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On Fourier Coefficients of Automorphic Forms of GL(n)

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It is a well-known theorem, due to J. Shalika and I. Piatetski-Shapiro, independently, that any nonzero cuspidal automorphic form on $\mathrm{GL}_n(\mathbb{A})$ is generic, that is, has a nonzero Whittaker–Fourier coefficient. Its proof follows from the Fourier expansion of the cuspidal automorphic form in terms of its Whittaker–Fourier coefficients. In this paper, we extend this Fourier expansion to the whole discrete spectrum of the space of all square-integrable automorphic forms of $\mathrm{GL}_n(\mathbb{A})$ and determine the Fourier coefficients of irreducible noncuspidal (residual) automorphic representations of $\mathrm{GL}_n(\mathbb{A})$ in terms of unipotent orbits.

1 Introduction

Let k be a number field and $\mathbb{A}$ be the ring of adeles of k. For the general linear group GL_n, it is a well-known theorem that any nonzero cuspidal automorphic form φ on $\mathrm{GL}_n(\mathbb{A})$ is globally generic, that is, has a nonzero Whittaker–Fourier coefficient (which will be defined in Section 2), which was proved by Shalika [12] and Piatetski-Shapiro [10], independently, using the Fourier expansion of the cuspidal automorphic form φ in terms of its Whittaker–Fourier coefficients (which will be recalled in Section 2). This important fact for cuspidal automorphic forms on $\mathrm{GL}_n(\mathbb{A})$ distinguishes the theory of GL_n from that

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of other reductive algebraic groups, since over a general reductive algebraic group, there do exist nongeneric cuspidal automorphic forms.

In general, Fourier coefficients of automorphic forms over $G(\mathbb{A})$, where G is a reductive algebraic group defined over k, may be defined in terms of unipotent orbits of $G(k)$. However, the Fourier coefficients in a Fourier expansion of automorphic forms over $G(\mathbb{A})$ may be different from the notion of Fourier coefficients attached to unipotent orbits, although when $G = \mathrm{GL}_n$ and the automorphic forms are cuspidal, they coincide. Hence it is important to study the relations between the two different notions of Fourier coefficients. In this paper, we do this for the noncuspidal discrete series automorphic forms on $\mathrm{GL}_n(\mathbb{A})$, with the hope that some of the ideas and methods may be extendable to the discrete spectrum of classical groups.

First, we extend the Fourier expansion to automorphic forms on $\mathrm{GL}_n(\mathbb{A})$, which occur in the discrete spectrum of square-integrable automorphic forms on $\mathrm{GL}_n(\mathbb{A})$. This is done in Section 3 (Theorem 3.3). A technical, but very useful lemma (Lemma 3.2) is proved in Section 4. Based on the Fourier expansion in Section 3, we determine the degenerate Whittaker–Fourier coefficients along the standard maximal unipotent subgroup U_n of GL_n with degenerate characters for all noncuspidal discrete series automorphic representations of $\mathrm{GL}_n(\mathbb{A})$, following the terminology used by Zelevinsky [14, Section 8.3]. Note that the notion of degenerate Whittaker–Fourier coefficients is easy to use when the group is GL_n, as one can see in the Fourier expansion in Section 3. However, for other reductive groups, there are cuspidal automorphic forms, which have no nonzero Whittaker–Fourier coefficients, and hence such degenerate Whittaker–Fourier coefficients are all zero. Therefore, it is natural to introduce the notion of Fourier coefficients attached to unipotent orbits for automorphic forms on general reductive groups [1, 2, 6].

In Section 5, we define the notion of Fourier coefficients attached to unipotent orbits for automorphic forms on $\mathrm{GL}_n(\mathbb{A})$, and determine the relation between the degenerate Whittaker–Fourier coefficients from the Fourier expansion and the Fourier coefficients attached to unipotent orbits for the residual spectrum of $\mathrm{GL}_n(\mathbb{A})$ (Theorem 5.4). We remark that Ginzburg [1] gives a sketch of a proof of this result [1, Proposition 5.3] with an argument combining local and global methods. We give here a global proof with full details. In Section 6, we show that the Fourier coefficient for any noncuspidal, discrete series automorphic form of $\mathrm{GL}_n(\mathbb{A})$ obtained from Theorem 5.4 is the biggest Fourier coefficient according to the partial ordering of unipotent orbits (Theorem 5.5). Finally, Theorem 5.6, which is the combination of Theorems 5.4 and 5.5 and Corollary 3.4, extends the results of Shalika and of Piatetski-Shapiro on cuspidal automorphic forms of

$\mathrm{GL}_n(\mathbb{A})$ to the whole discrete spectrum of $\mathrm{GL}_n(\mathbb{A})$. In other words, we prove the following main result (Theorem 5.6) of this paper.

Theorem 1.1 (Fourier coefficients for discrete spectrum of GL_n)**.** Let τ be an irreducible unitary cuspidal automorphic representation of $\mathrm{GL}_a(\mathbb{A})$ and let $n=ab$ with $b\geq 1$. Define the residual representation $\mathcal{E}_{(\tau,b)}$ of $\mathrm{GL}_n(\mathbb{A})$ as in Section 2.1. Let $\underline{p}=[p_1 p_2 \cdots p_r]$ be a partition of n with $p_1 \geq p_2 \geq \cdots \geq p_r > 0$ and denote by $[a^b]$ the partition of all parts equal to a. Then the following hold.

(1) The residual representation $\mathcal{E}_{(\tau,b)}$ has a nonzero $\psi_{[a^b]}$-Fourier coefficient, whose definition is given in Section 5.

(2) For any partition $\underline{p}=[p_1 p_2 \cdots p_r]$ of n, if $p_1 > a$, then the residual representation $\mathcal{E}_{(\tau,b)}$ has no nonzero $\psi_{\underline{p}}$-Fourier coefficients. □

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2 Discrete Spectrum of GL_n

We first recall from [7] the structure of the discrete spectrum of GL_n and from [10, 12] the Fourier expansion of any cuspidal automorphic form on GL_n.

2.1 Structure of discrete spectrum

Take $n=ab$ with $a, b \geq 1$ integers. It was a conjecture of Jacquet [4] and then a theorem of Moeglin and Waldspurger [7] that an irreducible automorphic representation π of $\mathrm{GL}_n(\mathbb{A})$ occurring in the discrete spectrum of the space of all square-integrable automorphic forms on $\mathrm{GL}_n(\mathbb{A})$ is parameterized by a pair (τ, b) with τ an irreducible unitary cuspidal automorphic representation of $\mathrm{GL}_a(\mathbb{A})$, for some pair a, b of integers such that $n=ab$. In particular, if π is also cuspidal, then $b=1$.

More precisely, we take the Borel subgroup $B_n=T_nU_n$ to be the subgroup of all upper triangular matrices in GL_n, where T_n consists of all diagonal matrices in GL_n. The triple $(\mathrm{GL}_n, B_n, T_n)$ determines the structure of the root system of GL_n. For $n=ab$ with $b>1$, take the standard parabolic subgroup $P_{a^b}=M_{a^b}N_{a^b}$ of GL_{ab}, with the Levi part M_{a^b} isomorphic to $\mathrm{GL}_a^{\times b}=\mathrm{GL}_a \times \cdots \times \mathrm{GL}_a (b-\text{times})$. Then $(P_{a^b}, \tau^{\otimes b})$ is a cuspidal datum of $\mathrm{GL}_{ab}(\mathbb{A})$. Following the theory of Langlands [5] and Moeglin and Waldspurger [8], there is an Eisenstein series $E(\phi_{\tau^{\otimes b}}, \underline{s}, g)$ attached to $(P_{a^b}, \tau^{\otimes b})$, where $\underline{s}=(s_1, \ldots, s_b)\in\mathbb{C}^b$. This

Eisenstein series converges absolutely for the real part of $\underline{s}$ belonging to a certain cone and has meromorphic continuation to the whole complex space $\mathbb{C}^b$. Moreover, it has an iterated residue at

$$\underline{s}_0 = \Lambda_b := \left(\frac{b-1}{2}, \frac{b-3}{2}, \ldots, \frac{1-b}{2}\right)$$

given by

$$E_{-1}(\phi_{\tau^{\otimes b}}, g) = \lim_{\underline{s} \to \Lambda_b} \prod_{i=1}^{b-1} (s_i - s_{i+1} - 1) E(\phi_{\tau^{\otimes b}}, \underline{s}, g), \tag{2.1}$$

which is square integrable, and hence belongs to the discrete spectrum of the space of all square integrable automorphic forms of $\mathrm{GL}_{ab}(\mathbb{A})$. Denote by $\mathcal{E}_{(\tau,b)}$ the automorphic representation generated by all the residues $E_{-1}(\phi_{\tau^{\otimes b}}, g)$. It is a theorem of Moeglin and Waldspurger [7] that $\mathcal{E}_{(\tau,b)}$ is irreducible, and any irreducible noncuspidal automorphic representation occurring in the discrete spectrum of $\mathrm{GL}_n(\mathbb{A})$ is of this form for some $a \geq 1$ and $b > 1$ such that $n = ab$, and has multiplicity one.

2.2 Fourier expansion for cuspidal automorphic forms

Recall $B_n = T_n U_n$ is the Borel subgroup fixed in Section 2.1. We write elements of U_n to be $u = (u_{i,j})$, which is upper triangular. Let ψ be a nontrivial character of $\mathbb{A}$, which is trivial on k. We define a nondegenerate character of $U_n(\mathbb{A})$ by

$$\psi_{U_n}(u) := \psi(u_{1,2} + u_{2,3} + \cdots + u_{n-1,n}). \tag{2.2}$$

It is clear that ψ_{U_n} is trivial on $U_n(k)$. For an automorphic form φ on $\mathrm{GL}_n(\mathbb{A})$, the (nondegenerate) Whittaker–Fourier coefficient of φ is given by

$$W^{\psi}(\varphi, g) := \int_{U_n(k)\backslash U_n(\mathbb{A})} \varphi(ug)\psi_{U_n}^{-1}(u)\, \mathrm{d}u. \tag{2.3}$$

When φ is cuspidal, the following well-known Fourier expansion of φ is proved independently in [10, 12]:

$$\varphi(g) = \sum_{\gamma \in U_{n-1}(k)\backslash \mathrm{GL}_{n-1}(k)} W^{\psi}(\varphi, \iota_n(\gamma)g), \tag{2.4}$$

where $\iota_n(\gamma) = \left(\begin{smallmatrix} \gamma & 0 \\ 0 & 1 \end{smallmatrix}\right)$. As a consequence, one deduces easily from this Fourier expansion that any nonzero cuspidal automorphic form φ has a nonzero Whittaker–Fourier coefficient, and hence is generic.

Now let us consider the residual representations of $\mathrm{GL}_n(\mathbb{A})$. Take $n = ab$. The residual representation $\mathcal{E}_{(\tau,b)}$ can also be generated by the residues at the point $(s_1, s_2) = \left(\frac{b-1}{2}, -\frac{1}{2}\right)$ of the Eisenstein series with support

$$\mathrm{GL}_a \times \mathrm{GL}_{a(b-1)}, \tau \otimes \mathcal{E}_{(\tau,b-1)}.$$

From the calculation of Shahidi [11, Chapter 7], it is clear to see that the residual representation $\mathcal{E}_{(\tau,b)}$ has a nonzero Whittaker–Fourier coefficient only if the residual representation $\mathcal{E}_{(\tau,b-1)}$ has a nonzero Whittaker–Fourier coefficient. By the induction argument, it is enough to show that $\mathcal{E}_{(\tau,2)}$ is not generic. This follows from [11, Theorem 7.1.2]. We summarize the discussion as in the following proposition.

Proposition 2.1. Any irreducible, noncuspidal, automorphic representation occurring in the discrete spectrum of $\mathrm{GL}_n(\mathbb{A})$ is nongeneric, that is, has no nonzero Whittaker–Fourier coefficients. □

We note that this global result can also be proved by using Zelevinsky classification theory of irreducible smooth representations of GL_n over a p-adic local field [14].

3 Fourier Expansion for the Discrete Spectrum

Fourier expansion for automorphic forms on $\mathrm{GL}_n(\mathbb{A})$ is an important tool to study the properties of automorphic forms. A general expansion is given in [13, Proposition 2.1.3]. However, it is not easy to use such a general expansion to establish an analog of the Fourier expansion (2.4) for the automorphic forms in the noncuspidal discrete spectrum of $\mathrm{GL}_n(\mathbb{A})$. In this section, we write the Eisenstein series in a more explicit form and study the vanishing and nonvanishing of certain Fourier coefficients of the residual representations to obtain the exact extension of (2.4) to the whole discrete spectrum of $\mathrm{GL}_n(\mathbb{A})$.

For $n = ab$ with $b > 1$, take the standard parabolic subgroup $P = P_{a^b} = MN$ of GL_{ab} with Levi part $M = \mathrm{GL}_a^{\times b}$. Consider the normalized induced representation

$$I(\tau, \underline{s}, b) = \mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})}(\tau|\cdot|^{s_1} \otimes \cdots \otimes \tau|\cdot|^{s_b}),$$

where τ is an irreducible unitary cuspidal automorphic representation of $\mathrm{GL}_a(\mathbb{A})$ and $\underline{s} = \{s_1, \ldots, s_b\} \in \mathbb{C}^b$.

For any section $\phi \in I(\tau, \underline{s}, b)$, let $\mathrm{i}\phi$ be a complex function over $N(\mathbb{A})M(k) \setminus G(\mathbb{A})$ defined by

$$\mathrm{i}\phi(g) = \phi(g)(I_a^b). \tag{3.1}$$

where I_a^b is the identity in the group M. Then, when the real part of $\underline{s}$ belongs to a certain cone, the Eisenstein series can be expressed as

$$E(\phi, \underline{s}, g) = \sum_{\gamma \in P(k) \backslash \mathrm{GL}_{ab}(k)} \mathrm{i}\phi(\gamma g). \tag{3.2}$$

For any parabolic subgroup $Q = LV$ of GL_n, the constant term of the Eisenstein series along Q is defined by

$$E(\phi, \underline{s}, g)_Q = \int_{V(k) \backslash V(\mathbb{A})} E(\phi, \underline{s}, vg)\, \mathrm{d}v. \tag{3.3}$$

Then the constant term of the residue $E_{-1}(\phi, g)$ along Q is given by

$$E_{-1}(\phi, g)_Q = \lim_{\underline{s} \to \Lambda_b} \prod_{i=1}^{b-1} (s_i - s_{i+1} - 1) E(\phi, \underline{s}, g)_Q. \tag{3.4}$$

It follows from [8, Proposition 2.1.7] that the constant term $E_{-1}(\phi, g)_Q$ is always zero unless $P \subseteq Q$.

3.1 Families of fourier coefficients

In order to study the Fourier expansion for the discrete spectrum of GL_n, we introduce two families of Fourier coefficients. Let α, β, γ, and δ be four nonnegative integers, such that

$$\alpha + \beta + \gamma \cdot \delta = n.$$

Consider the standard parabolic subgroup

$$Q_{\alpha, 1^{(n-\alpha)}} = L_{\alpha, 1^{(n-\alpha)}} V_{\alpha, 1^{(n-\alpha)}}$$

of GL_n with the Levi part $L_{\alpha, 1^{(n-\alpha)}} = \mathrm{GL}_\alpha \times \mathrm{GL}_1^{\times (n-\alpha)}$. We define $Q^0_{\alpha, 1^{(n-\alpha)}}$ to be the subgroup of $Q_{\alpha, 1^{(n-\alpha)}} = (q_{i,j})$ with $q_{i,i} = 1$ for all $i > \alpha$. Note that

$$V_{1, 1^{(n-1)}} = V_{0, 1^n} = U_n$$

is the standard maximal unipotent subgroup of GL_n.

For nonnegative integers α, β, γ, and δ as given above, we define two types of (degenerate) characters $\psi^n_{\beta^+;\gamma\cdot\delta}$ and $\psi^n_{\beta;\gamma\cdot\delta}$ of $V_{\alpha,1^{(n-\alpha)}}$ as follows:

$$\begin{aligned}\psi^n_{\beta^+;\gamma\cdot\delta}(v) := \psi(&v_{\alpha,\alpha+1} + v_{\alpha+1,\alpha+2} + \cdots + v_{\alpha+\beta-1,\alpha+\beta})\\ &\cdot \psi(v_{\alpha+\beta+1,\alpha+\beta+2} + \cdots + v_{\alpha+\beta+\delta-1,\alpha+\beta+\delta})\\ &\cdot \psi(v_{\alpha+\beta+\delta+1,\alpha+\beta+\delta+2} + \cdots + v_{\alpha+\beta+2\delta-1,\alpha+\beta+2\delta})\\ &\vdots\\ &\cdot \psi(v_{\alpha+\beta+(\gamma-1)\cdot\delta+1,\alpha+\beta+(\gamma-1)\cdot\delta+2} + \cdots + v_{n-1,n})\end{aligned} \tag{3.5}$$

and

$$\begin{aligned}\psi^n_{\beta;\gamma\cdot\delta}(v) := \psi(&v_{\alpha+1,\alpha+2} + \cdots + v_{\alpha+\beta-1,\alpha+\beta})\\ &\cdot \psi(v_{\alpha+\beta+1,\alpha+\beta+2} + \cdots + v_{\alpha+\beta+\delta-1,\alpha+\beta+\delta})\\ &\cdot \psi(v_{\alpha+\beta+\delta+1,\alpha+\beta+\delta+2} + \cdots + v_{\alpha+\beta+2\delta-1,\alpha+\beta+2\delta})\\ &\vdots\\ &\cdot \psi(v_{\alpha+\beta+(\gamma-1)\cdot\delta+1,\alpha+\beta+(\gamma-1)\cdot\delta+2} + \cdots + v_{n-1,n}).\end{aligned} \tag{3.6}$$

Note that $\psi^n_{\beta^+;\gamma\cdot\delta}(v) = \psi(v_{\alpha,\alpha+1}) \cdot \psi^n_{\beta;\gamma\cdot\delta}(v)$. The corresponding Fourier coefficients of the residue $E_{-1}(\phi, g)$ are given by:

$$E_{-1}^{\psi^n_{\beta^+;\gamma\cdot\delta}}(\phi, g) := \int_{[V_{\alpha,1^{(n-\alpha)}}]} E_{-1}(\phi, vg)\psi^n_{\beta^+;\gamma\cdot\delta}(v)^{-1}\,\mathrm{d}v, \tag{3.7}$$

$$E_{-1}^{\psi^n_{\beta;\gamma\cdot\delta}}(\phi, g) := \int_{[V_{\alpha,1^{(n-\alpha)}}]} E_{-1}(\phi, vg)\psi^n_{\beta;\gamma\cdot\delta}(v)^{-1}\,\mathrm{d}v, \tag{3.8}$$

where $[V_{\alpha,1^{(n-\alpha)}}] := V_{\alpha,1^{(n-\alpha)}}(k) \setminus V_{\alpha,1^{(n-\alpha)}}(\mathbb{A})$. For simplicity of notation, if there is no confusion, we also use $\psi_{\beta^+;\gamma\cdot\delta}$ for $\psi^n_{\beta^+;\gamma\cdot\delta}$, and $\psi_{\beta;\gamma\cdot\delta}$ for $\psi^n_{\beta;\gamma\cdot\delta}$.

