Flattening and analytic continuation of affinoid morphisms: remarks on a paper of Gardener and Schoutens

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Abstract. We give an example of an affinoid curve without analytic continuation. We use this to produce an example of an affinoid morphism that cannot be flattened by a finite sequence of local blow-ups. Thus the global rigid analogue, [7], Theorem 2.3, of Hironaka's complex analytic flattening theorem is not true. Since this is a key step in the proof of the affinoid elimination theorem, [7], Theorem 3.12, that proof contains a serious gap. We also give an example of an affinoid subset of the plane that is not the image under a proper rigid analytic map of a set that is globally semianalytic in the domain of that map. This clarifies the relationship among several natural categories of rigid subanalytic sets.

1 Introduction

In this paper, we give an example of a curve V defined in an affinoid subpolydisc A of $X := \operatorname{Sp} T_2$ such that V is not the restriction to A of any affinoid set Z defined in a larger polydisc. This is an example, in other words, of a curve without analytic continuation. We use this example to clarify some questions (i) about flattening affinoid morphisms, and (ii) about rigid semi- and sub-analytic sets (see the corresponding paragraphs, below.)

The theory of semi- and sub-analytic sets and the flattening of analytic morphisms, as well as the relation among these concepts, were first developed over the real and complex (complete, Archimedean) fields ([5], [8], [9].) To carry out a similar development over algebraically closed, complete, non-Archimedean fields (rigid analytic or affinoid geometry) required the use of various G-topologies and corresponding analytic spaces ([2], [4]) because the corresponding metric local topology is neither compact nor connected. Indeed, shifting from considering finite rational covers of spaces of points in maximal ideal spectra of affinoid algebras to wide open covers of spaces of analytic points in valuation spectra

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of affinoid algebras ([16], [17] and [1]) recovers to a certain extent the classical setting of (locally) compact, connected analytic spaces. Unlike the classical Archimedean case, in the rigid category there are different ways to pass from local to global yielding different global (sheaf-theoretic and other) properties. We use the example of Section 4 to clarify some issues that arise due to differences in the various global topological settings when transferring the real variables theory to the rigid category.

(i) In [9], Hironaka developed the theory of real subanalytic sets (Boolean combinations of images of proper real analytic maps) making use of properties of flat analytic morphisms, his local flattening theorem and the voûte étoilée. The global affinoid analog of the theory of [9] would sharpen the quantifier simplification theorem of [13] to a full quantifier elimination theorem, completing the analogy between the theories of real and p-adic subanalytic sets [3] on the one hand, and affinoid subanalytic sets on the other. What was needed was the existence of rigid analytic flatificators in wide neighborhoods of analytic points, and in [6], [20] and [7], an effort was made to carry out that program. The proof of [20], Theorem A.2 is not correct. In Theorem 5.3 and Theorem 5.4, below, we use the example of Section 4 to give an affinoid morphism $\varphi: Y \to X$ and an analytic point P of $\varphi(Y)$ such that φ neither has a flatificator defined in a wide affinoid neighborhood of P, nor can φ be flattened by a finite sequence of local blow-ups. Thus the global rigid analogue, [7], Theorem 2.3, of Hironaka's complex analytic flattening theorem is not true. Since this is a key step in the proof of the affinoid elimination theorem, [7], Theorem 3.12, that proof contains a serious gap.

(ii) There are several classes of rigid subanalytic sets ([18], [11], [13]) that result from applying the approach of [3] to different categories of rigid analytic functions. There are, in addition, different classes of rigid semianalytic sets that depend both on the G-topology and the category of functions. In Section 6, we give examples to clarify some of the relations among these classes of sets.

In Section 2, we establish notational conventions.

In Section 3, we study covers of neighborhoods of affinoid plane curves. The main results involve the concept of a special tubular domain, which is a tubular neighborhood of an analytic point on the curve V, mentioned above, with, roughly speaking, a semi-wide projection onto the first coordinate. In other words, if $\pi: \operatorname{Sp} T_2 \to \operatorname{Sp} T_1$ denotes projection on the first coordinate, then a special tubular domain W has the property that $\pi(W) \cup \pi(V)$ is a wide affinoid neighborhood of $\pi(P)$ for any analytic point P on V. (See Definitions 3.1 and 3.2.) The main results of this section are Propositions 3.4 and Lemma 3.6. These results establish conditions under which an affinoid neighborhood of an analytic point P of the curve V contains a special tubular domain.

In Section 4, we discuss the problem of analytic continuation of plane affinoid curves, and give an example of a plane curve V with no analytic continuation (see Theorem 4.3.)

In Section 5, we establish the counter-example mentioned above.

In Section 6, we discuss the relationships among various classes of rigid semi- and sub-analytic sets.

2 Notation

Let K be an algebraically closed field of characteristic zero, complete in the non-Archimedean absolute value $|\cdot|\colon K \to \mathbb{R}_+$. By K° denote the valuation ring of K with maximal ideal $K^{\circ\circ}$ and residue field $\widetilde{K} = K^\circ/K^{\circ\circ}$.

The ring of strictly convergent power series over K in the variables $\xi = (\xi_1, \dots, \xi_n)$ is

$$T_n = K\langle \xi_1, \dots, \xi_n \rangle := \left\{ \sum a_\nu \xi^\nu : |a_\nu| \to 0 \text{ as } |\nu| \to \infty \right\}.$$

A K-affinoid algebra is a quotient of some T_n. For $\alpha = (\alpha_1, \dots, \alpha_n)$ with $\alpha_i \in |K \setminus \{0\}|$,

$$T_{n,\alpha} := \left\{ \sum a_\nu \xi^\nu : |a_\nu|\alpha^\nu \to 0 \text{ as } |\nu| \to \infty \right\}$$

is an affinoid algebra; it is the affinoid algebra of the rational polydisc of polyradius α. We call an element f of $T_{n,\alpha}$ an *overconvergent element of* $T_{n,\alpha}$ if there is an α' with $\alpha'_i > \alpha_i$ for each i such that $f \in T_{n,\alpha'}$.

Let A be an affinoid algebra, and, as in [2], by $\operatorname{Sp} A$ denote the affinoid variety X associated to $(\operatorname{Max} A, A)$. In particular, an open affinoid subdomain U of X has a natural K-affinoid algebra $\mathcal{O}(U) = \mathcal{O}_X(U)$. Furthermore, an affinoid subset Z of U also has a natural K-affinoid algebra, which is a quotient of $\mathcal{O}(U)$, and which we denote $\mathcal{O}(Z) = \mathcal{O}_Z(Z)$.

The Gauss norm on T_n is denoted $\|\cdot\|$, and $\|\cdot\|_{\sup}$ denotes the supremum semi-norm. The set over which the supremum is taken will be clear from the context or will be indicated by a subscript. Indeed, $\|\cdot\|_{\sup}$ is defined on any K-affinoid algebra A.

For functions $f \in T_n$, $V(f) = \operatorname{Sp} T_n/(f) \subset \operatorname{Sp} T_n$ denotes the zero-set of f.

For the definition of the flatificator of a rigid analytic map at a point, we refer the reader to [20]. The definition of the flatificator of a complex analytic morphism can be found in [9] and [10].

For the definition of analytic and geometric points of a rigid analytic variety, we refer the reader to [17] (see also [16] and [1]). We use $\operatorname{Max} A$ to denote the set of geometric points associated to the K-affinoid algebra A, and since an analytic point is determined by the collection of open affinoids to which it belongs, we regard the analytic points as "belonging to" the locally ringed space $\operatorname{Sp} A$.

