REAL CLOSED FIELDS WITH NONSTANDARD AND STANDARD ANALYTIC STRUCTURE

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Abstract. We consider the ordered field which is the completion of the Puiseux series field over $\mathbb{R}$ equipped with a ring of analytic functions on $[-1,1]^n$ which contains the standard subanalytic functions as well as functions given by t-adically convergent power series, thus combining the analytic structures from [DD] and [LR3]. We prove quantifier elimination and o–minimality in the corresponding language. We extend these constructions and results to rank n ordered fields $\mathbb{R}_n$ (the maximal completions of iterated Puiseux series fields). We generalize the example of Hrushovski and Peterzil [HP] of a sentence which is not true in any o–minimal expansion of $\mathbb{R}$ (shown in [LR3] to be true in an o–minimal expansion of the Puiseux series field) to a tower of examples of sentences σ_n, true in $\mathbb{R}_n$, but not true in any o–minimal expansion of any of the fields $\mathbb{R}, \mathbb{R}_1, \dots, \mathbb{R}_{n-1}$.

1. Introduction

In [LR3] it is shown that the ordered field K_1 of Puiseux series in the variable t over $\mathbb{R}$, equipped with a class of t–adically overconvergent functions such as $\sum_n (n+1)!(tx)^n$ has quantifier elimination and is o–minimal in the language of ordered fields enriched with function symbols for these functions on $[-1,1]^n$. This was motivated (indirectly) by the observation of Hrushovski and Peterzil, [HP], that there are sentences true in this structure that are not satisfiable in any o–minimal expansion of $\mathbb{R}$. This in turn was motivated by a question of van den Dries. See [HP] for details.

In [DMM1] it was observed that if K is a maximally complete, non–archimedean real closed field with divisible value group, and if f an element of $\mathbb{R}[[\xi]]$ with radius of convergence > 1, then f extends naturally to an "analytic" function $I^n \to K$, where $I = \{x \in K\colon -1 \leq x \leq 1\}$. Hence if $\mathcal{A}$ is the ring of real power series with radius of convergence > 1 then K has $\mathcal{A}$–analytic structure i.e. this extension preserves all the algebraic properties of the ring $\mathcal{A}$. In particular the real quantifier

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elimination of [DD] works in this context so K has quantifier elimination, is o–minimal in the analytic field language, and is even elementarily equivalent to $\mathbb{R}$ with the subanalytic structure. See [DMM2] and [DMM3] for extensions.

In Sections 2 and 3 below we extend the results of [LR3] by proving quantifier elimination and o–minimality for K_1 in a larger language that contains the overconvergent functions together with the usual analytic functions on $[-1,1]^n$. In Section 4 we extend these results to a larger class of non-archimedean real-closed fields, including fields $\mathbb{R}_m$, of rank $m = 1, 2, 3, \ldots$, and in Section 5 we show that the idea of the example of [HP] can be iterated so that for each m there is a sentence true in $\mathbb{R}_m$ but not satisfiable in any o–minimal expansion of $\mathbb{R} = \mathbb{R}_0, \mathbb{R}_1, \cdots, \mathbb{R}_{m-1}$. In a subsequent paper we will give a more comprehensive treatment of both Henselian fields with analytic structure and real closed fields with analytic structure, see [CL].

2. Notation and Quantifier Elimination

In this section we establish notation and prove quantifier elimination (Theorem 2.16) for the field of Puiseux series over $\mathbb{R}$ in a language which contains function symbols for all the standard analytic functions on $[-1,1]^n$ and all the t–adically overconvergent functions on this set.

Definition 2.1.

$$\begin{aligned} K_1 &:= \bigcup_n \mathbb{R}((t^{1/n})), \text{ the field of Puiseux series over } \mathbb{R}, \\ K &:= \widehat{K}_1, \text{ the } t\text{–adic completion of } K_1. \end{aligned}$$

K is a real closed, nonarchimedean normed field. We shall use $\|\cdot\|$ to denote the (nonarchimedean) t–adic norm on K, and $<$ to denote the order on K that comes from the real closedness of K. We will use $|\cdot|$ to denote the corresponding absolute value, $|x| = \sqrt{x^2}$.

$$\begin{aligned} K^\circ &:= \{x \in K \colon \|x\| \le 1\}, \text{ the finite elements of } K \\ K^{\circ\circ} &:= \{x \in K \colon \|x\| < 1\}, \text{ the infinitesimal elements of } K \\ K_{alg} &:= K[\sqrt{-1}], \text{ the algebraic closure of } K \\ A_{n,\alpha} &:= \{f \in \mathbb{R}[[\xi_1, \ldots, \xi_n]] \colon \text{ radius of convergence of } f > \alpha\}, 0 < \alpha \in \mathbb{R} \\ \mathcal{R}_{n,\alpha} &:= A_{n,\alpha} \widehat{\otimes}_{\mathbb{R}} K = (A_{n,\alpha} \otimes_{\mathbb{R}} K)\widehat{\ }, \text{ where } \widehat{\ } \text{ stands for the } t\text{-adic completion} \\ \mathcal{R}^\circ{}_{n,\alpha} &:= A_{n,\alpha} \widehat{\otimes}_{\mathbb{R}} K^\circ = (A_{n,\alpha} \otimes_{\mathbb{R}} K^\circ)\widehat{\ } \\ \mathcal{R}_n &:= \bigcup_{\alpha > 1} \mathcal{R}_{n,\alpha} \\ \mathcal{R} &:= \bigcup_n \bigcup_{\alpha > 1} \mathcal{R}_{n,\alpha} \\ I &:= [-1,1] = \{x \in K \colon |x| \le 1\}. \end{aligned}$$

Remark 2.2. *(i) K has $\mathcal{R}$–analytic structure – in* [DMM1] *it is explained how the functions of $A_{n,\alpha}$ are defined on I^n. For example, if $f \in A_{1,\alpha}$, $\alpha > 1$, $a \in \mathbb{R} \cap [-1,1]$, $\beta \in K^{\circ\circ}$, then $f(a+\beta) := \sum_n f^{(n)}(a) \frac{\beta^n}{n!}$. The extension to functions in $\mathcal{R}_{n,\alpha}$ is clear from the completeness of K. We define these functions to be zero outside*

$[-1,1]^n$. *This extension also naturally works for maximally complete fields and fields of LE-series, see also* [DMM2] *and* [DMM3].

(ii) $\mathcal{R}$ *contains all the "standard" real analytic functions on* $[-1,1]^n$ *and all the* t*–adically overconvergent functions in the sense of* [LR3].

(iii) The elements of $A_{n,\alpha}$ *in fact define complex analytic functions on the complex polydisc* $\{x \in \mathbb{C} \colon |x_i| \le \alpha$ *for* $i = 1, \dots, n\}$*, and hence the elements of* $\mathcal{R}_{n,\alpha}$ *define "*K_{alg}*–analytic" functions on the corresponding* K_{alg}*–polydisc* $\{x \in K_{alg} \colon |x_i| \le \alpha$ *for* $i = 1, \dots, n\}$.

(iv) We could as well work with K_1 *instead of* K*. Then we must replace* $\mathcal{R}_{n,\alpha}$ *by*

$$\mathcal{R}'_{n,\alpha} := \bigcup_m A_{n,\alpha} \widehat{\otimes}_{\mathbb{R}} \mathbb{R}((t^{1/m})) = \bigcup_m \big((A_{n,\alpha} \otimes_{\mathbb{R}} \mathbb{R}((t^{1/m})))^{\widehat{}}\big).$$

(v) If $\beta \le \alpha$ *and* $m \le n$ *then* $\mathcal{R}_{\alpha,m} \subset \mathcal{R}_{\beta,n}$.