3.2 **Fourier expansion: Step 1**

We consider first a preliminary version of the Fourier expansion for the residue $E_{-1}(\phi, g)$, following the idea of the Fourier expansion for cuspidal automorphic forms on $\mathrm{GL}_{ab}(\mathbb{A})$ ($n = ab$ from now on) given in [10, 12].

Consider the standard parabolic subgroup $Q_{ab-1,1} = L_{ab-1,1}V_{ab-1,1}$ of GL_{ab}. The unipotent radical $V_{ab-1,1}$ is abelian and is isomorphic to $k^{\oplus(ab-1)}$. Hence we have the following Fourier expansion for the residue $E_{-1}(\phi, g)$ along $V_{ab-1,1}(k) \setminus V_{ab-1,1}(\mathbb{A})$:

$$E_{-1}(\phi, g) = E_{-1}(\phi, g)_{Q_{ab-1,1}} + \sum_{\gamma_{ab} \in Q^0_{ab-2,1^{(2)}}(k) \setminus Q^0_{ab-1,1}(k)} E_{-1}^{\psi_{1^+;0\cdot 0}}(\phi, \gamma_{ab} g),$$

with $\alpha = ab - 1$, $\beta = 1$, and $\gamma = \delta = 0$.

If $a = 1$, then $b = n$ and the residual representation $\mathcal{E}_{\tau,n}$ is one-dimensional. Hence it has no nontrivial Fourier coefficients. That is, $E_{-1}^{\psi_{1^+;0\cdot 0}}(\phi, g)$ is identically zero (this will be generalized in Lemma 3.2), which implies that

$$E_{-1}(\phi, g) = E_{-1}(\phi, g)_{Q_{ab-1,1}} = E_{-1}^{\psi_{1;0\cdot 0}}(\phi, g) = E_{-1}^{\psi_{0;1\cdot 1}}(\phi, g). \tag{3.9}$$

If $a > 1$, then $Q_{ab-1,1}$ does not contain P. By the cuspidal support of the residue, the constant term $E_{-1}(\phi, g)_{Q_{ab-1,1}}$ is always zero. Hence we obtain

$$E_{-1}(\phi, g) = \sum_{\gamma_{ab} \in Q^0_{ab-2,1^{(2)}}(k) \setminus Q^0_{ab-1,1}(k)} E_{-1}^{\psi_{1^+;0\cdot 0}}(\phi, \gamma_{ab} g). \tag{3.10}$$

Note that $E_{-1}^{\psi_{1^+;0\cdot 0}}(\phi, g)$ is left $\iota_{ab-1,ab}(Q^0_{ab-2,1}(k))$ invariant, where the subgroup $Q^0_{ab-2,1}$ is the GL_{ab-1}-analog of subgroup $Q^0_{ab-1,1}$ of GL_{ab} and

$$\iota_{ab-1,ab}(h) := \begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix} \tag{3.11}$$

for $h \in \mathrm{GL}_{ab-1}$. Next consider the unipotent radical $V_{ab-2,1}$ of $Q^0_{ab-2,1}$, which is abelian and isomorphic to $k^{\oplus(ab-2)}$. Hence we have the Fourier expansion of $E_{-1}^{\psi_{1^+;0\cdot 0}}(\phi, g)$ along $V_{ab-2,1}(k) \setminus V_{ab-2,1}(\mathbb{A})$:

$$E_{-1}^{\psi_{1^+;0\cdot 0}}(\phi, g) = E_{-1}^{\psi_{1^+;0\cdot 0}}(\phi, g)_{\iota_{ab-1,ab}(Q_{ab-2,1})} + \sum_{\gamma_{ab-1} \in Q^0_{ab-3,1^{(2)}}(k) \setminus Q^0_{ab-2,1}(k)} E_{-1}^{\psi_{2^+;0\cdot 0}}(\phi, \gamma_{ab-1} g).$$

It is easy to see from the definition that the constant term

$$E_{-1}^{\psi_{1^+;0\cdot 0}}(\phi, g)_{\iota_{ab-1,ab}(Q_{ab-2,1})} = E_{-1}^{\psi_{0;1\cdot 2}}(\phi, g).$$

If $a=2$, we show that the Fourier coefficient $E_{-1}^{\psi_{2^+;0\cdot 0}}(\phi,\gamma_{ab-1}g)$ is zero (Lemma 3.2). Hence, when $a=2$, from (3.10), we obtain

$$E_{-1}(\phi,g)=\sum_{\gamma_{ab}\in Q^0_{2(b-1),1^{(2)}}(k)\backslash Q^0_{2b-1,1}(k)} E_{-1}^{\psi_{0;1\cdot 2}}(\phi,\gamma_{ab}g). \tag{3.12}$$

If $a>2$, then the term $E_{-1}^{\psi_{0;1\cdot 2}}(\phi,g)$ contains as an inner integration the constant term $E_{-1}(\phi,g)_{P_{ab-2,2}}$ of $E_{-1}(\phi,g)$ along the standard maximal parabolic subgroup $P_{ab-2,2}$ of GL_{ab} with Levi part isomorphic to $\mathrm{GL}_{ab-2}\times\mathrm{GL}_2$. Since $P_{ab-2,2}$ does not contain P when $a>2$, the constant term $E_{-1}(\phi,g)_{P_{ab-2,2}}$ must be zero. Hence, when $a>2$, we have

$$E_{-1}^{\psi_{1^+;0\cdot 0}}(\phi,g)=\sum_{\gamma_{ab-1}\in Q^0_{ab-3,1^{(2)}}(k)\backslash Q^0_{ab-2,1}(k)} E_{-1}^{\psi_{2^+;0\cdot 0}}(\phi,\gamma_{ab-1}g). \tag{3.13}$$

With (3.10), we obtain, when $a>2$, that the residue $E_{-1}(\phi,g)$ is equal to

$$\sum_{\gamma_{ab}}\ \sum_{\gamma_{ab-1}\in Q^0_{ab-3,1^{(2)}}(k)\backslash Q^0_{ab-2,1}(k)} E_{-1}^{\psi_{2^+;0\cdot 0}}(\phi,\iota_{ab-1,ab}(\gamma_{ab-1})\gamma_{ab}g),$$

where γ_{ab} runs over $Q^0_{ab-2,1^{(2)}}(k)\setminus Q^0_{ab-1,1}(k)$. Note that

$$Q^0_{ab-2,1^{(2)}}=\iota_{ab-1,ab}(Q^0_{ab-2,1})V_{ab-1,1}$$

and $\iota_{ab-1,ab}(Q^0_{ab-2,1})$ normalizes $V_{ab-1,1}$. Note also that

$$\iota_{ab-2,ab}(Q^0_{ab-3,1^{(2)}})V_{ab-1,1}=Q^0_{ab-3,1^{(3)}},$$

where $\iota_{ab-2,ab}:=\iota_{ab-2,ab-1}\circ\iota_{ab-1,ab}$. We obtain, when $a>2$, the following Fourier expansion for the residue $E_{-1}(\phi,g)$:

$$E_{-1}(\phi,g)=\sum_{\gamma_{ab}\in Q^0_{ab-3,1^{(3)}}(k)\backslash Q^0_{ab-1,1}(k)} E_{-1}^{\psi_{2^+;0\cdot 0}}(\phi,\gamma_{ab}g). \tag{3.14}$$

We continue with the expansion (3.14) and repeat the above argument, and finally we obtain the following expansion:

$$E_{-1}(\phi,g)=\sum_{\gamma_{ab}\in Q^0_{a(b-1),1^{(a)}}(k)\backslash Q^0_{ab-1,1}(k)} E_{-1}^{\psi_{0;1\cdot a}}(\phi,\gamma_{ab}g), \tag{3.15}$$

which uses Lemma 3.2. Note that this expansion generalizes (3.9) and (3.12), from $a=1,2$ to general a. We record the above discussion as follows.

Proposition 3.1. Let $E_{-1}(\phi,g)$ be the residue of the Eisenstein series $E(\phi,\underline{s},g)$ as defined in (2.1).

(1) It has the following Fourier expansion:

$$E_{-1}(\phi,g)=\sum_{\gamma\in Q^0_{a(b-1),1^{(a)}}(k)\backslash Q^0_{ab-1,1}(k)} E_{-1}^{\psi^{ab}_{0;1\cdot a}}(\phi,\gamma g).$$

(2) The above Fourier expansion is absolutely convergent and uniformly converges on any compact set in g. □

Note that Part (2) is clear since, in each step, the Fourier expansion in the argument is absolutely convergent and uniformly converges on any compact set in g. Also, if we assume that $b=1$, then $\mathcal{E}_{\tau,1}=\tau$, and the Fourier expansion in Part (1) recovers the Fourier expansion in (2.4) for any cuspidal automorphic form φ_τ in the space of τ, given in [10, 12].

Now we state the technical lemma, which will be proved in Section 4.

Lemma 3.2. Let $E_{-1}(\phi,g)$ be the residue of the Eisenstein series $E(\phi,\underline{s},g)$ as defined in (2.1). The Fourier coefficient $E_{-1}^{\psi_{\alpha^+;\gamma\cdot a}}(\phi,g)$ is identically zero, for $\alpha=a(b-\gamma-1)$, $\beta=\delta=a$, and $\gamma=0,1,2,\ldots,b-2$. □

3.3 Fourier expansion: Step 2

As we remarked after Proposition 3.1, if $b=1$, then $\mathcal{E}_{\tau,1}=\tau$, and we finish the Fourier expansion. If $b>1$, we are able to do further Fourier expansion from the Fourier expansion in Part (1) of Proposition 3.1.

Consider the subgroup $Q^0_{a(b-1)-1,1}$ of $\mathrm{GL}_{a(b-1)}$, which is the $\mathrm{GL}_{a(b-1)}$-analog of $Q^0_{ab-1,1}$. In this case, $V_{a(b-1)-1,1}$ is the unipotent radical of $Q^0_{a(b-1)-1,1}$, which is isomorphic to $k^{\oplus(a(b-1)-1)}$. It is clear that the Fourier coefficient $E_{-1}^{\psi_{0;1\cdot a}}(\phi,g)$ in (3.15) is left $\iota_{a(b-1),ab}(V_{a(b-1)-1,1}(k))$ invariant, where for $h\in\mathrm{GL}_{a(b-1)}$, $\iota_{a(b-1),ab}(h)$ is the block-diagonal matrix $\mathrm{diag}(h,I_a)$ in GL_{ab}. Hence we have the following Fourier expansion for $E_{-1}^{\psi'_{a(b-1),1^{(a)}}}(\phi, g)$:

$$E_{-1}^{\psi_{0;1\cdot a}}(\phi, g) = E_{-1}^{\psi_{0;1\cdot a}}(\phi, g)_{\iota_{a(b-1),ab}(Q^0_{a(b-1)-1,1})} + \sum_{\gamma} E_{-1}^{\psi^{ab}_{0;1\cdot a};\psi^{a(b-1)}_{1^+;0\cdot 0}}(\phi, \iota_{a(b-1),ab}(\gamma)g),$$

where γ runs over $Q^0_{a(b-1)-2,1^{(2)}}(k) \setminus Q^0_{a(b-1)-1,1}(k)$ and

$$E_{-1}^{\psi^{ab}_{0;1\cdot a};\psi^{a(b-1)}_{1^+;0\cdot 0}}(\phi, g) := \int_v E_{-1}^{\psi^{ab}_{0;1\cdot a}}(\phi, vg)\psi^{a(b-1)}_{1^+;0\cdot 0}(v)^{-1}\,\mathrm{d}v. \tag{3.16}$$

Here the integration dv is over

$$\iota_{a(b-1),ab}(V_{a(b-1)-1,1}(k)) \setminus \iota_{a(b-1),ab}(V_{a(b-1)-1,1}(\mathbb{A})).$$

As long as $a > 1$, which we always assume from now on, the constant term $E_{-1}^{\psi_{0;1\cdot a}}(\phi, g)_{\iota_{a(b-1),ab}(Q^0_{a(b-1)-1,1})}$ contains as an inner integration the constant term of the residue $E_{-1}(\phi, g)$ along the maximal parabolic subgroup $P_{a(b-1)-1,a+1}$ of GL_{ab}, which is identically zero since $P_{a(b-1)-1,a+1}$ does not contain P. Hence we obtain ($a > 1$)

$$E_{-1}^{\psi_{0;1\cdot a}}(\phi, g) = \sum_{\gamma} E_{-1}^{\psi^{ab}_{0;1\cdot a};\psi^{a(b-1)}_{1^+;0\cdot 0}}(\phi, \iota_{a(b-1),ab}(\gamma)g), \tag{3.17}$$

where γ runs over $Q^0_{a(b-1)-2,1^{(2)}}(k) \setminus Q^0_{a(b-1)-1,1}(k)$. As in Section 3.1, we can continue and obtain the following Fourier expansion

$$E_{-1}^{\psi_{0;1\cdot a}}(\phi, g) = \sum_{\gamma} E_{-1}^{\psi^{ab}_{0;1\cdot a};\psi^{a(b-1)}_{0;1\cdot a}}(\phi, \iota_{a(b-1),ab}(\gamma)g), \tag{3.18}$$

where γ runs over $Q^0_{a(b-2),1^{(a)}}(k) \setminus Q^0_{a(b-1)-1,1}(k)$, and

$$E_{-1}^{\psi^{ab}_{0;1\cdot a};\psi^{a(b-1)}_{0;1\cdot a}}(\phi, g) = \int_v E_{-1}^{\psi^{ab}_{0;1\cdot a}}(\phi, vg)\psi^{a(b-1)}_{0;1\cdot a}(v)^{-1}\,\mathrm{d}v.$$

Here the integration dv is over

$$\iota_{a(b-1),ab}(V_{a(b-2),1^{(a)}}(k)) \setminus \iota_{a(b-1),ab}(V_{a(b-2),1^{(a)}}(\mathbb{A})).$$

Note that the proof here uses the technical lemma (Lemma 3.2 again).

Since $\iota_{a(b-1),ab}(V_{a(b-2),1^{(a)}}V_{a(b-1),1^{(a)}})=V_{a(b-2),1^{(2a)}}$, we see that

$$E_{-1}^{\psi^{ab}_{0;1\cdot a};\psi^{a(b-1)}_{0;1\cdot a}}(\phi,g)=\int_v E_{-1}(\phi,vg)\psi^{ab}_{0;2\cdot a}(v)^{-1}\,\mathrm{d}v$$
$$=E_{-1}^{\psi^{ab}_{0;2\cdot a}}(\phi,g), \tag{3.19}$$

where the integration $\mathrm{d}v$ is over $V_{a(b-2),1^{(2a)}}(k)\setminus V_{a(b-2),1^{(2a)}}(\mathbb{A})$. Note here that $\alpha=a(b-2)$, $\beta=0$, $\gamma=2$, and $\delta=a$.

From the above discussion, we obtain

$$E_{-1}(\phi,g)=\sum_{\gamma_1}\sum_{\gamma_2}E_{-1}^{\psi_{0;2\cdot a}}(\phi,\gamma_2\gamma_1 g), \tag{3.20}$$

where γ_1 runs over $Q^0_{a(b-1),1^{(a)}}(k)\setminus Q^0_{ab-1,1}(k)$ and γ_2 runs over

$$\iota_{a(b-1),ab}(Q^0_{a(b-2),1^{(a)}}(k))\setminus\iota_{a(b-1),ab}(Q^0_{a(b-1)-1,1}(k)).$$

For general $b>1$, such that $n=ab$, we repeat the above argument and obtain, by means of the inductive argument and Lemma 3.2 for each step, the following Fourier expansion for the residue $E_{-1}(\phi,g)$.

$$E_{-1}(\phi,g)=\sum_{\gamma_1}\cdots\sum_{\gamma_b}E_{-1}^{\psi_{0;b\cdot a}}(\phi,\gamma_b\cdots\gamma_1 g), \tag{3.21}$$

where γ_i runs over

$$\iota_{a(b-i+1),ab}(Q^0_{a(b-i),1^{(a)}}(k))\setminus\iota_{a(b-i+1),ab}(Q^0_{a(b-i+1)-1,1}(k))$$

for $i=1,2,\ldots,b$. Note here that $\alpha=\beta=0$, $\gamma=b$, and $\delta=a$. We state this as follows.

