Subspaces, Basis, Dimension, and Rank

Definition. A *subspace* of $\mathbb{R}^n$ is any collection S of vectors in $\mathbb{R}^n$ such that

1. The zero vector $\vec{0}$ is in S.
2. If $\vec{u}$ and $\vec{v}$ are in S, then $\vec{u}+\vec{v}$ is in S (that is, S is *closed under addition*).
3. If $\vec{u}$ is in S and c is a scalar, then $c\vec{u}$ is in S (that is, S is *closed under multiplication by scalars*).

Remark. Property 1 is needed only to ensure that S is non-empty; for non-empty S property 1 follows from property 3, as $0\vec{a} = \vec{0}$.

Theorem 3.19. Let $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k$ be vectors in $\mathbb{R}^n$. Then span $(\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k)$ is a subspace of $\mathbb{R}^n$.

Subspaces Associated with Matrices

Definition. Let A be an $m \times n$ matrix.

1. The *row space* of A is the subspace row(A) of $\mathbb{R}^n$ spanned by the rows of A.
2. The *column space* of A is the subspace col(A) of $\mathbb{R}^m$ spanned by the columns of A.

If we need to determine if $\vec{b}$ belongs to col(A), this is actually the same problem as whether $\vec{b} \in$ span of the columns of A; see the method on p. 9.

If we need to determine if $\vec{b}$ belongs to row(A), then we can apply the same method as above to the columns $\vec{b}^T$ and col(A^T). Another method for the same task is described in Example 3.41 in the Textbook.

Theorem 3.20. Let B be any matrix that is row equivalent to a matrix A. Then row(B) =row(A).

See the theorem on p. 13.

Theorem 3.21. Let A be an $m \times n$ matrix and let N be the set of solutions of the homogeneous linear system $A\vec{x} = \vec{0}$. Then N is a subspace of $\mathbb{R}^n$.

Definition. Let A be an $m \times n$ matrix. The *null space* of A is the subspace of $\mathbb{R}^n$ consisting of solutions of the homogeneous linear system $A\vec{x} = \vec{0}$. It is denoted by null(A).

Theorem. Let B be any matrix that is row equivalent to a matrix A. Then null(B) =null(A).

This is the Fund. Th. on e.r.o.s, see p. 4.

E.g., the set $\{[x_1, x_2, x_3] \mid x_1 + x_2 + x_3 = 0\}$ is automatically a subspace of $\mathbb{R}^3$ — no need to verify those closedness properties 1, 2, 3, as this is the null space of the homogeneous system $x_1 + x_2 + x_3 = 0$ (consisting of one equation).

Basis

Definition. A *basis* for a subspace S of $\mathbb{R}^n$ is a set of vectors in S that

1. spans S and
2. is linearly independent.

Remark. It can be shown that this definition is equivalent to each of the following two definitions:

Definition′. A *basis* for a subspace S of $\mathbb{R}^n$ is a set of vectors in S that spans S and is minimal with this property (that is, any proper subset does not span S).

Definition″. A *basis* for a subspace S of $\mathbb{R}^n$ is a set of vectors in S that is linearly independent and is maximal with this property (that is, adding any other vector in S to this subset makes the resulting set linearly dependent).

Method for finding a basis of row(A). Reduce A to r.r.e.f. R by e.r.o.s. (We know row(A) = row(R).) The non-zero rows of R, say, $\vec{b}_1, \dots, \vec{b}_r$, form a basis of row($R$) = row($A$). Indeed, they clearly span row(R), as zero rows contribute nothing. The fact that the non-zero rows are linearly independent can be seen from columns with leading 1s: in a linear combination $\sum c_i \vec{b}_i$ the coordinate in the column of the 1st leading 1 is c_1, since there are only zeros above and below this leading 1; also the coordinate in the column of the 2nd leading 1 is c_2, since there are only zeros above and below this leading 1; and so on. If $\sum c_i \vec{b}_i = \vec{0}$, then we must have all $c_i = 0$.

