

A RECIPROCAL BRANCHING PROBLEM FOR AUTOMORPHIC REPRESENTATIONS AND GLOBAL VOGAN PACKETS

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ABSTRACT. Let G be a group and H be a subgroup of G . The classical branching rule (or symmetry breaking) asks: For an irreducible representation π of G , determine the occurrence of an irreducible representation σ of H in the restriction of π to H . The reciprocal branching problem of this classical branching problem is to ask: For an irreducible representation σ of H , find an irreducible representation π of G such that σ occurs in the restriction of π to H . For automorphic representations of classical groups, the branching problem has been addressed by the well-known global Gan-Gross-Prasad conjecture. In this paper, we investigate the reciprocal branching problem for automorphic representations of special orthogonal groups using the twisted automorphic descent method as developed in [17]. The method may be applied to other classical groups as well.

1. INTRODUCTION

The classical *branching rule* or the so called *symmetry breaking* in representation theory is to ask a question that can be formulated in an over-simplified way as follows: Let G be a group and H be a subgroup. For an irreducible representation π of G , the problem is to ask which irreducible representation σ of H occurs in the restriction of π to H . A refinement of this classical problem is to ask if an irreducible representation σ of H with extra condition can appear in the decomposition when π is restricted to H . Such a classical problem has been successfully studied in many different contents. For automorphic representations of classical groups, the global Gan-Gross-Prasad conjecture addresses this classical problem in a certain format, which will be described with some details below.

The objective of this paper is to consider the automorphic version of the problem reciprocal to the refined classical branching rule problem. The reciprocal branching problem could be formulated as follows: For an irreducible representation σ of H , with a certain extra property, find an irreducible representation π (possibly with a certain extra property) of G , which contains H as a subgroup, such that σ occurs in the restriction of π to H . It is clear that without those extra conditions on π or σ ,

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the usual Frobenius reciprocity law suggests that one may take π to be the induced representation from σ . However, with those extra conditions, the usual induced representation may not be enough for such refined problems. This paper is to understand this reciprocal problem in the theory of automorphic representations of special orthogonal groups in terms of global Vogan packets and with connection to the global Gan-Gross-Prasad conjecture. We believe that the method used in this paper should be applicable to other classical groups as well.

1.1. The branching problem and the global Gan-Gross-Prasad conjecture.

Before introducing the precise problem we will consider, we recall from [7] the global Gan-Gross-Prasad conjecture for the case closely relevant to the topics discussed in this paper.

Let F be a number field and $\mathbb{A} = \mathbb{A}_F$ be the ring of adèles of F . Following [2], we denote by $G_n^* := \mathrm{SO}_{2n}^\epsilon$ an F -quasi-split even special orthogonal group, and denote by $H_m^* := \mathrm{SO}_{2m+1}^*$ the F -split odd special orthogonal group. Here $\mathrm{SO}_{2n}^\epsilon$ ($\epsilon \in F^\times / (F^\times)^2$) is the F -quasi-split special orthogonal group determined by $(n-1)$ hyperbolic planes and $E_\epsilon = F[X]/(X^2 - \epsilon)$, and is F -split if $\epsilon \in (F^\times)^2$. As in [2] and [7], we denote by G_n and H_m pure inner forms of G_n^* and H_m^* , respectively. Note that G_n and G_n^* share the same Langlands L -group, and so do H_m and H_m^* . We recall the global Arthur parameters from [2], and consider the generic ones mostly. A generic global Arthur parameter for G_n^* is given as a formal sum

$$(1.1) \quad \phi = (\tau_1, 1) \boxplus \cdots \boxplus (\tau_r, 1),$$

where τ_i with $i = 1, 2, \dots, r$ is an irreducible unitary, self-dual cuspidal automorphic representation of $\mathrm{GL}_{a_i}(\mathbb{A})$, and $\tau_i \not\cong \tau_j$ if $i \neq j$. Moreover, each τ_i is of orthogonal type in the sense that the symmetric square L -function $L(s, \tau_i, \mathrm{Sym}^2)$ has a pole at $s = 1$. Note that one must have $2n = \sum_{i=1}^r a_i$. The global Arthur parameter ϕ as in (1.1) can be realized as an irreducible automorphic representation of $\mathrm{GL}_{2n}(\mathbb{A})$. The set of generic global Arthur parameters of G_n^* is denoted by $\tilde{\Phi}_2(G_n^*)$. For each Arthur parameter $\phi \in \tilde{\Phi}_2(G_n^*)$, the associated global Arthur packet is denoted by $\tilde{\Pi}_\phi(G_n^*)$. Since G_n shares the Langlands L -group with G_n^* , an Arthur parameter $\phi \in \tilde{\Phi}_2(G_n^*)$ may be regarded as an Arthur parameter of G_n . However, the associated global Arthur packet $\tilde{\Pi}_\phi(G_n)$ could be empty. If the global Arthur packet $\tilde{\Pi}_\phi(G_n)$ is not empty, we call the global Arthur parameter ϕ of G_n^* is G_n -relevant. The set of all G_n -relevant, generic global Arthur parameters of G_n^* is denoted by $\tilde{\Phi}_2(G_n^*)_{G_n}$. Now the global Vogan packet associated to $\phi \in \tilde{\Phi}_2(G_n^*)$ is given by

$$(1.2) \quad \tilde{\Pi}_\phi[G_n^*] := \cup_{G_n} \tilde{\Pi}_\phi(G_n),$$

where G_n runs over all pure inner forms of G_n^* over F . Similarly, we have global Arthur parameters ϕ' for H_m^* as in (1.1),

$$(1.3) \quad \phi' = (\tau'_1, 1) \boxplus \cdots \boxplus (\tau'_{r'}, 1),$$

with only difference that $\tau'_1, \dots, \tau'_{r'}$ are now of symplectic type in the sense that the exterior square L -function $L(s, \tau'_i, \wedge^2)$ has a pole at $s = 1$. The set of generic global

Arthur parameters of H_m^* is denoted by $\tilde{\Phi}_2(H_m^*)$. Accordingly, we have the global Arthur packet $\tilde{\Pi}_{\phi'}(H_m^*)$ and the global Vogan packet $\tilde{\Pi}_{\phi'}[H_m^*]$.

Assume that (G_n^*, H_m^*) ($m < n$) is a relevant pair in the sense of the Gan-Gross-Prasad conjecture in [7], and $G_n \times H_m$ is a relevant pure inner form of $G_n^* \times H_m^*$ over F . The global Vogan packet for $\phi \times \phi'$ is given by

$$(1.4) \quad \tilde{\Pi}_{\phi \times \phi'}[G_n^* \times H_m^*] := \cup_{G_n \times H_m} \tilde{\Pi}_{\phi \times \phi'}(G_n \times H_m),$$

where $G_n \times H_m$ runs through all relevant pure inner forms of $G_n^* \times H_m^*$ over F . The tensor product L -function associated to the pair ϕ and ϕ' is defined to be

$$(1.5) \quad L(s, \phi \times \phi') := \prod_{i=1}^r \prod_{j=1}^{r'} L(s, \tau_i \times \tau'_j).$$

The global Gan-Gross-Prasad conjecture (or GGP conjecture for short) ([7]) asserts that the central value $L(\frac{1}{2}, \phi \times \phi')$ is non-zero if and only if there exists a pair (π_0, σ_0) in the global Vogan packet $\tilde{\Pi}_{\phi \times \phi'}[G_n^* \times H_m^*]$ with a non-zero Bessel period for (π_0, σ_0) . Here the Bessel period is a certain period integral $\mathcal{P}(\varphi_\pi, \varphi_\sigma)$ ($\varphi_\pi \in \pi, \varphi_\sigma \in \sigma$), which relies on the automorphic realizations. We refer the readers to §6 for the precise definition. Since, for the even special orthogonal groups, the multiplicities of irreducible cuspidal automorphic representations in the discrete spectrum $L_{\text{disc}}^2(G_n(F) \backslash G_n(\mathbb{A}))$ could be two (see [2, §1.5]), here we say that $(\pi_0, \sigma_0) \in \tilde{\Pi}_{\phi \times \phi'}[G_n^* \times H_m^*]$ has a non-zero Bessel period means that there exists an automorphic realization of π such that $\mathcal{P}(\varphi_\pi, \varphi_\sigma) \neq 0$ for some $\varphi_\pi \in \pi$ and $\varphi_\sigma \in \sigma$. The uniqueness of such a pair (π_0, σ_0) in the corresponding Vogan packet $\tilde{\Pi}_{\phi \times \phi'}[G_n^* \times H_m^*]$ will follow from the local GGP conjecture (one may refer to [7, §17]). For the case of special orthogonal groups over non-Archimedean local fields, the local GGP conjecture was proved by the works of Waldspurger ([27]) and Mœglin-Waldspurger ([20]). When such a pair exists, we call it the *Gan-Gross-Prasad pair* or *GGP pair* for short.