Theorem 3.3 (Fourier expansion). Let $E_{-1}(\phi,g)$ be the residue of the Eisenstein series $E(\phi,\underline{s},g)$ as defined in (2.1).

(1) It has the following Fourier expansion:

$$E_{-1}(\phi,g)=\sum_{\gamma_1}\cdots\sum_{\gamma_b}E_{-1}^{\psi^{ab}_{0;b\cdot a}}(\phi,\gamma_b\cdots\gamma_1 g),$$

where γ_i runs over

$$\iota_{a(b-i+1),ab}(Q^0_{a(b-i),1^{(a)}}(k)) \setminus \iota_{a(b-i+1),ab}(Q^0_{a(b-i+1)-1,1}(k))$$

for $i = 1, 2, \ldots, b$.

(2) The above Fourier expansion is absolutely convergent and uniformly converges on any compact set in g. □

As a consequence, we obtain the following corollary.

Corollary 3.4. Let $E_{-1}(\phi, g)$ be the residue of the Eisenstein series $E(\phi, \underline{s}, g)$ as defined in (2.1) with cuspidal datum $(P_{a^b}, \tau^{\otimes b})$. Then the degenerate Whittaker–Fourier coefficient $E_{-1}^{\psi^{ab}_{0;b\cdot a}}(\phi, g)$ is nonzero for some choice of ϕ. □

More generally, there is a notion of Fourier coefficients of an automorphic form on $\mathrm{GL}_n(\mathbb{A})$ parameterized by partitions of n or unipotent orbits, which works better for general reductive groups and will be introduced for GL_n in Section 5. By using the ordering of partitions, we will show that the Fourier coefficient $E_{-1}^{\psi^{ab}_{0;b\cdot a}}(\phi, g)$ is essentially the one attached to the biggest partition among all partitions to which the residue $E_{-1}(\phi, g)$ can have nonzero Fourier coefficients attached.

4 Proof of Lemma 3.2

In order to prove Lemma 3.2, we need Lemma 4.1, which will also be used in the proofs of Theorem 5.4 and Lemma 6.1. In the following, we take the standard section for the Eisenstein series $E(\phi, \underline{s}, g)$, via the natural isomorphism of vector spaces: $I(\tau, \underline{0}, b) \cong I(\tau, \underline{s}, b)$, which takes ϕ to $\phi(\underline{s})$, for any section $\phi \in I(\tau, \underline{0}, b)$, canonically. In this way, the Eisenstein series is given by, when the real part of $\underline{s}$ belongs to a certain cone,

$$E(\phi, \underline{s}, g) = \sum_{\gamma \in P(k)\backslash \mathrm{GL}_{ab}(k)} \mathrm{i}\phi(\underline{s})(\gamma g).$$

Lemma 4.1. Let $Q_i := Q_{ai,a(b-i)} = L_i V_i$ be the standard maximal parabolic subgroup of GL_{ab} with Levi part $L_i \cong \mathrm{GL}_{ai} \times \mathrm{GL}_{a(b-i)}$, where $1 \leq i \leq b-1$. Then there is a section

$$f \in \mathrm{Ind}_{Q_i(\mathbb{A})}^{\mathrm{GL}_{ab}(\mathbb{A})}(|\cdot|^{-\frac{b-i}{2}} \mathcal{E}_{(\tau,i)} \otimes |\cdot|^{\frac{i}{2}} \mathcal{E}_{(\tau,b-i)}),$$

such that

$$E_{-1}(\phi, g)_{Q_i} = f(g)(I_{ai} \times I_{a(b-i)}).$$ □

Proof. We first calculate the constant term of the Eisenstein series $E(\phi, \underline{s}, g)$ along the parabolic subgroup Q_i. This Eisenstein series has cuspidal support on the standard parabolic subgroup $P_{a^b} = M_{a^b} N_{a^b}$. In this proof, we use $P = MN$ to simplify the notation.

To do this we introduce the set ${}_M W_{L_i}^c$, which consists of elements $\omega^{-1} \in W(GL_{ab})$ (the Weyl group of GL_{ab}), with the properties that $\omega(\alpha) > 0$ for any $\alpha \in \Phi_M^+$, the positive roots in M, $\omega^{-1}(\beta) > 0$ for any $\beta \in \Phi_{L_i}^+$, the positive roots in L_i, and $\omega M \omega^{-1} \subseteq L_i$. By Moeglin and Waldspurger [8, Proposition 2.1.7(2)], we have

$$\begin{aligned} E(\phi, \underline{s}, g)_{Q_i} &= \sum_{\omega^{-1} \in {}_M W_{L_i}^c} \sum_{\gamma \in (\omega P \omega^{-1} \cap L_i)(k) \backslash L_i(k)} iM(\omega, \underline{s}) \phi(\underline{s})(\gamma g) \\ &= \sum_{\omega^{-1} \in {}_M W_{L_i}^c} E^{Q_i}(M(\omega, \underline{s})\phi(\underline{s}), \omega \underline{s}, g), \end{aligned}$$

where $M(\omega, \underline{s})$ is the intertwining operator corresponding to ω.

Note that ${}_M W_{L_i}^c$ has total C_b^i elements which are of following forms: for any i numbers $\{l_1, \ldots, l_i\}$ in $\{1, \ldots, b\}$ with increasing order, and the complement $b - i$ numbers $\{m_1, \ldots, m_{b-i}\} = \{1, \ldots, b\} \setminus \{l_1, \ldots, l_i\}$ with increasing order, the corresponding element $\omega^{-1} \in {}_M W_{L_i}^c$ is defined as

$$\omega^{-1} : \ j \mapsto l_j \quad \text{and} \quad i + f \mapsto m_f \tag{4.1}$$

for $j \in \{1, 2, \ldots, i\}$ and $f \in \{1, 2, \ldots, b - i\}$. Note here that by $e \mapsto n$ we mean that ω^{-1} takes the eth block to the nth block (with the block size $a \times a$).

To compute $E_{-1}(\phi, g)_{Q_i}$, we use the fact that the multi-residue operator $\lim_{\underline{s} \to \Lambda_b} \prod_{i=1}^{b-1}(s_i - s_{i+1} - 1)$ (where Λ_b is defined in (2.1)) and the constant term operator are interchangeable. Using the same argument as in the proof of Offen and Sayag [9, Lemma 2.4], we deduce that after applying the multi-residue operator, the only term left is the one corresponding to ω_i^{-1}, where $\omega_i = \begin{pmatrix} 0 & I_{ai} \\ I_{a(b-i)} & 0 \end{pmatrix}$.

Indeed, given an element ω^{-1} as in (4.1), let

$$\Delta^1(\omega) = \{1 \le j \le b - 1 | \omega(j) > \omega(j + 1)\},$$

$$\Delta^2(\omega) = \{1 \le j \le b - 1 | \omega(j + 1) - \omega(j) = 1\} \setminus \{\omega^{-1}(i)\}.$$

Then the normalized intertwining operator

$$N(\omega, \underline{s}) := \prod_{j \in \Delta^1(\omega)} (s_j - s_{j+1} - 1) M(\omega, \underline{s})$$

is holomorphic at Λ_b, and the normalized Eisenstein series

$$\prod_{j\in\Delta^2(\omega)}(s_j - s_{j+1} - 1)E^{Q_i}(N(\omega, \underline{s})\phi(\underline{s}), \omega\underline{s}, g)$$

is holomorphic at Λ_b. Therefore, the term corresponding to ω^{-1} survives after applying the multi-residue operator if and only if ω^{-1} has the property that

$$\Delta^1(\omega) \cup \Delta^2(\omega) = \{1, \ldots, b-1\},$$

which is equivalent to that $\omega \neq I_{ab}$, and there is no $1 \le j \le b-1$, such that $\omega(j+1) - \omega(j) > 1$. Note that if $\omega = I_{ab}$, then

$$\Delta^1(\omega) \cup \Delta^2(\omega) = \{1, \ldots, b-1\} \setminus \{i\} \neq \{1, \ldots, b-1\}.$$

Since the property that there is no $1 \le j \le b-1$, such that $\omega(j+1) - \omega(j) > 1$, implies that

$$\omega : \; j \mapsto i + j \quad \text{and} \quad b - i + f \mapsto f \tag{4.2}$$

for $j \in \{1, 2, \ldots, b-i\}$ and $f \in \{1, 2, \ldots, i\}$. This means that $\omega = \begin{pmatrix} 0 & I_{ai} \\ I_{a(b-i)} & 0 \end{pmatrix}$, or $\omega = I_{ab}$. After applying the multi-residue operator, the only term left is the one corresponding to ω_i^{-1}, where $\omega_i = \begin{pmatrix} 0 & I_{ai} \\ I_{a(b-i)} & 0 \end{pmatrix}$. Therefore, we prove the following identity

$$E_{-1}(\phi, g)_{Q_i} = E_{-1}^{Q_i}(M_{-1}(\omega_i)\phi, \mu_{ai,a(b-i)}, g),$$

where $\mu_{ai,a(b-i)} = \left(-\frac{b-i}{2}; \frac{i}{2}\right) \in \mathbb{C}^2$. We embed $\mathbb{C}^2$ to $\mathbb{C}^b$ by $(s_1, s_2) \hookrightarrow (s_1, \ldots, s_1; s_2, \ldots, s_2)$ with i-copies of s_1 and $(b-i)$-copies of s_2, and identify $\mathbb{C}^2$ with the image. Note that $\left(\Lambda_b = \left(\frac{b-1}{2}, \frac{b-3}{2}, \ldots, \frac{1-b}{2}\right)\right)$

$$\begin{aligned}
\omega_i \Lambda_b &= \left(\frac{2i-b-1}{2}, \frac{2i-b-3}{2}, \ldots, \frac{1-b}{2}; \frac{b-1}{2}, \frac{b-3}{2}, \ldots, \frac{2i-b+1}{2}\right) \\
&= \left(\frac{i-1}{2}, \ldots, \frac{1-i}{2}; \frac{(b-i)-1}{2}, \ldots, \frac{1-(b-i)}{2}\right) + \left(-\frac{b-i}{2}; \frac{i}{2}\right) \\
&= \Lambda_{ai,a(b-i)} + \mu_{ai,a(b-i)},
\end{aligned}$$

where

$$\Lambda_{ai,a(b-i)} = \left(\frac{i-1}{2}, \frac{i-3}{2}, \ldots, \frac{1-i}{2}; \frac{(b-i)-1}{2}, \frac{(b-i)-3}{2}, \ldots, \frac{1-(b-i)}{2}\right).$$

As discussed in [9, Page 10] (or in the proof of the Proposition 2.3 [9]), $E_{-1}^{Q_i}(M_{-1}(\omega_i)\phi, \mu_{ai,a(b-i)}, g)$ defines a surjective intertwining operator from $I(\tau, \Lambda_b, b)$ onto $\mathrm{Ind}_{Q_i(\mathbb{A})}^{\mathrm{GL}_{ab}(\mathbb{A})}(|\cdot|^{-\frac{b-i}{2}}\mathcal{E}_{(\tau,i)} \otimes |\cdot|^{\frac{i}{2}}\mathcal{E}_{(\tau,b-i)})$. Hence there exists a section

$$f \in \mathrm{Ind}_{Q_i(\mathbb{A})}^{\mathrm{GL}_{ab}(\mathbb{A})}\left(|\cdot|^{-\frac{b-i}{2}}\mathcal{E}_{(\tau,i)} \otimes |\cdot|^{\frac{i}{2}}\mathcal{E}_{(\tau,b-i)}\right)$$

such that

$$E_{-1}(\phi, g)_{Q_i} = E_{-1}^{Q_i}(M_{-1}(\omega_i)\phi, \mu_{Q_{ai,a(b-i)}}, g) = f(g)(I_{ai} \times I_{a(b-i)}).$$

This finishes the proof. ∎

Now, we are ready to prove the technical Lemma 3.2. This is to show that for the residue $E_{-1}(\phi, g)$ of the Eisenstein series $E(\phi, \underline{s}, g)$ as defined in (2.1), the Fourier coefficient $E_{-1}^{\psi^{ab}_{a^+;\gamma\cdot a}}(\phi, g)$ is identically zero, for $\alpha = a(b-\gamma-1)$, $\beta = \delta = a$, and $\gamma = 0, 1, 2, \ldots, b-2$. Note that when $\gamma = b-1$,

$$\psi^{ab}_{a^+;(b-1)\cdot a} = \psi^{ab}_{a;(b-1)\cdot a} = \psi^{ab}_{0;b\cdot a}.$$

By using the cuspidal support of the residue $E_{-1}(\phi, g)$, we have the following Fourier expansion for the Fourier coefficient $E_{-1}^{\psi^{ab}_{a^+;\gamma\cdot a}}(\phi, g)$

$$\begin{aligned} E_{-1}^{\psi^{ab}_{a^+;\gamma\cdot a}}(\phi, g) = & \sum_{\epsilon \in Q^0_{a(b-\gamma-2),1^*}(k)\backslash Q^0_{a(b-\gamma-1)-1,1^*}(k)} E_{-1}^{\psi^{ab}_{2a;\gamma\cdot a}}(\phi, \epsilon g) \\ & + \sum_{\epsilon^+ \in Q^0_{a(b-\gamma-2)-1,1^*}(k)\backslash Q^0_{a(b-\gamma-1)-1,1^*}(k)} E_{-1}^{\psi^{ab}_{2a^+;\gamma\cdot a}}(\phi, \epsilon^+ g). \end{aligned} \tag{4.3}$$

Here we use $Q^0_{m,1^*} := Q^0_{m,1^{ab-m}}$ to simplify the notation.

We show that the Fourier coefficient $E_{-1}^{\psi^{ab}_{2a;\gamma\cdot a}}(\phi, g)$ is identically zero. Based on this, the vanishing of $E_{-1}^{\psi^{ab}_{a^+;\gamma\cdot a}}(\phi, g)$ is equivalent to the vanishing of $E_{-1}^{\psi^{ab}_{2a^+;\gamma\cdot a}}(\phi, \epsilon^+ g)$. By using the inductive argument on la for $l = 1, 2, \ldots, b-\gamma$, which is based on the vanishing of the Fourier coefficient $E_{-1}^{\psi^{ab}_{la;\gamma\cdot a}}(\phi, g)$ for each $l = 2, 3, \ldots, b-\gamma$, it follows that the vanishing of $E_{-1}^{\psi^{ab}_{a^+;\gamma\cdot a}}(\phi, g)$ is equivalent to the vanishing of $E_{-1}^{\psi^{ab}_{(b-\gamma)a^+;\gamma\cdot a}}(\phi, \epsilon^+ g)$, which is the same as $E_{-1}^{\psi^{ab}_{(b-\gamma)a;\gamma\cdot a}}(\phi, \epsilon^+ g)$.

Hence, in order to prove Lemma 3.2, it is enough to prove the following lemma.

Lemma 4.2. For $\gamma = 0, 1, 2, \ldots, b-2$ and $l = 2, 3, \ldots, b-\gamma$, the Fourier coefficient $E_{-1}^{\psi^{ab}_{la;\gamma\cdot a}}(\phi, g)$ is identically zero. □

Proof. First, when $\gamma = 0$ and $l = b-\gamma = b$, the Fourier coefficient $E_{-1}^{\psi^{ab}_{ba;0\cdot a}}(\phi, g)$ is exactly the Whittaker–Fourier coefficient of the residue $E_{-1}(\phi, g)$, which is identically zero by Proposition 2.1.

In the following, we use Lemma 4.1 twice to reduce the general case to the above special case with lower rank. Hence those Fourier coefficients must all be zero identically.

We assume that $l < b-\gamma$, and we show, by using Lemma 4.1, that this will reduce to the case $l = b-\gamma$, which will be treated next.

Recall the parabolic subgroup $Q_{b-\gamma-l} = L_{b-\gamma-l}V_{b-\gamma-l}$ of GL_{ab} from Lemma 4.1, with $L_{b-\gamma-l} = \mathrm{GL}_{a(b-\gamma-l)} \times \mathrm{GL}_{a(\gamma+l)}$. By the definition of the Fourier coefficient $E_{-1}^{\psi^{ab}_{la;\gamma\cdot a}}(\phi, g)$, the constant term of the residue $E_{-1}(\phi, g)$ along $Q_{b-\gamma-l}$ is an inner integration of $E_{-1}^{\psi^{ab}_{la;\gamma\cdot a}}(\phi, g)$. More precisely, we have

$$E_{-1}^{\psi^{ab}_{la;\gamma\cdot a}}(\phi, g) = [E_{-1}(\phi, g)_{Q_{b-\gamma-l}}]^{\psi^{a(\gamma+l)}_{la;\gamma\cdot a}}. \tag{4.4}$$

Note here that the $\psi^{a(\gamma+l)}_{la;\gamma\cdot a}$-Fourier coefficient is taken from the subgroup $\mathrm{GL}_{\gamma+l}$, which is the second factor in the Levi subgroup $L_{b-\gamma-l}$.