Lemma 2.3. *Every nonzero* $f \in \mathcal{R}_{n,\alpha}$ *has a unique representation*

$$f = \sum_{i \in J \subset \mathbb{N}} f_i t^{\gamma_i}$$

where the $f_i \in A_{n,\alpha}$*,* $f_i \ne 0$*,* $\gamma_i \in \mathbb{Q}$*, the* γ_i *are increasing,* $0 \in J$*, and, either* J *is finite of the form* $\{0, \dots, n\}$*, or* $\gamma_i \to \infty$ *as* $i \to \infty$ *and* $J = \mathbf{N}$*. The function* f *is a unit in* $\mathcal{R}_{n,\alpha}$ *exactly when* f_0 *is a unit in* $A_{n,\alpha}$.

Proof. Observe that if $a \in K, a \ne 0$ then a has a unique representation $a = \sum_{i \in J \subset \mathbb{N}} a_i t^{\gamma_i}$, where the $0 \ne a_i \in \mathbb{R}, \gamma_i \in \mathbb{Q}$ and, either J is finite, or $\gamma_i \to \infty$. If f_0 is a unit in $A_{n,\alpha}$, then $f \cdot f_0^{-1} \cdot t^{-\gamma_0} = 1 + \sum_{i=1}^{\infty} f_i \cdot f_0^{-1} \cdot t^{\gamma_i - \gamma_0}$ and $\gamma_i - \gamma_0 > 0$. ☐

Definition 2.4. (i) *In the notation of the previous lemma,* f_0 *is called the* top slice *of* f.
(ii) *We call* f regular in ξ_n of degree s at $a \in [I \cap \mathbb{R}]^n$ *if, in the classical sense,* f_0 *is regular in* ξ_n *of degree* s *at* a.
(iii) *We shall abuse notation and use* $\|\cdot\|$ *to denote the* t*–adic norm on* K*, and the corresponding gauss–norm on* $\mathcal{R}_{n,\alpha}$*, so, with* f *as in the above lemma,* $\|f\| = \|t^{\gamma_0}\|$.

The standard Weierstrass Preparation and Division Theorems for $A_{n,\alpha}$ extend to corresponding theorems for $\mathcal{R}_{n,\alpha}$.

Theorem 2.5. *(Weierstrass Preparation and Division). If* $f \in \mathcal{R}_{n,\alpha}$ *with* $\|f\| = 1$ *is regular in* ξ_n *of degree* s *at 0, then there is a* $\delta \in \mathbb{R}$*,* $\delta > 0$*, such that there are unique* $A_1, \dots, A_s, U$ *satisfying*

$$f = [\xi_n^s + A_1(\xi')\xi_n^{s-1} + \ldots + A_s(\xi')]U(\xi)$$

and

$$A_1, \dots, A_s, \in \mathcal{R}_{n-1,\delta}, \text{ and } U \in \mathcal{R}_{n,\delta} \text{ a unit.}$$

Then automatically

$$\|A_1\|, \dots, \|A_s\|, \|U\| \le 1, \quad \|A_1(0)\|, \dots, \|A_s(0)\| < 1, \quad \text{and} \quad \|U(0)\| = 1.$$

Furthermore, if $g \in \mathcal{R}_{n,\alpha}$ then there are unique $Q \in \mathcal{R}_{n,\delta}$ and $R_0(\xi'), \ldots, R_{s-1}(\xi') \in \mathcal{R}_{n-1,\delta}$, satisfying

$$\|Q\|, \|R_i\| \leq \|g\|$$

and

$$g = Qf + R_0(\xi') + R_1(\xi')\xi_n + \ldots + R_{s-1}(\xi')\xi_n^{s-1}.$$

Proof. We may assume that $f = \sum_{\gamma \in I} f_\gamma t^\gamma$, where $I \subset \mathbb{Q}^+, 0 \in I, I = I + I$ and I is well ordered. (We do not require that $f_\alpha \neq 0$ for all $\alpha \in I$, but we do require $f_0 \neq 0$.) We prove the Preparation Theorem. The proof of the Division Theorem is similar. We shall produce, inductively on $\gamma \in I$, monic polynomials $P_\gamma[\xi_n]$ with coefficients from $\mathcal{R}_{n-1,\delta}$, and units $U_\gamma \in \mathcal{R}_{n,\delta}$ such that, writing γ' for the successor of γ in I, we have

$$f \equiv P_\gamma \cdot U_\gamma \bmod t^{\gamma'}$$

and if $\gamma < \beta$

$$P_\gamma \equiv P_\beta \text{ and } U_\gamma \equiv U_\beta \bmod t^{\gamma'}.$$

Using [GR], Theorem II.D.1 (p.80), or the proof on pp. 142-144 of [ZS], we see that there is a $0 < \delta \leq \alpha$ such that for every $g \in A_{n,\delta}$ the Weierstrass data on dividing g by f_0 are in $A_{n,\delta}$.

P_0 and U_0 are the classical Weierstrass data for f_0, i.e. $f_0 = U_0 P_0$, where $P_0 \in A_{n-1,\delta}[\xi_n]$ is monic of degree s, and $U \in A_{n,\delta}$ is a unit. Suppose P_γ and U_γ have been found. Then

$$f \equiv P_\gamma \cdot U_\gamma \bmod t^{\gamma'}$$

so we have

$$U_\gamma{}^{-1} \cdot f \equiv P_\gamma + g_{\gamma'} t^{\gamma'} + o(t^{\gamma'}),$$

where $g_{\gamma'} \in A_{n,\delta}$ and we write $o(t^{\gamma'})$ to denote terms of order $> \gamma'$. By classical Weierstrass division we can write

$$g_{\gamma'} = P_0 \cdot Q_{\gamma'} + R_{\gamma'},$$

where $Q_{\gamma'} \in A_{n,\delta}$ and $R_{\gamma'} \in A_{n-1,\delta}[\xi_n]$ has degree $< s$ in ξ_n. Let

$$P_{\gamma'} := P_\gamma + t^{\gamma'} R_{\gamma'}.$$

Then

$$\begin{aligned} U_\gamma{}^{-1} \cdot f &= P_\gamma + t^{\gamma'}(P_0 Q_{\gamma'} + R_{\gamma'}) + o(t^{\gamma'}) \\ &= (P_{\gamma'} + t^{\gamma'} P_{\gamma'} Q_{\gamma'}) + t^{\gamma'}(P_0 - P_{\gamma'})Q_{\gamma'} + o(t^{\gamma'}) \\ &= P_{\gamma'}(1 + t^{\gamma'} Q_{\gamma'}) + o(t^{\gamma'}), \end{aligned}$$

since $P_0 - P_{\gamma'} = o(1)$, i.e. it has positive order. Take $U_{\gamma'} := U_\gamma(1 + t^{\gamma'} Q_{\gamma'})$. The uniqueness of the A_i and U follows from the same induction. □

Remark 2.6. *We remark, for use in a subsequent paper (*[CL]*), that the argument of the previous proof works in the more general context that I is a well ordered subset of the value group Γ of a suitably complete field, for example a maximally complete field.*

From the above proof or by direct calculation we have

Corollary 2.7. *If $g \in \mathcal{R}_1$, $\beta \in [-1, 1]$ and $g(\beta) = 0$ then $\xi_1 - \beta$ divides g in $\mathcal{R}_1$.*

Remark 2.8. *Let $f(\xi,\eta)$ be in $\mathcal{R}_{m+n,\alpha}$. Then there are unique $\overline{f}_\mu$ in $\mathcal{R}_{m,\alpha}$ such that*

$$f(\xi,\eta)=\sum_\mu \overline{f}_\mu(\xi)\eta^\mu.$$

The following Lemma is used to prove Theorem 2.10.

