Selected Homework Problems

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1 Method of variation of parameters

1.1 Problem 1

Use the method of variation of parameters to determine a particular solution to the given equation.

$$y''' - 3y'' + 4t = e^{2x}$$

1.2 Solution

Factoring the auxiliary polynomial, $r^3 - 3r^2 + 4$, yields

$$r^3 - 3r^2 + 4 = (r+1)(r-2)^2.$$

Therefore, $y_1 = e^{-x}$, $y_2 = e^{2x}$ and $y_3 = xe^{x}$ form a fundamental solution set. According to method of variation of parameters method, we seek for a particular solution of the form

$$\begin{aligned} y_p(x) &= v_1(x)y_1(x) + v_2(x)y_2(x) + v_3(x)y_3(x) \\ &= v_1(x)e^{-x} + v_2(x)e^{2x} + v_3(x)xe^{2x}. \end{aligned}$$

To find functions v_j's, we need four determinants: the Wronskian $W[y_1, y_2, y_3](x)$, $W_1(x)$, $W_2(x)$, $W_3(x)$ given in formula (10) the page 277 in the text. Thus, we compute

$$W[e^{-x}, e^{2x}, xe^{2x}](x) = \begin{bmatrix} e^{-x} & e^{2x} & xe^{2x} \\ -e^{-x} & 2e^{2x} & (1+2x)e^{2x} \\ e^{-x} & 4e^{2x} & (4+4x)e^{2x} \end{bmatrix} = e^{-x}e^{2x}e^{2x} \begin{bmatrix} 1 & 1 & x \\ -1 & 2 & (1+2x) \\ 1 & 4 & (4+4x) \end{bmatrix} = 9e^{3x},$$

$$W_1(x) = (-1)^{3+1}W[e^{2x}, xe^{2x}](x) = \begin{bmatrix} e^{2x} & xe^{2x} \\ 2e^{2x} & (1+2x)e^{2x} \end{bmatrix} = e^{4x},$$

$$W_2(x) = (-1)^{3+2}W[e^{-x}, xe^{2x}](x) = \begin{bmatrix} e^{-x} & xe^{2x} \\ -e^{-x} & (1+2x)e^{2x} \end{bmatrix} = -(1+3x)e^{x},$$

$$W_3(x) = (-1)^{3+3}W[e^{-x}, e^{2x}](x) = \begin{bmatrix} e^{-x} & e^{2x} \\ -e^{-x} & 3e^{2x} \end{bmatrix} = 3e^{x}.$$

Substituting these expressions into the formula (11), we obtain

$$v_1(x) = \int \frac{g(x)W_1(x)}{W[e^{-x}, e^{2x}, xe^{2x}]}dx = \int \frac{e^{2x}e^{4x}}{9e^{3x}}dx = \frac{1}{27}e^{3x},$$

$$v_2(x) = \int \frac{g(x)W_2(x)}{W[e^{-x}, e^{2x}, xe^{2x}]} dx = \int \frac{-e^{2x}(1+3x)e^x}{9e^{3x}} dx = -\frac{x}{9} - \frac{x^2}{6},$$
$$v_3(x) = \int \frac{g(x)W_3(x)}{W[e^{-x}, e^{2x}, xe^{2x}]} dx = \int \frac{e^{2x}e^{3x}}{9e^{3x}} dx = \frac{x}{3}.$$

Thus, the formula (12) in the text gives a particular solution that

$$y_p(x) = \frac{1}{27}e^{3x}e^{-x} - (\frac{x}{9} + \frac{x^2}{6})e^{2x} + \frac{x}{3}xe^{2x} = \frac{e^{2x}}{27} - \frac{xe^{2x}}{9} + \frac{x^2e^{2x}}{6}.$$

We note that the first two terms in $y_p(x)$ are solutions to the corresponding homogeneous equation. Thus, another (and simpler) answer is $y_p(x) = \frac{x^2e^{2x}}{6}$.