By Lemma 4.1, there exists a section f belonging to

$$\mathrm{Ind}^{\mathrm{GL}_{ab}(\mathbb{A})}_{Q_{b-\gamma-l}(\mathbb{A})}(|\cdot|^{-\frac{\gamma+l}{2}}\mathcal{E}_{(\tau,b-\gamma-l)} \otimes |\cdot|^{\frac{b-\gamma-l}{2}}\mathcal{E}_{(\tau,\gamma+l)}),$$

such that

$$E_{-1}(\phi, g)_{Q_{b-\gamma-l}} = f(g)(I_{a(b-\gamma-l)} \times I_{a(\gamma+l)}).$$

Since the $\psi^{a(\gamma+l)}_{la;\gamma\cdot a}$-Fourier coefficient of the constant term $E_{-1}(\phi, g)_{Q_{b-\gamma-l}}$ is taken from the subgroup $\mathrm{GL}_{\gamma+l}$, it suffices to show that the residual representation $\mathcal{E}_{(\tau,\gamma+l)}$ of $\mathrm{GL}_{a(\gamma+l)}(\mathbb{A})$ has no nonzero $\psi^{a(\gamma+l)}_{la;\gamma\cdot a}$-Fourier coefficients. This reduces the problem from GL_{ab} to $\mathrm{GL}_{a(\gamma+l)}$. Note that this reduces the general case $l < b-\gamma$ to the case $l = b-\gamma$ for $b = \gamma + l$.

Now take $E_{-1}(\phi_{\tau^{\otimes(\gamma+l)}}, g)$ from the space of the residual representation $\mathcal{E}_{(\tau,\gamma+l)}$ of $\mathrm{GL}_{a(\gamma+l)}(\mathbb{A})$. Consider the standard maximal parabolic subgroup $Q_{l,\gamma} = L_{l,\gamma} V_{l,\gamma}$ of $\mathrm{GL}_{a(\gamma+l)}$ with Levi part $L_{l,\gamma} = \mathrm{GL}_{la} \times \mathrm{GL}_{\gamma a}$. By the definition of the $\psi^{a(\gamma+l)}_{la;\gamma\cdot a}$-Fourier coefficient of $E_{-1}(\phi_{\tau^{\otimes(\gamma+l)}}, g)$, the constant term of the residue $E_{-1}(\phi_{\tau^{\otimes(\gamma+l)}}, g)$ along $Q_{l,\gamma}$ occurs as an inner integration in the $\psi^{a(\gamma+l)}_{la;\gamma\cdot a}$-Fourier coefficient of $E_{-1}(\phi_{\tau^{\otimes(\gamma+l)}}, g)$. As before, we write it more precisely as follows:

$$[E_{-1}(\phi_{\tau^{\otimes(\gamma+l)}}, g)]^{\psi^{a(\gamma+l)}_{la;\gamma\cdot a}} = [E_{-1}(\phi_{\tau^{\otimes(\gamma+l)}}, g)_{Q_{l,\gamma}}]^{\psi^{la}_{la;0\cdot a};\psi^{\gamma a}_{0;\gamma\cdot a}}.$$

After taking the constant term along $Q_{l,\gamma}$, $E_{-1}(\phi_{\tau^{\otimes(\gamma+l)}}, g)_{Q_{l,\gamma}}$ is an automorphic function over $\mathrm{GL}_{la}(\mathbb{A}) \times \mathrm{GL}_{\gamma a}(\mathbb{A})$. Note here that the $\psi^{la}_{la;0\cdot a}$-Fourier coefficient is taking on $\mathrm{GL}_{la}(\mathbb{A})$ and the $\psi^{\gamma a}_{0;\gamma\cdot a}$-Fourier coefficient is taken on $\mathrm{GL}_{\gamma a}(\mathbb{A})$.

By Lemma 4.1 again (applied to $\mathrm{GL}_{a(\gamma+l)}$), it is enough to show that the residual representation $\mathcal{E}_{(\tau,l)}$ has no nonzero $\psi^{la}_{2a;0\cdot a}$-Fourier coefficients or the residual representation $\mathcal{E}_{(\tau,\gamma)}$ has no nonzero $\psi^{\gamma a}_{0;\gamma\cdot a}$-Fourier coefficients. It is clear that the character $\psi^{la}_{la;0\cdot a}$ is exactly the Whittaker character of $\mathrm{GL}_{la}(\mathbb{A})$. By Proposition 2.1, the residual representation $\mathcal{E}_{(\tau,l)}$ is not generic, and hence it has no nonzero $\psi^{la}_{la;0\cdot a}$-Fourier coefficients. ■

This completes the proof of Lemma 3.2.

5 Fourier Coefficients for GL_n

In Theorem 3.3 and Corollary 3.4, we show that the residue $E_{-1}(\phi, g)$ of the Eisenstein series $E(\phi, \underline{s}, g)$, with cuspidal datum $(P_{a^b}, \tau^{\otimes b})$, has a nonzero degenerate Whittaker–Fourier coefficient $E_{-1}^{\psi^{ab}_{0;b\cdot a}}(\phi, g)$. In this section, we give the definition of Fourier coefficients of automorphic forms attached to unipotent orbits or partitions of n, and show that this degenerate Whittaker–Fourier coefficient for the residue $E_{-1}(\phi, g)$ is analogous to the Whittaker–Fourier coefficient for the cuspidal automorphic forms on $\mathrm{GL}_n(\mathbb{A})$, as remarked at the end of Section 3. In other words, we show that, according to the partial ordering of partitions or unipotent orbits, the Fourier coefficient $E_{-1}^{\psi^{ab}_{0;b\cdot a}}(\phi, g)$ is equivalent (for the nonvanishing property) to the biggest possible Fourier coefficient that the residue $E_{-1}(\phi, g)$ can possibly have.

5.1 Fourier coefficients for GL_n

We consider the Fourier coefficients of automorphic forms of $\mathrm{GL}_n(\mathbb{A})$ attached to unipotent k-orbits under the $\mathrm{GL}_n(k)$-adjoint action, following the idea of Ginzburg et al. [2] and Ginzburg [1] for the global theory and of Moeglin et al. [6] for the local theory.

When $G = \mathrm{GL}_n$, each unipotent k-orbit $\mathcal{O}$ of $\mathrm{GL}_n(k)$ has an element in the standard Jordan form, which is unique up to permutation (conjugation by a certain Weyl group element), and hence is characterized by a standard partition of n: $\underline{p} = [p_1 p_2 \cdots p_r]$ with $p_1 \geq p_2 \geq \cdots \geq p_r > 0$ and $n = \sum_{i=1}^{r} p_i$. We denote by $\mathcal{J}_{\underline{p}}$ the unipotent Jordan matrix in the unipotent k orbit $\mathcal{O}_{\underline{p}}$ determined by the partition $\underline{p}$. Since $\mathcal{J}_{\underline{p}}$ is of standard Jordan form, there is a one-dimensional toric subgroup $\mathcal{H}_{\underline{p}}$ of $\mathrm{GL}_n(k)$:

$$\mathcal{H}_{\underline{p}}(t) := \mathrm{diag}(\mathcal{H}_{[p_1]}(t), \mathcal{H}_{[p_2]}(t), \ldots, \mathcal{H}_{[p_r]}(t)) \tag{5.1}$$

with $\mathcal{H}_{[p_i]}(t) := \mathrm{diag}(t^{p_i - 1}, t^{p_i - 3}, \ldots, t^{3-p_i}, t^{1-p_i})$ for $i = 1, 2, \ldots, r$, and $t \in k^\times$, such that

$$\forall\, t \in k^\times,\ \mathrm{Ad}(\mathcal{H}_{\underline{p}}(t))(\mathcal{J}_{\underline{p}}) = t^2 \mathcal{J}_{\underline{p}}.$$

Take $\mathcal{J}_{\underline{p}}^-$ to be the opposite to $\mathcal{J}_{\underline{p}}$. It is clear that

$$\{\mathcal{J}_{\underline{p}}, \mathcal{H}_{\underline{p}}, \mathcal{J}_{\underline{p}}^-\}$$

generates the k-SL_2 attached to the k-orbit $\mathcal{O}_{\underline{p}}$. Under the adjoint action, the Lie algebra $\mathfrak{gl}_n(k)$ of $\mathrm{GL}_n(k)$ decomposes into a direct sum of $\mathrm{Ad}(\mathcal{H}_{\underline{p}})$-eigenspaces:

$$\mathfrak{gl}_n(k) = \mathfrak{g}_{-m} \oplus \cdots \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \cdots \oplus \mathfrak{g}_m \tag{5.2}$$

for some m, where $\mathfrak{g}_l := \{X \in \mathfrak{gl}_n(k) \mid \mathrm{Ad}(\mathcal{H}_{\underline{p}}(t))(X) = t^l \cdot X\}$.

Let $V_{\underline{p},j}(k)$ (with $j = 1, 2, \ldots, m$) denote the unipotent subgroup of $\mathrm{GL}_n(k)$ whose Lie algebra is $\oplus_{l=j}^{m} \mathfrak{g}_l$. Let $L_{\underline{p}}(k)$ be the algebraic subgroup of $\mathrm{GL}_n(k)$ such that its Lie algebra is $\mathfrak{g}_0$. It is easy to check that $\mathcal{J}_{\underline{p}}$ belongs to $V_{\underline{p},2}(k)$. Under the adjoint action, the set $\mathrm{Ad}(L_{\underline{p}}(k))(\mathcal{J}_{\underline{p}})$ is Zariski open dense in the affine space $V_{\underline{p},2}(k)/V_{\underline{p},3}(k)$. Hence one may use the representative $\mathcal{J}_{\underline{p}}$ of the k-orbit $\mathcal{O}_{\underline{p}}$ to define a (generic) character $\psi_{\underline{p}}$ of $V_{\underline{p},2}(k)$. Let $Q_{\underline{p}}$ be the standard parabolic subgroup of GL_n corresponding to the partition $\underline{p}$. The Levi subgroup $M_{\underline{p}}$ is $\mathrm{GL}_{p_1} \times \mathrm{GL}_{p_2} \times \cdots \times \mathrm{GL}_{p_r}$. It is clear that the intersection $M_{\underline{p}} \cap V_{\underline{p},2}$ is $U_{p_1} \times U_{p_2} \times \cdots \times U_{p_r}$, where U_{p_i} is the standard maximal unipotent subgroup (the radical of the standard Borel subgroup) of GL_{p_i}. We define a character of $V_{\underline{p},2}$ as follows: for any $v \in V_{\underline{p},2}$,

$$\begin{aligned}\psi_{\underline{p}}(v) &:= \psi(\mathrm{tr}((\mathcal{J}_{\underline{p}}^- - I_n)\log(v)))\\ &= \psi(v_{1,2} + \cdots + v_{p_1-1,p_1}\end{aligned}$$

$$
\begin{aligned}
&\cdot \psi(v_{p_1+1,p_1+2} + \cdots + v_{p_1+p_2-1,p_1+p_2}) \\
&\vdots \\
&\cdot \psi(v_{p_1+\cdots+p_{r-2}+1,p_1+\cdots+p_{r-2}+2} + \cdots + v_{p_1+\cdots+p_{r-1}-1,p_1+\cdots+p_{r-1}}) \\
&\cdot \psi(v_{p_1+\cdots+p_{r-1}+1,p_1+\cdots+p_{r-1}+2} + \cdots + v_n).
\end{aligned} \tag{5.3}
$$

Note that $\psi_{\underline{p}}$ also defines a nondegenerate (Whittaker) character of $M_{\underline{p}}$.

Let ϕ be an automorphic form on $\mathrm{GL}_n(\mathbb{A})$. We define the $\psi_{\underline{p}}$-Fourier coefficient of ϕ attached to the partition $\underline{p}$ or the unipotent orbit $\mathcal{O}_{\underline{p}}$ by the following integral:

$$
\phi^{\psi_{\underline{p}}}(g) := \int_{V_{\underline{p},2}(k)\backslash V_{\underline{p},2}(\mathbb{A})} \phi(vg)\psi_{\underline{p}}^{-1}(v)\,\mathrm{d}v. \tag{5.4}
$$

Note that the definition of the $\psi_{\underline{p}}$-Fourier coefficient of ϕ depends on the choice of the representative $\mathcal{J}_{\underline{p}}$ (and the semisimple element $\mathcal{H}_{\underline{p}}$).

According to the k-rational version of the Jacobson–Morozov Theorem [6], the Fourier coefficient of ϕ can be defined by means of any choice of representatives in the unipotent k-orbit $\mathcal{O}_{\underline{p}}$. Since ϕ is automorphic, the vanishing or nonvanishing of the $\psi_{\underline{p}}$-Fourier coefficient of ϕ depends only on the k-orbit $\mathcal{O}_{\underline{p}}$.

Let π be an irreducible automorphic representation of $\mathrm{GL}_n(\mathbb{A})$ occurring as a subspace of the discrete spectrum of square-integrable automorphic functions on $\mathrm{GL}_n(\mathbb{A})$. We say that π has a $\psi_{\underline{p}}$-Fourier coefficient if there is a function $\phi \in \pi$ such that $\phi^{\psi_{\underline{p}}}(g)$ is nonzero. As discussed above, the property that π has a $\psi_{\underline{p}}$-Fourier coefficient depends only on the k-orbit $\mathcal{O}_{\underline{p}}$.

For $\mathfrak{g}_i$, as defined in (5.2), let G_i^+ (G_i^-, respectively) be the union of all one-parameter subgroups $X_\alpha(x)$ whose Lie algebra is in $\mathfrak{g}_i$, with positive (negative, respectively) roots α, in the root system determined by $(\mathrm{GL}_n, B_n, T_n)$. It is easy to see that both G_1^+ and G_1^- have group structures and are abelian. In the following, by saying that one entry in GL_n is in G_i^+ or G_i^-, we mean that the corresponding element in the associated one-parameter subgroup is in G_i^+ or G_i^-.

Recall that $V_{\underline{p},1}(k)$ is the unipotent subgroup of $\mathrm{GL}_n(k)$ whose Lie algebra is $\mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \cdots \oplus \mathfrak{g}_m$. Following [6], we define

$$
(\mathcal{J}_{\underline{p}}^-)^\sharp = \{X \in \mathfrak{g} \mid \mathrm{tr}((\mathcal{J}_{\underline{p}}^- - I_n)[X, X']) = 0, \forall X' \in \mathfrak{g}\}.
$$

Define $V'_{\underline{p},2} = \exp(\mathfrak{g}_1 \cap (\mathcal{J}^-_{\underline{p}})^\sharp)V_{\underline{p},2}$, which is a normal subgroup of $V_{\underline{p},1}(k)$. From the definition of $(\mathcal{J}^-_{\underline{p}})^\sharp$, it is easy to see that the character $\psi_{\underline{p}}$ on $V_{\underline{p},2}$ can be trivially extended to $V'_{\underline{p},2}$, which we still denote by $\psi_{\underline{p}}$. It turns out that $V_{\underline{p},1}/\ker_{V'_{\underline{p},2}}(\psi_{\underline{p}})$ has a Heisenberg structure $W \oplus Z$ (see [6, Section 1.7]), where $W \cong V_{\underline{p},1}/V'_{\underline{p},2}$ and $Z \cong V'_{\underline{p},2}/\ker_{V'_{\underline{p},2}}(\psi_{\underline{p}})$. Note that the symplectic form on W is the one inherited from the Lie algebra bracket, that is, for $w_1, w_2 \in W$ (here, we identify $w \in W$ with any of it's representatives in $V_{\underline{p},1}$ such that $\log(w) \in \mathfrak{g}_1$),

$$\begin{aligned}\langle w_1, w_2\rangle &= \mathrm{tr}((\mathcal{J}^-_{\underline{p}} - I_n)\log([w_1, w_2]))\\ &= \mathrm{tr}((\mathcal{J}^-_{\underline{p}} - I_n)[\log(w_1), \log(w_2)]).\end{aligned}$$

The nondegeneracy of this symplectic form can be checked easily as following: for fixed $w_1 \in W$, if $\langle w_1, w_2\rangle \equiv 0$, for any $w_2 \in W$, that is, $\mathrm{tr}((\mathcal{J}^-_{\underline{p}} - I_n)[\log(w_1), \log(w_2)]) \equiv 0$, for any $w_2 \in W$, that is,

$$\mathrm{tr}((\mathcal{J}^-_{\underline{p}} - I_n)[\log(w_1), X']) \equiv 0,$$

for any $X' \in \mathfrak{g}_1$, which implies that $\mathrm{tr}((\mathcal{J}^-_{\underline{p}} - I_n)[\log(w_1), X']) \equiv 0$, for any $X' \in \mathfrak{g}$, that is, $\log(w_1) \in (\mathcal{J}^-_{\underline{p}})^\sharp$, that is, $w_1 = 0 \in V_{\underline{p},1}/V'_{\underline{p},2}$.

Lemma 5.1. $V'_{\underline{p},2} = V_{\underline{p},2}$. □

Proof. As discussed at the beginning of this subsection, the partition $\underline{p} = [p_1 p_2 \cdots p_r]$ gives rise to the SL_2-triple

$$\{\mathcal{J}_{\underline{p}}, \mathcal{H}_{\underline{p}}, \mathcal{J}^-_{\underline{p}}\}.$$

If, under the adjoint action of $\mathcal{H}_{\underline{p}}$ on the Lie algebra $\mathfrak{gl}_n(k)$, the space $\mathfrak{g}_1$ as defined in (5.2) is zero, there is nothing to prove. In the following we assume that $\mathfrak{g}_1$ is not zero. To prove $V'_{\underline{p},2} = V_{\underline{p},2}$, it suffices to prove that $\mathfrak{g}_1 \cap (\mathcal{J}^-_{\underline{p}})^\sharp = \{0\}$.