The *branching problem* for this case is to ask: For any π in the global Vogan packet $\tilde{\Pi}_\phi[G_n^*]$, is there any σ in the global Vogan packet $\tilde{\Pi}_{\phi'}[H_m^*]$ such that $\pi \otimes \sigma$ belongs to the global Vogan packet $\tilde{\Pi}_{\phi \times \phi'}[G_n^* \times H_m^*]$ and (π, σ) forms the GGP pair, i.e. (π, σ) has a non-zero Bessel period. The global GGP conjecture is to characterize the occurrence of σ in the branching decomposition of π with respect to relevant pair (G_n, H_m) in terms of the central value of the L -function, $L(\frac{1}{2}, \phi \times \phi')$, and the symplectic local root numbers associated to the pair (π, σ) . On the other hand, because of the uniqueness of the pair (π_0, σ_0) in the Vogan packet $\tilde{\Pi}_{\phi \times \phi'}[G_n^* \times H_m^*]$, it also makes sense to ask the following question:

Reciprocal Branching Problem: For $\sigma \in \tilde{\Pi}_{\phi'}(H_m)$ with $\sigma \not\cong \sigma_0$, how to find some group G_\square , an irreducible cuspidal automorphic representation π_\square of $G_\square(\mathbb{A})$ with a generic global Arthur parameter ϕ_\square , and certain automorphic realization of π_\square , such that G_\square and H_m form a relevant pair and (π_\square, σ) has a non-zero Bessel period?

1.2. The reciprocal branching problem and the twisted automorphic descent. We are going to study this reciprocal branching problem for automorphic

representations of special orthogonal groups within the general framework of global Vogan packets and the global Gan-Gross-Prasad conjecture, by means of the twisted automorphic descents.

Assume that σ belongs to the global Arthur packet $\tilde{\Pi}_{\phi'}(H_m)$ for some pure inner form H_m of H_m^* over F . To present an answer to the reciprocal branching problem in this situation, one may take a generic global Arthur parameter $\phi \in \tilde{\Phi}[G_n^*]$ for some quasi-split form G_n^* at first. By the implication of the global GGP conjecture, we may also assume $L(\frac{1}{2}, \phi \times \phi') \neq 0$. Assume that (π_0, σ_0) is the unique GGP pair in the global Vogan packet $\tilde{\Pi}_{\phi \times \phi'}[G_n^* \times H_m^*]$, which has a non-zero Bessel period, as given by the global GGP conjecture. If $\sigma \simeq \sigma_0$, then the global GGP conjecture predicts that the member π_0 in the global Vogan packet $\tilde{\Pi}_{\phi}[G_n^*]$ gives an answer to the reciprocal branching problem. The twisted automorphic descent of [17] provides an explicit construction of this π_0 in terms of the generic global Arthur parameter ϕ and σ . Note that the construction also gives a specific automorphic realization of π_0 .

In this paper we consider the reciprocal branching problem for any σ belonging to the global Vogan packet $\tilde{\Pi}_{\phi'}[H_m^*]$ but not equivalent to σ_0 . The interesting part of this situation is that, by the uniqueness property in the GGP conjecture, for the fixed parameter ϕ of dimension $2n$, one does not expect that there exists $\pi \in \tilde{\Pi}_{\phi}[G_n^*]$ such that π and σ have a non-zero Bessel period. The idea is to find an (specific) even special orthogonal group G_{n+k} for some $k \geq 1$, which is a pure inner form of some quasi-split form G_{n+k}^* and is relevant to H_m , and construct explicitly (with concrete automorphic realization) an irreducible cuspidal automorphic representation π_{n+k} of $G_{n+k}(\mathbb{A})$ with the properties that π_{n+k} has a generic global Arthur parameter and has a non-zero Bessel period with the given σ . We expect that the integer k should be determined by the *first occurrence index* of σ (see [17] or §2.3 for the definition) in the tower of the Bessel descents from σ . Hence the construction of G_{n+k} and π_{n+k} should depend on the structure of the Bessel-Fourier coefficients of σ , which reflects the natural relation between the Bessel-Fourier coefficients and the twisted automorphic descents.

We will give an answer to this reciprocal branching problem when $m = 1$. We fix a non-trivial additive character $\psi : F \backslash \mathbb{A} \rightarrow \mathbb{C}^\times$ and fix a positive integer n . Let

$$(1.6) \quad \tau = \tau_1 \boxplus \tau_2 \boxplus \cdots \boxplus \tau_r$$

be an isobaric sum automorphic representation of $\mathrm{GL}_{2n}(\mathbb{A})$. Here τ_i is an irreducible unitary cuspidal automorphic representation of $\mathrm{GL}_{n_i}(\mathbb{A})$ such that $\sum_{i=1}^r n_i = 2n$, and $\tau_i \not\cong \tau_j$ if $i \neq j$. Assume that τ corresponds to the generic Arthur parameter $\phi_\tau = \phi_{\tau_1} \boxplus \cdots \boxplus \phi_{\tau_r}$. Moreover, we assume that each τ_i is of orthogonal type, i.e. the L -function $L(s, \tau_i, \mathrm{Sym}^2)$ has a pole at $s = 1$. Hence ϕ_τ is a generic global Arthur parameter of some G_n^* .

On the other hand, we assume that τ_0 is an irreducible unitary cuspidal representation of $\mathrm{GL}_2(\mathbb{A})$ of symplectic type and has the property that

$$L\left(\frac{1}{2}, \tau \times \tau_0\right) \neq 0.$$

Let V_0 be a quadratic space over F of dimension 3 and $H_1^{V_0} = \mathrm{SO}(V_0)$ be the corresponding special orthogonal group. In any case, the F -split group is $H_1^* = \mathrm{SO}_3$. We also denote by J_{V_0} the quadratic form of V_0 . Let σ be an irreducible cuspidal automorphic representation of $H_1^{V_0}(\mathbb{A})$ parametrized by ϕ_{τ_0} . We make the following assumption for τ and σ :

Assumption 1.1. *The Bessel period $\mathcal{P}(\varphi_\pi, \varphi_\sigma)$ (see §6) vanishes for all relevant pure inner forms $G_n \times H_1^{V_0}$ such that ϕ_τ is G_n -relevant, all cuspidal automorphic representations $\pi \in \tilde{\Pi}_{\phi_\tau}(G_n)$, all cuspidal automorphic forms φ_π in any automorphic realizations of π , and all $\varphi_\sigma \in \sigma$.*

The goal of this paper is, under Assumption 1.1, to construct an even special orthogonal group G_{n+1} such that $G_{n+1} \times H_1^{V_0}$ is a relevant pure inner form of $G_{n+1}^* \times H_1^*$, and construct an irreducible cuspidal automorphic representation π_{n+1} of $G_{n+1}(\mathbb{A})$ such that π_{n+1} has a generic Arthur parameter and has a non-zero Bessel period with respect to σ . A more precise explanation is in order.

For the given representations (τ, σ) , we construct in §2 a square-integrable residual automorphic representation $\mathcal{E}_{\tau \otimes \sigma}$ on $\mathrm{SO}_{4n+3}^{V_0}(\mathbb{A})$ with support $(P, \tau \otimes \sigma)$, where $P = MN$ is the standard parabolic subgroup of $\mathrm{SO}_{4n+3}^{V_0}$ such that $M \simeq \mathrm{GL}_{2n} \times H_1^{V_0}$. A tower of *automorphic descents of τ twisted by σ* , which we denoted by $\pi_{\ell, \beta} = \mathcal{D}_{\psi_{\ell, \beta}}(\mathcal{E}_{\tau \otimes \sigma})$ with $\beta \in F^\times$, can be constructed by taking the Bessel-Fourier coefficients of certain depth ℓ (see §2) of the residual representation $\mathcal{E}_{\tau \otimes \sigma}$. For each ℓ , if the twisted descent $\pi_{\ell, \beta} = \mathcal{D}_{\psi_{\ell, \beta}}(\mathcal{E}_{\tau \otimes \sigma})$ is non-zero, then it consists of certain automorphic functions on $G_{2n-\ell+1, \beta}(\mathbb{A}) = \mathrm{SO}_{4n-2\ell+2, \beta}(\mathbb{A})$, with moderate growth. After choosing a suitable basis, the group $G_{2n-\ell+1, \beta}$ can be arranged to associate with the symmetric matrix

$$\tilde{J} = \begin{pmatrix} & & w_{2n-\ell-1} \\ & J_{V_{0, \beta}} & \\ w_{2n-\ell-1} & & \end{pmatrix},$$

where w_i is an $(i \times i)$ -matrix with only 1's on its anti-diagonal. Note that it could be either F -split, quasi-split or non-quasi-split, where $J_{V_{0, \beta}} = \begin{pmatrix} J_{V_0} & \\ & -\beta \end{pmatrix}$ is a (4×4) -symmetric matrix that defines the 4-dimensional quadratic space $V_{0, \beta}$. We also denote by $\eta_{V_{0, \beta}} : F^\times \backslash \mathbb{A}^\times \rightarrow \{\pm 1\}$ the quadratic character associated to the quadratic space $V_{0, \beta}$. We usually call $G_{2n-\ell+1, \beta}$ the *target group* of the descent module $\mathcal{D}_{\psi_{\ell, \beta}}(\mathcal{E}_{\tau \otimes \sigma})$. The point here is that under the Assumption 1.1, the first occurrence in this tower is at the depth $\ell^* = n$. In this case, we denote the resulting representation by $\pi_\beta = \mathcal{D}_{\psi_{n, \beta}}(\mathcal{E}_{\tau \otimes \sigma})$, which consists of certain automorphic functions on $G_{n+1, \beta}(\mathbb{A}) = \mathrm{SO}_{2n+2, \beta}(\mathbb{A})$, with moderate growth. The main results of this paper can be summarized as follows.