3 Special neighborhoods of analytic points

In the following, we use two sorts of affinoid subdomains of K°, the disc D about 0 of some fixed K-rational radius $\varepsilon < 1$, and certain sets which we term "special annuli," though strictly speaking, they are not annuli. A special annulus U is a disc D' of radius $\delta > \varepsilon$ that is modified by removing finitely many "open" discs of radius ε about points of the smaller disc D. Since the "holes" of U are "contained" in D, $D \cup U$ is an admissible affinoid cover (see [2]) of the disc D'.

Definition 3.1 *A* special annulus U of K° *of* outer radius δ *and* inner radius ε, $0 < \varepsilon < \delta \le 1$, *is a finite intersection of sets of the form*

$$\{x \in K^\circ : |x| \le \delta \text{ and } |x - a| \ge \varepsilon\}$$

where $a \in K^\circ$ and $|a| \le \varepsilon$. Note that any special annulus is a rational domain.

The following observation will be used several times below. Let U be a special annulus. Since a non-zero element of $\mathcal{O}(U)$ can vanish at only finitely many points of U (see [4], Theorem I.2.2), $\mathcal{O}(U)$ must be an integral domain.

Let D be the closed disc $\{\xi \in K : |\xi| \le \varepsilon\}$ of radius ε with $\varepsilon < 1$ and $\varepsilon \in |K \setminus \{0\}|$. Let $f \in \mathcal{O}(D)$. Theorem 4.3 shows that for some choices of f, the graph of f does not extend to an affinoid set over a disc of radius greater than ε. Nonetheless, a tubular neighborhood of the graph of f (i.e., a set $A := \{(\xi, \eta) : |\eta - f(\xi)| \le \gamma\}$) depends only on a suitable truncation $\bar{f}$ of f (i.e., in the definition of A, we may replace f by any $\bar{f} \in K[\xi]$ such that $\|f - \bar{f}\|_{\sup,D} \le \gamma$). If $\|\bar{f}\|_{\sup,D} < 1$ (which will be the case if $\gamma < \|f\|_{\sup,D} < 1$), then there is a $\delta \in |K \setminus \{0\}|$ with $\delta > \varepsilon$ such that $\|\bar{f}\|_{\sup,U} \le 1$, where U is any special annulus of outer radius $\le \delta$. Thus in this case, a tubular neighborhood of the graph of f over D extends to a tubular neighborhood of the graph of $\bar{f}$ over U. This is captured in the following definition.

Definition 3.2 *Fix $\varepsilon \in |K \setminus \{0\}|$ with $\varepsilon < 1$ and put*

$$D := \{\xi \in K : |\xi| \le \varepsilon\},$$

the closed disc of radius ε. Fix $f \in \mathcal{O}(D)$ with $\|f\|_{\sup} < 1$. Let π be the coordinate projection

$$\pi \colon (K^\circ)^2 \to K^\circ \colon (\xi, \eta) \mapsto \xi.$$

A set $W \subset (K^\circ)^2$ is called a special tubular domain *in $(K^\circ)^2$ iff it is a set of the form*

$$W = \pi^{-1}(U) \cap \{(\xi, \eta) \in (K^\circ)^2 : |\eta - \bar{f}(\xi)| \le \gamma\},$$

where

(i) $U \subset K^\circ$ is a special annulus of outer radius $\delta > \varepsilon$ and inner radius ε,

(ii) $\bar{f} \in K[\xi]$ is a polynomial with $\|f - \bar{f}\|_{\sup,D} \le \gamma < \|f\|_{\sup,D} < 1$, and

(iii) $\|\bar{f}\|_{\sup,U} \le 1$.

Observe that if these conditions are satisfied, then $\pi(W) = U$. Furthermore, since $\|f\|_{\sup,D} < 1$, there is a $\delta_0 \in |K \setminus \{0\}|$ with $\delta_0 > \varepsilon$ such that these conditions can be satisfied for any $\delta \in |K \setminus \{0\}|$ with $\varepsilon < \delta \le \delta_0$.

Let W be a special tubular domain. Since $\pi(W) = U$, it follows from the definition of W that $\mathcal{O}(W)$ is isomorphic to $\mathcal{O}(U \times K^\circ)$ via the unique K-algebra morphism that sends ξ to ξ and η to $\frac{\eta - \bar{f}(\xi)}{c}$, where $c \in K$ and $|c| = \gamma$. As we remarked above, $\mathcal{O}(U)$ is an integral domain; therefore, $\mathcal{O}(W) \cong \mathcal{O}(U \times K^\circ) = \mathcal{O}(U)\langle\eta\rangle$ must also be an integral domain.

Definition 3.3 *Fix $\varepsilon \in |K|$, $0 < \varepsilon < 1$, let $D \subset K^\circ$ be the closed disc of radius ε and fix $f \in \mathcal{O}(D)$, with $\|f\|_{\sup} < 1$. Let p be the analytic point of D given by*

$$p(h) := \|h\|_{\sup,D}$$

for all $h \in \mathcal{O}(D)$, and let P be the analytic point of $\pi^{-1}(D)$ determined by

$$P(h(\xi,\eta)) := p(h(\xi, f(\xi))) = \|h(\xi, f(\xi))\|_{\sup}.$$

We regard p as an analytic point of $\operatorname{Sp} T_1$ via the natural inclusion $T_1 \hookrightarrow \mathcal{O}(D)$ and we regard P as an analytic point of $\operatorname{Sp} T_2$ via the natural inclusion $T_2 \hookrightarrow \mathcal{O}(\pi^{-1}(D))$.

Proposition 3.4 *Keep the notation of the above definitions. Let $W \subset (K^\circ)^2$ be a wide affinoid neighborhood of P. Then there is a special tubular domain $W' \subset W$ (which by Definitions 3.2 and 3.3 contains P).*

Proof. By definition of wide neighborhood, the collection of geometric points of W must contain a finite intersection of sets A given by conditions of the form

$$|h(\xi,\eta)| \ \overline{\Box}\ \gamma,$$

where $\gamma \in |K \setminus \{0\}|$, $P(h) \Box \gamma$, $\Box \in \{<,>\}$ and $\overline{\Box} \in \{\leq,\geq\}$ is the corresponding weak inequality, $h \in T_2$, and $\|h\| = 1$. Take $\bar{f} \in K[\xi]$ such that

$$\|f - \bar{f}\|_{\sup,D} := \gamma' < \min\{\gamma, \|f\|_{\sup,D}\}.$$

Then A contains the wide affinoid neighborhood given by the conditions

$$|h(\xi, \bar{f}(\xi))| \ \overline{\Box}\ \gamma \text{ and } |\eta - \bar{f}(\xi)| \leq \gamma'. \tag{1}$$

Since a finite intersection of special tubular domains is a special tubular domain, it suffices to show that a set defined by a condition of the form (1) contains a special tubular domain. For this, it suffices to show that if $P(h(\xi,\eta)) \Box \gamma$, then the set defined by

$$|h(\xi, \bar{f}(\xi))| \ \overline{\Box}\ \gamma$$

contains a special annulus of outer radius $> \varepsilon$ and inner radius ε. Equivalently, it suffices to show, for any monic polynomial $h(\xi) \in K[\xi]$, that if $p(h) \Box \gamma$ then the set defined by

$$|h(\xi)| \ \overline{\Box}\ \gamma \tag{2}$$

contains a special annulus of outer radius $> \varepsilon$ and inner radius ε. Since h is monic and K is algebraically closed,

$$h(\xi) = \prod_{i=1}^{N_1}(\xi - a_i) \prod_{j=1}^{N_2}(\xi - b_j) \prod_{k=1}^{N_3}(\xi - c_k)$$

where $|a_i| < \varepsilon$, $|b_j| = \varepsilon$ and $|c_k| > \varepsilon$. Observe that

$$p(h) = \varepsilon^{N_1+N_2} \prod_{k=1}^{N_3} |c_k| \,\square\, \gamma,$$

and, similarly, for any $\xi \in K^\circ$ with $|\xi| \geq \varepsilon$, $|\xi| < \min\{|c_k|\}$ and $\min\{|\xi - b_j|\} \geq \varepsilon$,

$$|h(\xi)| = |\xi|^{N_1+N_2} \prod_{k=1}^{N_3} |c_k|.$$

Case 1. $\square$ is $<$.