First let us describe $V_{\underline{p},2}$. Elements in $V_{\underline{p},2}$ have the following form:

$$v = \begin{pmatrix} \ddots & & & \\ & n_i & q^{i,j} & \\ & & \ddots & \\ & 0 & n_j & \\ & & & \ddots \end{pmatrix}\begin{pmatrix} \ddots & & & \\ & I_{p_i} & 0 & \\ & & \ddots & \\ & p^{i,j} & I_{p_j} & \\ & & & \ddots \end{pmatrix},$$

where $n_i \in U_{p_i}$ ($n_j \in U_{p_j}$, respectively), the maximal unipotent radical of GL_{p_i} (GL_{p_j}, respectively), $q^{i,j} \in M_{p_i \times p_j}$, and $p^{i,j} \in M_{p_j \times p_i}$ satisfy some conditions. Since in this lemma, we only need to care about (i, j) such that p_i and p_j are of different parity, we describe the conditions for $q^{i,j}$ and $p^{i,j}$ only for the case that p_i and p_j are of different parity: $q^{i,j} \in M_{p_i \times p_j}$ with $q^{i,j}_{l,m} = 0$, for $l \geq m + \frac{p_i - p_j - 1}{2}$, $p^{i,j} \in M_{p_j \times p_i}$ with $p^{i,j}_{l,m} = 0$, for $m \leq l + \frac{p_i - p_j - 1}{2} + 1$.

According to the structure of the space $\mathfrak{g}_1$, we define abelian groups Y and X, which are given by

$$Y = \prod_{1 \leq i < j \leq r,\, p_i \text{ and } p_j \text{ are of different parity}} Y^{i,j},$$
$$Y^{i,j} = \prod_{l=1}^{p_j} X_{\alpha_l^{i,j}}(y_l^{i,j}), \tag{5.5}$$

where $\alpha_l^{i,j} = e_{\sum_{m=1}^{i-1} p_m + \frac{p_i - p_j - 1}{2} + l} - e_{\sum_{m=1}^{j-1} p_m + l}$; and

$$X = \prod_{1 \leq i < j \leq r,\, p_i \text{ and } p_j \text{ are of different parity}} X^{i,j},$$
$$X^{i,j} = \prod_{l=1}^{p_j} X_{\beta_l^{i,j}}(x_l^{i,j}), \tag{5.6}$$

where $\beta_l^{i,j} = e_{\sum_{m=1}^{j-1} p_m + l} - e_{\sum_{m=1}^{i-1} p_m + \frac{p_i - p_j - 1}{2} + l + 1}$. Then, we can see that $\mathfrak{g}_1 = \log(X) \oplus \log(Y)$. Therefore, to show that $\mathfrak{g}_1 \cap (\mathcal{J}_{\underline{p}}^-)^\sharp = \{0\}$, we only need to show that $(\log(X) \oplus \log(Y)) \cap (\mathcal{J}_{\underline{p}}^-)^\sharp = \{0\}$. It suffices to show that for $1 \leq i < j \leq r$, such that p_i and p_j are of different parity, and for any $l = 1, \ldots, p_j$, both $\log(X_{\alpha_l^{i,j}}(y_l^{i,j}))$ and $\log(X_{\beta_l^{i,j}}(x_l^{i,j}))$ are not in $(\mathcal{J}_{\underline{p}}^-)^\sharp$, where $0 \neq y_l^{i,j}, x_l^{i,j} \in k^\times$. This is true, since by direct calculation, when $y_l^{i,j}, x_l^{i,j} \neq 0$,

$$\mathrm{tr}((\mathcal{J}_{\underline{p}}^- - I_n)[\log(X_{\beta_l^{i,j}}(x_l^{i,j})), \log(X_{\alpha_l^{i,j}}(y_l^{i,j}))]) = -x_l^{i,j} y_l^{i,j} \neq 0.$$

This completes the proof of the lemma. ■

Therefore, by Lemma 5.1 and the discussion above, $V_{\underline{p},1}/\ker_{V_{\underline{p},2}}(\psi_{\underline{p}})$ has a Heisenberg structure $W \oplus Z$, where $Z \cong V_{\underline{p},2}/\ker_{V_{\underline{p},2}}(\psi_{\underline{p}})$, and $X \oplus Y$ is a polarization of W, where X, Y are defined in (5.6) and (5.5).

In more explicit calculations of Fourier coefficients of automorphic forms, there is a very useful lemma, which has been used in many occasions and is now formulated in a general term in [3, Corollary 7.1]. In order to fit it better for our use in this paper, we reformulate it in a slightly different way and use a slightly different argument to prove the GL_n-analog of the useful lemma.

Let C be an F-subgroup of a maximal unipotent subgroup of GL_n, and let ψ_C be a nontrivial character of $[C]=C(F)\setminus C(\mathbb{A})$. $\tilde{X}$, $\tilde{Y}$ are two unipotent F-subgroups, satisfying the following conditions:

(1) $\tilde{X}$ and $\tilde{Y}$ normalize C;
(2) $\tilde{X}\cap C$ and $\tilde{Y}\cap C$ are normal in $\tilde{X}$ and Y, respectively, $(\tilde{X}\cap C)\setminus\tilde{X}$ and $(\tilde{Y}\cap C)\setminus\tilde{Y}$ are abelian;
(3) $\tilde{X}(\mathbb{A})$ and $\tilde{Y}(\mathbb{A})$ preserve ψ_C;
(4) ψ_C is trivial on $(\tilde{X}\cap C)(\mathbb{A})$ and $(\tilde{Y}\cap C)(\mathbb{A})$;
(5) $[\tilde{X},\tilde{Y}]\subset C$;
(6) there is a nondegenerate pairing $(\tilde{X}\cap C)(\mathbb{A})\times(\tilde{Y}\cap C)(\mathbb{A})\to\mathbb{C}^*$, given by $(x,y)\mapsto\psi_C([x,y])$, which is multiplicative in each coordinate, and identifies $(\tilde{Y}\cap C)(F)\setminus\tilde{Y}(F)$ with the dual of $\tilde{X}(F)(\tilde{X}\cap C)(\mathbb{A})\setminus\tilde{X}(\mathbb{A})$, and $(\tilde{X}\cap C)(F)\setminus\tilde{X}(F)$ with the dual of $\tilde{Y}(F)(\tilde{Y}\cap C)(\mathbb{A})\setminus\tilde{Y}(\mathbb{A})$.

Let $B=C\tilde{Y}$ and $D=C\tilde{X}$, and extend ψ_C trivially to characters of $[B]=B(F)\setminus B(\mathbb{A})$ and $[D]=D(F)\setminus D(\mathbb{A})$, which will be denoted by ψ_B and ψ_D, respectively. Here is the reformulation of the useful lemma, the proof of which is valid for the general group $H(\mathbb{A})$ as in [3].

Lemma 5.2. Assume the quadruple $(C,\psi_C,\tilde{X},\tilde{Y})$ satisfies the above conditions. Let f be an automorphic form on $\mathrm{GL}_n(\mathbb{A})$. Then

$$\int_{[C]} f(cg)\psi_C^{-1}(c)\,\mathrm{d}c\equiv 0,\ \forall g\in\mathrm{GL}_n(\mathbb{A}),$$

if and only if

$$\int_{[D]} f(ug)\psi_D^{-1}(u)\,\mathrm{d}u\equiv 0,\ \forall g\in\mathrm{GL}_n(\mathbb{A}),$$

if and only if

$$\int_{[B]} f(vg)\psi_B^{-1}(v)\,\mathrm{d}v\equiv 0,\ \forall g\in\mathrm{GL}_n(\mathbb{A}).\qquad\square$$

Proof. By symmetry, we only need to show that

$$\int_{[C]} f(cg)\psi_C^{-1}(c)\,\mathrm{d}c\equiv 0,\ \forall g\in\mathrm{GL}_n(\mathbb{A}),$$

if and only if

$$\int_{[D]} f(ug)\psi_D^{-1}(u)\,\mathrm{d}u \equiv 0, \forall g \in \mathrm{GL}_n(\mathbb{A}).$$

Since

$$\int_{[D]} f(ug)\psi_D^{-1}(u)\,\mathrm{d}u = \int_{\tilde{X}(F)(\tilde{X}\cap C)(\mathbb{A})\backslash \tilde{X}(\mathbb{A})} \int_{[C]} f(cxg)\psi_C^{-1}(c)\,\mathrm{d}c\,\mathrm{d}y,$$

we know that if

$$\int_{[C]} f(cg)\psi_C^{-1}(c)\,\mathrm{d}c \equiv 0, \forall g \in \mathrm{GL}_n(\mathbb{A}),$$

then

$$\int_{[D]} f(ug)\psi_D^{-1}(u)\,\mathrm{d}u \equiv 0, \forall g \in \mathrm{GL}_n(\mathbb{A}).$$

On the other hand, by Formula [3, (7.5)],

$$\int_{[C]} f(cg)\psi_C^{-1}(c)\,\mathrm{d}c = \sum_{y' \in (\tilde{Y}\cap C)(F)\backslash \tilde{Y}(F)} \int_{[D]} f(uy'g)\psi_D^{-1}(u)\,\mathrm{d}u,$$

which implies that if

$$\int_{[D]} f(ug)\psi_D^{-1}(u)\,\mathrm{d}u \equiv 0, \forall g \in \mathrm{GL}_n(\mathbb{A}),$$

then

$$\int_{[C]} f(cg)\psi_C^{-1}(c)\,\mathrm{d}c \equiv 0, \forall g \in \mathrm{GL}_n(\mathbb{A}).$$

This completes the proof of the lemma. ■

Note that when we apply this lemma in the remaining of the paper, we always denote ψ_B and ψ_D by ψ_C for convenience.

The following corollary gives an important property of the $\psi_{\underline{p}}$-Fourier coefficients for automorphic forms on $\mathrm{GL}_n(\mathbb{A})$. The corresponding case for symplectic group is given in [2, Lemma 1.1].

Corollary 5.3. Let $\underline{p} = [p_1 p_2 \cdots p_r]$ be a standard partition of n, that is, $p_1 \geq p_2 \geq \cdots \geq p_r > 0$ and $n = \sum_{i=1}^r p_i$. Let ϕ be an automorphic form on $\mathrm{GL}_n(\mathbb{A})$. Then $\phi^{\psi_{\underline{p}}}$, the $\psi_{\underline{p}}$-Fourier coefficient of ϕ is nonvanishing if and only if the following integral is nonvanishing:

$$\int_{[Y]}\int_{[V_{\underline{p},2}]}\phi(vyg)\psi_{\underline{p}}^{-1}(v)\,\mathrm{d}v\,\mathrm{d}y,$$

where the subgroup Y is defined in (5.5). □

Proof. This is a consequence of Lemmas 5.1 and 5.2. In fact, by Lemma 5.1 and the discussion before it, we know that $V_{\underline{p},1}/\ker_{V_{\underline{p},2}}(\psi_{\underline{p}})$ has a Heisenberg structure $W\oplus Z$, where $Z\cong V_{\underline{p},2}/\ker_{V_{\underline{p},2}}(\psi_{\underline{p}})$ and $X\oplus Y$ is a polarization of W, where X, Y are defined in (5.6) and (5.5). This implies directly that the quadruple $(V_{\underline{p},2}, \psi_{\underline{p}}, X, Y)$ satisfies all the conditions for Lemma 5.2. ■

5.2 Fourier coefficients for the discrete spectrum of GL_n

Recall from [10, 12] that any nonzero irreducible cuspidal automorphic representation π of $\mathrm{GL}_n(\mathbb{A})$ is generic, that is, has a nonzero Whittaker–Fourier coefficient. From the definition, the Whittaker–Fourier coefficient of π is the one attached to the partition $\underline{p}=[n]$.

In the following, we assume that $n=ab$ with $b>1$, and consider $\pi=\mathcal{E}_{(\tau,b)}$, the residual representation of $\mathrm{GL}_{ab}(\mathbb{A})$ with cuspidal support $(P_{a^b}, \tau^{\otimes b})$.

Theorem 5.4. For any residue $E_{-1}(\phi,\cdot)$ in the residual representation $\mathcal{E}_{(\tau,b)}$ of $\mathrm{GL}_{ab}(\mathbb{A})$ with cuspidal support $(P_{a^b}, \tau^{\otimes b})$, the $\psi^{ab}_{0;b\cdot a}$-Fourier coefficient of $E_{-1}(\phi,\cdot)$, denoted by $E_{-1}^{\psi^{ab}_{0;b\cdot a}}(\phi, g)$, is nonvanishing for some choice of data if and only if the $\psi_{[a^b]}$-Fourier coefficient of $E_{-1}(\phi,\cdot)$, denoted by $E_{-1}^{\psi_{[a^b]}}(\phi, g)$, is nonvanishing for some choice of data. □

Proof. If $a=1$, then $b=n$ and the residual representation $\mathcal{E}_{\tau,b}$ is just $\chi\circ\det$, a character of $\mathrm{GL}_n(\mathbb{A})$, which of course has only the trivial Fourier coefficient attached to the partition $[1^n]$ of n. The theorem holds for this case.

When $n=a$, $E_{-1}^{\psi^{n}_{0;1\cdot n}}(\phi, g)=E_{-1}^{\psi_{[n]}}(\phi, g)$. We are done for this case, since in this case the parabolic subgroup is trivial, that is, the whole group GL_n, and hence the automorphic form considered is cuspidal.

We only need to consider the case $1<a<n$. In order to use the induction argument, we assume that for $n=a, 2a, \ldots, a(b-1)$, the equivalence of the nonvanishing of both Fourier coefficients holds. We are going to prove that it will also be true for $n=ab$.

We start from $E_{-1}^{\psi_{[a^b]}}(\phi, g)$, the $\psi_{[a^b]}$-Fourier coefficient of the residue $E_{-1}(\phi, \cdot)$. In order to apply Lemma 5.2 to the following integral

$$E_{-1}^{\psi_{[a^b]}}(\phi, g) = \int_{V_{[a^b],2}(k)\backslash V_{[a^b],2}(\mathbb{A})} E_{-1}(\phi, vg)\psi_{[a^b]}^{-1}(v)\,\mathrm{d}v, \tag{5.7}$$

we conjugate it by the Weyl element ω of GL_n which conjugates the one-parameter toric subgroup $\mathcal{H}_{[a^b]}$ in (5.1) corresponding to the partition $[a^b]$ to the following toric subgroup:

$$\mathrm{diag}(\mathcal{H}_{[a]}(t); t^{a-1}I_{b-1}, t^{a-3}I_{b-1}, \ldots, t^{3-a}I_{b-1}, t^{1-a}I_{b-1}),$$

where $\mathcal{H}_{[a]}(t) = \mathrm{diag}(t^{a-1}, t^{a-3}, \ldots, t^{3-a}, t^{1-a})$. Note that ω is of the form $\mathrm{diag}(I_a, \omega')$, where ω' permutes the toric subgroup $\mathcal{H}_{[a^{b-1}]}$ in (5.1) corresponding to the partition $[a^{b-1}]$ to the toric subgroup in $\mathrm{GL}_{a(b-1)}$:

$$\mathrm{diag}(t^{a-1}I_{b-1}, t^{a-3}I_{b-1}, \ldots, t^{3-a}I_{b-1}, t^{1-a}I_{b-1}).$$

Let $U_{[a^b],2} = \omega V_{[a^b],2}\omega^{-1}$. Then any element of $U_{[a^b],2}$ has the following form:

$$u = \begin{pmatrix} n_1 & q \\ 0 & n_2 \end{pmatrix} \begin{pmatrix} I_a & 0 \\ p & I_{a(b-1)} \end{pmatrix},$$

where $n_1 \in U_a$, the maximal unipotent radical of GL_a; $n_2 \in U_{[a^{b-1}],2} := \omega' V_{[a^{b-1}],2}\omega'^{-1}$; $q \in M_{a\times a(b-1)}$ with $q_{l,m} = 0$, for $m \le l(b-1)$; and $p \in M_{a(b-1)\times a}$ with $p_{l,m} = 0$, for $l > (m-1)(b-1)$. We define

$$\psi_{U_{[a^{b-1}],2}}(u) := \psi_{V_{[a^{b-1}],2}}(\omega'^{-1}u\omega').$$

Therefore, after conjugating by ω, the integral (5.7) becomes

$$\int_* E_{-1}\left(\phi, \begin{pmatrix} n_1 & q \\ 0 & n_2 \end{pmatrix} \begin{pmatrix} I_a & 0 \\ p & I_{a(b-1)} \end{pmatrix} \omega g\right) \psi_{[a]}^{-1}(n_1)\psi_{U_{[a^{b-1}],2}}^{-1}(n_2)\mathrm{d}*, \tag{5.8}$$

where $\int_* = \int_p \int_q \int_{n_2} \int_{n_1}$, $\mathrm{d}* = \mathrm{d}n_1\,\mathrm{d}n_2\,\mathrm{d}q\,\mathrm{d}p$, $\psi_{[a]}$ is a nondegenerate character of GL_a.

We are going to apply Lemma 5.2 consecutively in order to replace the integration on the variable p by corresponding integration on the variable q. To do so, we define a sequence of unipotent subgroups of GL_{ab} (R for "row," C for "column"). For $1 \le s \le a-1$, define

$$R_s = \left\{ \begin{pmatrix} I_a & q \\ 0 & I_{a(b-1)} \end{pmatrix} \right\},$$

where $q \in M_{a \times a(b-1)}$ with the property that $q_{l,m} = 0$ if $l \neq s$ or $l = s$, $m > l(b-1)$. For $2 \le s \le a$, define

$$C_s = \left\{ \begin{pmatrix} I_a & 0 \\ p & I_{a(b-1)} \end{pmatrix} \right\},$$

where $p \in M_{a(b-1) \times a}$ with the property that $p_{l,m} = 0$ if $m \neq s$ or $m = s$, $l > (m-1)(b-1)$. It is easy to see that for $1 \le s \le a-1$, $R_s(k) \cong C_{s+1}(k) \cong k^{s(b-1)}$, as abelian groups.

Write $U_{[a^b],2} = \tilde{U}_{[a^b],2} \prod_{s=2}^{a} C_s$, where $\tilde{U}_{[a^b],2}$ consists of elements in $U_{[a^b],2}$ with p-part (as indicated in the subgroup C_s) being zero. For $1 \le s \le a-1$, write $R_s = \prod_{i=1}^{s} R_s^i$, where $R_s^i \subset G^+_{-2(i-1)}$. For $2 \le s \le a$, write $C_s = \prod_{i=1}^{s-1} C_s^i$, where $C_s^i \subset G^-_{2i}$.