Theorem 1.2. *Let $\tau = \tau_1 \boxplus \cdots \boxplus \tau_r$ be an irreducible isobaric sum automorphic representation of $\mathrm{GL}_{2n}(\mathbb{A})$ such that each τ_i is an irreducible unitary cuspidal representation of $\mathrm{GL}_{n_i}(\mathbb{A})$ of orthogonal type, and $\tau_i \not\cong \tau_j$ if $i \neq j$. Let V_0 be a quadratic space of dimension 3 over F , and σ be an irreducible cuspidal automorphic representation of $H_1^{V_0}(\mathbb{A})$ which lies in the global Vogan packet $\tilde{\Pi}_{\phi_{\tau_0}}[H_1^*]$ for some irreducible cuspidal*

automorphic representation τ_0 of $\mathrm{GL}_2(\mathbb{A})$ of symplectic type. Assume that

$$L\left(\frac{1}{2}, \tau \times \tau_0\right) \neq 0.$$

Then we have:

- (a) There exists $\beta \in F^\times$ such that the twisted descent $\pi_\beta = \mathcal{D}_{\psi_n, \beta}(\mathcal{E}_{\tau \otimes \sigma}) \neq 0$.
- (b) Suppose that the representations τ and σ also satisfy Assumption 1.1, then

$$\pi_\beta = \bigoplus_i \pi_\beta^{(i)}$$

is a multiplicity free direct sum of irreducible cuspidal automorphic representations $\pi_\beta^{(i)}$ of $\mathrm{SO}_{2n+2, \beta}(\mathbb{A})$, such that $(\pi_\beta^{(i)}, \sigma)$ has a non-zero Bessel period for each i .

- (c) If one of the following two conditions is satisfied:

- (1) $\omega_\tau = 1$ and the $\beta \in F^\times$ in Part (a) is not a square;
- (2) $\omega_\tau \neq 1$ and the $\beta \in F^\times$ in Part (a) is a square;

then π_β is automatically cuspidal with a similar decomposition as in Part (b), without Assumption 1.1. Moreover, under the condition (1), each irreducible component $\pi_\beta^{(i)}$ of π_β belongs to a global Arthur packet with a generic global Arthur parameter $\phi^{(i)}$. The parameter $\phi^{(i)}$ has the central character $\eta_{V_{0, \beta}}$, and has the property that $L(\frac{1}{2}, \phi^{(i)} \times \phi_{\tau_0}) \neq 0$ and $(\pi_\beta^{(i)}, \sigma)$ has a non-zero Bessel period.

Theorem 1.2 gives an answer to the reciprocal branching problem for σ , and one may refer to Theorem 6.3 at the end of this paper.

Part (a) of Theorem 1.2 provides a base for establishing the main results in this paper and will be proved in §5. It is always the hardest part in the theory of automorphic descent to prove the global non-vanishing of the construction. In this paper, we find a new argument to do so. It makes use of two recent results. One of them is the result in [11] on relations between degenerate Whittaker models and generalized Whittaker models of representations (see §5.1 for definitions). Another one is the result in [13] on raising nilpotent orbits in the wave front set of representations. They are indispensable for this new way to establish the global non-vanishing of the construction of the twisted automorphic descents. More explicitly, first, using results in [11], we show that $\mathcal{E}_{\tau \otimes \sigma}$ has a non-zero generalized Whittaker-Fourier coefficient attached to the partition $[(2n)^2 1^3]$, which is not special. Then, using results in [13], we show that $\mathcal{E}_{\tau \otimes \sigma}$ has a non-zero generalized Whittaker-Fourier coefficient attached to the partition $[(2n+1)(2n-1)1^3]$, which is the smallest orthogonal special partition bigger than $[(2n)^2 1^3]$. This eventually implies that there exists $\beta \in F^\times$, such that $\mathcal{D}_{\psi_n, \beta}(\mathcal{E}_{\tau \otimes \sigma})$ is non-zero. We note that the method we use here is different from that previously used in the theory of automorphic descent (see, for example, [10, 14]), and is more conceptual and valid for more general situation (see Remark 5.5).

Part (c) of Theorem 1.2 is a connection of the descent construction to the reciprocal branching problem we are considering. Briefly the argument is as follows. Let $\pi_\beta^{(i)}$

be an irreducible component of the descent π_β , which is cuspidal. By the unramified structure and the endoscopic classification of Arthur ([2]), $\pi_\beta^{(i)}$ has a generic Arthur parameter $\phi^{(i)}$. By unfolding of Eisenstein series as used in [17], we can show that the Bessel period for $(\pi_\beta^{(i)}, \sigma)$ is non-zero. Note that the main result of [17] also tells that the central value $L(\frac{1}{2}, \phi^{(i)} \times \phi_{\tau_0})$ is also non-zero (see §6).

The content of this paper is organized as following. In §2, we introduce the construction of the automorphic descent in the case considered in this paper, and study some of its local structures in §3. In §4 – §5, we study some global aspects of the descent. In particular, we show the cuspidality of the descent in §4, and the non-vanishing of the descent in §5. Hence we obtain Part (a) and the cuspidality in Part (b) of Theorem 1.2. Finally in §6, we prove other statements in Part (b) and also Part (c) of Theorem 1.2.

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2. RESIDUAL REPRESENTATIONS AND THE DESCENT CONSTRUCTION

2.1. Notation. We set up some general notation that will be used throughout this paper. Let F be a field of characteristic 0, and V be a quadratic space of dimension $4n + 3$ defined over F , with quadratic form denoted by $\langle \cdot, \cdot \rangle$. Let \tilde{m} be the Witt index of V . Then V has a polar decomposition

$$V = V^+ \oplus W \oplus V^-,$$

where V^+ has dimension \tilde{m} and is a maximal totally isotropic subspace of V , and W is anisotropic of dimension $4n + 3 - 2\tilde{m}$. Fix a maximal flag

$$\mathcal{F} : \quad \{0\} \subset V_1^+ \subset V_2^+ \subset \cdots \subset V_{\tilde{m}}^+ = V^+$$

in V^+ , and choose a basis $\{e_1, \dots, e_{\tilde{m}}\}$ of V^+ over F such that $V_i^+ = \text{Span}\{e_1, \dots, e_i\}$, for $1 \leq i \leq \tilde{m}$. Let $\{e_{-1}, \dots, e_{-\tilde{m}}\}$ be a basis for V^- , which is dual to $\{e_1, \dots, e_{\tilde{m}}\}$, that is,

$$\langle e_i, e_{-j} \rangle = \delta_{i,j}, \text{ for } 1 \leq i, j \leq \tilde{m}.$$

We have two cases to consider: $\tilde{m} = 2n$ or $\tilde{m} = 2n + 1$. If $\tilde{m} = 2n$, take a basis $\{e_0^{(1)}, e_0^{(2)}, e_0^{(3)}\}$ of W such that the anisotropic quadratic form on W is associated with $J_W = \begin{pmatrix} 1 & & \\ & \delta & \\ & & \alpha \end{pmatrix}$, where $\delta, \alpha \in F^\times$. In this case, the ternary quadratic form $x^2 + \delta y^2 + \alpha z^2$ does not represent 0 in F , and the form of V is associated with

$J = \begin{pmatrix} & & w_{2n} \\ & J_W & \\ w_{2n} & & \end{pmatrix}$, where w_i is an $(i \times i)$ -matrix with 1's on the anti-diagonal

and zero's elsewhere. If $\tilde{m} = 2n + 1$, take a basis $\{e_0\}$ of W with $\langle e_0, e_0 \rangle = 1$. Then the form of V is associated with $J = w_{4n+3}$. For each case, let $H = \mathrm{SO}(V)$, which could be F -split or F -non-split, according to the above two cases.