Lemma 2.9. *Let $f(\xi,\eta)=\sum_\mu \overline{f}_\mu(\xi)\eta^\mu \in \mathcal{R}_{m+n,\alpha}$. Then the $\overline{f}_\mu \in \mathcal{R}_{m,\alpha}$ and there is an integer $d\in\mathbb{N}$, a constant $\beta>0$, $\beta\in\mathbb{R}$ and $g_\mu \in \mathcal{R}^\circ_{m+n,\beta}$ for $|\mu|<d$, such that*

$$f=\sum_{|\mu|<d}\overline{f}_\mu(\xi)g_\mu(\xi,\eta),$$

in $\mathcal{R}_{m+n,\beta}$.

Proof. We may assume that $\|f\|=1$, and choose a ν_0 such that $\|\overline{f}_{\nu_0}\|=1$. Making an $\mathbb{R}$-linear change of variables, and shrinking α if necessary, we may assume that $\overline{f}_{\nu_0}$ is regular at 0 in ξ_m of degree s, say. Write ξ' for $(\xi_1,\dots,\xi_{m-1})$. By Weierstrass Division (Theorem 2.5) there is a $\beta>0$ and there are $Q(\xi,\eta)\in\mathcal{R}_{m+n,\beta}$ and $R(\xi,\eta)=R_0(\xi',\eta)+\cdots+R_{s-1}(\xi',\eta)\xi_m^{s-1}\in\mathcal{R}_{m+n-1,\beta}[\xi_m]$ such that

$$f(\xi,\eta)=\overline{f}_{\nu_0}(\xi)Q(\xi,\eta)+R(\xi,\eta).$$

By induction on m, we may write

$$R_0=\sum_{|\mu|<d}\overline{R}_{0\mu}(\xi')g_\mu(\xi',\eta),$$

for some $d\in\mathbb{N}$, some $\beta>0$ and $g_\mu(\xi',\eta)\in\mathcal{R}^\circ_{m+n-1,\beta}$. Writing $R=\sum_\nu \overline{R}_\nu(\xi)\eta^\nu$, observe that each $\overline{R}_\nu$ is an $\mathcal{R}^\circ_{m,\beta}-$linear combination of the $\overline{f}_\nu$, since, taking the coefficient of η^ν on both sides of the equation $f(\xi,\eta)=\overline{f}_{\nu_0}(\xi)Q(\xi,\eta)+R(\xi,\eta)$, we have

$$\overline{f}_\nu=\overline{f}_{\nu_0}\overline{Q}_\nu+\overline{R}_\nu.$$

Consider

$$\begin{aligned} f-\overline{f}_{\nu_0}Q-\sum_{|\mu|<d}\overline{R}_\mu(\xi)g_\mu(\xi',\eta) &=: S_1\xi_m+S_2{\xi_m}^2+\cdots+S_{s-1}\xi_m^{s-1}\\ &= \xi_m[S_1+S_2\xi_m+\cdots+S_{s-1}\xi_m^{s-2}]\\ &=: \xi_m\cdot S,\ \text{say,}\end{aligned}$$

where the $S_i\in\mathcal{R}^\circ_{m+n-1,\beta}$. Again, observe that each $\overline{S}_\nu$ is an $\mathcal{R}^\circ_{m,\beta}-$linear combination of the $\overline{f}_{\nu'}$. Complete the proof by induction on s, working with S instead of R. □

Theorem 2.10. *(Strong Noetherian Property). Let $f(\xi,\eta)=\sum_\mu \overline{f}_\mu(\xi)\eta^\mu\in\mathcal{R}_{m+n,\alpha}$. Then the $\overline{f}_\mu\in\mathcal{R}_{m,\alpha}$ and there is an integer $d\in\mathbb{N}$, a constant $\beta>0$, $\beta\in\mathbb{R}$ and units $U_\mu(\xi,\eta)\in\mathcal{R}^\circ_{m+n,\beta}$ for $|\mu|<d$, such that*

$$f=\sum_{\mu\in J}\overline{f}_\mu(\xi)\eta^\mu U_\mu(\xi,\eta)$$

in $\mathcal{R}_{m+n,\beta}$, where J is a subset of $\{0,1,\dots,d\}^n$.

Proof. It is sufficient to show that there are an integer d, a set $J \subset \{0, 1, \ldots, d\}^n$, and $g_\mu \in \mathcal{R}^\circ_{m+n,\beta}$ such that

$$f = \sum_{\mu \in J} \overline{f}_\mu(\xi)\eta^\mu g_\mu(\xi, \eta), \tag{2.11}$$

since then, rearranging the sum if necessary, we may assume that each g_μ is of the form $1 + h_\mu$ where $h_\mu \in (\eta)\mathcal{R}^\circ_{m+n,\beta}$. Shrinking β if necessary will guarantee that the g_μ are units. But then it is in fact sufficient to prove (2.11) for f replaced by

$$f_{I_i} := \sum_{\mu \in I_i} \overline{f}_\mu(\xi)\eta^\mu$$

for each I_i in a finite partition $\{I_i\}$ of $\mathbb{N}^n$.

By Lemma 2.9 there is an integer $d \in \mathbb{N}$, a constant $\beta > 0$, $\beta \in \mathbb{R}$ and $g_\mu \in \mathcal{R}^\circ_{m+n,\beta}$ for $|\mu| \leq d$, such that

$$f = \sum_{|\mu| \leq d} \overline{f}_\mu(\xi) g_\mu(\xi, \eta).$$

Rearranging, we may assume for $\nu, \mu \in \{1, \ldots, d\}^n$ that $(\bar{g}_\mu)_\nu$ equals 1 if $\mu = \nu$ and that it equals 0 otherwise.

Focus on $f_{I_1}(\xi, \eta)$, defined as above by

$$f_{I_1}(\xi, \eta) = \sum_{\mu \in I_1} \overline{f}_\mu(\xi)\eta^\mu$$

with

$$I_1 := \{0, \ldots, d\}^n \cup \{\mu \colon \mu_i \geq d \text{ for all } i\}$$

and note that

$$f_{I_1}(\xi, \eta) = \sum_{|\mu| \leq d} \overline{f}_\mu(\xi) g_{\mu, I_1}(\xi, \eta) \tag{2.12}$$

with $g_{\mu,I_1}(\xi, \eta) \in \mathcal{R}^\circ_{m+n,\beta}$ defined by the corresponding sum

$$g_{\mu,I_1}(\xi, \eta) = \sum_{\nu \in I_1} \overline{g}_{\mu,\nu}(\xi)\eta^\nu.$$

It is now clear that g_{μ,I_1} is of the form $\eta^\mu(1 + h_\mu)$ where $h_\mu \in (\eta)\mathcal{R}^\circ_{m+n,\beta}$.

One now proceeds by noting that $f - f_{I_1}$ is a finite sum of terms of the form f_{I_j} for $j > 1$ and $\{I_j\}_j$ a finite partition of $\mathbb{N}^n$ and where each f_{I_j} for $j > 1$ is of the form $\eta_i^\ell q(\xi, \eta')$ where η' is $(\eta_1, \ldots, \eta_{i-1}, \eta_{i+1}, \ldots, \eta_n)$ and q is in $\mathcal{R}^\circ_{m+n-1,\beta}$. These terms can be handled by induction on n.