In this case, the set defined by Condition (2) contains the special annulus $\{\xi \in K : \varepsilon \leq |\xi| \leq \delta\}$ for any $\delta \in |K|$ with $\varepsilon < \delta < \left(\frac{\gamma}{\prod |c_k|}\right)^{\frac{1}{N_1+N_2}}$ and $\delta < \min\{|c_k|\}$.

Case 2. $\square$ is $>$.

Let δ satisfy $\varepsilon < \delta < \min\{|c_k|\}$. Then the intersection of the set $\{\xi \in K^\circ : \varepsilon \leq |\xi| \leq \delta\}$ with the sets $\{\xi \in K^\circ : |\xi - b_j| \geq \varepsilon\}$ is a special annulus contained in the set defined by Condition (2). $\square$

Lemma 3.5 *Let $A_1, \dots, A_m$ be affinoid subdomains of $(K^\circ)^2$, let $A := A_1 \cup \dots \cup A_m$, and suppose $P \notin A$. Then there is a special tubular domain W (which by Definitions 3.2 and 3.3 contains P) such that $W \cap A = \varnothing$.*

Proof. By [2], Corollary 7.3.5.3, we may assume that each A_i is a rational domain. Since the intersection of a finite number of special tubular domains is a special tubular domain, it suffices to treat the case $m = 1$. Write

$$A = \{(\xi, \eta) : |g_0(\xi, \eta)| \geq |g_j(\xi, \eta)|,\ j = 1, \dots, n\},$$

where the $g_j \in K[\xi, \eta]$ and $(g_0, \dots, g_n)T_2 = T_2$. Since $P \notin A$, for some j, say $j = 1$, $P(g_0) < P(g_j)$. Since g_0 is a unit on A, we may put $0 \neq \delta := P(g_0) < P(g_1)$. Pick $\alpha \in K$ with $|\alpha| > 1$ and $|\alpha| P(g_0) < P(g_1)$, and consider the set

$$W := \{(\xi, \eta) : |g_1(\xi, \eta)| \geq \delta/2 \text{ and } |g_1(\xi, \eta)| \geq |\alpha g_0(\xi, \eta)|\}.$$

This is a wide affinoid neighborhood of P which, since $|\alpha| > 1$, is disjoint from A. Now apply Proposition 3.4. $\square$

Lemma 3.6 *Suppose $A_1, \dots, A_m$ are affinoid subdomains of $(K^\circ)^2$ such that $A_1 \cup \dots \cup A_m$ contains a special tubular domain W. Then there is a special tubular domain $W' \subset W$ and an i, $1 \leq i \leq m$, such that $W' \subset A_i$. (Note that W' by definition contains P.) In particular, suppose $A_1 \cup \dots \cup A_m = (K^\circ)^2$ covers $(K^\circ)^2$, then for some i, A_i contains a special tubular domain W.*

Proof. We reduce to the case that each A_i (and W) is a *tube;* i.e., a set of the form

$$\pi^{-1}(U) \cap \{(\xi, \eta) \in (K^\circ)^2 : |\eta - \bar{f}(\xi)| \leq \gamma\},$$

where $\gamma \in |K \setminus \{0\}|$, $\bar{f} \in K[\xi]$ with $\|f - \bar{f}\|_{\sup} \le \gamma$, and U is an affinoid subdomain of K°. By [2], Corollary 7.3.5.3, we may first reduce to the case that each A_i is a rational domain, so we may write

$$A_i = \{(\xi,\eta) \in (K^\circ)^2 : |g_{i0}(\xi,\eta)| \ge |g_{ij}(\xi,\eta)|,\ j = 1,\dots,n_i\},$$

where the $g_{ij} \in K[\xi,\eta]$ and $(g_{i0},\dots,g_{in_i})T_2 = T_2$, $1 \le i \le m$.

Since each g_{i0} is a unit on A_i, there is a $\gamma \in |K|$ such that $0 < \gamma < |g_{i0}(\xi,\eta)|$ for all $(\xi,\eta) \in A_i$, $1 \le i \le m$. Write

$$W = \pi^{-1}(U) \cap \{(\xi,\eta) \in (K^\circ)^2 : |\eta - \bar{f}(\xi)| \le \alpha\},$$

where U is a special annulus in K° with outer radius $> \varepsilon$, and $\bar{f} \in K[\xi]$ with $\|f - \bar{f}\|_{\sup} \le \alpha$. Without loss of generality, we may assume $\alpha < \gamma$. Put

$$A'_i := A_i \cap \{(\xi,\eta) \in (K^\circ)^2 : |\eta - \bar{f}(\xi)| \le \gamma\},\ 1 \le i \le m,$$

so that $W \subset A'_1 \cup \cdots \cup A'_m$. Observe that

$$\begin{aligned} A'_i &= \{(\xi,\eta) : |\eta - \bar{f}(\xi)| \le \gamma \text{ and } \gamma \le |g_{i0}(\xi,\bar{f}(\xi))| \ge |g_{ij}(\xi,\bar{f}(\xi))|,\ 1 \le j \le n_i\} \\ &= \pi^{-1}(U_i) \cap \{(\xi,\eta) \in (K^\circ)^2 : |\eta - \bar{f}(\xi)| \le \gamma\}, \end{aligned}$$

where

$$U_i := \{\xi \in K^\circ : \gamma \le |g_{i0}(\xi,\bar{f}(\xi))| \ge |g_{ij}(\xi,\bar{f}(\xi))|,\ 1 \le j \le n_i\}.$$

It suffices to show that one of the U_i contains a special annulus of outer radius $> \varepsilon$. By [12], Theorem 4.5, each U_i is a Boolean combination of discs, so we may assume that each U_i is a closed disc about 0 with finitely many open discs removed. Since U is a special annulus of outer radius $> \varepsilon$ and $U \subset U_1 \cup \cdots \cup U_n$, by shrinking the outer radius, it follows that one of the U_i is a special annulus and that $\|\bar{f}\|_{\sup,U_i} \le 1$. □

4 Analytic continuation of affinoid plane curves

Let D, as usual, be the disc of K-rational radius ε, so $\mathcal{O}(D) = T_{1,\varepsilon}$. It is easy to find an $f \in \mathcal{O}(D)$ that is not overconvergent; for example, take f as in Definition 5.1. It is not obvious, however, that the graph of f is a curve that cannot be analytically continued (see Theorem 4.3). By an analytic continuation of the graph V of f, we mean an affinoid subset Z in a rational domain W such that $V \subset W$, $V = Z \cap \pi^{-1}(D)$ and $\pi(W)$ contains a disc of radius $> \varepsilon$. Indeed, this condition is equivalent to the slightly weaker condition given in Lemma 4.2.