For $1 \leq \ell \leq \tilde{m}$, let Q_ℓ be the (maximal) parabolic subgroup fixing the flag

$$0 \subset V_\ell^+ \subset V^+.$$

Then Q_ℓ has a Levi decomposition $M'_\ell \cdot U_\ell$ with Levi subgroup $M'_\ell \simeq \mathrm{GL}_\ell \times \mathrm{SO}(V^{(\ell)})$, where $V^{(\ell)}$ is the subspace which sits into the decomposition

$$V = V_\ell^+ \oplus V^{(\ell)} \oplus V_\ell^-.$$

For simplicity, when $\ell = 2n$, let $P = Q_{2n}$, $M = M'_{2n}$, $U = U_{2n}$, and $V_0 = V^{(2n)}$. Then $M \simeq \mathrm{GL}_{2n} \times \mathrm{SO}(V_0)$, and $\mathrm{SO}(V_0)$ is an F -split or F -non-split special orthogonal group with $\dim V_0 = 3$. Accordingly, we may sometimes denote $H = \mathrm{SO}(V)$ by $\mathrm{SO}_{4n+3}^{V_0}$ if we want to indicate the structure of the group. Note that if $\tilde{m} = 2n$, we have $V_0 = W$, the anisotropic kernel of V (and hence $J_{V_0} = J_W$).

For $1 \leq \ell \leq \tilde{m}$, we also let P_ℓ be the parabolic subgroup of H which stabilizes the partial flag

$$\mathcal{F}_\ell: \quad 0 \subset V_1^+ \subset V_2^+ \subset \cdots \subset V_\ell^+.$$

It has a Levi decomposition $P_\ell = M_\ell \cdot N_\ell$ with Levi subgroup $M_\ell \simeq \mathrm{GL}_\ell^\ell \times \mathrm{SO}(V^{(\ell)})$.

2.2. The residual representations. Now we take F to be a number field and denote $\mathbb{A} = \mathbb{A}_F$ to be its ring of adèles. Let $\tau = \tau_1 \boxplus \tau_2 \boxplus \cdots \boxplus \tau_r$ be an isobaric sum automorphic representation of $\mathrm{GL}_{2n}(\mathbb{A})$, and σ an irreducible unitary cuspidal representation of $\mathrm{SO}(V_0)(\mathbb{A})$. Here τ_i 's are irreducible unitary cuspidal automorphic representations of $\mathrm{GL}_{n_i}(\mathbb{A})$, satisfying $\sum_{i=1}^r n_i = 2n$, and $\tau_i \not\cong \tau_j$ if $i \neq j$. For $s \in \mathbb{C}$ and an automorphic function on $H(\mathbb{A})$

$$\phi_{\tau \otimes \sigma} \in \mathcal{A}(M(F)U(\mathbb{A}) \backslash H(\mathbb{A}))_{\tau \otimes \sigma},$$

following [19, §II.1], one defines $\lambda_s \phi_{\tau \otimes \sigma}$ to be $(\lambda_s \circ m_P) \phi_{\tau \otimes \sigma}$, where $\lambda_s \in X_M^H \simeq \mathbb{C}$ (see [19, §I.1] for the definition of X_M^H and the map m_P), and defines the corresponding Eisenstein series

$$E(s, h, \phi_{\tau \otimes \sigma}) = \sum_{\gamma \in P(F) \backslash H(F)} \lambda_s \phi_{\tau \otimes \sigma}(\gamma h),$$

which converges absolutely for $\mathrm{Re}(s) \gg 0$ and has meromorphic continuation to the whole complex plane ([19, §IV]). Under the normalization of Shahidi ([21]), one may take $\lambda_s = s\tilde{\alpha}$, where α is the unique reduced root of the maximal F -split torus of H in U .

When $L(s, \tau_i, \mathrm{Sym}^2)$ has a pole at $s = 1$ for all $i = 1, \dots, r$ (i.e. τ is of orthogonal type), and $L(\frac{1}{2}, \tau \times \sigma) \neq 0$, the Eisenstein series $E(s, h, \phi_{\tau \otimes \sigma})$ has a pole at $s = \frac{1}{2}$ of order r (see [17, Proposition 5.3]). Let $\mathcal{E}_{\tau \otimes \sigma}$ denote the automorphic representation of $H(\mathbb{A})$ generated by the iterated residues $\mathrm{Res}_{s=\frac{1}{2}} E(s, h, \phi_{\tau \otimes \sigma})$ for all $\phi_{\tau \otimes \sigma} \in \mathcal{A}(M(F)U(\mathbb{A}) \backslash H(\mathbb{A}))_{\tau \otimes \sigma}$. It is square integrable by the L^2 -criterion in [19].

Moreover, the residual representation $\mathcal{E}_{\tau \otimes \sigma}$ is irreducible (see [18, Theroem A]). Note that the global Arthur parameter ([2]) for $\mathcal{E}_{\tau \otimes \sigma}$ is $(\tau_1, 2) \boxplus (\tau_2, 2) \boxplus \cdots \boxplus (\tau_r, 2) \boxplus \phi_\sigma$ ([17, §6]).

2.3. The twisted automorphic descent. The twisted automorphic descents was introduced in [17], which extends the automorphic descent of Ginzburg-Rallis-Soudry ([10]) to much more general situation. Following [17] and [10], we introduce a family of Bessel-Fourier coefficients that define the tower of automorphic descents.

For $1 \leq \ell < 2n$, we consider the parabolic subgroup $P_\ell = M_\ell N_\ell$ of H defined in §2.1. The nilpotent subgroup $N_\ell(F)$ consists of elements of the form

$$u = u_\ell(z, x, y) = \begin{pmatrix} z & z \cdot x & y \\ & I_{4n+3-2\ell} & x' \\ & & z^* \end{pmatrix},$$

where $z \in Z_\ell(F)$, $x \in \text{Mat}_{\ell \times (4n+3-2\ell)}(F)$, and $y \in \text{Mat}_{\ell \times \ell}(F)$. Here Z_ℓ is the maximal upper-triangular unipotent subgroup of GL_ℓ , and $x' = -J^{(\ell)t} x w_\ell$, where $J^{(\ell)} = \begin{pmatrix} & & w_{2n-\ell} \\ & J_{V_0} & \\ w_{2n-\ell} & & \end{pmatrix}$. Take an anisotropic vector $w_0 \in V^{(\ell)}$ with $\langle w_0, w_0 \rangle$ in a given square class of F^\times , and define a homomorphism $\chi_{\ell, w_0} : N_\ell \rightarrow \mathbb{G}_a$ by

$$\chi_{\ell, w_0}(u) = \sum_{i=2}^{\ell} \langle u \cdot e_i, e_{-(i-1)} \rangle + \langle u \cdot w_0, e_{-\ell} \rangle.$$

Here $u \cdot e_i$ and $u \cdot w_0$ are multiplications of matrices, where e_i 's and w_0 are viewed as column vectors. Define also a character

$$(2.1) \quad \psi_{\ell, w_0} = \psi \circ \chi_{\ell, w_0} : N_\ell(\mathbb{A}) \rightarrow \mathbb{C}^\times,$$

with $\psi : F \backslash \mathbb{A} \rightarrow \mathbb{C}^\times$ being a fixed non-trivial additive character. It is clear that the character ψ_{ℓ, w_0} is trivial on $N_\ell(F)$. Now the adjoint action of M_ℓ on N_ℓ induces an action of $\text{SO}(V^{(\ell)})$ on the set of all such characters ψ_{ℓ, w_0} . The stabilizer L_{ℓ, w_0} of χ_{ℓ, w_0} in $\text{SO}(V^{(\ell)})$ is equal to $\text{SO}(w_0^\perp \cap V^{(\ell)})$.

Let Π be an automorphic representation of $H(\mathbb{A})$. For $f \in V_\Pi$ and $h \in H(\mathbb{A})$, we define the ψ_{ℓ, w_0} -Fourier coefficients of f by

$$(2.2) \quad f^{\psi_{\ell, w_0}}(h) = \int_{[N_\ell]} f(vh) \psi_{\ell, w_0}^{-1}(v) dv,$$

where $[N_\ell]$ denotes the quotient $N_\ell(F) \backslash N_\ell(\mathbb{A})$. This is one of the Fourier coefficients of f associated to the partition $[(2\ell + 1)1^{4n+2-2\ell}]$. As ℓ varies, it produces a family of *Bessel-Fourier coefficients* of f needed in this paper. It is clear that $f^{\psi_{\ell, w_0}}(h)$ is left $L_{\ell, w_0}(F)$ -invariant, and is of moderate growth on the Siegel domain of $L_{\ell, w_0}(F) \backslash L_{\ell, w_0}(\mathbb{A})$.