The key point here is to apply Lemma 5.2 to the integration on the variables in $\prod_{s=2}^{a} C_s$. To do this, we will deal with the subgroups C_s for $s = 2, 3, \ldots, a$, one by one.

First we apply Lemma 5.2 to the integration on C_2-part. To do so, consider the quadruple

$$\left(\tilde{U}_{[a^b],2} \prod_{s=3}^{a} C_s,\, \psi_{[a]} \psi_{U_{[a^{b-1}],2}},\, R_1^1,\, C_2^1 \right).$$

Note that both R_1^1 and C_2^1 normalize $\tilde{U}_{[a^b],2}$ and preserve $\psi_{[a]} \psi_{U_{[a^{b-1}],2}}$, $R_1^1 \subset G_0^+$, the conjugation by R_1^1 will change some entries in G_i^+ with $i \ge 2$, but not attached to any character, and $C_2^1 \subset G_2^-$, the conjugation by C_2^1 only changes some entries in G_i^+ or G_i^- with $i \ge 4$. It is easy to see that the quadruple $(\tilde{U}_{[a^b],2} \prod_{s=3}^{a} C_s, \psi_{[a]} \psi_{U_{[a^{b-1}],2}}, R_1^1, C_2^1)$ satisfies all the other conditions for Lemma 5.2. By Lemma 5.2, the integral (5.8) is nonvanishing if and only if the following integral is nonvanishing

$$\int_* E_{-1}\left(\phi, \begin{pmatrix} n_1 & q \\ 0 & n_2 \end{pmatrix} \begin{pmatrix} I_a & 0 \\ p & I_{a(b-1)} \end{pmatrix} g\right) \psi_{[a]}^{-1}(n_1) \psi_{U_{[a^{b-1}],2}}^{-1}(n_2)\, \mathrm{d}*, \tag{5.9}$$

where $\int_* = \int_p \int_q \int_{n_2} \int_{n_1}$, $\mathrm{d}* = \mathrm{d}n_1\, \mathrm{d}n_2\, \mathrm{d}q\, \mathrm{d}p$, $\left(\begin{smallmatrix} I_a & 0 \\ p & I_{a(b-1)} \end{smallmatrix}\right) \in \prod_{s=3}^{a} C_s$, and $\left(\begin{smallmatrix} n_1 & q \\ 0 & n_2 \end{smallmatrix}\right) \in \tilde{U}_{[a^b],2} R_1$.

The next step is to apply Lemma 5.2 to the integration on variables in the subgroup C_3. Since $C_3 = C_3^1 \cdot C_3^2$, we have to consider the quadruples

$$\left(\tilde{U}_{[a^b],2} R_1 \prod_{s=4}^{a} C_s C_3^2,\, \psi_{[a]} \psi_{U_{[a^{b-1}],2}},\, R_2^1,\, C_3^1 \right)$$

and

$$\left(\tilde{U}_{[a^b],2} R_1 R_2^1 \prod_{s=4}^{a} C_s,\ \psi_{[a]}\psi_{U_{[a^{b-1}],2}},\ R_2^2,\ C_3^2\right).$$

After finishing this step, we come to consider $C_4 = C_4^1 \cdot C_4^2 \cdot C_4^3$. This time we consider consecutively three quadruples

$$\left(\tilde{U}_{[a^b],2} R_1 R_2 \prod_{s=5}^{a} C_s C_4^2 C_4^3,\ \psi_{[a]}\psi_{U_{[a^{b-1}],2}},\ R_3^1,\ C_4^1\right),$$

$$\left(\tilde{U}_{[a^b],2} R_1 R_2 R_3^1 \prod_{s=5}^{a} C_s C_4^3,\ \psi_{[a]}\psi_{U_{[a^{b-1}],2}},\ R_3^2,\ C_4^2\right),$$

and

$$\left(\tilde{U}_{[a^b],2} R_1 R_2 R_3^1 R_3^2 \prod_{s=5}^{a} C_s,\ \psi_{[a]}\psi_{U_{[a^{b-1}],2}},\ R_3^3,\ C_4^3\right).$$

By repeating the same procedure, we end up considering the subgroup $C_a = \prod_{i=1}^{a-1} C_a^i$. To finish this step, we apply Lemma 5.2 to the following $(a-1)$ quadruples

$$\left(\tilde{U}_{[a^b],2} R_1 \cdots R_{a-2} \prod_{l=1}^{i-1} R_{a-1}^l \prod_{j=i+1}^{a-1} C_a^i,\ \psi_{[a]}\psi_{U_{[a^{b-1}],2}},\ R_{a-1}^i,\ C_a^i\right)$$

with $i = 1, 2, \ldots, a-1$. After finishing all the steps, we obtain that the integral (5.9) is nonvanishing if and only if the following integral is nonvanishing

$$\int_* E_{-1}\left(\phi, \begin{pmatrix} n_1 & q \\ 0 & n_2 \end{pmatrix} g\right) \psi_{[a]}^{-1}(n_1)\psi_{U_{[a^{b-1}],2}}^{-1}(n_2)\mathrm{d}*, \tag{5.10}$$

where $\int_* = \int_q \int_{n_2} \int_{n_1}$, $\mathrm{d}* = \mathrm{d}n_1\, \mathrm{d}n_2\, \mathrm{d}q$, and

$$\begin{pmatrix} n_1 & q \\ 0 & n_2 \end{pmatrix} \in \tilde{U}_{[a^b],2} R_1 R_2 \cdots R_{a-1}.$$

Rewrite the integral (5.10) as follows:

$$\int_{n_2}\int_{n_1,q} E_{-1}\left(\phi, \begin{pmatrix} n_1 & q \\ 0 & I_{a(b-1)} \end{pmatrix}\begin{pmatrix} I_a & 0 \\ 0 & n_2 \end{pmatrix} g\right) \psi_{[a]}^{-1}(n_1)\psi_{U_{[a^{b-1}],2}}^{-1}(n_2)\mathrm{d}* \tag{5.11}$$

by changing the variable $q \cdot n_2^{-1} \mapsto q$. Note that $q \cdot n_2^{-1}$ has the same structure as q.

Now consider the inner integral

$$\int_{n_1,q} E_{-1}\left(\phi, \begin{pmatrix} n_1 & q \\ 0 & I_{a(b-1)} \end{pmatrix} g\right) \psi_{[a]}^{-1}(n_1)\, dn_1\, dq, \tag{5.12}$$

which is exactly $E_{-1}^{\psi_{V_{1^{a-1},ab-a+1}}}(\phi, g)$, using the notation in Lemma 6.2, since the n_1 is integrated over $[U_a]$, the maximal unipotent subgroup of GL_a, q is integrated over $\left[M'_{a\times a(b-1)}\right]$, where

$$M'_{a\times a(b-1)} = \{q = (q_{i,j}) \in M_{a\times a(b-1)} \mid q_{a,j} = 0, \forall 1 \le j \le a(b-1)\},$$

and $\psi_{[a]}(n_1)$ is the Whittaker character of U_a. Then by Lemma 6.2, the integral (5.12) is actually equal to $E_{-1}^{\tilde{\psi}_{V_{1^a,ab-a}}}(\phi, g)$ (for notation, see Lemma 6.2), that is, the following integral

$$\int_{n_1,q} E_{-1}\left(\phi, \begin{pmatrix} n_1 & q \\ 0 & I_{a(b-1)} \end{pmatrix} g\right) \psi_{[a]}^{-1}(n_1)\, dn_1\, dq,$$

where any entry in any row of q is integrated over $k\backslash\mathbb{A}$. Therefore, the integral (5.11) becomes

$$\int_{n_2}\int_{n_1,q} E_{-1}\left(\phi, \begin{pmatrix} n_1 & q \\ 0 & I_{a(b-1)} \end{pmatrix}\begin{pmatrix} I_a & 0 \\ 0 & n_2 \end{pmatrix} g\right) \psi_{[a]}^{-1}(n_1)\psi^{-1}_{U_{[a^{b-1}],2}}(n_2) d*, \tag{5.13}$$

where any entry in any row of q is integrated over $k\backslash\mathbb{A}$.

Note that the inner integral (5.13)

$$\int_{n_1,q} E_{-1}\left(\phi, \begin{pmatrix} n_1 & q \\ 0 & I_{a(b-1)} \end{pmatrix} g\right) \psi_{[a]}^{-1}(n_1)\, dn_1\, dq = \int_{n_1} E_{-1}\left(\phi, \begin{pmatrix} n_1 & 0 \\ 0 & I_{a(b-1)} \end{pmatrix} g\right)_{Q_{a,a(b-1)}} \psi_{[a]}^{-1}(n_1)\, dn_1,$$

where $Q_{a,a(b-1)}$ is the parabolic subgroup of GL_{ab} with its Levi subgroup $\mathrm{GL}_a \times \mathrm{GL}_{a(b-1)}$. By Lemma 4.1, there is a section

$$f \in \mathrm{Ind}_{Q_{a,a(b-1)}(\mathbb{A})}^{\mathrm{GL}_{ab}(\mathbb{A})}(|\cdot|^{-\frac{b-1}{2}}\tau \otimes |\cdot|^{\frac{1}{2}}\mathcal{E}_{(\tau,b-1)}),$$

such that

$$E_{-1}\left(\phi, \begin{pmatrix} n_1 & 0 \\ 0 & I_{a(b-1)} \end{pmatrix} g\right)_{Q_{a,a(b-1)}} = f\left(\begin{pmatrix} n_1 & 0 \\ 0 & I_{a(b-1)} \end{pmatrix} g\right)(I_a \times I_{a(b-1)}).$$

Recall the standard Iwasawa decomposition

$$\mathrm{GL}_{ab}(\mathbb{A}) = Q_{a,a(b-1)}(\mathbb{A}) \cdot K,$$

with $K = \prod_v K_v$ being the standard maximal compact subgroup of $\mathrm{GL}_{ab}(\mathbb{A})$. Then for any $g \in \mathrm{GL}_{ab}(\mathbb{A})$, we write $g = h_1(g)h_2(g)v(g)k(g)$, where $h_1(g) \in \mathrm{GL}_a(\mathbb{A})$ is identified with $\mathrm{diag}(h_1(g), I_{a(b-1)})$, $h_2(g) \in \mathrm{GL}_{a(b-1)}(\mathbb{A})$ is identified with $\mathrm{diag}(I_a, h_2(g))$, $v(g) \in V_{a,a(b-1)}(\mathbb{A})$, and $k(g) \in K$.

Now, for $n_1 \in U_a$, write

$$\begin{pmatrix} n_1 & 0 \\ 0 & I_{a(b-1)} \end{pmatrix} g = n_1 h_1(g)h_2(g)v(g)k(g).$$

Then we have

$$E_{-1}\left(\phi, \begin{pmatrix} n_1 & 0 \\ 0 & I_{a(b-1)} \end{pmatrix} g\right)_{Q_{a,a(b-1)}} = f(n_1 h_1(g)h_2(g)k(g))(I_a \times I_{a(b-1)}).$$

By definition, we have

$$f(n_1 h_1(g)h_2(g)k(g)) = |h_1(g)|^{-\frac{b-1}{2}} |h_2(g)|^{\frac{1}{2}} \cdot (\tau(n_1 h_1(g)) \otimes \mathcal{E}_{(\tau,b-1)}(h_2(g)))(f(k(g))).$$

When $k(g) \in K$, $f(k(g))$ is a vector in the space of $\tau \otimes \mathcal{E}_{(\tau,b-1)}$. Since the sections defining the Eisenstein series are of K-finite, we may assume that

$$f(k(g)) = \sum_{j=1}^{n^{k(g)}} f_j^{k(g)} \otimes \phi_j^{k(g)},$$

where $f_j^{k(g)} \in \tau$, and $\phi_j^{k(g)} \in \mathcal{E}_{(\tau,b-1)}$. Hence we have

$$(\tau(n_1 h_1(g)) \otimes \mathcal{E}_{(\tau,b-1)}(h_2(g)))(f(k(g))) = \sum_{j=1}^{n^{k(g)}} \tau(n_1 h_1(g))(f_j^{k(g)}) \otimes \mathcal{E}_{(\tau,b-1)}(h_2(g))(\phi_j^{k(g)}).$$

By definition, we have

$$\tau(n_1 h_1(g))(f_j^{k(g)})(I_a) = f_j^{k(g)}(n_1 h_1(g))$$

and

$$\mathcal{E}_{(\tau,b-1)}(h_2(g))(\phi_j^{k(g)})(I_{a(b-1)}) = \phi_j^{k(g)}(h_2(g)).$$

It follows that

$$E_{-1}\left(\phi, \begin{pmatrix} n_1 & 0 \\ 0 & I_{a(b-1)} \end{pmatrix} g\right)_{Q_{a,a(b-1)}} = \sum_{j=1}^{n^{k(g)}} |h_1(g)|^{-\frac{b-1}{2}} |h_2(g)|^{\frac{1}{2}} f_j^{k(g)}(n_1 h_1(g)) \phi_j^{k(g)}(h_2(g)).$$

Therefore, we have

$$\begin{aligned}
&\int_{n_1} E_{-1}\left(\phi, \begin{pmatrix} n_1 & 0 \\ 0 & I_{a(b-1)} \end{pmatrix} g\right)_{Q_{a,a(b-1)}} \psi_{[a]}^{-1}(n_1)\,\mathrm{d}n_1 \\
&= \int_{n_1} \sum_{j=1}^{n^{k(g)}} |h_1(g)|^{-\frac{b-1}{2}} |h_2(g)|^{\frac{1}{2}} f_j^{k(g)}(n_1 h_1(g)) \phi_j^{k(g)}(h_2(g)) \psi_{[a]}^{-1}(n_1)\,\mathrm{d}n_1 \\
&= \sum_{j=1}^{n^{k(g)}} \int_{n_1} |h_1(g)|^{-\frac{b-1}{2}} f_j^{k(g)}(n_1 h_1(g)) \psi_{[a]}^{-1}(n_1)\,\mathrm{d}n_1 |h_2(g)|^{\frac{1}{2}} \phi_j^{k(g)}(h_2(g)).
\end{aligned}$$

Hence the integral (5.13) becomes

$$\sum_{j=1}^{n^{k(g)}} |h_1(g)|^{-\frac{b-1}{2}} \int_{n_1} f_j^{k(g)}(n_1 h_1(g)) \psi_{[a]}^{-1}(n_1)\,\mathrm{d}n_1 \cdot |h_2(g)|^{\frac{1}{2}} \int_{n_2} \phi_j^{k(g)}(n_2 h_2(g)) \psi_{U_{[a^{b-1}],2}}^{-1}(n_2)\,\mathrm{d}n_2. \tag{5.14}$$

By the induction assumption, the integral

$$\int_{n_2} \phi_j^{k(g)}(n_2 h_2(g)) \psi_{U_{[a^{b-1}],2}}^{-1}(n_2)\,\mathrm{d}n_2$$

is nonvanishing if and only if the following integral is nonvanishing:

$$\int_{[U_{a(b-1)}]} \phi_j^{k(g)}(n_2 h_2(g)) \psi_{0;(b-1)\cdot a}^{a(b-1)}(n_2)^{-1}\,\mathrm{d}n_2.$$

Therefore, the integral (5.14), hence the integral (5.13), is nonvanishing if and only if the following integral is nonvanishing:

$$\int_{[U_{a(b-1)}]}\int_{n_1,q} E_{-1}\left(\phi, \begin{pmatrix} n_1 & q \\ 0 & I_{a(b-1)} \end{pmatrix}\begin{pmatrix} I_a & 0 \\ 0 & n_2 \end{pmatrix} g\right)\psi_{[a]}^{-1}(n_1)\psi_{0;(b-1)\cdot a}^{a(b-1)}(n_2)^{-1}\,\mathrm{d}n_1\,\mathrm{d}q\,\mathrm{d}n_2$$

$$= \int_{[U_{ab}]} E_{-1}(\phi, vg)\psi_{0;b\cdot a}^{ab}(u)^{-1}\,\mathrm{d}u.$$

This finishes the proof of the theorem. ■

Furthermore, we will prove the following theorem in the next section.

Theorem 5.5. Let $\underline{p}=[p_1 p_2 \cdots p_r]$ be a standard partition of n, that is, $p_1 \geq p_2 \geq \cdots \geq p_r > 0$. If $p_1 > a$, then the residual representation $\mathcal{E}_{(\tau,b)}$ of $\mathrm{GL}_{ab}(\mathbb{A})$ with cuspidal support $(P_{a^b}, \tau^{\otimes b})$ has no nonzero $\psi_{\underline{p}}$-Fourier coefficients, that is, for any $E_{-1}(\phi, g) \in \mathcal{E}_{(\tau,b)}$, the $\psi_{\underline{p}}$-Fourier coefficient $E_{-1}^{\psi_{\underline{p}}}(\phi, g)$ is identically zero. □

Combining Theorems 5.4 and 5.5 with Corollary 3.4, we obtain the following extension to the residual spectrum of $\mathrm{GL}_n(\mathbb{A})$ of the theorem of Shalika [12] and of Piatetski-Shapiro [10], independently, that all nonzero irreducible cuspidal automorphic representations of $\mathrm{GL}_n(\mathbb{A})$ are generic, that is, they have nonzero Whittaker–Fourier coefficients.

