Series

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    Just because the n-th term goes to zero does not mean the series necessarily converges. We need more tests to determine if a series converges.

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- Sometimes you can not apply it directly: $\sum_{i=0}^{\infty} \frac{8n!}{e^{n^2}}$

Convergence of Series

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Convergence of Series

- What tests you can use after the first step (test for divergence)?
- Geometric Series
- Integral Test
- p-series
- Direct Comparison Test
- Limit Comparison Test (Taylor series)
- Alternating Series Test
- Ratio Test
- Root Test

Geometric Series

- This formula not only tell you the convergence, but also can give you the **exact value** of series or function.
- Find the sum of series $\sum\limits_{i=0}^{\infty}$
- To what simple function $f(x)$ does $\sum\limits_{n=1}^{\infty} \frac{x^n}{n2^n}$ converge near $x = 0$?
- Find the sum of the series $\sum\limits_{n=1}^{\infty} \frac{n}{2^n}$

Integral Test

- When a_n contains some $\ln(n)$ or $\ln(\ln(n))$ terms
- Determine whether the series is convergent or divergent

$$\sum_{n=2}^{\infty} \frac{3}{\ln(n)n}$$

$$\sum_{n=2}^{\infty} \frac{3}{\ln(\ln(n))\ln(n)n}$$

$$\sum_{n=2}^{\infty} \frac{3}{\ln^2(n)n}$$

Note: $\lim_{n\to\infty} \ln(n) = \infty$, $\frac{1}{\infty} = 0$

Comparison Test

- Combine the comparison test with $p-$series and Taylor series.
- Determine whether the series is convergent or divergent

$$\sum_{n=1}^{\infty} \frac{\arctan(n)}{n^{1.6}}$$

$$\sum_{n=2}^{\infty} \frac{2n + 7^n}{3n + 8^n}$$

$$\sum_{n=2}^{\infty} \frac{2n + (0.5c)^n}{3n + c^n}$$

Alternating Series Test

- a_n has the alternating term ($a_n = (-1)^n b_n$ where $b_n > 0$)
- Determine whether the series is conditionally convergent, absolutely convergent or divergent.

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{4n+5}$$

$$\sum_{n=2}^{\infty} \frac{(-1)^n(3n+2)}{4n^5+n^3}$$

$$\sum_{n=2}^{\infty} \frac{(-1)^n 3n}{4n^2+5}$$

- Estimate the sum with given error

$$|S - S_n| < b_{n+1}$$

Ratio Test

- a_n has factorial term like $n!$, $(2n)!$ or x^n.
- Determine whether the series is convergent or divergent

$$\sum_{n=1}^{\infty} \frac{n^n}{n!}$$

$$\sum_{n=2}^{\infty} \frac{n!}{e^{n^2}}$$

$$\sum_{n=2}^{\infty} \frac{(n!)^2}{(kn)!}$$

Root Test

- a_n has term like x^n, x^{n^2}.
- Determine whether the series is convergent or divergent

$$\sum_{n=1}^{\infty} \left(\frac{2n+3}{3n+2} \right)^n$$

$$\sum_{n=2}^{\infty} 7(1 + \frac{1}{n})^{n^2}$$

Power Series

- Series $\sum a_n(x)$ is a function $f(x)$.
  $\sum x^n = \frac{1}{1-x}$
- Series $\sum a_n$ is a number.
  $\sum (1/2)^n = \frac{1}{1-1/2} = 2$

Thus, for series containing variable x, there are convergence interval or radius.

And for series only witout x, there are only convergent or divergent.

Taylor Series

If f has a power series representation at a, that is, if

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n, \ |x-a| < R$$

then its coeffients are given by the formula $c_n = \frac{f^{(n)}(a)}{n!}$.

- Find a Taylor or Maclaurin series for a function
- Find a binomial series.
- Use a basic list of Taylor series to find other Taylor series or evaluate the limit ot integrand.

Taylor Series

Find a Taylor or Maclaurin series for a function.

- Use a list to find the pattern after successive differentiation of $f(x)$
- Find the associated radius of convergence R. (Ratio test)
- Find the Taylor series for $f(x) = \frac{4}{x}$ centered at the given value of $a = 2$

Taylor Series

Find a binomial series.

- Use the binomial series to expand the function as a power series.
- Follow up question wich is based on the answer you have computed.
- (a1)Use the binomial series to expand $\frac{7}{\sqrt{(1-x^2}}$
  (b1)Use part (a1) to find the Maclaurin series for $7\arcsin(x)$
- (a2)Use the binomial series to expand $\frac{5}{\sqrt[4]{1+x}}$
  (b2)Use part (a2) to estimate $\frac{5}{\sqrt[4]{1.1}}$ correct to three decimal places.

Taylor Series

Use a basic list of Taylor series to find other Taylor series or evaluate the limit ot integrand.

- Given a table of Maclaurin series to obtain the Maclaurin series for given function.

$$f(x) = 3\sin 2(x)$$

- Evaluate the indefinite integral as an infinite series.

$$\int \frac{e^x - 1}{7x}\,dx$$

- Use series to evaluate the limit.

$$\lim_{x\to 0} \frac{1-\cos(5x)}{1+5x-e^{5x}}$$

Work

Since work=force×distance, we can rewrite the work as $W = \int_a^b f(x)dx$, when the force is a function about distance x.

- Stretch a spring.
  - Hooke's Law $f(x) = k(x - x_0) = k\delta x$.
- Pump water out of the tank.
  - Find the volume of layer and distance to the top.

Vector

- Dot Product $a \cdot b = |a||b|\cos(\theta)$
  - $a \cdot b = 0$ if and only if a and b are orthogonal.
- Cross Product $c = a \times b =< \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}, - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} >, \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}.$
  - Right-hand Rule to find the direction of c.
  - $a \times b = 0$ if and only if a and b are parallel.
- Triple Products $a \cdot (b \times c)$
  - $V = |a \cdot (b \times c)|$

Arc Length

- Given $y = f(x)$
  - $L = \int_a^b \sqrt{1 + [f'(x)]^2} dx$
- Given $x = g(y)$
  - $L = \int_c^d \sqrt{1 + [g'(y)]^2} dy$
- Given $x = x(t)$, $y = y(t)$
  - $L = \int_a^b \sqrt{[x'(t)]^2 + [y'(t)]^2} dt$
- Given $r = f(\theta)$
  - $L = \int_a^b \sqrt{r^2 + (\frac{dr}{d\theta})^2} d\theta$