As explained in [14, §2] and [17, §2.4], for $1 \leq \ell < 2n$, we may take

$$w_0 = y_\beta = e_{2n} + \frac{\beta}{2} e_{-2n}$$

for some $\beta \in F^\times$ as a precise choice of w_0 . We have $\langle y_\beta, y_\beta \rangle = \beta$, and also denote $\psi_{\ell,\beta} = \psi_{\ell,y_\beta}$ for simplicity. Then in matrix form, we have

$$\begin{aligned} \psi_{\ell,\beta}(u_\ell(z, x, y)) &= \psi(z_{1,1} + \cdots + z_{\ell-1,\ell} + x_{\ell,2n-\ell} + \frac{\beta}{2}x_{\ell,2n+4-\ell}) \\ &= \psi_{Z_\ell}(z)\psi(x_{\ell,2n-\ell} + \frac{\beta}{2}x_{\ell,2n+4-\ell}). \end{aligned}$$

It follows that the stabilizer of the character is $L_{\ell,\beta} := L_{\ell,y_\beta} = \mathrm{SO}_{4n+2-2\ell,\beta}$, which is an even special orthogonal group associated to the symmetric matrix

$$\tilde{J} = \begin{pmatrix} & & w_{2n-\ell-1} \\ & J_{V_{0,\beta}} & \\ w_{2n-\ell-1} & & \end{pmatrix},$$

where $J_{V_{0,\beta}} = \begin{pmatrix} J_{V_0} & \\ & -\beta \end{pmatrix}$. This group can be split, quasi-split or non-quasi-split over F , depending on the choice of V_0 and β (Proposition 2.5 of [17]). We also denote by $V_{0,\beta}$ the 4-dimensional quadratic space over F associated to $J_{V_{0,\beta}}$. Define $\mathcal{D}_{\psi_{\ell,\beta}}(\Pi)$ to be the space of $L_{\ell,\beta}(\mathbb{A})$ -span of the automorphic functions $f^{\psi_{\ell,w_0}}|_{L_{\ell,\beta}(\mathbb{A})}$ with all $f \in V_\Pi$, where the group $L_{\ell,\beta}(\mathbb{A})$ acts by right translation.

Following [17], the twisted automorphic descent construction in this paper is to take $\Pi = \mathcal{E}_{\tau \otimes \sigma}$ in the above construction, here $\mathcal{E}_{\tau \otimes \sigma}$ is the residual representation we have introduced before. In this case, we have a family of automorphic $L_{\ell,\beta}(\mathbb{A})$ -modules:

$$\pi_{\ell,\beta} = \mathcal{D}_{\psi_{\ell,\beta}}(\mathcal{E}_{\tau \otimes \sigma}).$$

It has been seen in many previous works (see [10, 14, 17]) that the $L_{\ell,\beta}(\mathbb{A})$ -modules $\pi_{\ell,\beta}$ satisfy the so-called *tower property* when the depth ℓ varies. That is, there exists an ℓ^* such that $\pi_{\ell^*,\beta} \neq 0$ for some choice of data, and $\pi_{\ell,\beta} = 0$ for all $\ell^* < \ell < \tilde{m}$. We call ℓ^* the *first occurrence index* (of σ) for the tower $\{\pi_{\ell,\beta}\}_\ell$. In particular, at the first occurrence index $\ell^* = \ell^*(\mathcal{E}_{\tau \otimes \sigma})$, the $L_{\ell^*,\beta}(\mathbb{A})$ -module $\pi_{\ell^*,\beta} = \mathcal{D}_{\psi_{\ell^*,\beta}}(\mathcal{E}_{\tau \otimes \sigma})$ consists of cuspidal automorphic functions $f^{\psi_{\ell^*,\beta}}(\cdot)$ (see [10, 14, 17]). In this case we denote $\pi_\beta = \pi_{\ell^*,\beta}$ for simplicity, and call it the *automorphic descent of τ to $L_{\ell^*,\beta}(\mathbb{A})$, twisted by σ* or *σ -twisted automorphic descent of τ* . To be compatible with the notation before, we denote $G_{2n-\ell+1,\beta} = L_{\ell,\beta}$, and call $G_{2n-\ell^*+1,\beta}(\mathbb{A})$ the *target group* of this descent construction.

In this paper, we will basically restrict ourselves to the pair (τ, σ) that satisfies Assumption 1.1 in §1. In this situation, the first occurrence index is $\ell^* = n$ (see §4–§5), and the target group is $G_{n+1,\beta}(\mathbb{A}) = \mathrm{SO}_{2n+2,\beta}(\mathbb{A})$, as stated in Theorem 1.2.

3. LOCAL ASPECTS OF THE DESCENT

In this section, we study the local analogue of the automorphic descent discussed in §2.3, which is certain twisted Jacquet module.

3.1. The twisted Jacquet modules. We define the twisted Jacquet modules in both cases we are considering. Let F be a p -adic field of characteristic 0, and fix a non-trivial additive character $\psi : F \rightarrow \mathbb{C}^\times$. Let $H = \mathrm{SO}(V)$ be a special orthogonal group over F with $\dim(V) = 4n + 3$, and suppose that the Witt index $\tilde{m} = 2n$ or $2n + 1$. Let $1 \leq \ell \leq \tilde{m}$ and take an anisotropic vector $w_0 \in V^{(\ell)}$. We define the character ψ_{ℓ, w_0} on $N_\ell(F)$ similar to (2.1) in the global setting. Note that if $\ell = \tilde{m}$, we take $w_0 \in W = V^{(\tilde{m})}$, the anisotropic kernel of V (see §2.1).

For an irreducible admissible representation Π of $H(F)$, one defines the *Jacquet module of Bessel type* (of depth ℓ) to be

$$(3.1) \quad J_{\psi_{\ell, w_0}}(\Pi) := \Pi / \mathrm{Span}\{\pi(u)\xi - \psi_{\ell, w_0}(u)\xi \mid u \in N_\ell(F), \xi \in \Pi\}.$$

Embed $\mathrm{SO}(V^{(\ell)})$ into H via the Levi subgroup M_ℓ , and set $L_{\ell, w_0} = \mathrm{Stab}_{\psi_{\ell, w_0}}(\mathrm{SO}(V^{(\ell)}))$ as before. By definition, $J_{\psi_{\ell, w_0}}(\Pi)$ is an $L_{\ell, w_0}(F)$ -module. As before, when $\ell < \tilde{m}$, we may take $w_0 = y_\beta = e_{2n} + \frac{\beta}{2}e_{-2n} \in V^{(\ell)}$ with $\beta \in F^\times$. Then we set $J_{\psi_{\ell, y_\beta}}(\Pi) = J_{\psi_{\ell, \beta}}(\Pi)$, and $L_{\ell, y_\beta} = L_{\ell, \beta}$. With a suitable choice of basis, $L_{\ell, \beta}(F)$ is determined by the symmetric matrix

$$\tilde{J} = \begin{cases} \begin{pmatrix} & & w_{2n-\ell-1} \\ & J_{W, \beta} & \\ w_{2n-\ell-1} & & \end{pmatrix}, & \text{if } \tilde{m} = 2n, \\ \begin{pmatrix} & & w_{2n-\ell} \\ & J_\beta & \\ w_{2n-\ell} & & \end{pmatrix}, & \text{if } \tilde{m} = 2n + 1, \end{cases}$$

where $J_\beta = \begin{pmatrix} 1 & \\ & -\beta \end{pmatrix}$. Recall that we have $V_0 = W$ if $\tilde{m} = 2n$ (see §2.1).

3.2. Some local unramified calculations of Jacquet modules. We keep some notation which are used in [10, Chapter 5]. For our purpose, here we just calculate the twisted Jacquet module (3.1) in some cases where all the data are unramified. In particular, in these cases the group $H = \mathrm{SO}_{4n+3}(F)$ is F -split and hence we have $\tilde{m} = 2n + 1$. The results will be used in §4 and §6 to study the global properties of automorphic descents.

Case I: trivial central character. Let τ be an irreducible, generic, admissible representation of $\mathrm{GL}_{2n}(F)$, which is of orthogonal type. We assume moreover that $\omega_\tau = 1$. To consider the unramified cases, we write τ as a fully induced representation from the Borel subgroup:

$$(3.2) \quad \tau = \mu_1 \times \cdots \times \mu_n \times \mu_n^{-1} \times \cdots \times \mu_1^{-1},$$

where μ_i 's are unramified characters of F^\times . Let $\sigma = \mathrm{Ind}_{\mathrm{BSO}_3(F)}^{\mathrm{SO}_3(F)} \xi$, here ξ is also an unramified character of F^\times . Let $\pi_{\tau \otimes \sigma}$ be the unramified constituent of the induced

representation $\text{Ind}_{P(F)}^{\text{SO}_{4n+3}(F)}(\tau|\det(\cdot)|^{1/2} \otimes \sigma)$. We want to study the unramified constituents of the twisted Jacquet module $J_{\psi_{\ell,\beta}}(\pi_{\tau \otimes \sigma})$.