□

Definition 2.13. *For $\gamma \in K^\circ_{alg}$ let γ° denote the closest element of $\mathbb{C}$, i.e. the unique element γ° of $\mathbb{C}$ such that $|\gamma - \gamma^\circ| \in K^{\circ\circ}_{alg}$.*

Lemma 2.14. *Let $f \in \mathcal{R}_1$. If $f(\gamma) = 0$ then $f_0(\gamma^\circ) = 0$ (f_0 is the top slice of f). Conversely, if $\beta \in \mathbb{R}$ (or $\mathbb{C}$) and $f_0(\beta) = 0$ there is a $\gamma \in K_{alg}$ with $\gamma^\circ = \beta$ and $f(\gamma) = 0$. Indeed, f_0 has a zero of order n at $\beta \in \mathbb{C}$ if, and only if, f has n zeros γ (counting multiplicity) with $\gamma^\circ = \beta$.*

Proof. Use Weierstrass Preparation and [BGR] Proposition 3.4.1.1. □

Corollary 2.15. *A nonzero* $f \in \mathcal{R}_{1,\alpha}$ *has only finitely many zeros in the set* $\{x \in K_{alg} \colon |x| \leq \alpha\}$. *Indeed, there is a polynomial* $P(x) \in K[x]$ *and a unit* $U(x) \in \mathcal{R}_{1,\alpha}$ *such that* $f(x) = P(x) \cdot U(x)$.

Proof. Observe that f_0 has only finitely many zeros in $\{x \in \mathbb{C} \colon |x| \leq \alpha\}$, that non-real zeros occur in complex conjugate pairs, and that f is a unit exactly when f_0 has no zeros in this set, i.e. when f_0 is a unit in $A_{1,\alpha}$, and use Lemma 2.14. □

Theorem 2.16 (Quantifier Elimination Theorem)**.** *Denote by* $\mathcal{L}$ *the language* $\langle +, \cdot, ^{-1}, 0, 1, <, \mathcal{R} \rangle$ *where the functions in* $\mathcal{R}_n$ *are interpreted to be zero outside* I^n. *Then* K *admits quantifier elimination in* $\mathcal{L}$.

Proof. This is a small modification of the real quantifier elimination of [DD] as in [DMM1], using the Weierstass Preparation Theorem and the Strong Noetherian Property above. Crucial is that, in Theorems 2.5 and 2.10, as in [DD], β and δ are positive real numbers so one can use the compactness of $[-1,1]^n$ in $\mathbb{R}^n$. □

3. *o*–minimality

In this section we prove the *o*–minimality of K in the language $\mathcal{L}$. Let $\alpha > 1$. As we remarked above, each $f \in A_{n,\alpha}$ defines a function from the poly-disc $(I_{\mathbb{C},\alpha})^n \to \mathbb{C}$, where $I_{\mathbb{C},\alpha} := \{x \in \mathbb{C} \colon |x| \leq \alpha\}$, and hence each $f \in \mathcal{R}_{n,\alpha}$ defines a function from $(I_{K_{alg},\alpha})^n \to K_{alg}$, where $I_{K_{alg},\alpha} := \{x \in K_{alg} \colon |x| \leq \alpha\}$. In general $A_{n,\alpha}$ is not closed under composition. However, if $F(\eta_1, \ldots, \eta_m) \in \mathcal{R}_{m,\alpha}$, $G_j(\xi) \in \mathcal{R}_{n,\beta}$ for $j = 1, \ldots, m$ and $|G_j(x)| \leq \alpha$ for all $x \in (I_{K_{alg},\beta})^n$, then $F(G_1(\xi), \ldots, G_m(\xi)) \in \mathcal{R}_{n,\beta}$. This is clear if $F \in A_{n,\alpha}$ and the $G_j \in A_{n,\beta}$. The general case follows easily.

For $c, r \in K,\ r > 0$, we denote the "closed interval" with center c and radius r by

$$I(c,r) := \{x \in K \colon |x - c| \leq r\}$$

and for $c, \delta, \varepsilon \in K$, $0 < \delta < \varepsilon$, we denote the "closed annulus" with center c, inner radius δ and outer radius ε by

$$A(c,\delta,\varepsilon) := \{x \in K \colon \delta \leq |x - c| \leq \varepsilon\}.$$

On occasion we will consider $I(c,r)$ as a disc in K_{alg} and $A(c,\delta,\varepsilon)$ as an annulus in K_{alg}, replacing K by K_{alg} in the definitions. No confusion should result. Note that these discs and annuli are defined in terms of the real-closed order on K, not the non–archimedean absolute value $\|\cdot\|$, and hence are not discs or annuli in the sense of [BGR],[LR3] or [FP], which we will refer to as *affinoid discs* and *affinoid annuli.* For $I = I(c,r)$, $A = A(c,\delta,\varepsilon)$ as above, we define the rings of analytic function on I and A as follows:

$$\begin{aligned} \mathcal{O}_I &:= \left\{ f\left(\frac{x-c}{r}\right) : f \in \mathcal{R}_1 \right\} \\ \mathcal{O}_A &:= \left\{ g\left(\frac{\delta}{x-c}\right) + h\left(\frac{x-c}{\varepsilon}\right) : g, h \in \mathcal{R}_1,\ g(0) = 0 \right\}. \end{aligned}$$

The elements of $\mathcal{O}_I$ (respectively, $\mathcal{O}_A$) are analytic functions on the corresponding K_{alg}–disc (respectively, annulus) as well.

Remark 3.1. *(i) Elements of $\mathcal{O}_A$ are multiplied using the relation $\frac{\delta}{x-c} \cdot \frac{x-c}{\varepsilon} = \frac{\delta}{\varepsilon}$ and the fact that $|\frac{\delta}{\varepsilon}| < 1$. Indeed, let $g(\xi_1) = \sum_i a_i \xi_1^i$, $h(\xi_2) = \sum_j b_j \xi_2^j$. Then, using the relation $\xi_1 \xi_2 = \frac{\delta}{\varepsilon}$, we have*

$$\begin{aligned} g \cdot h &= \sum_{j<i} a_i b_j \big(\frac{\delta}{\varepsilon}\big)^j \xi_1^i + \sum_{i \leq j} a_i b_j \big(\frac{\delta}{\varepsilon}\big)^{j-i} \xi_2^{j-i} \\ &= f_1(\xi_1) + f_2(\xi_2). \end{aligned}$$

If $g,\ h \in A_{1,\alpha}$ and $\frac{\delta}{\varepsilon} \in \mathbb{R}$ then $f_1,\ f_2 \in A_{1,\alpha}$, and this extends easily to the case $g,\ h \in \mathcal{R}_{1,\alpha}$ and $\frac{\delta}{\varepsilon} \in K^\circ$. Lemma 3.6 will show that in fact the only case of an annulus that we must consider is when $\frac{\delta}{\varepsilon} \in K^{\circ\circ}$.

(ii) We define the gauss-norm on $\mathcal{O}_I$ by $\|f\big(\frac{x-c}{r}\big)\| := \|f(\xi)\|$, and on $\mathcal{O}_A$ by $\|g\big(\frac{\delta}{x-c}\big)+h\big(\frac{x-c}{r}\big)\| := \max\{\|g(\xi_1)\|, \|h(\xi_2)\|\}$. It is clear that the gauss-norm equals the supremum norm.

(iii) If $f \in \mathcal{O}_I$ then $\|\big(\frac{x-c}{r}\big)f\| = \|f\|$. If $f \in \mathcal{O}_A$ then $\|\big(\frac{x-c}{\varepsilon}\big)f\| \leq \|f\|$ and if $\frac{\delta}{\varepsilon} \in K^{\circ\circ}$, (i.e. is infinitesimal) then $\|\big(\frac{x-c}{\varepsilon}\big)g\big(\frac{\delta}{x-c}\big)\| = \|\frac{\delta}{\varepsilon}\| \cdot \|g\big(\frac{\delta}{x-c}\big)\| < \|g\big(\frac{\delta}{x-c}\big)\|$ and $\|\big(\frac{x-c}{\varepsilon}\big)h\big(\frac{x-c}{\varepsilon}\big)\| = \|h\big(\frac{x-c}{\varepsilon}\big)\|$.

