by g'. Let R' be the R-algebra $\mathcal{O}_{X',x'}$ and let $J' := JR'$. Set $\mathcal{F}' := g^*\mathcal{F}$. As in (4.2.1), we can identify $H_y^n(\mathcal{F})$ and $H_{y'}^n(\mathcal{F}')$.

We claim that the following two diagrams (in which unlabeled maps are the obvious ones) commute:

(4.5.2)

$$
\begin{array}{ccc}
H_y^n(\mathcal{F}) \otimes_A H_{\mathbf{t}R}^d\big((f^{!}\mathcal{O}_Y)_x\big) & \longrightarrow & H_{y'}^n(\mathcal{F}') \otimes_{A'} H_{\mathbf{t}R'}^d\big((f'^{!}\mathcal{O}_{Y'})_{x'}\big) \\
\simeq\downarrow \qquad \textcircled{1} & & \downarrow\simeq \\
H_y^n(\mathcal{F}) \otimes_A H_{\mathbf{t}R}^d(\omega_{R/A}^d) & \longrightarrow & H_{y'}^n(\mathcal{F}') \otimes_{A'} H_{\mathbf{t}R'}^d(\omega_{R'/A'}^d) \\
{\scriptstyle 1\otimes\mathrm{Res}_{R/A}^J}\downarrow \qquad \textcircled{2} & & \downarrow{\scriptstyle 1\otimes\mathrm{Res}_{R'/A'}^{J'}} \\
H_y^n(\mathcal{F}) \otimes_A A' & \longrightarrow & H_{y'}^n(\mathcal{F}') \otimes_{A'} A' \\
\downarrow \qquad \textcircled{3} & & \downarrow \\
H_y^n(\mathcal{F}) & = & H_{y'}^n(\mathcal{F}')
\end{array}
$$

(4.5.3)

$$
\begin{array}{ccc}
H_y^n(\mathcal{F}) \otimes_A H_J^d\big((f^{!}\mathcal{O}_Y)_x\big) & \longrightarrow & H_{y'}^n(\mathcal{F}') \otimes_{A'} H_{J'}^d\big((f'^{!}\mathcal{O}_{Y'})_{x'}\big) \\
\mu\downarrow \qquad \textcircled{4} & & \downarrow\mu \\
H_x^{n+d}(f^*\mathcal{F} \otimes_{\mathcal{O}_X} f^{!}\mathcal{O}_Y) & \longrightarrow & H_{x'}^{n+d}(f'^*\mathcal{F}' \otimes_{\mathcal{O}_{X'}} f'^{!}\mathcal{O}_{Y'}) \\
\eta\downarrow \qquad \textcircled{5} & & \downarrow\eta \\
H_x^{n+d}(f^{!}\mathcal{F}) & \longrightarrow & H_{x'}^{n+d}(f'^{!}\mathcal{F}') \\
{\scriptstyle t_x^f}\downarrow \qquad \textcircled{6} & & \downarrow{\scriptstyle t_{x'}^{f'}} \\
H_y^n(\mathcal{F}) & = & H_{y'}^n(\mathcal{F}')
\end{array}
$$

21

Commutativity of subdiagram ① follows from that of (4.5.1) and the (easily proved) fact that for R'-modules, the functors $H_{\mathbf{t}R}^d$ and $H_{\mathbf{t}R'}^d$ are naturally isomorphic. Commutativity of ② results from the fact that the local homomorphism $R \to R'$ extends to an isomorphism of completions. (By [HK1, Definition 2.1], $\mathrm{Res}_{R/A}^J$ depends only on $\hat{R}$.) Commutativity of ③ is clear. Commutativity of ④ is left to the reader. Commutativity of ⑤ follows from (2.2.6). Subdiagram ⑥ can be expanded as follows, with $\mathcal{E} := f^!\mathcal{F}$ and $\mathcal{E}' := f'^!\mathcal{F}' \cong g'^*\mathcal{E}$, cf. (2.2.4), so that there is a natural map $\mathcal{E} \to g'_*\mathcal{E}'$:

$$
\begin{array}{ccccccc}
H_x^{n+d}(\mathcal{E}) & \longrightarrow & H_E^{n+d}(\mathcal{E}) & \xrightarrow{\delta^{-1}} & H_y^n(R^d f_*\mathcal{E}) & \xrightarrow{H_y^n(t_f)} & H_y^n(\mathcal{F}) \\
\downarrow & & \downarrow & & \downarrow & & \| \\
H_x^{n+d}(g'_*\mathcal{E}') & \longrightarrow & H_E^{n+d}(g'_*\mathcal{E}') & \xrightarrow{\delta^{-1}} & H_y^n(R^d f_* g'_*\mathcal{E}') & \quad ⑨ & \\
\downarrow & ⑦ & \downarrow & ⑧ & \downarrow{\scriptstyle\gamma} & & \| \\
H_{x'}^{n+d}(\mathcal{E}') & \longrightarrow & H_{E'}^{n+d}(\mathcal{E}') & \xrightarrow[\delta^{-1}]{} & H_{y'}^n(R^d f'_*\mathcal{E}') & \xrightarrow[H_{y'}^n(t_{f'})]{} & H_{y'}^n(\mathcal{F}')
\end{array}
$$

(The map γ arises via the natural functorial map $R^d f_* g'_* \to g_* R^d f'_*$.) Commutativity of ⑧ is an exercise on derived functors. It follows, for example, from commutativity of the following natural derived-category diagram (keep in mind that the maps g and g' are affine, so that, for instance, $g'_*\mathcal{E}' \xrightarrow{\sim} \mathbb{R}g'_*\mathcal{E}'$):

$$
\begin{array}{ccccc}
\mathbb{R}\Gamma_E \mathbb{R}g'_* & \xrightarrow{\sim} & \mathbb{R}\Gamma_y \mathbb{R}f_* \mathbb{R}g'_* & \longrightarrow & \mathbb{R}\Gamma_y \tau_{\geq d} \mathbb{R}f_* \mathbb{R}g'_* \\
& & \simeq\downarrow & & \downarrow{\scriptstyle\simeq} \\
& & \mathbb{R}\Gamma_y \mathbb{R}g_* \mathbb{R}f'_* & \longrightarrow & \mathbb{R}\Gamma_y \tau_{\geq d} \mathbb{R}g_* \mathbb{R}f'_* \\
{\scriptstyle\simeq}\downarrow & & \| & & \downarrow \\
& & \mathbb{R}\Gamma_y \mathbb{R}g_* \mathbb{R}f'_* & \longrightarrow & \mathbb{R}\Gamma_y \mathbb{R}g_* \tau_{\geq d} \mathbb{R}f'_* \\
& & \simeq\downarrow & & \downarrow{\scriptstyle\simeq} \\
\mathbb{R}\Gamma_{E'} & \xrightarrow{\sim} & \mathbb{R}\Gamma_{y'} \mathbb{R}f'_* & \longrightarrow & \mathbb{R}\Gamma_{y'} \tau_{\geq d} \mathbb{R}f'_*
\end{array}
$$

Commutativity of ⑦ is left to the reader. Finally, since by the definition of (2.2.2) there is a natural commutative diagram of functorial maps

$$
\begin{array}{ccc}
R^d f_* & \longrightarrow & R^d f_* g'_* g'^* \\
\downarrow & & \downarrow \\
g_* g^* R^d f_* & \longrightarrow & g_* R^d f'_* g'^*
\end{array}
$$

22

therefore subdiagram ⑨ commutes by [Km, pp. 44-45, Thm. 5(i)]. So (4.5.2) and (4.5.3) do indeed commute.

Since (4.2.2) asserts the equality of the maps which arise by composing the columns on the left of (4.5.2) and (4.5.3) respectively, it will be enough now to show equality for the columns on the right, so that we can replace Y by Y'. Thus we can assume that A is complete and that $Y = \mathrm{Spec}(A)$.