Proposition 3.1. *With the data chosen as above, the following hold:*

- (1) *Suppose that $\beta \in F^\times - (F^\times)^2$. Then the twisted Jacquet module $J_{\psi_{\ell,\beta}}(\pi_{\tau \otimes \sigma}) = 0$ for all $n+1 \leq \ell \leq 2n+1$. And when $\ell = n$, any unramified constituent of $J_{\psi_{n,\beta}}(\pi_{\tau \otimes \sigma})$ is a subquotient of $\text{Ind}_{B_{\text{SO}_{2n+2,\beta}}(F)}^{\text{SO}_{2n+2,\beta}(F)} \mu_1 \otimes \cdots \otimes \mu_n \otimes 1$.*
- (2) *Suppose that $\beta \in (F^\times)^2$. Then the twisted Jacquet module $J_{\psi_{\ell,\beta}}(\pi_{\tau \otimes \sigma}) = 0$ for all $n+2 \leq \ell \leq 2n+1$. When $\ell = n+1$, any unramified constituent of $J_{\psi_{n+1,\beta}}(\pi_{\tau \otimes \sigma})$ is a subquotient of $\text{Ind}_{B_{\text{SO}_{2n}}(F)}^{\text{SO}_{2n}(F)} \mu_1 \otimes \cdots \otimes \mu_n$, and when $\ell = n$, any unramified constituent of $J_{\psi_{n,\beta}}(\pi_{\tau \otimes \sigma})$ is a subquotient of*

$$\bigoplus_{t=1}^n \text{Ind}_{P_{\text{SO}_{2n+2}}^{1,\dots,1,2,1,\dots,1}(F)}^{\text{SO}_{2n+2}(F)} \mu_1 \otimes \cdots \otimes \mu_{t-1} \otimes \mu_t(\det_{\text{GL}_2}) \otimes \mu_{t+1} \otimes \cdots \otimes \mu_n,$$

or

$$\text{Ind}_{B_{\text{SO}_{2n+2}}(F)}^{\text{SO}_{2n+2}(F)} \mu_1 \otimes \cdots \otimes \mu_n \otimes \lambda,$$

where λ is an unramified character of F^\times .

Proof. We will apply [10, Theorem 5.1] to calculate $J_{\psi_{\ell,\beta}}(\pi_{\tau \otimes \sigma})$, and we also follow the same notation there. By conjugation of some Weyl element, it suffices to consider the unramified constituent of the induced representation $\text{Ind}_{Q_{2n}(F)}^{\text{SO}_{4n+3}(F)} \tau' \otimes \sigma$ instead of $\pi_{\tau \otimes \sigma}$, here

$$\tau' = \text{Ind}_{P_{2,\dots,2}(F)}^{\text{GL}_{2n}(F)} \mu_1(\det_{\text{GL}_2}) \otimes \cdots \otimes \mu_n(\det_{\text{GL}_2}).$$

Note that the derivative (see [3] and [10, §5]) $\tau'^{(\ell)}$ of τ' vanishes for $\ell \geq n+1$. Now applying [10, Theorem 5.1 (i)] with $j = 2n < \tilde{m} = 2n+1$, one can see that the corresponding twisted Jacquet module vanishes for all $n+2 \leq \ell \leq 2n+1$.

When $\ell = n+1$, by the formula in [10, Theorem 5.1 (1)] we have

$$\begin{aligned} & J_{\psi_{n+1,\beta}}(\text{Ind}_{Q_{2n}(F)}^{\text{SO}_{4n+3}(F)} \tau' \otimes \sigma) \\ & \equiv \delta_\beta \cdot \left(\text{ind}_{Q'_{n,+}(F)}^{\text{SO}_{2n,\beta}(F)} |\det(\cdot)|^{\frac{1-n}{2}} \tau'^{(n)} \oplus \text{ind}_{Q'_{n,-}(F)}^{\text{SO}_{2n,\beta}(F)} |\det(\cdot)|^{\frac{1-n}{2}} \tau'^{(n)} \right), \end{aligned}$$

here $\delta_\beta = 1$ if $\beta \in (F^\times)^2$, and is zero otherwise, and $Q'_{n,\pm}$ are defined as in [10, Remark 5.1]. Note that $|\det(\cdot)|^{\frac{1-n}{2}} \tau'^{(n)} = \mu_1 \times \cdots \times \mu_n$, we obtain all the statements for $n+1 \leq \ell \leq 2n+1$ in part (1) and (2) of the proposition.

Finally we take $\ell = n$. Also by [10, Theorem 5.1 (1)], we have

$$\begin{aligned} J_{\psi_{n,\beta}}(\text{Ind}_{Q_{2n}(F)}^{\text{SO}_{4n+3}(F)} \tau' \otimes \sigma) & \equiv \left(\text{ind}_{Q'_n(F)}^{\text{SO}_{2n+2,\beta}(F)} |\cdot|^{\frac{1-n}{2}} \tau'^{(n)} \otimes J_{\psi'_{0,\beta}}(\sigma) \right) \oplus \\ & \delta_\beta \cdot \left(\text{ind}_{Q'_{n,+}(F)}^{\text{SO}_{2n+2,\beta}(F)} |\det(\cdot)|^{\frac{2-n}{2}} \tau'^{(n-1)} \otimes J_{\text{Wh},+}(\sigma) \right. \\ & \left. \oplus \text{ind}_{Q'_{n,-}(F)}^{\text{SO}_{2n+2,\beta}(F)} |\det(\cdot)|^{\frac{2-n}{2}} \tau'^{(n-1)} \otimes J_{\text{Wh},-}(\sigma) \right), \end{aligned}$$

where $J_{\text{Wh},\pm}(\sigma)$ are Jacquet modules with respect to certain Whittaker characters (see [10, Proposition 5.5]). Here we have used the fact that $\tau'_{(n)} = 0$ (see, for example, [10, page 99]), and Q'_n is defined in [10, (5.16)]. Note also that here Q'_n contains $B_{\text{SO}_{2n+2,\beta}}$. Moreover, we have $J_{\text{Wh},\pm}(\sigma) \simeq \mathbb{C}$ since σ is generic. Since $\delta_\beta = 0$ if $\beta \notin (F^\times)^2$, we can conclude (1) from the above formula. And if $\beta \in (F^\times)^2$, we take $(n-1)$ -th derivative of τ' (as in [3, §4]), and then the last expression stated in the proposition follows as desired. The unramified character λ in Part (2) of the proposition comes from the Jacquet module $J_{\psi'_{0,\beta}}(\sigma)$, which is exactly the restriction of σ to the F -split torus $\text{SO}_2(F)$ (where all unramified characters will occur). \square

Case II: non-trivial central character. Now let λ_0 be the unique non-trivial unramified quadratic character of F^\times , and let

$$(3.3) \quad \tau = \mu_1 \times \cdots \times \mu_{n-1} \times 1 \times \lambda_0 \times \mu_{n-1}^{-1} \times \cdots \times \mu_1^{-1},$$

where μ_i 's are unramified characters of F^\times (recall that τ is of orthogonal type). Let $\sigma = \text{Ind}_{B_{\text{SO}_3(F)}}^{\text{SO}_3(F)} \xi$, where ξ is also an unramified character of F^\times . As before, we denote $\pi_{\tau \otimes \sigma}$ to be the unramified constituent of $\text{Ind}_{P(F)}^{\text{SO}_{4n+3}(F)} (\tau | \det(\cdot)|^{1/2} \otimes \sigma)$.