1.3 Problem 2

Find a general solution to the Cauchy-Euler equation

$$x^3y''' - 3xy' + 3y = x^4\cos(X), \quad x > 0.$$

1.4 Solution

First, we find a fundamental solution set for the corresponding homogeneous equation,

$$x^3y''' - 3xy' + 3y = 0.$$

We involve the procedure of solving Cauchy-Euler equation. We look for solutions of the form $y = x^r$. Substituting and simplifying yields

$$x^3r(r-1)(r-2)x^{r-3} - 3xrx^{r-1} + 3x^r = (r^3 - 3r^2 - r + 3)x^r = 0.$$

Factoring the auxiliary equation, we obtain

$$r^3 - 3r^2 - r + 3 = r^2(r-3) - (r-3) = (r-1)(r+1)(r-3) = 0.$$

Therefore, $r_1 = 1$, $r_2 = -1$, and $r_3 = 3$ and the functions

$$y_1(x) = x, y_2(x) = x^{-1}, y_3(x) = x^3$$

form a fundamental solution set for the homogeneous equation. Next, we apply the variation of parameters method to find a particular solution to the original equation of the form

$$y_p(x) = v_1(x)x + v_2(x)x^{-1} + v_3(x)x^3.$$

To find functions $v_1(x)$, $v_2(x)$, and $v_3(x)$m we compute

$$W[x, x^{-1}, x^3](x) = \begin{bmatrix} x & x^{-1} & x^3 \\ 1 & -x^{-2} & 3x^2 \\ 0 & 2x^{-3} & 6x \end{bmatrix} = -16,$$

$$W_1(x) = (-1)^{3+1}W[x^{-1}, x^3](x) = 4x,$$
$$W_2(x) = (-1)^{3+2}W[x, x^3](x) = -2x^3,$$
$$W_3(x) = (-1)^{3+3}W[x, x^{-1}](x) = -2x^{-1}.$$

Writing the given equation in standard form,

$$y''' - \frac{3}{x^2}y' + \frac{3}{x^3}y = x\cos(x),$$

we see that the nonhomogeneity term is $g(x) = x\cos(x)$. Then,

$$v_1(x) = \int \frac{x\cos(x)(4x)}{-16}dx = -\frac{1}{4}\int x^2\cos(x)dx = -\frac{1}{4}(x^2\sin(x) + 2x\cos(x) - 2\sin(x)),$$

$$v_2(x) = \int \frac{x\cos(x)(-2x^3)}{-16}dx = \frac{1}{8}(x^4\sin(x) + 4x^3\cos(x) - 12x^2\sin(x) - 24x\cos(x) + 24\sin(x)),$$

$$v_3(x) = \int \frac{x\cos(x)(-2x^{-1})}{-16}dx = \frac{1}{8}\int \cos(x)dx = \frac{1}{8}\sin(x).$$

Substituting these functions into $y_p(x)$ and simplifying yields

$$y_p(x) = -x\sin(x) - 3\cos(x) + 3x^{-1}\sin(x).$$

Thus, the answer is

$$y(x) = y_h(x) + y_p(x) = c_1xc_2x^{-1} + c_3x^3 - x\sin(x) - 3\cos(x) + 3x^{-1}\sin(x).$$

2 Matrix methods for linear system

2.1 RLC network

This problem is the hand-graded homework 36 (#21 in page 332).

2.2 Solution

Substituting $t = 0$ into the system we get remaining initial values. Namely $I_3(0) = 10/13$, $I_2(0) = -10/13$, $I_2'(0) = 5/2$. Differentiating the first two equations and replacing in the results $q_1' = I_1$ by $I_1 = I_2 + I_3$ yields the system

$$\begin{aligned} I_2'' &= -13I_2 - 13I_3 \\ I_3' &= -4I_2 - 4I_3. \end{aligned}$$

To obtain a system in normal form, we denote $x_1 = I_2$, $x_2 = I_2'$, and $x_3 = I_3$. Then,

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= -13x_1 - 13x_3 \\ x_3' &= -4x_1 - 4x_3 \end{aligned}$$

with initial conditions

$$x_1(0) = -\frac{10}{13}, x_2(0) = \frac{5}{2}, x_3(0) = \frac{10}{13}.$$

Solving for eigenvalues of A, we get

$$r_1 = 0, r_2 = -2 + 3i, r_3 = -2 - 3i.$$

Accordingly, we have eigenvectors

$$\boldsymbol{u_1} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \boldsymbol{u_2} = \begin{bmatrix} -2 \\ 13 \\ 4 \end{bmatrix} + i \begin{bmatrix} -3 \\ 0 \\ 0 \end{bmatrix}, \boldsymbol{u_3} = \begin{bmatrix} -2 \\ 13 \\ 4 \end{bmatrix} - i \begin{bmatrix} -3 \\ 0 \\ 0 \end{bmatrix}.$$