(II) Arguing as in [HK1, 4.10] (or cf. [EGA II, (6.2.5)]), we can find an affine neighborhood $U := \mathrm{Spec}(T)$ of x, and a sequence $\mathbf{t}' := (t'_1, \ldots, t'_d)$ in T whose image in R is $\mathbf{t}$, and such that the scheme $Z := \mathrm{Spec}(T/\mathbf{t}'T)$ is closed in X and finite over Y; and we may assume furthermore that $T/\mathbf{t}'T$ is a local ring, so that for any $r > 0$, the natural map $T/\mathbf{t}'^r T \to R/\mathbf{t}^r R$ is an isomorphism (where $\mathbf{t}^r := (t^r_1, \ldots, t^r_d)$, and similarly for $\mathbf{t}'^r$). It follows from the identification $H^d_{\mathbf{t}'T}(D) = \varinjlim(D/\mathbf{t}'^r D)$ that for any T-module D,

$$H^d_{\mathbf{t}'T}(D) = H^d_{\mathbf{t}R}(D \otimes_T R);$$

and by (3.2.3), for example, there is a natural isomorphism

$$H^d_{\mathbf{t}'T}(D \otimes_T R) \xrightarrow{\sim} H^d_{\mathbf{t}R}(D \otimes_T R).$$

For any quasi-coherent $\mathcal{O}_X$-Module $\mathcal{D}$, let D be the T-module $\Gamma(U, \mathcal{D})$. By excision, and the preceding remarks, we have

$$H^d_Z(X, \mathcal{D}) = H^d_Z(U, \mathcal{D}|U) = H^d_{\mathbf{t}'T}(D) = H^d_{\mathbf{t}R}(D \otimes_T R) = H^d_{\mathbf{t}R}(\mathcal{D}_x).$$

We conclude (cf. [HK1, Prop. 4.2]):

(4.5.4) LEMMA. *With preceding notation, the higher direct image with support in Z, $R^d_Z f_* \mathcal{D}$, is the quasi-coherent $\mathcal{O}_Y$-Module associated to the A-module $H^d_{\mathbf{t}R}(\mathcal{D}_x)$.*

(III) For any closed subscheme Z of X, any $\mathcal{O}_X$-Module $\mathcal{E}$, and any $\mathcal{O}_Y$-Module $\mathcal{F}$, consider the natural composition

$$G_Z(\mathcal{F}, \mathcal{E}) \colon \mathcal{F} \otimes_{\mathcal{O}_Y} R^d_Z f_* \mathcal{E} \to f_* f^* \mathcal{F} \otimes_{\mathcal{O}_Y} R^d_Z f_* \mathcal{E} \to R^d_Z f_*(f^* \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{E})$$

where the second map arises from a cup product. The case $Z = X$ was treated in §2; and as in (2.1) we can show that $G_Z(\mathcal{F}, \mathcal{E})$ is an isomorphism. For present purposes, the following description of the cup product is needed—and using this description one can forget about the term "cup product" both here and in §2.

For any $\mathcal{O}_X$-Module $\mathcal{D}$, then, we define a bifunctorial map of $\mathcal{O}_Y$-Modules

$$f_*(\mathcal{D}) \otimes R^d_Z f_*(\mathcal{E}) \to R^d_Z f_*(\mathcal{D} \otimes \mathcal{E})$$

or, equivalently,
$$f_*(\mathcal{D}) \to \mathcal{H}om_{\mathcal{O}_Y}\left(R_Z^d f_*(\mathcal{E}),\, R_Z^d f_*(\mathcal{D} \otimes \mathcal{E})\right),$$
to be, over each open subset U of Y, with $V := f^{-1}U$, the natural map

$$\begin{aligned}
\Gamma_U f_*(\mathcal{D}) = \Gamma_V(\mathcal{D}) &= \mathrm{Hom}_V(\mathcal{O}_X,\, \mathcal{D}) \\
&\to \mathrm{Hom}_U\left(R_Z^d(f|V)_*(\mathcal{O}_X \otimes \mathcal{E}),\, R_Z^d(f|V)_*(\mathcal{D} \otimes \mathcal{E})\right) \\
&= \Gamma_U \mathcal{H}om_{\mathcal{O}_Y}\left(R_Z^d f_*(\mathcal{E}),\, R_Z^d f_*(\mathcal{D} \otimes \mathcal{E})\right).
\end{aligned}$$