(iv) If $\|f\| < 1$ then $1-f$ is a unit in $\mathcal{O}_A$. (In fact it is a strong unit *– a unit u satisfying $\|1-u\| < 1$.)*

Definition 3.2. *(i) We say that $f \in \mathcal{O}_{I(0,1)} = \mathcal{R}_1$* has a zero close to $a \in I(0,1)$ *if f, as a K_{alg}–function defined on the K_{alg}–disc $\{|x| \colon |x| \leq \alpha\}$ for some $\alpha > 1$ has a K_{alg}–zero b with $a-b$ infinitesimal in K_{alg}. We say f* has a zero close to $I(0,1)$ *if it has a zero close to a for some $a \in I(0,1)$. For an arbitrary interval $I = I(c,r)$ we say that $f = F(\frac{x-c}{r}) \in \mathcal{O}_I$* has a zero close to $a \in I$ *if F has a zero close to $\frac{a-c}{r} \in I(0,1)$, and that f* has a zero close to $I(c,r)$ *if F has a zero close to $I(0,1)$.*

(ii) For $0 < a, b \in K^\circ$ we write $a \sim b$ if $\frac{a}{b}, \frac{b}{a} \in K^\circ$ and we write $a << b$ if $\frac{a}{b} \in K^{\circ\circ}$.

(iii) Let X be an interval or an annulus, and let f be defined on a superset of X. We shall write $f \in \mathcal{O}_X$ to mean that there is a function $g \in \mathcal{O}_X$ such that

$$f|_X = g.$$

Lemma 3.3. *If $f \in \mathcal{O}_{I(c,r)}$ has no zero close to $I(c,r)$, then there is a cover of $I(c,r)$ by finitely many closed intervals $I_j = I(c_j, r_j)$ such that $\frac{1}{f} \in \mathcal{O}_{I_j}$ for each j.*

Proof. It is sufficient to consider the case $I(c,r) = I(0,1)$. Cover $I(0,1)$ by finitely many intervals $I(c_j, r_j)$, $c_j, r_j \in \mathbb{R}$ such that f has no K_{alg}–zero in the K_{alg}–disc $I(c_j, r_j)$. Finally use Corollary 2.15. □

Remark 3.4. *The function $f(x) = 1 + x^2$ has no zeros close to $I(0,1)$. It is not a unit in $\mathcal{O}_{I(0,1)}$, but it is a unit in both $\mathcal{O}_{I(-\frac{1}{2},\frac{1}{2})}$ and $\mathcal{O}_{I(\frac{1}{2},\frac{1}{2})}$. The function $g(x) = x$ is not a unit in $\mathcal{O}_{I(\delta,\varepsilon)}$ for any $0 < \delta \in K^{\circ\circ}$ and $0 < \varepsilon \in K^\circ \setminus K^{\circ\circ}$. It is of course a unit in $\mathcal{O}_{A(0,\delta,\varepsilon)}$. The function $\frac{1}{g} = \frac{1}{x} \in A(0,\delta,1)$ for all $\delta > 0$ but is not in $\mathcal{O}_I$*

for $I = I(\frac{1+\delta}{2}, \frac{1-\delta}{2})$, for any $\delta \in K^{\circ\circ}$. Thus we see that if $X_1 \subset X_2$ are annuli or intervals it does not necessarily follow that $\mathcal{O}_{X_2} \subset \mathcal{O}_{X_1}$. However the following are clear. If $I_1 \subset I_2$ are intervals, then $\mathcal{O}_{I_2} \subset \mathcal{O}_{I_1}$. If $0 < \delta \in K^{\circ\circ}$, $0 < r \in K^\circ \setminus K^{\circ\circ}$, $r < 1$, and $A = A(0, \delta, 1)$, $I = I(\frac{1+r}{2}, \frac{1-r}{2})$ then $\mathcal{O}_A \subset \mathcal{O}_I$. If $A_1 \subset A_2$ are annuli that have the same center, then $\mathcal{O}_{A_2} \subset \mathcal{O}_{A_1}$. If $0 < \delta << c$ and $0 < r < \frac{c}{\alpha}$ for some $1 < \alpha \in \mathbb{R}$, and $I = I(c, r) \subset A(0, \delta, 1) = A$, then $\mathcal{O}_A \subset \mathcal{O}_I$. (Writing $x = c - y$, $|y| \le r$ we see that $\frac{\delta}{x} = \frac{\delta}{c-y} = \frac{\delta/c}{1-y/c} + \frac{\delta}{c}\sum(\frac{y}{c})^k = \frac{\delta}{c}\sum(\frac{r}{c}\frac{y}{r})^k$).

Restating Corollary 2.15 we have

Corollary 3.5. *If $f \in \mathcal{O}_{I(c,r)}$ there is a polynomial $P \in K[\xi]$ and a unit $U \in \mathcal{O}_{I(c,r)}$ such that $f(\xi) = P(\xi) \cdot U(\xi)$.*

Lemma 3.6. *If $\varepsilon < N\delta$ for some $N \in \mathbb{N}$, then there is a covering of $A(c, \delta, \varepsilon)$ by finitely many intervals I_j such that for every $f \in \mathcal{O}_{A(c,\delta,\xi)}$ and each j, $f \in \mathcal{O}_{I_j}$.*

Proof. Use Lemma 3.3 or reduce directly to the case $\varepsilon = 1$, $0 < \delta = r \in \mathbb{R}$ and the two intervals $[-1, -r] = I(-\frac{1+r}{2}, \frac{1-r}{2})$ and $[r, 1] = I(\frac{1+r}{2}, \frac{1-r}{2})$. □

The following Lemma is key for proving *o*–minimality.

Lemma 3.7. *Let $f \in A(c, \delta, \varepsilon)$. There are finitely many intervals and annuli X_j that cover $A(c, \delta, \varepsilon)$, polynomials P_j and units $U_j \in \mathcal{O}_{X_j}$ such that for each j we have $f|_{X_j} = (P_j \cdot U_j)|_{X_j}$.*

Proof. By the previous lemma, we may assume that $c = 0$, $\varepsilon = 1$ and $\delta \in K^{\circ\circ}$ (i.e. δ is infinitesimal, say $\delta = t^\gamma$ for some $\gamma > 0$). Let

$$f(x) = g\left(\frac{\delta}{x}\right) + h(x) \quad \text{with} \quad g(\xi),\ h(\xi) \in \mathcal{R}_1,\ g(0) = 0,$$

and

$$g(\xi) = \sum_{i \in I \subset \mathbf{N}} t^{\alpha_i} \xi^{n_i} g_i(\xi),$$

with $n_i > 0$, $g_i(0) \neq 0$, $g_i \in A_{1,\alpha}$ for some $\alpha > 1$. Observe that

$$xg\left(\frac{\delta}{x}\right) = \sum_{i \in I} (t^{\alpha_i}\delta)\left(\frac{\delta}{x}\right)^{n_i - 1} g_i\left(\frac{\delta}{x}\right).$$