Theorem 5.6. Let $\underline{p}=[p_1 p_2 \cdots p_r]$ be a partition of n with $p_1 \geq p_2 \geq \cdots \geq p_r > 0$ and denote by $[a^b]$ the partition of all parts equal to a. For the residual representation $\mathcal{E}_{(\tau,b)}$ with cuspidal support $(P_{a^b}, \tau^{\otimes b})$, belonging to the discrete spectrum of $\mathrm{GL}_n(\mathbb{A})$, the following hold.

(1) The residual representation $\mathcal{E}_{(\tau,b)}$ has a nonzero $\psi_{[a^b]}$-Fourier coefficient.

(2) For any partition $\underline{p}=[p_1 p_2 \cdots p_r]$ of n, if $p_1 > a$, then the residual representation $\mathcal{E}_{(\tau,b)}$ has no nonzero $\psi_{\underline{p}}$-Fourier coefficients. □

It is clear that Part (1) follows from Theorem 5.4 and Corollary 3.4, and Part (2) is from Theorem 5.5. Note that if we use the notation of Ginzburg [1], Theorem 5.6 implies that $\mathcal{O}(\mathcal{E}_{(\tau,b)}) = \{\mathcal{O}_{[a^b]}\}$.

6 Proof of Theorem 5.5

In this section, we will prove the vanishing property of Fourier coefficients of the residue $E_{-1}(\phi, g)$ attached to the partitions either bigger than or not related to the partition $[a^b]$. To this end, we need to prove the following two key lemmas.

Lemma 6.1. Let $\underline{p} = [p_1 p_2 \cdots p_r]$ be a standard partition of n, that is, $p_1 \geq p_2 \geq \cdots \geq p_r > 0$. If $p_1 > a$, then the $\psi_{\underline{p}}^{\epsilon_1,\ldots,\epsilon_{r-1}}$-Fourier coefficient

$$E_{-1}^{\psi_{\underline{p}}^{\epsilon_1,\ldots,\epsilon_{r-1}}}(\phi, g) := \int_{[U_{ab}]} E_{-1}(\phi, ug)\psi_{\underline{p}}^{\epsilon_1,\ldots,\epsilon_{r-1}}(u)^{-1}\,\mathrm{d}u \equiv 0, \tag{6.1}$$

where

$$\begin{aligned}
\psi_{\underline{p}}^{\epsilon_1,\ldots,\epsilon_{r-1}}(u) := \psi(&u_{1,2} + \cdots + u_{p_1-1,p_1} + \epsilon_1 u_{p_1,p_1+1}) \\
&\cdot \psi(u_{p_1+1,p_1+2} + \cdots + u_{p_1+p_2-1,p_1+p_2} + \epsilon_2 u_{p_1+p_2,p_1+p_2+1}) \\
&\vdots \\
&\cdot \psi(u_{p_1+\cdots+p_{r-2}+1,p_1+\cdots+p_{r-2}+2} + \cdots + u_{p_1+\cdots+p_{r-1}-1,p_1+\cdots+p_{r-1}} \\
&+ \epsilon_{r-1} u_{p_1+\cdots+p_{r-1},p_1+\cdots+p_{r-1}+1}) \\
&\cdot \psi(u_{p_1+\cdots+p_{r-1}+1,p_1+\cdots+p_{r-1}+2} + \cdots + u_{ab}),
\end{aligned}$$

and $\epsilon_i \in \{0, 1\}$, $i = 1, \ldots, r-1$. □

Proof. We separate the proof into two steps: (1) $\epsilon_1 = 0$; and (2) $\epsilon_1 = 1$.

Step (I). $\epsilon_1 = 0$. Since $p_1 > a$, there are two cases to be considered: (1) $p_1 \neq as$ for all $1 < s \leq b$; and (2) $p_1 = as$ for some $1 < s \leq b$.

Case (1). Let $Q_{p_1,ab-p_1}$ be the parabolic subgroup of GL_{ab} with Levi isomorphic to $\mathrm{GL}_{p_1} \times \mathrm{GL}_{ab-p_1}$. By the definition of the $\psi_{\underline{p}}^{\epsilon_1,\ldots,\epsilon_{r-1}}$-Fourier coefficient, $E_{-1}^{\psi_{\underline{p}}^{\epsilon_1,\ldots,\epsilon_{r-1}}}(\phi, g)$ has the constant term of the residue $E_{-1}(\phi, g)$ along $Q_{p_1,ab-p_1}$ as an inner integral. More precisely,

$$E_{-1}^{\psi_{\underline{p}}^{\epsilon_1,\ldots,\epsilon_{r-1}}}(\phi, g) = [E_{-1}(\phi, g)_{Q_{p_1,ab-p_1}}]^{\psi_{[p_1]};\psi_{[p_2\cdots p_r]}^{\epsilon_2,\ldots,\epsilon_{r-1}}}.$$

Since $p_1 \neq s \cdot a$ for all $1 \leq s \leq b$, $P \nsubseteq Q_{ab-p_r,p_r}$, which implies that $E_{-1}(\phi, g)_{Q_{p_1,ab-p_1}} = 0$. Therefore, $E_{-1}^{\psi_{\underline{p}}^{\epsilon_1,\ldots,\epsilon_{r-1}}}(\phi, g) = 0$.

Case (2). If $p_1 = ab$, then $E_{-1}^{\psi_{\underline{p}}^{\epsilon_1,\ldots,\epsilon_{r-1}}}(\phi, g) = 0$, since $\mathcal{E}_{(\tau,b)}$ is not generic, and $E_{-1}^{\psi_{\underline{p}}^{\epsilon_1,\ldots,\epsilon_{r-1}}}(\phi, g)$ is a Whittaker–Fourier coefficient. From now on, we assume that $p_1 = as$, with $1 < s < b$.

Recall from Lemma 4.1 that $Q_{as,a(b-s)}$ is the parabolic subgroup of GL_{ab} with Levi isomorphic to $\mathrm{GL}_{as} \times \mathrm{GL}_{a(b-s)}$. By the definition of the $\psi_{\underline{p}}^{\epsilon_1,\ldots,\epsilon_{r-1}}$-Fourier coefficient, $E_{-1}^{\psi_{\underline{p}}^{\epsilon_1,\ldots,\epsilon_{r-1}}}(\phi, g)$ has the constant term of the residue $E_{-1}(\phi, g)$ along $Q_{as,a(b-s)}$ as an inner integral. As before, we have

$$E_{-1}^{\psi_{\underline{p}}^{\epsilon_1,\ldots,\epsilon_{r-1}}}(\phi, g) = [E_{-1}(\phi, g)_{Q_{p_1,ab-p_1}}]^{\psi_{[p_1]};\psi_{[p_2\cdots p_r]}^{\epsilon_2,\ldots,\epsilon_{r-1}}}.$$

After taking the constant term along $Q_{as,a(b-s)}$, $E_{-1}(\phi, g)_{Q_{as,a(b-s)}}$ is an automorphic function over $\mathrm{GL}_{sa}(\mathbb{A}) \times \mathrm{GL}_{a(b-s)}(\mathbb{A})$. Note here that the $\psi_{[p_1]}$-Fourier coefficient is taken on $\mathrm{GL}_{sa}(\mathbb{A})$ and the $\psi_{[p_2\cdots p_r]}^{\epsilon_2,\ldots,\epsilon_{r-1}}$-Fourier coefficient is taken on $\mathrm{GL}_{a(b-s)}(\mathbb{A})$.

By Lemma 4.1, it is enough to show that the residual representation $\mathcal{E}_{(\tau,s)}$ has no nonzero $\psi_{[p_1]}$-Fourier coefficients or the residual representation $\mathcal{E}_{(\tau,b-s)}$ has no nonzero $\psi_{[p_2\cdots p_r]}^{\epsilon_2,\ldots,\epsilon_{r-1}}$-Fourier coefficients. It is clear that the character $\psi_{[p_1]}$ is exactly the Whittaker character of $\mathrm{GL}_{sa}(\mathbb{A})$. By Proposition 2.1, the residual representation $\mathcal{E}_{(\tau,s)}$ is not generic, and hence it has no nonzero $\psi_{[p_1]}$-Fourier coefficients.

Hence, if $p_1 > a$ and $\epsilon_1 = 0$, then $E_{-1}^{\psi_{\underline{p}}^{\epsilon_1,\ldots,\epsilon_{r-1}}}(\phi, g) = 0$.

Step (II). We assume that $\epsilon_1 = 1$. If $\epsilon_i = 1$, for all $1 \le i \le r-1$, then $\psi_{\underline{p}}^{\epsilon_1,\ldots,\epsilon_{r-1}}$ is a nondegenerate character of GL_{ab}, and hence $E_{-1}^{\psi_{\underline{p}}^{\epsilon_1,\ldots,\epsilon_{r-1}}}(\phi, g) = 0$.

So, we may assume $i < r-1$ to be the first number such that $\epsilon_i = 0$. By applying the proof of *Step* (I) to the partition $[(\sum_{j=1}^{i} p_j)p_{i+1}\cdots p_r]$, which is still either bigger than or not related to the partition $[a^b]$, we deduce that $E_{-1}^{\psi_{\underline{p}}^{\epsilon_1,\ldots,\epsilon_{r-1}}}(\phi, g) = 0$.

This completes the proof of the lemma. ■

Lemma 6.2. Let $V_{1^{m-1},ab-m+1}$ be the unipotent radical of the parabolic subgroup $Q_{1^{m-1},ab-m+1}$ with Levi part $\mathrm{GL}_1^{\times(m-1)} \times \mathrm{GL}_{ab-m+1}$. Let

$$\psi_{V_{1^{m-1},ab-m+1}}(v) = \psi(v_{1,2} + \cdots + v_{m-1,m})$$

and

$$\tilde{\psi}_{V_{1^{m-1},ab-m+1}}(v) = \psi(v_{1,2} + \cdots + v_{m-2,m-1})$$

be two characters of $V_{1^{m-1},ab-m+1}$. Define $E_{-1}^{\psi_{V_{1^{m-1},ab-m+1}}}(\phi, g)$ by

$$\int_{[V_{1^{m-1},ab-m+1}]} E_{-1}(\phi, vg)\psi^{-1}_{V_{1^{m-1},ab-m+1}}(v)\,\mathrm{d}v. \tag{6.2}$$

Then, if $m > a$, $E_{-1}^{\psi_{V_{1^{m-1},ab-m+1}}}(\phi, g) \equiv 0$; and if $m = a$,

$$E_{-1}^{\psi_{V_{1^{m-1},ab-m+1}}}(\phi, g) = E_{-1}^{\tilde{\psi}_{V_{1^m,ab-m}}}(\phi, g). \qquad \square$$

Proof. For $i = 1, \ldots, ab-1$, let R_i be the subgroup of U_{ab} such that any element $u = (u_{j,l}) \in R_i$, $u_{j,l} = 0$, unless $j = i$.

Since $E_{-1}^{\psi_{V_{1^{m-1},ab-m+1}}}(\phi, g)$ is left $R_m(k)$-invariant, we take Fourier expansion of $E_{-1}^{\psi_{V_{1^{m-1},ab-m+1}}}(\phi, g)$ along $[R_m] = R_m(k) \setminus R_m(\mathbb{A})$:

$$E_{-1}^{\psi_{V_{1^{m-1},ab-m+1}}}(\phi, g) = E_{-1}^{\tilde{\psi}_{V_{1^m,ab-m}}}(\phi, g) + \sum_{\gamma \in Q^0_{1,ab-m-1}(k)\setminus GL_{ab-m}(k)} E_{-1}^{\psi_{V_{1^m,ab-m}}}(\phi, \mathrm{diag}(I_m, \gamma)g). \tag{6.3}$$

Since both $E_{-1}^{\tilde{\psi}_{V_{1^m,ab-m}}}(\phi, g)$ and $E_{-1}^{\psi_{V_{1^m,ab-m}}}(\phi, g)$ are left $R_{m+1}(k)$-invariant, we can take the Fourier expansion of them along $[R_{m+1}] = R_{m+1}(k) \setminus R_{m+1}(\mathbb{A})$. We repeat this process for each term in the Fourier expansion of $E_{-1}^{\tilde{\psi}_{V_{1^m,ab-m}}}(\phi, g)$ or $E_{-1}^{\psi_{V_{1^m,ab-m}}}(\phi, g)$ along the following sequence $[R_{m+2}], \ldots, [R_{ab-1}]$. After plugging back all these Fourier expansion to (6.3), we can see that $E_{-1}^{\psi_{V_{1^{m-1},ab-m+1}}}(\phi, g)$ can be written as a summation, each term of which is of the form (6.1), and is identically zero, if $m > a$, by Lemma 6.1.

If $m = a$, then from (6.3), we can see that

$$E_{-1}^{\psi_{V_{1^{m-1},ab-m+1}}}(\phi, g) = E_{-1}^{\tilde{\psi}_{V_{1^m,ab-m}}}(\phi, g),$$

since $E_{-1}^{\psi_{V_{1^m,ab-m}}}(\phi, g) \equiv 0$ from the above discussion.

This finishes the proof of the lemma. ■

Before proving the general case of Theorem 5.5, we prove the vanishing of Fourier coefficients of $E_{-1}(\phi, \cdot)$ corresponding to the orbits $[p_1 1^{ab-p_1}]$ with $p_1 > a$. The idea of the proof for this special case is applicable to the general case. Note that in the proof of Ginzburg [1, Proposition 5.3], the vanishing of Fourier coefficients of $E_{-1}(\phi, \cdot)$ corresponding to the general bigger than or not related orbits is sketched by reducing to the proof of that corresponding to the special partition $[(a+1)1^{ab-a-1}]$, which is then proved by using local argument. We prove it below using global argument.

Proposition 6.3. The Fourier coefficient of $E_{-1}(\phi,\cdot)$ corresponding to the orbit $\mathcal{O}=[p_1 1^{ab-p_1}]$, $p_1>a$ is identically zero. □

Proof. We separate the proof into two cases: (1) p_1 is odd and (2) p_1 is even.

Case (1). We assume that p_1 is odd. From the definition, any element in $V_{[p_1 1^{ab-p_1}],2}$ has the following form:

$$u=\begin{pmatrix} n_1 & q \\ 0 & I_{ab-p_1}\end{pmatrix}\begin{pmatrix} I_{p_1} & 0 \\ p & I_{ab-p_1}\end{pmatrix},$$

where $n_1\in U_{p_1}$, the maximal unipotent radical of GL_{p_1}, $q\in M_{p_1\times ab-p_1}$ with $q_{l,m}=0$, for $l\geq \frac{p_1-1}{2}+1$, and $p\in M_{ab-p_1\times p_1}$ with $p_{l,m}=0$, for $m\leq \frac{p_1-1}{2}+1$.

The $\psi_{[p_1 1^{ab-p_1}]}$-Fourier coefficient of $E_{-1}(\phi,\cdot)$, $E_{-1}^{\psi_{[p_1 1^{ab-p_1}]}}(\phi,g)$, can be rewritten as

$$\int_p\int_q\int_{n_1} E_{-1}\left(\phi,\begin{pmatrix} n_1 & q \\ 0 & I_{ab-p_1}\end{pmatrix}\begin{pmatrix} I_{p_1} & 0 \\ p & I_{ab-p_1}\end{pmatrix}g\right)\psi^{-1}_{[p_1]}(n_1)\,\mathrm{d}n_1\,\mathrm{d}q\,\mathrm{d}p, \tag{6.4}$$

where $\psi_{[p_1]}(n_1)$ is a nondegenerate character of GL_{p_1}.

For $\frac{p_1-1}{2}+1\leq s\leq p_1-1$, define the following unipotent subgroup of GL_{ab}:

$$R_s=\left\{\begin{pmatrix} I_{p_1} & q \\ 0 & I_{ab-p_1}\end{pmatrix}: q\in M_{p_1\times ab-p_1}, q_{l,m}=0, l\neq s\right\}.$$

For $\frac{p_1-1}{2}+2\leq s\leq p_1$, define the following unipotent subgroup of GL_{ab}:

$$C_s=\left\{\begin{pmatrix} I_{p_1} & 0 \\ p & I_{ab-p_1}\end{pmatrix}: p\in M_{ab-p_1\times p_1}, q_{l,m}=0, m\neq s\right\}.$$

Then we can see that $R_s(k)\backslash R_s(\mathbb{A})\cong C_s(k)\backslash C_s(\mathbb{A})\cong(k\backslash\mathbb{A})^{ab-p_1}$. Note that $R_s\subset G^+_{-2(s-\frac{p_1-1}{2}-1)}$ and $C_s\subset G^-_{2(s-\frac{p_1-1}{2}-1)}$.

Write $V_{[p_1 1^{ab-p_1}],2}=\tilde{V}_{[p_1 1^{ab-p_1}],2}\prod_{s=\frac{p_1-1}{2}+2}^{p_1} C_s$, where $\tilde{V}_{[p_1 1^{ab-p_1}],2}$ consists of elements in $V_{[p_1 1^{ab-p_1}],2}$ with p-part zero.

Now we are ready to apply Lemma 5.2 to the integral in (6.4) first with the quadruple

$$\left(\tilde{V}_{[p_1 1^{ab-p_1}],2} \prod_{s=\frac{p_1-1}{2}+3}^{p_1} C_s,\ \psi_{[p_1]},\ R_{\frac{p_1-1}{2}+1},\ C_{\frac{p_1-1}{2}+2}\right),$$

then with the quadruple

$$\left(\tilde{V}_{[p_1 1^{ab-p_1}],2} R_{\frac{p_1-1}{2}+1} \prod_{s=\frac{p_1-1}{2}+4}^{p_1} C_s,\ \psi_{[p_1]},\ R_{\frac{p_1-1}{2}+2},\ C_{\frac{p_1-1}{2}+3}\right),$$

and keep doing the same thing until the final step with the quadruple

$$(\tilde{V}_{[p_1 1^{ab-p_1}],2} R_{\frac{p_1-1}{2}+1} \cdots R_{p_1-2},\ \psi_{[p_1]},\ R_{p_1-1},\ C_{p_1}).$$

This calculation shows that the integral (6.4) is identically zero if and only if the following integral is identically zero

$$\int_q \int_{n_1} E_{-1}\left(\phi, \begin{pmatrix} n_1 & q \\ 0 & I_{ab-p_1} \end{pmatrix} g\right) \psi_{[p_1]}^{-1}(n_1)\, \mathrm{d}n_1\, \mathrm{d}q\, \mathrm{d}p, \tag{6.5}$$

where $\left(\begin{smallmatrix} n_1 & q \\ 0 & I_{ab-p_1} \end{smallmatrix}\right) \in \tilde{V}_{[p_1 1^{ab-p_1}],2} R_{\frac{p_1-1}{2}+1} \cdots R_{p_1-1}$, that is, all first (p_1-1)-rows of q are integrated over $k \backslash \mathbb{A}$, and the last row of q is zero.