Proposition 3.2. *With the data chosen as above, we have $J_{\psi_{\ell,\beta}}(\pi_{\tau \otimes \sigma}) = 0$ for all $n+2 \leq \ell \leq 2n+1$. Moreover, define that $\delta_\beta = 1$ if $\beta \in (F^\times)^2$ and $\delta_\beta = 0$ otherwise. Then we have:*

- (1) *When $\ell = n+1$, any unramified constituent of $J_{\psi_{n+1,\beta}}(\pi_{\tau \otimes \sigma})$ is a subquotient of*

$$(1 - \delta_\beta) \cdot \text{Ind}_{B_{\text{SO}_{2n,\beta}(F)}}^{\text{SO}_{2n,\beta}(F)} \mu_1 \otimes \cdots \otimes \mu_{n-1} \otimes 1.$$

- (2) *When $\ell = n$, any unramified constituent of $J_{\psi_{n,\beta}}(\pi_{\tau \otimes \sigma})$ is a subquotient of*

$$\begin{aligned} & \bigoplus_{t=1}^{n-1} \text{Ind}_{P_{\text{SO}_{2n+2,\beta}}^{1,\dots,1,2,1,\dots,1}(F)}^{\text{SO}_{2n+2,\beta}(F)} \mu_1 \otimes \cdots \otimes \mu_{t-1} \otimes \mu_t(\det_{\text{GL}_2}) \otimes \mu_{t+1} \otimes \cdots \otimes \mu_{n-1} \otimes 1 \\ & \bigoplus_{t=1}^{n-1} \delta_\beta \cdot \text{Ind}_{P_{\text{SO}_{2n+2,\beta}}^{1,\dots,1,2,1,\dots,1}(F)}^{\text{SO}_{2n+2,\beta}(F)} \mu_1 \otimes \cdots \otimes \mu_{t-1} \otimes \mu_t(\det_{\text{GL}_2}) \otimes \mu_{t+1} \otimes \cdots \otimes \mu_{n-1} \otimes \lambda_0 \\ & \oplus \text{Ind}_{B_{\text{SO}_{2n+2,\beta}(F)}}^{\text{SO}_{2n+2,\beta}(F)} \mu_1 \otimes \cdots \otimes \mu_{n-1} \otimes \xi \otimes 1 \\ & \oplus \text{Ind}_{B_{\text{SO}_{2n+2,\beta}(F)}}^{\text{SO}_{2n+2,\beta}(F)} \mu_1 \otimes \cdots \otimes \mu_{n-1} \otimes \lambda_0 | \cdot |^{\frac{1}{2}} \otimes 1 \\ & \oplus \delta_\beta \cdot \left(\text{Ind}_{B_{\text{SO}_{2n+2,\beta}(F)}}^{\text{SO}_{2n+2,\beta}(F)} \mu_1 \otimes \cdots \otimes \mu_{n-1} \otimes \xi \otimes \lambda_0 \right. \\ & \quad \left. \oplus \text{Ind}_{B_{\text{SO}_{2n+2,\beta}(F)}}^{\text{SO}_{2n+2,\beta}(F)} \mu_1 \otimes \cdots \otimes \mu_{n-1} \otimes | \cdot |^{\frac{1}{2}} \otimes \lambda_0 \right) \\ & \oplus (1 - \delta_\beta) \cdot \text{Ind}_{B_{\text{SO}_{2n+2,\beta}(F)}}^{\text{SO}_{2n+2,\beta}(F)} \mu_1 \otimes \cdots \otimes \mu_{n-1} \otimes | \cdot |^{\frac{1}{2}} \otimes 1. \end{aligned}$$

Proof. By conjugation of some Weyl element, it suffices to consider the unramified constituent of the induced representation $\text{Ind}_{Q_{2n-2}(F)}^{\text{SO}_{4n+3}(F)} \tau'_1 \otimes \sigma_1$, here

$$\tau'_1 = \text{Ind}_{P_{2, \dots, 2, 1, 1}(F)}^{\text{GL}_{2n}(F)} \mu_1(\det_{\text{GL}_2}) \otimes \cdots \otimes \mu_{n-1}(\det_{\text{GL}_2}),$$

and $\sigma_1 = \text{Ind}_{Q_2^{\text{SO}_7(F)}}^{\text{SO}_7(F)} \tilde{\tau}_1 \otimes \mathbf{1}_{\text{SO}_3}$, with $\tilde{\tau}_1 = \lambda_0 |\cdot|^{1/2} \times \xi$ being a representation of $\text{GL}_2(F)$.

Note that σ_1 is *non-generic* and $\tau'^{(\ell)} = 0$ for $\ell \geq n$. Applying [10, Theorem 5.1 (1)] with $j = 2n - 2 < \tilde{m} = 2n + 1$, we see that if $n + 2 \leq \ell \leq 2n + 1$, this twisted Jacquet module is always zero.

Now we consider the case $\ell = n + 1$. Applying [10, Theorem 5.1 (1)] with $j = 2n - 2 < \tilde{m} = 2n + 1$, we have

$$J_{\psi_{n+1, \beta}}(\text{Ind}_{Q_{2n-2}(F)}^{\text{SO}_{4n+3}(F)} \tau'_1 \otimes \sigma_1) \equiv \text{ind}_{Q'_{n-3}(F)}^{\text{SO}_{2n, \beta}(F)} |\det(\cdot)|^{-\frac{n-2}{2}} \tau'^{(n-1)} \otimes J_{\psi'_{2, \beta}}(\sigma_1).$$

To calculate the Jacquet module $J_{\psi'_{2, \beta}}(\sigma_1)$, applying [10, Theorem 5.1 (1)] again with $j = 2$ and $\ell = 2$, we have

$$J_{\psi'_{2, \beta}}(\sigma_1) \equiv \text{ind}_{Q'_0(F)}^{\text{SO}_{2, \beta}(F)} |\cdot|^{-\frac{1}{2}} \tilde{\tau}_1^{(2)} \otimes J_{\psi'_{0, \beta}}(\mathbf{1}_{\text{SO}_3}).$$

Hence by taking derivatives $\tau_1'^{(n-1)}$ and $\tilde{\tau}_1^{(2)}$, we can see that any unramified constituent of $J_{\psi_{n+1, \beta}}(\text{Ind}_{Q_{2n-2}(F)}^{\text{SO}_{4n+3}(F)} \tau'_1 \otimes \sigma_1)$ is a subquotient of

$$(3.4) \quad \text{Ind}_{B_{\text{SO}_{2n, \beta}(F)}}^{\text{SO}_{2n, \beta}(F)} \mu_1 \otimes \cdots \otimes \mu_{n-1} \otimes 1.$$

On the other hand, we may also write $\sigma_1 = \text{Ind}_{Q_2^{\text{SO}_7(F)}}^{\text{SO}_7(F)} \tilde{\tau}_2 \otimes \sigma_{\lambda_0}$, with $\tilde{\tau}_2 = |\cdot|^{1/2} \times \xi$ and σ_{λ_0} is the unique irreducible quotient in $\text{Ind}_{B_{\text{SO}_3(F)}}^{\text{SO}_3(F)} \lambda_0 |\cdot|^{1/2}$. Recall that λ_0 is a quadratic character, then σ_{λ_0} can be viewed as a character of $\text{PGL}_2(F)$ by composing it with the determinant, modulo squares. If $\beta \notin (F^\times)^2$, similarly we can see that any unramified constituent of $J_{\psi_{n+1, \beta}}(\text{Ind}_{Q_{2n-2}(F)}^{\text{SO}_{4n+3}(F)} \tau'_1 \otimes \sigma_1)$ also lies in (3.4). If $\beta \in (F^\times)^2$, applying [10, Theorem 5.1 (2)] with $j = 1$ and $\ell = 0$, we can see that

$$J_{\psi'_{0, \beta}}(\sigma_{\lambda_0}) \equiv \lambda_0,$$

and hence any unramified constituent of $J_{\psi_{n+1, \beta}}(\text{Ind}_{Q_{2n-2}(F)}^{\text{SO}_{4n+3}(F)} \tau'_1 \otimes \sigma_1)$ is a subquotient of

$$(3.5) \quad \text{Ind}_{B_{\text{SO}_{2n, \beta}(F)}}^{\text{SO}_{2n, \beta}(F)} \mu_1 \otimes \cdots \otimes \mu_{n-1} \otimes \lambda_0.$$

It follows that when $\beta \in (F^\times)^2$, any unramified constituent of $J_{\psi_{n+2, \beta}}(\pi_{\tau \otimes \sigma})$ is a subquotient of both (3.4) and (3.5). If it is non-zero, then the two sets $\{\mu_1^\pm, \mu_2^\pm, \dots, \mu_{n-1}^\pm, 1\}$ and $\{\mu_1^\pm, \mu_2^\pm, \dots, \mu_{n-1}^\pm, \lambda_0\}$ will be equal, which is impossible. This shows that any irreducible constituent must be 0, and this finishes the proof of Part (1) of the proposition.