Lemma 4.1 *Let $f \in T_{1,\varepsilon}$ with $\|f\|_{\sup} \le 1$ and let $g \in T_{2,(\delta,\alpha)}$, with $\alpha, \delta \in |K|$, $\delta > \varepsilon$, $\alpha > \|f\|_{\sup}$. If*

$$V(\eta - f(\xi)) = V(g) \cap \operatorname{Sp} T_{2,(\varepsilon,\|f\|_{\sup})}$$

then f is overconvergent; i.e., f is an element of $T_{1,\delta'}$ for some $\delta' \in |K|$, $\delta' > \varepsilon$.

Proof. After a linear change of variables, we may assume that $f \in T_1$ with $\|f\| = 1$, $g \in T_{2,(\alpha,\alpha)}$, with $\alpha \in |K|$, $\alpha > 1$ and

$$V(\eta - f(\xi)) = V(g) \cap \operatorname{Sp} T_2. \tag{3}$$

We will show that f is overconvergent; i.e., f is an element of $T_{1,\alpha'}$ for some $\alpha' \in |K|$, $\alpha' > 1$.

Since $\eta - \widetilde{f}$ is a prime element of $\widetilde{T}_2$, $\eta - f$ is prime in T_2. Hence, if $g = g_1 \cdot g_2$ for some $g_1, g_2 \in T_{2,(\alpha,\alpha)}$, condition (3) must hold for at least one of g_1 or g_2 in place of g. Since $T_{2,(\alpha,\alpha)}$ is isomorphic to T_2, it is a UFD ([2], Theorem 5.2.6.1), so replacing g by one of its prime factors, we may assume that g is prime in $T_{2,(\alpha,\alpha)}$.

By the Nullstellensatz ([2], Theorem 7.1.2.3), $\eta - f$ belongs to the radical of the ideal $g \cdot T_2$; i.e., we may write

$$(\eta - f)^n = v \cdot g$$

for some integer n and some $v \in T_2$ that, since $\eta - f$ is prime, we may assume is not divisible by $\eta - f$. Then condition (3) and the Nullstellensatz imply that v is a unit of T_2. Moreover, since g is prime in $T_{2,(\alpha,\alpha)}$, n must be equal to 1. This follows immediately, once we see that g and $\frac{\partial g}{\partial \eta}$ can have at most finitely many common zeros. Let $\mathfrak{q}$ be a prime ideal containing the ideal $\left(g, \frac{\partial g}{\partial \eta}\right) \cdot T_{2,(\alpha,\alpha)}$. Since $\dim T_{2,(\alpha,\alpha)} = 2$, either $\mathfrak{q}$ is the prime ideal generated by g or $\mathfrak{q}$ is a maximal ideal. Since the ideal $\left(g, \frac{\partial g}{\partial \eta}\right) \cdot T_{2,(\alpha,\alpha)}$ has only finitely many minimal prime divisors, it suffices to show that $\mathfrak{q} \neq (g)$; i.e., that $\frac{\partial g}{\partial \eta}$ is not a multiple of g. To see this, assume that $g(0,0) = 0$, which, by the Nullstellensatz, is no loss of generality since K is algebraically closed. Since v is a unit and $\eta - f(\xi)$ is prime, g is not a multiple of ξ. Thus we may write

$$g(0, \eta) = \sum a_i \eta^i,$$

where $a_0 = 0$ and not all the $a_i = 0$. Since $\operatorname{Char} K = 0$, the order of $\left.\frac{\partial g}{\partial \eta}\right|_{\xi=0}$ must be strictly less than the order of $g(0, \eta)$. It follows that $\frac{\partial g}{\partial \eta}$ is not a multiple of g, so

$$vg = \eta - f(\xi), \quad v \text{ a unit of } T_2. \tag{4}$$

Claim. *There are $\alpha', \alpha'' \in |K|$, $1 < \alpha', \alpha'' < \alpha$, such that g is regular in η of degree 1 as an element of the ring $T_{2,(\alpha',\alpha'')}$; i.e., $g(a\xi, b\eta) \in T_2$ is regular in η of degree 1 for any $a, b \in K$ with $|a| = \alpha'$ and $|b| = \alpha''$.*

Let $g = \sum g_i(\xi)\eta^i$. Multiplying by a nonzero constant, we may assume that $\|g\| = 1$, as an element of T_2. By equation (4), g is regular in η of degree 1 as an element of T_2. Hence, g_1 is a unit of T_2, $\|g_1\| = 1$, and for $i > 1$, $\|g_i \eta^i\| < 1$, as elements of T_2. Since g is overconvergent, for all $a, b \in K$ with $\alpha > |a|, |b| > 1$ but sufficiently near 1,

$$\|g_0(a\xi)\| = |a^\ell| \|g_0\|,$$

$$g_1(a\xi) \text{ is a unit of } T_2,\ \|g_1(a\xi)\| = 1,$$
$$\text{and for all } i > 1,\ \|g_i(a\xi)(b\eta)^i\| < 1.$$

Thus, taking $|b| = |a^\ell|$ while the above conditions are satisfied, we have that $g(a\xi, b\eta)$ is regular of degree 1 in η as an element of T_2. In other words, $g(\xi, \eta)$ is regular of degree 1 in η as an element of $T_{2,(\alpha',\alpha'')}$, where $\alpha' = |a|$, $\alpha'' = |b|$. This proves the claim.

Finally, using the claim, by Weierstrass Preparation, we write

$$g = (\eta - h(\xi))u(\xi, \eta)$$

for some unit $u \in T_{2,(\alpha',\alpha'')}$ and some $h \in T_{1,\alpha'}$. By (3) it follows that $h(\xi) = f(\xi)$ as elements of T_1; i.e., f is overconvergent. □

In the following lemma, we give a condition under which the graph of f has an analytic continuation Z (in the sense of the introduction to this section.) The condition given is slightly weaker than the condition given above, in that we do not require the graph of f to be equal to $Z \cap \pi^{-1}(D)$, only to be contained in $Z \cap \pi^{-1}(D)$.

Lemma 4.2 *As in Definitions 3.1 and 3.2, let $\varepsilon \in |K \setminus \{0\}|$ with $\varepsilon < 1$, let*

$$D := \{\xi \in K : |\xi| \leq \varepsilon\}$$

be the disc of radius ε, let $f \in \mathcal{O}(D)$ with $\|f\|_{\sup} < 1$ and let

$$W := \pi^{-1}(U) \cap \{(\xi, \eta) \in (K^\circ)^2 : |\eta - \bar{f}(\xi)| \leq \gamma\}$$

be a special tubular domain (where, by definition, U is a special annulus of inner radius ε and outer radius $\delta > \varepsilon$.) Suppose Z is an affinoid subset of W of dimension 1 such that

$$V(\eta - f(\xi)) \cap \pi^{-1}(U \cap D) \subset Z \cap \pi^{-1}(U \cap D)$$

then there are: a special annulus $U' \subset U$ of inner radius ε and outer radius δ', $\varepsilon < \delta' \leq \delta$, and a $\gamma' \in |K \setminus \{0\}|$ with $\gamma' \leq \gamma$ such that

$$V(\eta - f(\xi)) \cap \pi^{-1}(U' \cap D) = Z \cap W' \cap \pi^{-1}(D),$$

where W' is the special tubular domain

$$W' := \pi^{-1}(U') \cap \{(\xi, \eta) \in (K^\circ)^2 : |\eta - \widehat{f}(\xi)| \leq \gamma'\},$$

and $\widehat{f} \in K[\xi]$ is a polynomial with $\|f - \widehat{f}\|_{\sup,D} \leq \gamma' < \|f\|_{\sup,D}$.