One checks then that the following diagram commutes:

(4.5.5)
$$\begin{array}{ccc}
H_{\mathbf{t}R}^d(\mathcal{F}_y \otimes_A \mathcal{E}_x) & \!\!=\!\!=\!\! & \left(R_Z^d f_*(f^*\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{E})\right)_y \\[2pt]
{\scriptstyle(3.3)}\Big\downarrow{\scriptstyle\simeq} & & {\scriptstyle\simeq}\Big\downarrow{\scriptstyle G_Z(\mathcal{F},\mathcal{E})} \\[2pt]
\mathcal{F}_y \otimes_A H_{\mathbf{t}R}^d(\mathcal{E}_x) & \!\!=\!\!=\!\! & \left(\mathcal{F} \otimes_{\mathcal{O}_Y} R_Z^d f_*(\mathcal{E})\right)_y
\end{array}$$

(IV) Let us write ω for $f^!\mathcal{O}_Y$. Our problem is now reduced to showing that the following diagram of natural maps commutes. (Keeping in mind the foregoing identification of $(R_Z^d f_*\omega)_y$ with $H_{\mathbf{t}R}^d(\omega_x) \cong H_{\mathbf{t}R}^d(\omega_{R/A}^d)$, note, using commutativity of (4.5.5), that the composition—call it c—in the left column equals the natural composition

$$\begin{aligned}
H_{\mathbf{s}R}^n\left(H_{\mathbf{t}R}^d(\mathcal{F}_y \otimes_A \omega_x)\right) &\xrightarrow{\ \sim\ } H_{\mathbf{s}R}^n\left(\mathcal{F}_y \otimes_A H_{\mathbf{t}R}^d(\omega_x)\right) \\
&\xrightarrow{\ \sim\ } H_{\mathbf{s}A}^n(\mathcal{F}_y) \otimes_A H_{\mathbf{t}R}^d(\omega_x) \\
&\xrightarrow{\ \sim\ } H_y^n(\mathcal{F}) \otimes_A (R_Z^d f_*\omega)_y\,.
\end{aligned}$$

Note also that the map μ in (4.2.2) is $(3.3) \circ c^{-1}$; and that the bottom row is $1 \otimes \mathrm{Res}_{R/A}^J$ by the Residue Theorem in [HS], together with [HK1, 1.7 and 2.6].)

$$\begin{array}{ccccc}
H_{\mathbf{s}R}^n\left(H_{\mathbf{t}R}^d((f^*\mathcal{F} \otimes \omega)_x)\right) & \xrightarrow{\ (3.3)\ } & H_x^{n+d}(f^*\mathcal{F} \otimes \omega) & \xrightarrow{\ \eta\ } & H_x^{n+d}(f^!\mathcal{F}) \\[4pt]
\Big\| & & \Big\downarrow & & \Big\downarrow \\[4pt]
& \textcircled{1} & H_E^{n+d}(f^*\mathcal{F} \otimes \omega) & \xrightarrow{\ \eta\ } & H_E^{n+d}(f^!\mathcal{F}) \\[4pt]
& & {\scriptstyle\delta^{-1}}\Big\downarrow & & \Big\downarrow{\scriptstyle\delta^{-1}} \\[4pt]
H_y^n\left(R_Z^d f_*(f^*\mathcal{F} \otimes \omega)\right) & \longrightarrow & H_y^n\left(R^d f_*(f^*\mathcal{F} \otimes \omega)\right) & \xrightarrow{\ \eta\ } & H_y^n(R^d f_* f^!\mathcal{F}) \\[4pt]
{\scriptstyle\simeq}\Big\downarrow & & \Big\downarrow{\scriptstyle\simeq} & \textcircled{2} & \Big\downarrow \\[4pt]
H_y^n(\mathcal{F} \otimes R_Z^d f_*\omega) & \longrightarrow & H_y^n(\mathcal{F} \otimes R^d f_*\omega) & \xrightarrow[{\scriptstyle 1 \otimes t_f}]{} & H_y^n(\mathcal{F}) \\[4pt]
{\scriptstyle\simeq}\Big\downarrow & & \Big\downarrow{\scriptstyle\simeq} & & \Big\downarrow{\scriptstyle\simeq} \\[4pt]
H_y^n(\mathcal{F}) \otimes (R_Z^d f_*\omega)_y & \longrightarrow & H_y^n(\mathcal{F}) \otimes (R^d f_*\omega)_y & \xrightarrow[{\scriptstyle 1 \otimes t_f}]{} & H_y^n(\mathcal{F}) \otimes A
\end{array}$$