Moreover, the same is true for any r.e.f. Q (not necessarily reduced r.e.f.): The non-zero rows of Q form a basis of row(Q)=row(A).

1st Method for finding a basis of col(A). Use the previous method applied to A^T.

2nd Method for finding a basis of col(A). We know that e.r.o.s do not alter the solution set of the homogeneous system $A\vec{x} = \vec{0}$. Every solution of it can be regarded as a dependence of the columns of A. Thus, after reducing by e.r.o.s to r.r.e.f. R, we shall have exactly the same dependences among the columns as for A. For R, the columns of leading 1s clearly form a basis of col(R). Then the corresponding columns of A will form a basis of col(A). (WARNING: col(R) $\neq$ col(A) in general!)

Again, the same is true for any r.e.f. Q (not necessarily reduced r.e.f.): the columns of leading entrie form a basis of col(Q) and therefore the corresponding columns of A form a basis of col(A).

Method for finding a basis of null(A). Express leading variables via free variables (independent parameters). Give these parameters values as in the columns of the identity matrix I_f, where f is the number of free variables, and compute the values of leading variables. The resulting f vectors form a basis of null(A).

E.g., suppose the general solution (=null(A)) is $\begin{bmatrix} s+2t+3u \\ s \\ t \\ 2u \\ u \end{bmatrix}$ (that is, the free variables are x_2, x_3, x_5, while leading are x_1, x_4). We give s, t, u the values $1, 0, 0$, then $0, 1, 0$, then $0, 0, 1$. The resulting vectors are

$$\begin{bmatrix} 1 \\ \mathbf{1} \\ \mathbf{0} \\ 0 \\ \mathbf{0} \end{bmatrix}; \quad \begin{bmatrix} 2 \\ \mathbf{0} \\ \mathbf{1} \\ 0 \\ \mathbf{0} \end{bmatrix}; \quad \begin{bmatrix} 3 \\ \mathbf{0} \\ \mathbf{0} \\ 2 \\ \mathbf{1} \end{bmatrix},$$ which form a basis of null(A).

Dimension and Rank

Theorem 3.23. The Basis Theorem

Let S be a subspace of $\mathbb{R}^n$. Then any two bases for S have the same number of vectors.

Warning: there is blunder in the textbook – the existence of a basis is not proven. A correct statement should be

Theorem 3.23+. The Basis Theorem

Let S be a **non-zero** subspace of $\mathbb{R}^n$. Then

(a) S **has a finite basis;**

(b) any two bases for S have the same number of vectors.

For the examination, no need to have proof. But, for the completeness of exposition, I give a proof of existence of basis, Theorem 3.23+(a), here.

The existence of basis. Let S be a non-zero subspace (that is, S does not consist of zero vector only) of $\mathbb{R}^n$. Then S has a basis.

Proof. Consider all linearly independent systems of vectors in S. Since S contains a non-zero vector $\vec{v} \neq \vec{0}$, there is at least one such system: $\vec{v}$. Now, if $\vec{v}_1, \ldots, \vec{v}_k$ is a system of linearly independent vectors in S, we have $k \leqslant n$ by Theorem 2.8.

We come to a crucial step of the proof: choose a system of linearly independent vectors $\vec{v}_1, \ldots, \vec{v}_k$ in such way that k is maximal possible and consider

$$U = \mathrm{span}(v_1, \ldots, v_k).$$

Observe that $U \subseteq S$. If $U = S$, then $\vec{v}_1, \ldots, \vec{v}_k$ is a basis of S by definition of the basis, and our theorem is proven. Therefore we can assume that $U \neq S$ and chose a vector $\vec{v} \in S \setminus U$ (in S but not in U).

The rest of proof of Theorem 3.23 can be taken from the textbook.

Definition. If S is a subspace of $\mathbb{R}^n$, then the number of vectors in a basis for S is called the *dimension* of S, denoted $\dim S$.