Proof. Note that the ring

$$\mathcal{O}(W \cap \pi^{-1}(D))/(\eta - f(\xi))$$

is isomorphic to the ring $\mathcal{O}(U \cap D)$ via the unique K-algebra morphism that sends ξ to ξ and η to $f(\xi)$. Indeed, since $\|f\|_{\sup} \leq 1$, f is power-bounded ([2], Proposition 6.2.3.1), hence this map is well-defined. Since $\mathcal{O}(U \cap D)$ is contained in $\mathcal{O}(W \cap \pi^{-1}(D))$, the map is surjective, and since $\eta - f(\xi)$ is Weierstrass-regular in η, the map is injective. As noted above, $\mathcal{O}(U \cap D)$ is an integral domain. Therefore, $\eta - f(\xi)$ is a prime element of $\mathcal{O}(W \cap \pi^{-1}(D))$.

We use an irreducible decomposition of $Z \cap \pi^{-1}(D)$ to obtain the δ', γ' and U' of the lemma. Let $Z = Z_1 \cup \cdots \cup Z_n$ be the irredundant irreducible decomposition of $Z \cap \pi^{-1}(D)$ in $\operatorname{Sp} \mathcal{O}(W \cap \pi^{-1}(D))$. Since $\eta - f$ is prime, $V(\eta - f)$ equals one of the Z_i, say Z_1. Since $\eta - f$ is prime and since $2 = \dim \mathcal{O}(W \cap \pi^{-1}(D))$, the intersection $Z_1 \cap (Z_2 \cup \cdots \cup Z_n)$ is equal to $\{(a_1, b_1), \ldots, (a_m, b_m)\}$ for some $a_i \in U \cap D$ not necessarily distinct, and we define a special annulus

$$U_0 := \{\xi \in U : |\xi - a_i| \geq \varepsilon,\ 1 \leq i \leq m\}.$$

It follows that $\eta - f$ is a unit of $\mathcal{O}(\pi^{-1}(U_0 \cap D) \cap (Z_2 \cup \cdots \cup Z_n))$. Take $\gamma' \in |K \setminus \{0\}|$ such that

$$\gamma' < \min \left\{ \begin{array}{l} \inf\{|\eta - f(\xi)| : (\xi, \eta) \in \pi^{-1}(U_0 \cap D) \cap (Z_2 \cup \cdots \cup Z_n)\} \\ \inf\{|\eta - f(\xi)| : (\xi, \eta) \in \pi^{-1}(U_0 \cap D) \setminus W\} \\ \|f\|_{\sup, D} \end{array} \right. .$$

Choose $\widehat{f} \in K[\xi]$ with $\|f - \widehat{f}\|_{\sup,D} \leq \gamma'$, find $\delta' \in |K \setminus \{0\}|$, $\varepsilon < \delta' \leq \delta$ such that

$$\sup\{|\widehat{f}(\xi)| : \xi \in K,\ \xi \leq \delta'\} < 1,$$

and put

$$U' := U_0 \cap \{(\xi \in K : |\xi| \leq \delta'\},$$

and

$$W' := \pi^{-1}(U') \cap \{(\xi, \eta) \in (K^\circ)^2 : |\eta - \widehat{f}(\xi)| \leq \gamma'\}.$$

□

Theorem 4.3 *Keep the notation of Lemma 4.2. Suppose $Z \subset W$ is an affinoid set of dimension 1 such that $V(\eta - f) \cap \pi^{-1}(D \cap U)$ is a subset of Z. Then there are δ', $\varepsilon < \delta' \leq \delta$ and an $F \in T_{1,\delta'}$ such that $f = F|_D$; i.e., f is overconvergent.*

Proof As we remarked in Section 3, $\mathcal{O}(W)$ is an integral domain; i.e., W is irreducible. Hence, since $Z = V(g_1, \ldots, g_n)$, without loss of generality, we may assume that $Z = V(g)$ for some $g \in \mathcal{O}(W)$. By Lemma 4.2, possibly replacing W by a smaller special tubular domain, we may assume that

$$V(\eta - f) \cap \pi^{-1}(D \cap U) = V(g) \cap \pi^{-1}(D \cap U).$$

Choose $a \in K$ such that $\|\bar{f}\|_{\sup,D} > |a| > \|f - \bar{f}\|_{\sup,D}$, and consider the map

$$\psi : (K^\circ)^2 \to (K^\circ)^2 : (\xi, \eta) \mapsto (\xi, a\eta + \bar{f}(\xi)).$$

Taking inverse images under ψ, we obtain:

$$V\left(\eta - \left(\frac{f - \bar{f}}{a}\right)\right) \cap \pi^{-1}(D \cap U) = V(g \circ \psi) \cap \pi^{-1}(D \cap U)$$

and $g \circ \psi \in \mathcal{O}(U)\langle\eta\rangle$. Let $D' := \{\xi \in K^\circ : |\xi| \leq \delta\}$. Then

$$\pi^{-1}(D) \cup \pi^{-1}(U) = \pi^{-1}(D')$$

is an admissible cover of $\pi^{-1}(D')$. Therefore, applying Kiehl's Coherence Theorem ([2], Corollary 9.5.2.8), there is an affinoid set Z' in $\pi^{-1}(D')$ such that

$$V(\eta - \frac{f - \bar{f}}{a}) \cup V(g \circ \psi) = Z'$$

is an admissible cover of Z'. Since each of $V(\eta - \frac{f-\bar{f}}{a})$ and $V(g \circ \psi)$ is of pure dimension 1, Z' is an affinoid set in $\pi^{-1}(D')$ of pure dimension 1. Thus, since $\mathcal{O}(\pi^{-1}(D')) \cong T_2$ is a UFD, and hence every height 1 prime is principal, there is an element $G \in \mathcal{O}(\pi^{-1}(D'))$ such that $Z' = V(G)$. Since $G \in T_{2,(\delta,1)}$ and $\|\frac{f-\bar{f}}{a}\|_{\sup} < 1$, by Lemma 4.1, $\frac{f-\bar{f}}{a}$ is overconvergent; i.e, there is a δ', $\varepsilon < \delta' \leq \delta$ such that $\frac{f-\bar{f}}{a} \in T_{1,\delta'}$, so $f \in T_{1,\delta'}$ is overconvergent, as claimed. □

5 Flatness of affinoid morphisms

Choose the curve V to be $V(\eta - f(\xi))$, where f is as in Definition 5.1, let $X := \operatorname{Sp} T_2$, and let Y be the affinoid variety obtained by pasting together, along a copy of the curve V, the variety $X \times \operatorname{Sp} T_1$ with a cylinder through the curve V. Define the map $\varphi: Y \to X$ to be the natural projection that collapses the cylinder through V to V and collapses the other component, $X \times \operatorname{Sp} T_1$, to X. Let P be the analytic point of V specified in Definition 3.3. Since φ has non-trivial torsion exactly above V, one calculates, as in Lemma 5.2, that the flatificator of φ at P is V. That the flatificator exists is an example of ([20], Theorem 4.7), which establishes, for any affinoid map $\varphi: Y \to X$ and any analytic point P of X, the existence of an affinoid neighborhood W of P and an affinoid subset Z of W, the germ of which is the flatificator of φ at P. Theorem 5.3 shows, however, that there is no *wide* affinoid neighborhood W' of the analytic point P of Definition 3.3 with an affinoid subset Z' of W' the germ of which is the flatificator of φ at P. Moreover, in Theorem 5.4, we show that φ cannot be (globally) flattened by applying finite sequences of local blow-ups, thereby providing the counter-example to [7], Theorem 2.3 that was discussed in the introduction. The domain of φ is not irreducible. In Remark 5.6, we indicate how the same methods yield a counter-example with irreducible domain.