Note that for each step, we can easily check the conditions for Lemma 5.2. For the first quadruple

$$\left(\tilde{V}_{[p_1 1^{ab-p_1}],2} \prod_{s=\frac{p_1-1}{2}+3}^{p_1} C_s,\ \psi_{[p_1]},\ R_{\frac{p_1-1}{2}+1},\ C_{\frac{p_1-1}{2}+2}\right),$$

the conjugation by $R_{\frac{p_1-1}{2}+1}$ will change some entries in G_i^+ with $i \geq 2$, but the changing of variables does not change the character, the conjugation by $C_{\frac{p_1-1}{2}+2}$ will change some entries in G_i^+ or G_i^- with $i \geq 4$. For $1 \leq j \leq \frac{p_1-1}{2} - 2$, when we consider the quadruple

$$\left(\tilde{V}_{[p_1 1^{ab-p_1}],2} \prod_{l=1}^{j+1} R_{\frac{p_1-1}{2}+l} \prod_{s=\frac{p_1-1}{2}+j+4}^{p_1} C_s,\ \psi_{[p_1]},\ R_{\frac{p_1-1}{2}+1+j+1},\ C_{\frac{p_1-1}{2}+1+j+2}\right),$$

the conjugation by $R_{\frac{p_1-1}{2}+1+j+1}$ will change some entries in

$$G_i^+ \cap \tilde{V}_{[p_1 1^{ab-p_1}],2} \prod_{l=1}^{j+1} R_{\frac{p_1-1}{2}+l} \prod_{s=\frac{p_1-1}{2}+j+4}^{p_1} C_s$$

with $i \geq -2j$, but the changing of variables does not change the character, the conjugation by $C_{\frac{p_1-1}{2}+1+j+2}$ will change some entries in G_i^+ with $i \geq 4$.

Note that the integral in (6.5) is actually $E_{-1}^{\psi_{V_{1^{p_1-1},ab-p_1+1}}}(\phi, g)$, which is identically zero by Lemma 6.2. This finishes the proof of the case of p_1 odd.

Case (2). Assume that p_1 is even. From the definition, any element in $V_{[p_1 1^{ab-p_1}],2}$ has the following form:

$$u = \begin{pmatrix} n_1 & q \\ 0 & I_{ab-p_1} \end{pmatrix} \begin{pmatrix} I_{p_1} & 0 \\ p & I_{ab-p_1} \end{pmatrix},$$

where $n_1 \in U_{p_1}$, the maximal unipotent radical of GL_{p_1}, $q \in M_{p_1 \times ab-p_1}$ with $q_{l,m} = 0$, for $l \geq \frac{p_1}{2}$, and $p \in M_{ab-p_1 \times p_1}$ with $p_{l,m} = 0$, for $m \leq \frac{p_1}{2} + 1$.

The $\psi_{[p_1 1^{ab-p_1}]}$-Fourier coefficient of $E_{-1}(\phi, \cdot)$, $E_{-1}^{\psi_{[p_1 1^{ab-p_1}]}}(\phi, g)$, can also be rewritten as

$$\int_p \int_q \int_{n_1} E_{-1}\left(\phi, \begin{pmatrix} n_1 & q \\ 0 & I_{ab-p_1} \end{pmatrix} \begin{pmatrix} I_{p_1} & 0 \\ p & I_{ab-p_1} \end{pmatrix} g\right) \psi_{[p_1]}^{-1}(n_1)\, \mathrm{d}n_1\, \mathrm{d}q\, \mathrm{d}p, \tag{6.6}$$

where $\psi_{[p_1]}(n_1)$ is a nondegenerate character of GL_{p_1}.

By Corollary 5.3, we only have to show that the following integral is identically zero:

$$\int_{y,p,q,n_1} E_{-1}\left(\phi, \begin{pmatrix} n_1 & q \\ 0 & I_{ab-p_1} \end{pmatrix} \begin{pmatrix} I_{p_1} & 0 \\ p & I_{ab-p_1} \end{pmatrix} yg\right) \psi_{[p_1]}^{-1}(n_1)\, \mathrm{d}n_1\, \mathrm{d}q\, \mathrm{d}p\, \mathrm{d}y, \tag{6.7}$$

where $y \in [Y]$.

For $\frac{p_1}{2} + 1 \leq s \leq p_1 - 1$, define the following unipotent subgroup of GL_{ab}:

$$R_s = \left\{ \begin{pmatrix} I_{p_1} & q \\ 0 & I_{ab-p_1} \end{pmatrix} : q \in M_{p_1 \times ab-p_1}, q_{l,m} = 0, l \neq s \right\}.$$

For $\frac{p_1}{2} + 2 \leq s \leq p_1$, define the following unipotent subgroup of GL_{ab}:

$$C_s = \left\{ \begin{pmatrix} I_{p_1} & 0 \\ p & I_{ab-p_1} \end{pmatrix} : p \in M_{ab-p_1 \times p_1}, q_{l,m} = 0, m \neq s \right\}.$$

Then we can see that $R_s(k) \setminus R_s(\mathbb{A}) \cong C_s(k) \setminus C_s(\mathbb{A}) \cong (k \setminus \mathbb{A})^{ab-p_1}$. Note that $R_s \subset G^+_{-2(s-\frac{p_1}{2})+1}$ and $C_s \subset G^-_{2(s-\frac{p_1}{2})+1}$.

Write $YV_{[p_1 1^{ab-p_1}],2} = Y\tilde{V}_{[p_1 1^{ab-p_1}],2} \prod_{s=\frac{p_1}{2}+2}^{p_1} C_s$, where $\tilde{V}_{[p_1 1^{ab-p_1}],2}$ consists of elements in $V_{[p_1 1^{ab-p_1}],2}$ with p-part zero.

Now we apply Lemma 5.2 to the integral (6.7) first with the quadruple

$$\left(Y\tilde{V}_{[p_1 1^{ab-p_1}],2} \prod_{s=\frac{p_1}{2}+3}^{p_1} C_s,\ \psi_{[p_1]},\ R_{\frac{p_1}{2}+1},\ C_{\frac{p_1}{2}+2} \right),$$

and then with the following quadruple

$$\left(Y\tilde{V}_{[p_1 1^{ab-p_1}],2} R_{\frac{p_1}{2}+1} \prod_{s=\frac{p_1}{2}+4}^{p_1} C_s,\ \psi_{[p_1]},\ R_{\frac{p_1}{2}+2},\ C_{\frac{p_1}{2}+3} \right),$$

and keep doing the same thing until the last step with the quadruple

$$(Y\tilde{V}_{[p_1 1^{ab-p_1}],2} R_{\frac{p_1}{2}+1} \cdots R_{p_1-2},\ \psi_{[p_1]},\ R_{p_1-1},\ C_{p_1}).$$

This calculation shows that the integral (6.7) is identically zero if and only if the following integral is identically zero

$$\int_y \int_q \int_{n_1} E_{-1}\left(\phi, \begin{pmatrix} n_1 & q \\ 0 & I_{ab-p_1} \end{pmatrix} yg\right) \psi^{-1}_{[p_1]}(n_1)\, dn_1\, dq\, dy, \tag{6.8}$$

where $\left(\begin{smallmatrix} n_1 & q \\ 0 & I_{ab-p_1} \end{smallmatrix}\right) \in Y\tilde{V}_{[p_1 1^{ab-p_1}],2} R_{\frac{p_1}{2}} \cdots R_{p_1-1}$, that is, all first (p_1-1)-rows of q are integrated over $k \setminus \mathbb{A}$, the last row of q is zero. For each step, the conditions for Lemma 5.2 can be easily checked as the case of p_1 odd.

Note that the integral in (6.8) is actually $E_{-1}^{\psi_{V_{1^{p_1-1},ab-p_1+1}}}(\phi, g)$, which is identically zero by Lemma 6.2.

This completes the proof of the proposition. ■

Now, we prove the general case of Theorem 5.5. As we mentioned, the idea will be similar to the case of special orbits $[p_1 1^{ab-p_1}]$, $p_1 > a$, in Proposition 6.3.

First, by Corollary 5.3, we only have to show that the following integral is identically zero:

$$\int_{[Y]}\int_{[V_{\underline{p},2}]} E_{-1}(\phi, vyg)\psi^{-1}_{V_{\underline{p},2}}(v)\,\mathrm{d}v\,\mathrm{d}y. \tag{6.9}$$

Note that we will use the notation introduced in Section 5 accordingly.

First we conjugate the integration variables in the integral (6.9) by the Weyl element ω of GL_n $(n=ab)$ which conjugates the toric subgroup $\mathcal{H}_{\underline{p}}$ of GL_{ab} in (5.1) attached to the partition $\underline{p}$ to the toric subgroup:

$$\mathrm{diag}(\mathcal{H}_{[p_1]}(t); t^{p_2-1}, \ldots, t^{1-p_2}),$$

where after the first block of size p_1, the exponents of t are of nonincreasing order. Note that ω is of the form $\mathrm{diag}(I_a, \omega')$, where ω' is a Weyl element of GL_{n-p_1}, which conjugates the toric subgroup $\mathcal{H}_{[p_2\cdots p_r]}$ of GL_{n-p_1} in (5.1) corresponding to the partition $[p_2\cdots p_r]$ to the toric subgroup of GL_{n-p_1}:

$$\mathrm{diag}(t^{p_2-1}, \ldots, t^{1-p_2}),$$

where the exponents of t are of nonincreasing order. For example, for the partition $[p_2\cdots p_r]=[(3^2)2]$, ω' is the Weyl element of GL_8, which conjugates the toric subgroup $\mathrm{diag}(t^2, 1, t^{-2}; t^2, 1, t^{-2}; t, t^{-1})$ to the toric subgroup: $\mathrm{diag}(t^2, t^2, t, 1, 1, t^{-1}, t^{-2}, t^{-2})$.

Let $U_{\underline{p},2}=\omega Y V_{\underline{p},2}\omega^{-1}$. Then any element of $U_{\underline{p},2}$ has the following form:

$$u=\begin{pmatrix} n_1 & q \\ 0 & n_2\end{pmatrix}\begin{pmatrix} I_{p_1} & 0 \\ p & I_{ab-p_1}\end{pmatrix},$$

where $n_1\in U_a$, the maximal unipotent radical of GL_a, and

$$n_2\in U_{[p_2\cdots p_r],2}:=\omega' Y_{[p_2\cdots p_r],2} V_{[p_2\cdots p_r],2}\omega'^{-1}$$

with $Y_{[p_2\cdots p_r],2}$ being the corresponding Y for the partition $[p_2\cdots p_r]$. Denote $\psi_{U_{\underline{p},2}}(u):=\psi_{V_{\underline{p},2}}(\omega^{-1}u\omega)$. Hence integral (6.9) equals

$$\int_{[U_{\underline{p},2}]} E_{-1}\left(\phi, \begin{pmatrix} n_1 & q \\ 0 & n_2\end{pmatrix}\begin{pmatrix} I_{p_1} & 0 \\ p & I_{ab-p_1}\end{pmatrix}\omega g\right)\psi^{-1}_{U_{\underline{p},2}}(u)\,\mathrm{d}u. \tag{6.10}$$

Consider the group $U_{\underline{p},2} \cap U_{p_1,ab-p_1}$, where $U_{p_1,ab-p_1}$ is the unipotent subgroup of the parabolic subgroup of G with Levi $\mathrm{GL}_{p_1} \times \mathrm{GL}_{ab-p_1}$. Let i be the index of the first row of $U_{\underline{p},2} \cap U_{p_1,ab-p_1}$ with zero entries. Let R_j be the subgroup of $U_{p_1,ab-p_1}$ with zeros everywhere except the complement of $U_{\underline{p},2} \cap U_{p_1,ab-p_1}$ in the jth row, $i \leq j \leq p_1 - 1$. Similarly, let C_j be the subgroup of $U^-_{p_1,ab-p_1}$ with zeros everywhere except the complement of $U_{\underline{p},2} \cap U^-_{p_1,ab-p_1}$ in the jth column, $i+1 \leq j \leq p_1$. Then, we can see that $R_j \cong C_{j+1}$.

Write $U_{\underline{p},2} = \tilde{U}_{\underline{p},2} \cdot \prod_{s=i+1}^{p_1} C_s$, where $\tilde{U}_{\underline{p},2}$ consists of elements in $U_{\underline{p},2}$ with $U^-_{p_1,ab-p_1}$-part zero. For $i \leq j \leq p_1 - 1$, write $R_j = \prod_{l=1}^{m_j} R^l_j$, where R^l_j consists of all the entries in $G^+_{k^l_j}$, with k^l_j decreasing. For $i+1 \leq j \leq p_1$, write $C_j = \prod_{l=1}^{m_{j-1}} C^l_j$, where C^l_j consists of all the entries in $G^-_{\tilde{k}^l_j}$, with $\tilde{k}^l_j$ increasing. Note that $R^l_j \cong C^l_{j+1}$.

Now we are ready to apply Lemma 5.2 to the integral (6.10) with a sequence of quadruples: $(\tilde{U}_{\underline{p},2} \prod_{s=i+2}^{p_1} C_s \prod_{l=2}^{m_i} C^l_{i+1}, \psi_{U_{\underline{p},2}}, R^1_i, C^1_{i+1})$, and then $(\tilde{U}_{\underline{p},2} R^1_i \prod_{s=i+2}^{p_1} C_s \prod_{l=3}^{m_i} C^l_{i+1}, \psi_{U_{\underline{p},2}}, R^2_i, C^2_{i+1})$, and keep going until $(\tilde{U}_{\underline{p},2} R^1_i \cdots R^{m_i-1}_i \prod_{s=i+2}^{p_1} C_s, \psi_{U_{\underline{p},2}}, R^{m_i}_i, C^{m_i}_{i+1})$. This finishes the first step. Then we go with a next sequence of quadruples

$$\left(\tilde{U}_{\underline{p},2} R_i \prod_{s=i+3}^{p_1} C_s \prod_{l=2}^{m_{i+1}} C^l_{i+2}, \psi_{U_{\underline{p},2}}, R^1_{i+1}, C^1_{i+2}\right)$$
$$\vdots$$
$$\left(\tilde{U}_{\underline{p},2} R_i R^1_{i+1} \cdots R^{m_{i+1}-1}_{i+1} \prod_{s=i+3}^{p_1} C_s, \psi_{U_{\underline{p},2}}, R^{m_{i+1}}_{i+1}, C^{m_{i+1}}_{i+2}\right);$$

and keep doing this until the last step with a sequence of quadruples

$$\left(\tilde{U}_{\underline{p},2} R_i \cdots R_{p_1-2} \prod_{l=2}^{m_{p_1-1}} C^l_{p_1}, \psi_{U_{\underline{p},2}}, R^1_{p_1-1}, C^1_{p_1}\right)$$
$$\vdots$$
$$(\tilde{U}_{\underline{p},2} R_i \cdots R_{p_1-2} R^1_{p_1-1} \cdots R^{m_{p_1-1}-1}_{p_1-1}, \psi_{U_{\underline{p},2}}, R^{m_{p_1-1}}_{p_1-1}, C^{m_{p_1-1}}_{p_1}).$$

Note that here for convenience, we denote all the characters in all the above quadruples by $\psi_{U_{\underline{p},2}}$. The above calculation shows that the integral (6.10) is identically zero if and only if the following integral is identically zero

$$\int_* \int_q \int_{n_1} E_{-1}\left(\phi, \begin{pmatrix} n_1 & q \\ 0 & u_{[p_2 \dots p_r]} \end{pmatrix} g\right) \psi^{-1}_{[p_1]}(n_1) \psi^{-1}_{[p_2 \dots p_r]}(u_{[p_2 \dots p_r]})\, \mathrm{d}n_1\, \mathrm{d}q\, \mathrm{d}u_*, \tag{6.11}$$

where $q \in U_{p_1, ab-p_1}(k) \setminus U_{p_1, ab-p_1}(\mathbb{A})$ with only the last row being zero, $\int_*$ means $\int_{U_{[p_2 \cdots p_r],2}}$ and du_* is $du_{[p_2 \cdots p_r]}$. For each step, the conditions for Lemma 5.2 can be checked easily as in the proof of Proposition 6.3.

Note that the integral (6.11) contains the following integral as an inner integral

$$\int_q \int_{n_1} E_{-1}(\phi, \begin{pmatrix} n_1 & q \\ 0 & I_{ab-p_1} \end{pmatrix} g) \psi_{[p_1]}^{-1}(n_1) \, dn_1 \, dq,$$

which is actually $E_{-1}^{\psi_{V_{1^{p_1-1}, ab-p_1+1}}}(\phi, g)$. By Lemma 6.2, the Fourier coefficient $E_{-1}^{\psi_{V_{1^{p_1-1}, ab-p_1+1}}}(\phi, g)$ is identically zero. This completes the proof of Theorem 5.5.

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