Commutativity of subdiagram ① is again a formal exercise on derived functors, perhaps best left to the reader, but in any case resulting from commutativity of the following diagram of functors on $\mathbf{D}_{\mathrm{qc}}^+(X)$ (where Γ_Z denotes "sheaf of sections with support in Z," where σ_x is the exact functor "stalk at x," and where τ_e stands for $\tau_{\geq e}$):

$$
\begin{array}{ccccc}
(\tau_{n+d}\mathbb{R}\Gamma_{\mathbf{s}R}) \circ (\tau_d\mathbb{R}\Gamma_{\mathbf{t}R}\sigma_x) & \xleftarrow{\;\sim\;} & \tau_{n+d}\mathbb{R}\Gamma_{\mathbf{s}R}\mathbb{R}\Gamma_{\mathbf{t}R}\sigma_x & \xleftarrow{\;\sim\;} & \tau_{n+d}\mathbb{R}\Gamma_x \\
\simeq \downarrow & & \downarrow \simeq & & \downarrow \simeq \\
(\tau_{n+d}\mathbb{R}\Gamma_y) \circ (\tau_d\mathbb{R}f_*\mathbb{R}\Gamma_Z) & \xleftarrow{\;\sim\;} & \tau_{n+d}\mathbb{R}\Gamma_y\mathbb{R}f_*\mathbb{R}\Gamma_Z & \xleftarrow{\;\sim\;} & \tau_{n+d}\mathbb{R}\Gamma_E\mathbb{R}\Gamma_Z \\
\downarrow & & \downarrow & & \downarrow \\
(\tau_{n+d}\mathbb{R}\Gamma_y) \circ (\tau_d\mathbb{R}f_*) & \xleftarrow{\;\sim\;} & \tau_{n+d}\mathbb{R}\Gamma_y\mathbb{R}f_* & \xleftarrow{\;\sim\;} & \tau_{n+d}\mathbb{R}\Gamma_E
\end{array}
$$

Commutativity of ② results from the definition of η, cf. (2.1.1); and commutativity of the remaining subdiagrams is obvious.

This completes our proof of Proposition (4.2.2), and of Theorem (4.1). □

(4.6) *Remark.* One can (should?) formulate and prove Theorem (4.1) without assuming that the maps f and g are proper. Without details, which would lead too far afield, here are some brief indications.

According to [HS, Cor. 1.7], one can define a functor $f^! \colon Y_{\mathrm{qc}} \to X_{\mathrm{qc}}$ for every finite-type map $f \colon X \to Y$ of noetherian schemes with fibres of dimension $\leq d$, in such a way that if f is proper then $f^!$ is the functor used throughout this paper, and that if $i \colon W \to X$ is an open immersion, then $(f \circ i)^! = i^* \circ f^!$. This $f^!$ behaves well with respect to flat base-changes $Y' \to Y$.

For maps $f \colon X \to Y$ in $\mathfrak{C}^d$ and $g \colon Y \to S$ in $\mathfrak{C}^n$, one can find a canonical isomorphism $f^!\omega_g^n \xrightarrow{\;\sim\;} \omega_{gf}^{n+d}$ (cf. [HS, §4] for the case when f is *proper*).