Definition 5.1 *Let $f \in \mathcal{O}(D) \setminus \bigcup_{\delta > \varepsilon} T_{1,\delta}$; i.e., suppose f converges on D and is not overconvergent. Suppose, in addition, that $\|f\|_{\sup} < 1$. For example, take*

$$f := \sum_{n \geq 1} a^{n-n^2} \xi^{n^2},$$

where $|a| = \varepsilon < 1$.

Let

$$Y := \operatorname{Sp} K\langle \xi, \eta, \zeta, \tau\rangle / (\eta - f(a\tau), \xi - a\tau) \cap (\zeta) \tag{5}$$
$$X := \operatorname{Sp} K\langle \xi, \eta\rangle, \tag{6}$$

and consider the map $\varphi: Y \to X: (\xi, \eta, \zeta, \tau) \mapsto (\xi, \eta)$.

The variety X is the unit K-affinoid polydisc of dimension 2. The variety Y is embedded in the unit K-affinoid polydisc of dimension 4. It has two components. The component $V(\eta - f(a\tau), \xi - a\tau)$ is a cylinder "in the ζ-direction" over the curve $\eta = f(\xi)$. The other component, $V(\zeta)$, is the K-affinoid unit polydisc of dimension 3. The component $V(\zeta)$ projects surjectively via φ onto X with one-dimensional linear fibers that are "parallel to the τ-axis." Note, in particular, that φ is a surjective morphism. The intersection of the two components is $V(\zeta, \eta - f(a\tau), \xi - a\tau)$, a curve in 4-space that projects isomorphically via φ to the curve $\eta = f(\xi)$. This creates non-trivial torsion in the induced maps on germs $\varphi_y^*: \mathcal{O}_{X,\varphi(y)} \to \mathcal{O}_{Y,y}$, when y is any point of $V(\zeta, \eta - f(a\tau), \xi - a\tau)$. Thus, as we shall see in Lemma 5.2, the flatificator of φ at P is the germ of the curve $\eta = f(\xi)$.

Let P be as above. We are interested in studying the germ at the analytic point P (defined using a non-overconvergent f as above) of a locally closed analytic subset $Z = \operatorname{Sp} S$ of X that contains P and that has the property that the restriction

$$\varphi|_{\varphi^{-1}(Z)}: \varphi^{-1}(Z) \to Z$$

is flat. Since we are interested in the germ of Z at P, it is natural to study the restriction $\varphi|_{\varphi^{-1}(W)}: \varphi^{-1}(W) \to W$, where $W := D \times K^\circ$. That is the situation of Lemma 5.2. Lemma 5.2 implies that the germ of Z at P is contained in the germ at P of the zero set of $\eta - f(\xi)$.

Lemma 5.2 *Let S be a K-affinoid $K\langle \xi, \eta\rangle$-algebra, and consider the natural map*

$$\operatorname{Sp} S \to \operatorname{Sp} K\langle \xi, \eta\rangle \xrightarrow{\pi} \operatorname{Sp} K\langle \xi\rangle.$$

Suppose that:

- *(i) the image of $\operatorname{Sp} S$ in $\operatorname{Sp} K\langle \xi\rangle$ is contained in D, the disc of radius ε, (so, in particular, by [2], Section 7.2.3, we may consider the image of $\eta - f(\xi)$ in S,)*
- *(ii) $(\eta - f(\xi)) \cdot S \neq S$; i.e., the image of $\operatorname{Sp} S$ in $\operatorname{Sp} K\langle \xi, \eta\rangle$ meets $V(\eta - f(\xi))$, and*
- *(iii) S is an integral domain.*

Then if the natural map $S \longrightarrow \mathcal{O}(Y)\widehat{\otimes}_{K\langle \xi, \eta\rangle} S$ is flat, the image of $\eta - f(\xi)$ in S must be zero.

Proof. Assume that $S \longrightarrow \mathcal{O}(Y)\widehat{\otimes}_{K\langle\xi,\eta\rangle}S$ is flat; then by (iii) it suffices to show that the image of $\eta - f(\xi)$ in $\mathcal{O}(Y)\widehat{\otimes}_{K\langle\xi,\eta\rangle}S$ is a zero-divisor. Indeed, we will show that the image of ζ in $\mathcal{O}(Y)\widehat{\otimes}_{K\langle\xi,\eta\rangle}S$ is non-zero, and that the image of $(\eta - f(\xi)) \cdot \zeta$ is zero.

Claim. *The image of ζ in $\mathcal{O}(Y)\widehat{\otimes}_{K\langle\xi,\eta\rangle}S$ is not zero.*

By (ii), there is some $\mathfrak{m} \in \operatorname{Sp} S \cap V(\eta - f(\xi))$. It suffices to show that the image of ζ in $\mathcal{O}(Y)\widehat{\otimes}_{K\langle\xi,\eta\rangle}S/\mathfrak{m}$ is non-zero. Since $\pi(\mathfrak{m}) \in D$, the equation $\tau = \xi(\mathfrak{m})/a$ determines a unique maximal ideal $\mathfrak{M}$ of $K\langle\xi,\eta,\tau\rangle$ that contains $\mathfrak{m}$, and $\xi - a\tau$, hence $\mathfrak{M}$ contains the ideal $(\eta - f(a\tau), \xi - a\tau) \cap (\zeta)$, and by Definition 5.1,

$$(\mathcal{O}(Y)\widehat{\otimes}_{K\langle\xi,\eta\rangle}S)/\mathfrak{M} \otimes 1 \cdot (\mathcal{O}(Y)\widehat{\otimes}_{K\langle\xi,\eta\rangle}S) \cong K\langle\zeta\rangle.$$

This proves the claim.

Claim. *The image of $(\eta - f(\xi)) \cdot \zeta$ in $\mathcal{O}(Y)\widehat{\otimes}_{K\langle\xi,\eta\rangle}S$ is zero.*

By (i), $S = \mathcal{O}(D)\widehat{\otimes}_{K\langle\xi\rangle}S$, hence by ([2], Proposition 2.1.7.7),

$$\mathcal{O}(Y)\widehat{\otimes}_{K\langle\xi,\eta\rangle}S = \mathcal{O}(Y)\widehat{\otimes}_{K\langle\xi,\eta\rangle}(\mathcal{O}(D)\widehat{\otimes}_{K\langle\xi\rangle}S) = (\mathcal{O}(Y)\widehat{\otimes}_{K\langle\xi\rangle}\mathcal{O}(D))\widehat{\otimes}_{K\langle\xi,\eta\rangle}S,$$

and it suffices to show that $(\eta - f(\xi)) \cdot \zeta$ is zero in $\mathcal{O}(Y)\widehat{\otimes}_{K\langle\xi\rangle}\mathcal{O}(D)$. We have:

$$\begin{aligned} \mathcal{O}(Y)\widehat{\otimes}_{K\langle\xi\rangle}\mathcal{O}(D) &= \mathcal{O}(Y)\widehat{\otimes}_{K\langle\xi\rangle}K\langle\xi,\lambda\rangle/(\xi - a\lambda) \\ &= K\langle\eta,\zeta,\tau,\lambda\rangle/((a\lambda - a\tau)\cdot\zeta, (\eta - f(a\tau))\cdot\zeta), \end{aligned}$$

and, since the image of $\xi - a\lambda$ is zero, it suffices to show that $(\eta - f(a\lambda)) \cdot \zeta$ belongs to the ideal generated by $(\lambda - \tau)\cdot\zeta$ and $(\eta - f(a\tau))\cdot\zeta$. To see this, note that $\lambda - \tau$ is regular in λ of degree 1, so by Weierstrass Division,

$$\eta - f(a\lambda) = q(\eta,\tau,\lambda)(\lambda - \tau) + \eta - f(a\tau).$$

It follows that

$$(\eta - f(a\lambda))\cdot\zeta = q(\eta,\tau,\lambda)(\lambda - \tau)\cdot\zeta + (\eta - f(a\tau))\cdot\zeta.$$

is zero in $\mathcal{O}(Y)\widehat{\otimes}_{K\langle\xi\rangle}\mathcal{O}(D)$, as desired. □