Remark. The zero vector $\vec{0}$ by itself is always a subspace of $\mathbb{R}^n$. (Why?) Yet any set containing the zero vector (and, in particular, $\{\vec{0}\}$) is linearly dependent, so $\{\vec{0}\}$ cannot have a basis. We define $\dim\{\vec{0}\}$ to be 0.

Examples. 1) As we know, the n standard unit vectors form a basis of $\mathbb{R}^n$; thus, $\dim \mathbb{R}^n = n$.

2) If $\vec{v}_1, \ldots, \vec{v}_k$ are linearly independent vectors, then they form a basis of span$(\vec{v}_1, \ldots, \vec{v}_k)$, so then dim span$(\vec{v}_1, \ldots, \vec{v}_k) = k$.

We shall need a slightly more general result:

Theorem 3.23++. (a) If $v_1, \ldots, v_k$ are linearly independent vectors in a subspace S, then they can be included in (complemented to) a basis of S; in particular, $k \leq \dim S$.

(b) If one subspace is contained in another, $S \subseteq T$, then $\dim S \leq \dim T$. If both $S \subseteq T$ and $\dim S = \dim T$, then $S = T$.

Example. If we have some n linearly independent vectors $\vec{v}_1, \ldots, \vec{v}_n$ in $\mathbb{R}^n$, they must also form a basis of $\mathbb{R}^n$, as the dimension of their span is n and we can apply Theorem 3.23++(b).

Theorem 3.24. The row and column spaces of a matrix A have the same dimension.

Definition The *rank* of a matrix A is the dimension of its row and column spaces and is denoted by rank(A).

Theorem 3.25. For any matrix A,

$$\text{rank }(A^T) = \text{ rank }(A)$$

Definition The *nullity* of a matrix A is the dimension of its null space and is denoted by nullity(A).

Theorem 3.26. The Rank–Nullity Theorem

If A is an $m \times n$ matrix, then

$$\text{rank }(A) + \text{ nullity }(A) = n$$

Theorem 3.27. The Fundamental Theorem of Invertible Matrices

Let A be an $n \times n$ matrix. The following statements are equivalent:

a. A is invertible.

b. $A\vec{x} = \vec{b}$ has a unique solution for every $\vec{b}$ in $\mathbb{R}^n$.

c. $A\vec{x} = \vec{0}$ has only the trivial solution.

d. The reduced row echelon form of A is I_n.

e. A is a product of elementary matrices.

f. rank(A)$= n$.

g. nullity(A)$= 0$.

h. The column vectors of A are linearly independent.

i. The column vectors of A span $\mathbb{R}^n$.

j. The column vectors of A form a basis for $\mathbb{R}^n$.

k. The row vectors of A are linearly independent.

l. The row vectors of A span $\mathbb{R}^n$.

m. The row vectors of A form a basis for $\mathbb{R}^n$.

Coordinates

Theorem 3.29. Let S be a subspace of $\mathbb{R}^n$ and let

$$\mathcal{B} = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$$

be a basis for S. For every vector $\vec{v}$ in S, there is exactly one way to write $\vec{v}$ as a linear combination of the basis vectors in $\mathcal{B}$:

$$\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k$$

Definition. Let S be a subspace of $\mathbb{R}^n$ and let

$$\mathcal{B} = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$$

be a basis for S. Let $\vec{v}$ be a vector in S, and write

$$\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k$$

Then $c_1, c_2, \ldots, c_k$ are called the *coordinates of $\vec{v}$ with respect to $\mathcal{B}$*, and column vector

$$[\vec{v}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{bmatrix}$$

is called the *coordinate vector of $\vec{v}$ with respect to $\mathcal{B}$*. (Note, although $\vec{v}$ has n "original" coordinates as a vector in $\mathbb{R}^n$, the same vector in the basis $\mathcal{B}$ of the subspace S of dimension k has k coordinates.)