Hence (see Remark 3.1) for suitable $n \in \mathbb{N}$, absorbing the constant terms into h, we have

$$x^n f(x) = \overline{g}\left(\frac{\delta}{x}\right) + \overline{h}(x)$$

where $\overline{g}(0) = 0$ and $\|\overline{g}\| < \|\overline{h}\|$. (For use in Section 4, below, note that this argument does not use that K is complete or of rank 1.) Multiplying by a constant, we may assume that $\|\overline{h}\| = 1$. Let

$$\overline{g}(\xi) = \sum_{i \in \overline{I}} t^{\beta_i} \xi^{m_i} \overline{g}_i(\xi), \text{ with } m_i > 0 \text{ and } \beta_i > 0 \text{ for each } i,$$

and

$$\overline{h}(\xi) = \xi^{k_0}\overline{h}_0(\xi) + \sum_{i \in J \subset \mathbf{N} \setminus 0} t^{\gamma_i} \overline{h}_i(\xi), \text{ with } \overline{h}_0(0) \neq 0 \text{ and the } \gamma_i > 0 \text{ increasing.}$$

Since $\|\overline{g}\| = \|t^{\beta_0}\| < 1$ there is a δ' with $\delta \leq \delta' \in K^{\circ\circ}$ (i.e. $\|\delta'\| < 1$) such that $\|t^{\gamma_1}\| < \|(\delta')^{k_0}\|$ and $\|\overline{g}\| < \|(\delta')^{k_0}\|$. Splitting off some intervals of the form $I(\frac{-1-r}{2}, \frac{-1+r}{2}) = [-1,-r]$ or $I(\frac{1+r}{2}, \frac{1-r}{2}) = [r,1]$ for $r > 0$, $r \in \mathbb{R}$ (on which the result reduces to Corollary 3.5 by Remark 3.4) and renormalizing, we may assume that $\overline{h}_0$, is a unit in $A_{1,\alpha}$, for some $\alpha > 1$. So

$$
\begin{aligned}
x^n f(x) &= \overline{g}(\frac{\delta}{x}) + \overline{h}(x) \\
&= \overline{h}_0(x)x^{k_0}\Big[1 + \sum_{i=1}^{\infty} (\frac{\delta'}{x})^{k_0} \frac{t^{\gamma_i}}{(\delta')^{k_0}} \frac{\overline{h}_i(x)}{\overline{h}_0(x)} + \frac{1}{\overline{h}_0(x)} (\frac{1}{\delta'})^{k_0} (\frac{\delta'}{x})^{k_0} g(\frac{\delta}{\delta'} \frac{\delta'}{x})\Big].
\end{aligned}
$$

By our choice of δ' and Remark 3.1 the quantity in square brackets is a (strong) unit. Hence we have taken care of an annulus of the form $A(0,\delta',1)$ for some δ' with $|\delta| \leq |\delta'|$ and $\|\delta'\| < 1$.

Observe that the change of variables $y = \frac{\delta}{x}$ interchanges the sets $\{x\colon \|x\| = \|\delta\|$ and $\delta \leq |x|\}$ and $\{y\colon \|y\| = 1$ and $|y| \leq 1\}$. Hence, as above, there is a $\delta'' \in K^{\circ\circ}$ with $\|\delta\| < \|\delta''\|$ and a covering of the annulus $\delta \leq |x| \leq \delta''$ by finitely many intervals and annuli with the required property.

It remains to treat the annulus $\delta'' \leq |x| \leq \delta'$. Using the terminology of [LR3], observe that on the much bigger *affinoid* annulus $\|\delta''\| \leq \|x\| \leq \|\delta'\|$ the function f is strictly convergent, indeed even overconvergent. Hence, as in [LR3] Lemma 3.6, on this affinoid annulus we can write

$$f = \frac{P(x)}{x^\ell} \cdot U(x)$$

where $P(x)$ is a polynomial and $U(x)$ is a strong unit (i.e. $\|U(x) - 1\| < 1$.) □

Corollary 3.8. *If X is an interval or an annulus and $f \in \mathcal{O}_X$, then the set $\{x \in X\colon f(x) \geq 0\}$ is semialgebraic (i.e. a finite union of (closed) intervals).*

Proof. This is an immediate corollary of 3.5 and 3.7 since units don't change sign on intervals and since an annulus has two intervals as connected components. □

Definition 3.9. *For $c = (c_1, \ldots, c_n)$, $r = (r_1, \ldots, r_n)$ we define the poly–interval $I(c,r) := \{x \in K^n\colon |x_i - c_i| \leq r_i,\ i = 1, \ldots, n\}$. This also defines the corresponding polydisc in $(K_{alg})^n$. The ring of analytic functions on this poly-interval (or polydisc) is*

$$\mathcal{O}_{I(c,r)} := \left\{ f\left(\frac{x_1 - c_1}{r_1}, \ldots, \frac{x_n - c_n}{r_n}\right) : f \in \mathcal{R}_n \right\}.$$

Lemma 3.10. *Let $\alpha, \beta > 1, F(\eta_1, \ldots, \eta_m) \in \mathcal{R}_{m,\alpha}$ and $G_j(\xi_1, \cdots, \xi_n) \in \mathcal{R}_{n,\beta}$ with $\|G_j\| \leq 1$ for $j = 1, \ldots, m$. Let $X = \{x \in [-1,1]^n\colon |G_j(x)| \leq 1$ for $j = 1, \ldots, m\}$. There are (finitely many) $c_i = (c_{i1}, \ldots, c_{in}) \in \mathbb{R}^n$, $\varepsilon_i \in \mathbb{R}$, $\varepsilon_i > 0$ with $\varepsilon_i < |c_{ij}|$ if $c_{ij} \neq 0$ such that the (poly) intervals $I_i = \{x \in K\colon |x_j - c_{ij}| < \varepsilon_i$ for $j = 1, \ldots, n\}$ cover X, $|G_j(x)| < \alpha$ for all $x \in I_i, j = 1, \ldots, m$, and there are $H_i \in \mathcal{O}_{I_i}$ such that*

$$F(G_1, \ldots, G_m)|_{I_i} = H_i|_{I_i}.$$

Proof. Use the compactness of $[-1,1]^n \cap \mathbb{R}^n$ and the following facts. If $\|G_j\| = 1$ then $|G_j(x) - G_{j0}(x)|$ is infinitesimal for all $x \in K^\circ$, where G_{j0} is the top slice of G_j. If $\|G_j\| < 1$ then $|G_j(x)| \in K^{\circ\circ}$ for all $x \in [-1,1]^n$. □

Corollary 3.11. *(i) Let I be an interval, $F(\eta_1,\ldots,\eta_m)\in\mathcal{R}_m$, and $G_j\in\mathcal{O}_I$ for $j=1,\ldots,m$. Then there are finitely many intervals I_i covering I and functions $H_i\in\mathcal{O}_{I_i}$ such that for each i*

$$F(G_1,\ldots,G_m)|_{I_i}=H_i|_{I_i}$$

(ii) Let A be an annulus, $F(\eta_1,\ldots,\eta_m)\in\mathcal{R}_m$, and $G_j\in\mathcal{O}_A$ for $j=1,\ldots,m$. Then there are finitely many X_i, each an interval or an annulus, covering A and $H_i\in\mathcal{O}_{X_i}$ such that for each i

$$F(G_1,\ldots,G_m)|_{X_i}=H_i|_{X_i}.$$

Proof. Part (i) reduces to Lemma 3.10 once we see that if $\|G_j\|>1$ we can use Corollary 2.15 to restrict to the subintervals of I around the zeros of G_j on which $|G_j|\le C$ for some $1<C\in\mathbb{R}$. On the rest of I, $F(G_1,\ldots,G_m)$ is zero.