The map η_f of §2 can also be defined for non-proper f. This is because η_f, as defined in §2 for proper f, is *local,* in the following sense. Let $f_k \colon X_k \to Y$ ($k = 1, 2$) be proper maps with fibres of dimension $\leq d$, and let $i_k \colon U \to X_k$ be open immersions. Then *there is a natural identification of the functorial maps* $i_1^*\eta_{f_1}$ *and* $i_2^*\eta_{f_2}$. The proof is rather long and technical, and follows the lines of the proof in [HS, 1.1] for the identification of $i_1^*f_1^!$ and $i_2^*f_2^!$ which underlies the above-mentioned definition of $f^!$ for non-proper f.

Given these facts, one can now state Theorem (4.1) without assuming f and g to be proper. To prove it, one notes that the question is local, and that locally, in view of the "quasinormalization" characterization of equidimensional maps [EGA IV, (13.3.1)(b)], and Zariski's Main Theorem [EGA IV, (18.12.13)], one can compactify the situation, i.e., arrange that it arise from the proper case by restriction to suitable open subsets. Thus the general case can be reduced to the proper case treated here.

REFERENCES

[GD] A. Grothendieck, J. Dieudonné, *Éléments de Géométrie Algébrique*, Springer-Verlag, Heidelberg, 1971.

[EGA II] ———, *Éléments de Géométrie Algébrique*, Publ. Math. IHES, 8 (1961).

[EGA III] ———, *Éléments de Géométrie Algébrique*, Publ. Math. IHES, 11 (1961).

[EGA IV] ———, *Éléments de Géométrie Algébrique*, Publ. Math. IHES, 24 (1965), 28 (1966), 32 (1967).

[RD] R. Hartshorne, *Residues and Duality*, Lecture Notes in Math. 20, Springer-Verlag, Heidelberg, 1966.

[Ha] ———, *Algebraic Geometry*, Springer-Verlag, Heidelberg, 1977.

[H1]   R. Hübl, *Traces of Differential Forms and Hochschild Homology*, Lecture Notes in Math. 1368, Springer-Verlag, Heidelberg, 1989.

[H2]   ———, *On the Transitivity of Regular Differential Forms*, Manuscripta Math., 65 (1989), 213–224.

[HK1]   R. Hübl, E.Kunz, *Integration of differential forms on schemes*, J. reine angew. Math., 410 (1990), 53–83.

[HK2]   ———, *Regular differential forms and duality for projective morphisms*, J. reine angew. Math., 410 (1990), 84–108.

[HS]   R. Hübl, P. Sastry, *Regular differential forms and relative duality*, preprint.

[Kf]   G. R. Kempf, *Some elementary proofs of basic theorems in the cohomology of quasi-coherent sheaves*, Rocky Mountain J. Math., 10 (1980), 637–645.

[Km]   S. L. Kleiman, *Relative duality for quasi-coherent sheaves*, Compositio Math., 41 (1980), 39–60.

[Kz]   E. Kunz, *Residuen von Differentialformen auf Cohen-Macaulay Varietäten*, Math. Z., 152 (1977), 165–189.

[KD]   ———, *Kähler Differentials*, Vieweg, Braunschweig, 1986.

[KW]   E. Kunz, R. Waldi, *Regular Differential Forms*, Contemporary Mathematics, vol 79, Amer. Math. Soc., Providence, 1988.

[L1]   J. Lipman, *Dualizing sheaves and residues on algebraic varieties*, Astérisque, 117 (1984).

[Ł2]   ———, *Residues and differential forms via Hochschild homology*, Contemporary Mathematics, vol. 61, Amer. Math. Soc., Providence, 1987.

[V]   J. L. Verdier, *Base change for twisted inverse image of coherent sheaves*, Algebraic Geometry (Bombay, 1968), Oxford University Press, London, 1969, pp. 393–408.