Denote $\boldsymbol{a} = \begin{bmatrix} -2 \\ 13 \\ 4 \end{bmatrix}$ and $\boldsymbol{b} = \begin{bmatrix} -3 \\ 0 \\ 0 \end{bmatrix}$, we have

$$\boldsymbol{x_1}(t) = \boldsymbol{u_1}, \boldsymbol{x_2}(t) = e^{-2t}(\boldsymbol{a}\cos(3t) - \boldsymbol{b}\sin(3t)), \boldsymbol{x_3}(t) = e^{-2t}(\boldsymbol{a}\sin(3t) + \boldsymbol{b}\cos(3t)).$$

And the general solution $\boldsymbol{x}(t) = c_1\boldsymbol{x_1}(t) + c_2\boldsymbol{x_2}(t) + c_2\boldsymbol{x_2}(t)$. Next, we utilize the initial conditions

$$c_1 = 0, c_2 = 5/26, c_3 = 5/39.$$

Therefore,

$$\boldsymbol{x}(t) = e^{-2t}\begin{bmatrix} (-10/13)\cos(3t) + (25/78)\sin(3t) \\ (5/2)\cos(3t) \\ (10/13)\cos(3t) + (20/39)\sin(3t) \end{bmatrix}$$

so that

$$\begin{aligned} I_2(t) = x_1(t) &= e^{-2t}\left(-\frac{10}{13}\cos(3t) + \frac{25}{78}\sin(3t)\right), \\ I_3(t) = x_3(t) &= e^{-2t}\left(\frac{10}{13}\cos(3t) + \frac{20}{39}\sin(3t)\right), \\ I_1(t) = I_2(t) + I_3(t) &= \frac{5}{6}e^{-2t}\sin(3t). \end{aligned}$$

2.3 Homogeneous Linear Systemss with Constant Coefficients

Solve the initial value problem

$$\boldsymbol{x}'(t) = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 4 & -4 & 5 \end{bmatrix}\boldsymbol{x}(t), \qquad \boldsymbol{x}(0) = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}.$$

Solution:
Step 1: Find the eigenvalues of A.

$$\begin{aligned} det|A - rI| &= \begin{bmatrix} 1-r & 2 & -1 \\ 1 & -r & 1 \\ 4 & -4 & 5-r \end{bmatrix} \\ &= (1-r)(r^2 - 5r + 4) - (10 - 2r - 4) + 4(2 - r) \\ &= (1-r)(r^2 - 5r + 4) - 6 + 2r + 8 - 4r \\ &= (1-r)(r^2 - 5r + 4) + 2(1-r) \\ &= (1-r)(r^2 - 5r + 6) \\ &= (1-r)(r-2)(r-3) = 0 \end{aligned}$$

Step 2: Find the eigenvectors accordingly.
For $r_1 = 1$, we set

$$(A - r_1 I)\boldsymbol{u_1} = 0,$$

then $\boldsymbol{u_1} = s \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$. Similarly, for $r_2 = 2$, $\boldsymbol{u_2} = s \begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix}$, and$r_3 = 3$, $\boldsymbol{u_3} = s \begin{bmatrix} -1 \\ 1 \\ 4 \end{bmatrix}$.

Step 3 : Write $\boldsymbol{x}(t)$ in the general form.
Apply following formula,

$$\boldsymbol{x}(t) = c_1 e^{r_1 t}\boldsymbol{u_1} + c_2 e^{r_2 t}\boldsymbol{u_2} + c_3 e^{r_3 t}\boldsymbol{u_3}.$$

We have

$$\begin{aligned} \boldsymbol{x}(t) &= c_1 e^t \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix} + c_3 e^{3t} \begin{bmatrix} -1 \\ 1 \\ 4 \end{bmatrix} \\ &= \begin{bmatrix} -e^t & -2e^{2t} & -e^{3t} \\ e^t & e^{2t} & e^{3t} \\ 2e^t & 4e^{2t} & 4e^{3t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \end{aligned}$$

Step 4 : Apply the initial condition.

$$\boldsymbol{x}(0) = \begin{bmatrix} -e1 & -2 & -1 \\ 1 & 1 & 1 \\ 2 & 4 & 4 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$$

Thus, $c_1 = 0$, $c_2 = 1$, $c_3 = -1$.