For (ii), we may assume that $A=A(0,\delta,1)$ with δ infinitesimal and that $G_j(x)=G_{j_1}(\frac{\delta}{x})+G_{j_2}(x)$. As in (i), we may reduce to the case that $\|G_j\|\le 1$, using Lemma 3.7 instead of Corollary 2.15, and using Lemma 3.6 and Remark 3.4. Apply Lemma 3.10 to the functions F and $G'_j(\xi_1,\xi_2)=G_{j_1}(\xi_1)+G_{j_2}(\xi_2)$. The case $c=(0,0)$ gives us the annulus $|\frac{\delta}{x}|\le\varepsilon$, $|x|\le\varepsilon$ i.e. $\frac{\delta}{\varepsilon}\le|x|\le\varepsilon$. The case $c=(0,c_2)$ with $c_2\neq 0$ gives us $\frac{\delta}{|x|}\le\varepsilon$ and $|x-c_2|\le\varepsilon$ with $\varepsilon<|c_2|$. This is equivalent to $|x-c_2|\le\varepsilon$ since $\varepsilon,c_1\in\mathbb{R}$, and hence equivalent to $-\varepsilon\le x-c_2\le\varepsilon$ or $c_2-\varepsilon\le x\le c_2+\varepsilon$ which is an interval bounded away from 0. The case $c=(c_1,0)$ gives us $|\frac{\delta}{x}-c_1|\le\varepsilon$, $|x|\le\varepsilon$ (since $\varepsilon<|c_1|$) which is equivalent to $|\frac{\delta}{x}-c_1|\le\varepsilon$ or $c_1-\varepsilon\le\frac{\delta}{x}\le c_1+\varepsilon$ or (considering the case $c_1>0$, the case $c_1<0$ is similar) $\frac{\delta}{c_1+\varepsilon}\le x\le\frac{\delta}{c_1-\varepsilon}$ which is part of an annulus that can be reduced to intervals using Lemma 3.6. The case $c=(c_1,c_2)$ with both $c_1,c_2\neq 0$ is vacuous since either x or $\frac{\delta}{x}$ is infinitesimal on A and $\varepsilon<|c_1|,|c_2|$. □

Lemma 3.12. *Let X be an interval or an annulus and let $f,g\in\mathcal{O}_X$. There are finitely many subintervals and subannuli $X_i\subset X$, $i=1,\ldots,\ell$ such that*

$$\{x\in X\colon |f(x)|\le|g(x)|\}\subset\bigcup X_i\subset X,$$

and, except at finitely many points,

$$\frac{f}{g}|_{X_i}\in\mathcal{O}_{X_i}.$$

Proof. We consider the case that X is an interval, I, and may take $I=I(0,1)=[-1,1]$. We may assume by Corollary 3.5 that f and g have no common zero. If g has no zeros close to $I(0,1)$, we are done by Lemma 3.3. Let $\alpha_1,\ldots,\alpha_n$ be the distinct elements α of $[-1,1]\cap\mathbb{R}$ such that g has at least one zero close to (i.e. within an infinitesimal of) α. Breaking into subintervals and making changes of variables we may assume that $n=1$ and that $\alpha_1=0$. Again making a change of variables (over K) we may assume that g has at least one zero with zero "real" part, i.e. of the form $a=\sqrt{-1}\alpha$ for some $\alpha\in K$. Let N denote the number of zeros of $f\cdot g$ close to 0 in $I(0,1)$. Let $\delta=3\cdot\max\{|x|\colon x\in K_{alg}$ close to 0 and $f(x)\cdot g(x)=0\}$. If $\delta=0$, then $a=\alpha=0$ and there is no other zero of $f\cdot g$ close to zero in $I(0,1)$. Then there is a $\delta'>0$ such that for $|x|<\delta'$ we have $|g(x)|<|f(x)|$. Then the interval $I(0,\delta')$ drops away, and on the annulus $A(0,\delta',1)$ the function g is a unit. If $\delta>0$, we consider the interval $I(0,\delta)$ and the annulus $A(0,\delta,1)$ separately, and proceed by induction on N. So suppose $\delta > 0$ and $f \cdot g$ has N zeros close to 0 in $I(0,1)$. Let α be as above. If $\alpha \sim \delta$ then the zero $a = \sqrt{-1}\alpha$ is not close to $I(0,\delta)$ and by restricting to $I(0,\delta)$ we have reduced N. If $|\alpha| << \delta$ then this zero is close to 0 in $I(0,\delta)$, but the largest zeros (those of size $\delta/3$) are not close to 0 in $I(0,\delta)$, and hence restricting to $I(0,\delta)$ again reduces N.

It remains to consider the case of the annulus $A(0,\delta,1) = \{x\colon \delta \le |x| \le 1\}$ where all the zeros of g are within $\delta/3$ of 0. By Lemma 3.7 we may assume that $g(x) = P(x)\cdot U(x)$ where U is a unit and all the zeros of P are within $\delta/3$ of 0. Let α_i, $i = 1,\ldots,\ell$ be these zeros. For $|x| \ge \delta$ we may write $\frac{1}{x-\alpha_i} = \frac{1}{x}\frac{1}{1-\alpha_i/x} = \frac{1}{x}\sum_{j=0}^{\infty}(\frac{\alpha_i}{x})^j = \frac{1}{x}\sum_{j=0}^{\infty}(\frac{\alpha_i}{\delta})^j(\frac{\delta}{x})^j$ and $|\frac{\alpha_i}{\delta}| \le \frac{1}{3}$. Hence $\frac{f}{g} \in \mathcal{O}_{A(0,\delta,1)}$. This completes the case that X is an interval.

The case that X is an annulus is similar – one can cover X with finitely many subannuli X_i and subintervals Y_j so that for each i $\;g|_{X_i}$ is a unit in $\mathcal{O}_{X_i}$ and for each j $\;f|_{Y_j}, g|_{Y_j} \in \mathcal{O}_{Y_j}$. □

From Corollary 3.11 and Lemma 3.12 we now have by induction on terms:

Proposition 3.13. *Let $f_1,\ldots,f_\ell$ be $\mathcal{L}$–terms in one variable, x. There is a covering of $[-1,1]$ by finitely many intervals and annuli X_i such that except for finitely many values of x, we have for each i and j that $f_i|_{X_j} \in \mathcal{O}_{X_j}$ (i.e. $f_i|_{X_j}$ agrees with an element of $\mathcal{O}_{X_j}$ except at finitely many points of X_j).*

This, together with Corollary 3.8 gives

Theorem 3.14. *K is o–minimal in $\mathcal{L}$.*

4. Further extensions

In this section we give extensions of the results of Sections 2 and 3 and the results of [LR3].

Let G be an (additive) ordered abelian group. Let t be a symbol. Then t^G is a (multiplicative) ordered abelian group. Following the notation of [DMM1] and [DMM2] (but not [DMM3] or [LR2]) we define $\mathbb{R}((t^G))$ to be the maximally-complete valued field with additive value group G (or multiplicative value group t^G) and residue field $\mathbb{R}$. So

$$\mathbb{R}((t^G)) := \Big\{ \sum_{g\in I} a_g t^g \colon a_g \in \mathbb{R} \text{ and } I \subset G \text{ well-ordered}\Big\}.$$

We shall be a bit sloppy about mixing the additive and multiplicative valuations. $I \subset G$ is well-ordered exactly when $t^I \subset t^G$ is reverse well-ordered. The field K of Puiseux series, or its completion, is a proper subfield of $\mathbb{R}_1 := \mathbb{R}((t^{\mathbb{Q}}))$. Considering $G = \mathbb{Q}^m$ with the lexicographic ordering, we define

$$\mathbb{R}_m := \mathbb{R}((t^{\mathbb{Q}^m})).$$

It is clear that if $G_1 \subset G_2$ as ordered groups, then $\mathbb{R}((t^{G_1})) \subset \mathbb{R}((t^{G_2}))$ as valued fields. Also, $\mathbb{R}((t^G))$ is Henselian and, if G is divisible, then $\mathbb{R}((t^G))$ is real-closed. We shall continue to use $<$ for the corresponding order on $\mathbb{R}((t^G))$.