Theorem 5.3 *Let the morphism φ and the analytic point P be as in Definitions 3.3 and 5.1. The map $\varphi\colon Y \to X$ has no flatificator at P that is defined on a wide affinoid neighborhood of P. In other words, if W is a rational domain, if Z is an affinoid set in W containing P, and if the restriction $\varphi|_{\varphi^{-1}(Z)}\colon \varphi^{-1}(Z) \to Z$ is flat, then W is not a wide neighborhood of P. Indeed, W does not contain a special tubular domain.*

Proof. In fact, by Proposition 3.4, it suffices to show that there is no flatificator Z of φ at P that is defined in a special tubular domain $W \subset (K^\circ)^2$. (Note that by definition, a flatificator of φ at P must contain P, so P is an analytic point of Z.) Suppose that there were such, and write

$$W = \pi^{-1}(U) \cap \{(\xi, \eta) : |\eta - \bar{f}(\xi)| \leq \gamma\},$$

where $\bar{f}$ is a polynomial with $\|f - \bar{f}\|_{\sup,D} \leq \gamma$ and U is a special annulus of K° of outer radius $\delta > \varepsilon$ and inner radius ε. By Lemma 5.2, the image of $\eta - f(\xi)$ is zero in $\mathcal{O}(Z \cap \pi^{-1}(U \cap D))$. Hence,

$$Z \cap \pi^{-1}(U \cap D) \subset V(\eta - f(\xi)) \cap \pi^{-1}(U \cap D).$$

Since P is not a geometric point, $\dim Z = 1$, and since, as we observed in the proof of Lemma 4.2, $\eta - f(\xi)$ is a prime element of $\mathcal{O}(W \cap \pi^{-1}(U \cap D))$,

$$Z \cap \pi^{-1}(U \cap D) = V(\eta - f(\xi)) \cap \pi^{-1}(U \cap D).$$

But then Theorem 4.3 implies that $f \in T_{1,\delta'}$ for some $\delta' > \varepsilon$, which is not possible by the choice of f. $\square$

Finally, we show that the map $\varphi: Y \to X$ of Definition 5.1 cannot be flattened by applying finite sequences of local blow-ups, thereby providing the counterexample to [7], Theorem 2.3 that was discussed in the introduction.

Theorem 5.4 *Let the map $\varphi: Y \to X$ be as in Definition 5.1. Suppose E is a finite collection of maps $\beta: X_\beta \to X$ such that:*

(i) each $\beta \in E$ is a composition $\beta_1 \circ \cdots \circ \beta_m$ of local blow-ups β_i with locally closed, nowhere dense centers, and

(ii) $\varphi(Y) \subset \bigcup_{\beta \in E} \beta(X_\beta)$.

Then for some $\beta \in E$, the map φ_β defined by the strict transform diagram

$$\begin{array}{ccc} Y_\beta & \xrightarrow{\ \theta\ } & Y \\ \downarrow{\scriptstyle \varphi_\beta} & & \downarrow{\scriptstyle \varphi} \\ X_\beta & \xrightarrow{\ \beta\ } & X \end{array}$$

is not flat.

Proof. The theorem is a special case of the next lemma. Indeed, suppose that we are given a collection E that satisfies conditions (i) and (ii) of Theorem 5.4. Fix any special tubular domain $W \subset (K^\circ)^2$, put $\Phi := \varphi|_{\varphi^{-1}(W)}$ and for each $\beta \in E$, put $\beta_W := \beta|_{\beta^{-1}(W)}$. If $\beta = \beta_1 \circ \cdots \circ \beta_m$, then β_W is the composition $\beta_1' \circ \cdots \circ \beta_m'$, where each β_i' is the restriction of β_i to some open affinoid, hence β_W is of the same form as β. Therefore, the strict transform Φ_{β_W} of Φ by β_W satisfies $\Phi_{\beta_W} = \varphi_\beta|_{\varphi_\beta^{-1}(W)}$, where φ_β is the strict transform of φ by β. By the next lemma, for some $\beta \in E$, the strict transform Φ_{β_W} is not flat. Flatness is a local property; therefore, if Φ_{β_W} is not flat, then φ_β is not flat, proving Theorem 5.4. $\square$

It remains to prove the following lemma.

Lemma 5.5 *Let the map $\varphi: Y \to X$ be as in Definition 5.1. Let W be a special tubular domain in $(K^\circ)^2$. Define $Y_W := \varphi^{-1}(W)$, and let Φ be the restriction $\Phi := \varphi|_{Y_W}$. Suppose E is a finite collection of maps $\beta: W_\beta \to W$ such that:*

- *(i) each $\beta \in E$ is a composition $\beta_1 \circ \cdots \circ \beta_m$ of local blow-ups β_i with locally closed, nowhere dense centers, and*
- *(ii) $\Phi(Y_W) \subset \bigcup\limits_{\beta \in E} \beta(W_\beta)$.*

Then for some $\beta \in E$, the map Φ_β defined by the strict transform diagram

$$\begin{array}{ccc} Y_{W,\beta} & \xrightarrow{\ \theta\ } & Y_W \\ \Big\downarrow{\scriptstyle \Phi_\beta} & & \Big\downarrow{\scriptstyle \Phi} \\ W_\beta & \xrightarrow{\ \beta\ } & W \end{array}$$

is not flat.

Proof. As we observed above, since flatness is a local property, if we prove the lemma for a special tubular domain W', then the lemma follows (by restricting the local blow-ups and strict transforms as above) for all special tubular domains $W \supset W'$. We prove the lemma by induction on the sum M of the lengths m of the sequences $\beta = \beta_1 \circ \cdots \circ \beta_m \in E$.

We introduce some notation. For a map $\beta \in E$ given as a composition $\beta = \beta_1 \circ \cdots \circ \beta_m$ of a finite sequence of local blow-ups, we define the open affinoid subdomain A_β of W and the closed affinoid subset Z_β of A_β as follows. Write

$$\beta_1: W_{\beta_1} \to A \subset W,$$

the local blow-up of the affinoid subdomain A with closed, nowhere dense, affinoid center Z and put

$$A_\beta := A \quad \text{and} \quad Z_\beta := Z.$$

Now induct on M. Since $\bigcup_{\beta \in E} A_\beta = W$, by Lemma 3.6, there is a special tubular domain $W' \subset W$ and a $\beta \in E$ such that $W' \subset A_\beta$. By Theorem 4.3, $P \notin Z_\beta$. Thus there is a $g \in \mathcal{O}(A_\beta)$ such that $P(g) > 0$ and $Z_\beta \subset V(g)$. Let $\alpha \in |K|$ such that $0 < \alpha < P(g)$, let $\bar{g} \in K[\xi, \eta]$ satisfy $\|g - \bar{g}\|_{\sup, A_\beta} < \alpha$ and put

$$A'_\beta := \{(\xi, \eta) \in (K^\circ)^2 : |\bar{g}(\xi, \eta)| \geq \alpha\}.$$