In analogy with Section 2, we define

Definition 4.1.

$$\begin{aligned}
\mathcal{R}_{n,\alpha}(G) &:= A_{n,\alpha}\widehat{\otimes}^*_{\mathbb{R}}\mathbb{R}((t^G)) \\
&:= \Big\{\sum_{g\in I} f_g t^g\colon f_g \in A_{n,\alpha} \textit{ and } I \subset G \textit{ well ordered}\Big\} \\
\mathcal{R}_n(G) &:= \bigcup_{\alpha>1} \mathcal{R}_{n,\alpha}(G) \\
\mathcal{R}(G) &:= \bigcup_{n}\bigcup_{\alpha>1} \mathcal{R}_{n,\alpha}(G).
\end{aligned}$$

As in Section 2, the elements of $\mathcal{R}_n(G)$ define functions from $I^n \subset \mathbb{R}((t^G))^n$ to $\mathbb{R}((t^G))$. Indeed this interpretation is a ring endomorphism. In other words, the field $\mathbb{R}((t^G))$ has analytic $\mathcal{R}(G)$–structure. (See [CLR] and especially [CL] for more about fields with analytic structure.) The elements of $\mathcal{R}_n(G)$ are interpreted as zero on $\mathbb{R}((t^G))^n \setminus I^n$. Let

$$\mathcal{L}_G := \langle +, \cdot, ^{-1}, 0, 1, <, \mathcal{R}(G)\rangle,$$

so $\mathbb{R}((t^G))$ is an $\mathcal{L}_G$-structure. Indeed, if $G_1 \subset G_2$, then $\mathbb{R}((t^{G_2}))$ is an $\mathcal{L}_{G_1}$-structure.

Theorem 4.2. *The Weierstrass Preparation Theorem (Theorem 2.5) and the Strong Noetherian Property (Theorem 2.10) hold with $\mathcal{R}$ replaced by $\mathcal{R}(G)$.*

Proof. Only minor modifications to the proofs of Theorems 2.5 and 2.10 are needed. □

The arguments of Sections 2 and 3 show

Theorem 4.3. *If G is divisible then $\mathbb{R}((t^G))$ admits quantifier elimination and is o–minimal in $\mathcal{L}_G$.*

Corollary 4.4. *If $G_1 \subset G_2$ are divisible, then $\mathbb{R}((t^{G_1})) \prec \mathbb{R}((t^{G_2}))$ in $\mathcal{L}_{G_1}$.*

We shall show in Section 5 that, though for $m < n$ we have $\mathbb{R}_m \prec \mathbb{R}_n$ in $\mathcal{L}_{\mathbb{Q}^m}$, there is a sentence of $\mathcal{L}_{\mathbb{Q}^n}$ that is true in $\mathbb{R}_n$ but is not true in any *o*–minimal expansion of $\mathbb{R}_m$.

The results of [LR3] also extend to this more general setting.

Definition 4.5. *We define the ring of* *strictly convergent power series over $\mathbb{R}((t^G))$ *as*

$$\mathbb{R}((t^G))^*\langle\xi\rangle := \{\sum_{g\in I} a_g(\xi)t^g\colon a_g(\xi) \in \mathbb{R}[\xi] \textit{ and } I \textit{ well ordered}\},$$

and the subring of *overconvergent power series over $\mathbb{R}((t^G))$ *as*

$$\begin{aligned}
\mathbb{R}((t^G))^*\langle\langle\xi\rangle\rangle &:= \{f\colon f(\gamma\xi) \in \mathbb{R}((t^G))^*\langle\xi\rangle \textit{ for some } \gamma \in \mathbb{R}((t^G)), \|\gamma\| > 1\}, \\
\mathcal{R}(G)_{over} &:= \bigcup_n \mathbb{R}((t^G))^*\langle\langle\xi_1, \cdots, \xi_n\rangle\rangle,
\end{aligned}$$

and the corresponding overconvergent language *as*

$$\mathcal{L}_{G,over} := \langle +, \cdot, ^{-1}, 0, 1, <, \mathcal{R}(G)_{over}\rangle.$$

As in [LR3] we have

Theorem 4.6. *If G is divisible then $\mathbb{R}((t^G))$ admits quantifier elimination and is o–minimal in $\mathcal{L}_{G,over}$.*

Of course the o–minimality follows immediately from Theorem 4.3.

5. Extensions of the example of Hrushovski and Peterzil [HP]

In this section we show that with minor modifications, the idea of the example of [HP] can be iterated to give a nested family of examples. This relates to a question of Hrushovski and Peterzil whether there exists a small class of o–minimal structures such that any sentence, true in some o–minimal structure, can be satisfied in an expansion of a model in the class. Combining with expansions with the exponential function, one perhaps can elaborate the tower of examples further.

Consider the functional equation

$$(*) \qquad F(\beta z) = \alpha z F(z) + 1,$$

and suppose that F is a "complex analytic" solution for $|z| \le 1$. By this we mean that, writing $z = x + \sqrt{-1}y$, $F(z) = f(x,y) + \sqrt{-1}g(x,y)$, $F(z)$ is differentiable as a function of z. This is a definable condition on the two "real" functions, f, g of the two "real" variables x, y. Then

$$F(z) = \sum_{k=0}^{\infty} a_k z^k$$

where

$$a_k = \frac{\alpha^k}{\beta^{\frac{k(k+1)}{2}}}.$$

(α, and β are parameters).

By this we mean that for each $n \in \mathbb{N}$ there is a constant A_n such that

$$(**) \qquad |F(z) - \sum_{k=0}^{n} a_k z^k| \le A_n |z^{n+1}|$$

is true for all z with $|z| \le 1$. Indeed, by [PS] Theorem 2.50, one can take $A_n = C \cdot 2^{n+1}$, for C a constant independent of n.

Consider the following statement: $F(z)$ is a complex analytic function (in the above sense) on $|z| \le 1$ that satisfies $(*)$; the number $\beta > 0$ is within the radius of convergence of the function $f(z) = \sum_{n=1}^{\infty} (n-1)! z^n$ and $\alpha > 0$.

This statement is not satisfiable by any functions in any o–minimal expansion of the field of Puiseux series K_1, or the maximally complete field $\mathbb{R}_1 = \mathbb{R}((t^{\mathbb{Q}}))$, because, if it were, we would have $\|\beta\| = \|t^\gamma\|$, $\|\alpha\| = \|t^\delta\|$, for some $0 < \gamma, \delta \in \mathbb{Q}$, and for suitable choice of n the condition $(**)$ would be violated. On the other hand, if we choose $\alpha, \beta \in \mathbb{R}_2$ with $ord(\alpha) = (1,0)$ and $ord(\beta) = (0,1)$ then $\Sigma a_k z^k \in \mathbb{R}_2\langle\langle z\rangle\rangle^*$ satisfies the statement on $\mathbb{R}_2$.

This process can clearly be iterated to give, in the notation of Section 4,

Proposition 5.1. *For each m there is a sentence of $\mathcal{L}_{\mathbb{Q}^m}$ true in $\mathbb{R}_m$ but not satisfiable in any o–minimal expansion of $\mathbb{R} = \mathbb{R}_0, \mathbb{R}_1, \ldots, \mathbb{R}_{m-1}$.*

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