Note that A'_β is a wide affinoid neighborhood of P, so by Proposition 3.4, there is a special tubular domain $W'' \subset A'_\beta$. Since the intersection of special tubular domains contains a special tubular domain, we may assume that $W'' \subset A'_\beta \cap W'$. Observe that $A'_\beta \cap Z_\beta = \varnothing$ implies that $W'' \cap Z_\beta = \varnothing$. Therefore, the local blow-up $\beta_1|_{\beta_1^{-1}(W'')}$ has empty center, and hence may be taken to be the identity map. If $m = 1$, then $\beta = \beta_1$, and since $\varphi|_{\varphi^{-1}(W'')}$ is not flat by Lemma 5.2, the strict transform Φ_β is not flat and the lemma is proved. If $m > 1$, then replace W by W'' and β by $\beta_2 \circ \cdots \circ \beta_m|_{\beta^{-1}(W'')}$, and we have reduced M, the sum of the lengths of the sequences of local blow-ups. The lemma follows by induction. □

Remark 5.6 *We thank Jan Denef for the following observation. Consider the open affinoid subdomain*

$$U := D \times \operatorname{Sp} T_2$$

of $\operatorname{Sp} T_3$*, where D is as in Definition 3.2, and consider the closed affinoid subset*

$$C := V(\eta - f(\xi), \zeta)$$

of U, where $f \in \mathcal{O}(D)$ is as in Definition 5.1. Note that C is of codimension 2 in U. Let $\pi\colon Y \to U$ be the blow-up of U with center C, and let φ be the composition of π with the natural inclusion of U in $\operatorname{Sp} T_3$*. As in Theorem 5.4, φ cannot be flattened by a finite sequence of local blow-ups. Furthermore, the variety Y is irreducible.*

6 Analytic classes

Several different classes of semianalytic sets have been discussed in the literature (see [14], Section 7 for a general setting.)

Definition 6.1 *Let X be a (reduced) affinoid variety and let U be a rational subdomain of X. A Boolean combination of sets of the form*

$$\{x \in U : |f(x)| \le |g(x)|\}, \qquad f, g \in \mathcal{O}(U),$$

is called globally semianalytic in U*. A subset S of X is called* wobbly semianalytic in X *if there are finitely many rational subdomains U_i of X and sets $S_i \subset U_i$, globally semianalytic in U_i, such that S is a Boolean combination of the S_i. A subset S of X is called* rigid semianalytic in X *if there is a finite cover of X by rational subdomains U_i such that $S \cap U_i$ is globally semianalytic in U_i.*

Proposition 6.2 *(i) There is a subset of $(K^\circ)^2$ that is rigid semianalytic, but not globally semianalytic in $(K^\circ)^2$.*

(ii) There is a subset of $(K^\circ)^2$ that is wobbly semianalytic, but not rigid semianalytic in $(K^\circ)^2$.

Proof. Let D' be the "closed" disc of radius ε^2 in K°, let D be the "closed" disc of radius ε in K° and let U be the annulus $\{x \in K^\circ : \varepsilon \leq |x| \leq 1\}$. Let $V := V(\eta - f(\xi)) \subset D \times K^\circ$ where $f \in \mathcal{O}(D)$ is as in Definition 5.1. Let $W := V \cap D' \times K^\circ$, and suppose $a \in |K|$ satisfies $|a| = \varepsilon$.

(i) Note that $\{D \times K^\circ, U \times K^\circ\}$ is a cover of $(K^\circ)^2$ by rational subdomains, $W \cap (U \times K^\circ) = \varnothing$ is globally semianalytic in $U \times K^\circ$, and

$$W \cap (D \times K^\circ) = \{(\xi, \eta) \in D \times K^\circ : |\xi| \leq |a|^2 \text{ and } |\eta - f(\xi)| \leq 0\}$$

is globally semianalytic in $D \times K^\circ$. Therefore W is rigid semianalytic in $(K^\circ)^2$. Suppose W is globally semianalytic in $(K^\circ)^2$; then there is a $F(\xi, \eta) \in T_2 \setminus \{0\}$ such that

$$W \subset V(F). \tag{7}$$

We will show that $V \subset V(F)$, contradicting Theorem 4.3. Since $\eta - f(\xi)$ is prime in $\mathcal{O}(D \times K^\circ)$, this follows by the Krull Intersection Theorem [15], Theorem 8.10, once we show that

$$F \in \bigcap_{n \in \mathbb{N}} (\xi, \eta)^n \left(\mathcal{O}(D \times K^\circ)/(\eta - f(\xi))\right). \tag{8}$$

By the Nullstellensatz, [2], Theorem 7.1.3.1, Formula 7 implies that

$$F \in (\eta - f(\xi))\mathcal{O}(D' \times K^\circ). \tag{9}$$

Since the completions $\mathcal{O}(D' \times K^\circ)_{(\xi,\eta)}\widehat{\ }$ and $\mathcal{O}(D \times K^\circ)_{(\xi,\eta)}\widehat{\ }$ are equal, Formula 8 follows from Formula 9. Thus W is rigid semianalytic, and not globally semianalytic in $(K^\circ)^2$.

(ii) Since $\eta - f(\xi) \in \mathcal{O}(D \times K^\circ)$, the set

$$V = \{(\xi, \eta) \in D \times K^\circ : |\eta - f(\xi)| \leq 0\}$$

is globally semianalytic in the rational subdomain $D \times K^\circ$ of $(K^\circ)^2$. Hence, V is wobbly semianalytic in $(K^\circ)^2$. An argument, based on Theorem 4.3, as in Part (i), shows that V is not rigid semianalytic in $(K^\circ)^2$. □

Definition 6.3 *(See [18].) Let X be a (reduced) affinoid variety. A subset S of X is called* strongly subanalytic in X *if there is an $n \in \mathbb{N}$, an $\alpha \in (|K|)^n$ with each $\alpha_i > 1$, and a globally semianalytic subset S' of $X \times \operatorname{Sp} T_{n,\alpha}$ such that $S' \subset X \times \operatorname{Sp} T_n$ and $S = \pi(S')$, where $\pi: X \times \operatorname{Sp} T_{n,\alpha} \to X$ is the projection on the first factor. (If we allow the $\alpha_i = 1$, then S is called an* affinoid subanalytic subset of X.*)*

In [18], it is shown that S is strongly subanalytic in X if, and only if, there is a proper analytic map $\varphi: Y \to X$ and a globally semianalytic subset S' of Y such that $S = \varphi(S')$.

Proposition 6.4 *There is an affinoid subanalytic subset of $(K^\circ)^2$ that is not strongly subanalytic in $(K^\circ)^2$.*

Proof. Let V be as in the proof of Proposition 6.2, and let

$$V' := \{(xi, \eta, \tau) \in (K^\circ)^3 : \xi - a\tau = 0 \text{ and } \eta - g(\tau) = 0\},$$

where $a \in K$ satisfies $|a| = \varepsilon$ and $g(\tau) := f(a\tau)$. Then since $V = \pi(V')$, where $\pi\colon (K^\circ)^3 \to (K^\circ)^2$ is the projection on the first two coordinates, V is affinoid subanalytic in $(K^\circ)^2$. We will show that V is not strongly subanalytic in $(K^\circ)^2$. If it were, then by [19], Theorem 3.2, it follows that V is a rigid semianalytic subset of $(K^\circ)^2$. But this contradicts Proposition 6.2(ii). □

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