Finally we take $\ell = n$. Applying [10, Theorem 5.1 (1)] with $j = 2n - 2 < \tilde{m} = 2n + 1$, we have

$$(3.6) \quad \begin{aligned} J_{\psi_{n,\beta}}(\text{Ind}_{Q_{2n-1}(F)}^{\text{SO}_{4n+3}(F)} \tau'_1 \otimes \sigma_1) &\equiv \text{ind}_{Q'_n(F)}^{\text{SO}_{2n+2,\beta}(F)} |\det(\cdot)|^{\frac{3-n}{2}} \tau'_1{}^{(n-2)} \otimes J_{\psi'_{2,\beta}}(\sigma_1) \\ &\oplus \text{ind}_{Q'_{n-1}(F)}^{\text{SO}_{2n+2,\beta}(F)} |\det(\cdot)|^{\frac{2-n}{2}} \tau'_1{}^{(n-1)} \otimes J_{\psi'_{1,\beta}}(\sigma_1). \end{aligned}$$

Now, arguing as in the case of $\ell = n + 1$, we can see that if an unramified constituent of $J_{\psi_{n,\beta}}(\text{Ind}_{Q_{2n-1}(F)}^{\text{SO}_{4n+3}(F)} \tau'_1 \otimes \sigma_1)$ comes from the first summand of (3.6), then it is a subquotient of

$$\begin{aligned} &\bigoplus_{t=1}^{n-1} \text{Ind}_{P_{\text{SO}_{2n+2,\beta}}^{1,\dots,1,2,1,\dots,1}(F)}^{\text{SO}_{2n+2,\beta}(F)} \mu_1 \otimes \cdots \otimes \mu_{t-1} \otimes \mu_t(\det_{\text{GL}_2}) \otimes \mu_{t+1} \otimes \cdots \otimes \mu_{n-1} \otimes 1 \\ &\bigoplus_{t=1}^{n-1} \delta_\beta \cdot \text{Ind}_{P_{\text{SO}_{2n+2,\beta}}^{1,\dots,1,2,1,\dots,1}(F)}^{\text{SO}_{2n+2,\beta}(F)} \mu_1 \otimes \cdots \otimes \mu_{t-1} \otimes \mu_t(\det_{\text{GL}_2}) \otimes \mu_{t+1} \otimes \cdots \otimes \mu_{n-1} \otimes \lambda_0. \end{aligned}$$

To compute the Jacquet module $J_{\psi'_{1,\beta}}(\sigma_1)$ in the second summand of (3.6), we apply [10, Theorem 5.1 (1)] with $j = 2$ and $\ell = 1$, and get

$$J_{\psi'_{1,\beta}}(\sigma_1) \equiv \text{ind}_{Q'_1(F)}^{\text{SO}_{4,\beta}(F)} \tilde{\tau}_1^{(1)} \otimes J_{\psi'_{0,\beta}}(\mathbf{1}_{\text{SO}_3}).$$

Moreover, one may also write $\sigma_1 = \text{Ind}_{Q_2^{\text{SO}_7(F)}}^{\text{SO}_7(F)} \tilde{\tau}_2 \otimes \sigma_{\lambda_0}$ and get

$$J_{\psi'_{1,\beta}}(\sigma_1) \equiv \text{ind}_{Q'_1(F)}^{\text{SO}_{4,\beta}(F)} \tilde{\tau}_2^{(1)} \otimes J_{\psi'_{0,\beta}}(\sigma_{\lambda_0}).$$

Recall that we have $J_{\psi'_{0,\beta}}(\sigma_{\lambda_0}) \equiv \lambda_0$ if $\beta \in (F^\times)^2$. Then we obtain that, if an unramified constituent of $J_{\psi_{n,\beta}}(\text{Ind}_{Q_{2n-1}(F)}^{\text{SO}_{4n+3}(F)} \tau'_1 \otimes \sigma_1)$ comes from the second summand of (3.6), then it is a subquotient of

$$\begin{aligned} &\text{Ind}_{B_{\text{SO}_{2n+2,\beta}}(F)}^{\text{SO}_{2n+2,\beta}(F)} \mu_1 \otimes \cdots \otimes \mu_{n-1} \otimes \xi \otimes 1 \\ &\oplus \text{Ind}_{B_{\text{SO}_{2n+2,\beta}}(F)}^{\text{SO}_{2n+2,\beta}(F)} \mu_1 \otimes \cdots \otimes \mu_{n-1} \otimes \lambda_0 | \cdot |^{\frac{1}{2}} \otimes 1 \\ &\oplus \delta_\beta \cdot \left(\text{Ind}_{B_{\text{SO}_{2n+2,\beta}}(F)}^{\text{SO}_{2n+2,\beta}(F)} \mu_1 \otimes \cdots \otimes \mu_{n-1} \otimes \xi \otimes \lambda_0 \right. \\ &\quad \left. \oplus \text{Ind}_{B_{\text{SO}_{2n+2,\beta}}(F)}^{\text{SO}_{2n+2,\beta}(F)} \mu_1 \otimes \cdots \otimes \mu_{n-1} \otimes | \cdot |^{\frac{1}{2}} \otimes \lambda_0 \right) \\ &\oplus (1 - \delta_\beta) \cdot \text{Ind}_{B_{\text{SO}_{2n+2,\beta}}(F)}^{\text{SO}_{2n+2,\beta}(F)} \mu_1 \otimes \cdots \otimes \mu_{n-1} \otimes | \cdot |^{\frac{1}{2}} \otimes 1, \end{aligned}$$

as desired. \square

We consider moreover a special case that $\ell = \tilde{m} = 2n + 1$ and $w_0 \in W$, which will be used in the proof of Lemma 4.1 later. The result is the following:

Proposition 3.3. *Let τ and σ be the same as in any case of Propositions 3.1–3.2, and let $w_0 \in W$. Then we have $J_{\psi_{\tilde{m},w_0}}(\pi_{\tau \otimes \sigma}) = 0$ if $n > 1$. If $n = 1$, the same result holds while $\omega_\tau = 1$.*

Proof. As before, conjugating by some Weyl element, it suffices to consider the unramified constituent of $J_{\psi_{\tilde{m}, w_0}}(\text{Ind}_{P(F)}^{H(F)}(\tau' \otimes \sigma))$, where

$$\tau' = \text{Ind}_{P_{2, \dots, 2}}^{\text{GL}_{2n}(F)} \mu_1(\det_{\text{GL}_2}) \otimes \cdots \otimes \mu_n(\det_{\text{GL}_2}),$$

or

$$\tau' = \text{Ind}_{P_{2, \dots, 2, 1, 1}(F)}^{\text{GL}_{2n}(F)} \mu_1(\det_{\text{GL}_2}) \otimes \cdots \otimes \mu_{n-1}(\det_{\text{GL}_2}) \otimes \lambda_0 |\cdot|^{1/2} \otimes |\cdot|^{1/2}.$$

By [10, Theorem 5.1 (3)], we have

$$J_{\psi_{\tilde{m}, w_0}}(\text{Ind}_{P(F)}^{H(F)}(\tau' \otimes \sigma)) = d_{\tau'} \cdot J_{\psi_{\tilde{m}-2n}}(\sigma),$$

where $d_{\tau'}$ is the dimension of the space of ψ -Whittaker functionals on τ' . By the construction of τ' , we can see that $d_{\tau'} = 0$ under the assumptions of the proposition, and this finishes the proof. \square

3.3. An additional case. In the end of this section, we consider an additional case, which is concerned with the Jacquet modules for a group of smaller size. It will be used in the proof of Proposition 4.3 later. We consider the induced representation

$$\text{Ind}_{R_{2n-1}}^{\text{SO}_{4n+1}(F)} \tau_1 |\det(\cdot)|^{1/2} \otimes \sigma$$

of the F -split group $\text{SO}_{4n+1}(F)$. Here $R_{2n-1} \subset \text{SO}_{4n+1}$ is the maximal parabolic subgroup whose Levi subgroup is isomorphic to $\text{GL}_{2n-1} \times \text{SO}_3$, $\sigma = \text{Ind}_{B_{\text{SO}_3}(F)}^{\text{SO}_3(F)} \xi$ with ξ being an unramified character of F^\times , and

$$(3.7) \quad \tau_1 = \mu_1 \times \cdots \times \mu_{n-1} \times \lambda \times \mu_{n-1}^{-1} \times \cdots \times \mu_1^{-1},$$

with μ_i 's and λ being unramified characters of F^\times and in addition λ being quadratic. For $1 \leq \ell \leq 2n$, let $N_\ell^\sharp \subset \text{SO}_{4n+1}$ and $\psi_{\ell, \beta}^\sharp : N_\ell^\sharp(F) \rightarrow \mathbb{C}^\times$ be similar to $N_\ell \subset H$ and $\psi_{\ell, \beta} : N_\ell(F) \rightarrow \mathbb{C}^\times$ in previous parts of this section. Let $\pi_{\tau_1 \otimes \sigma}$ be the unramified constituent of $\text{Ind}_{R_{2n-1}}^{\text{SO}_{4n+1}(F)} \tau_1 |\cdot|^{1/2} \otimes \sigma$, and we are going to consider the twisted Jacquet module $J_{\psi_{\ell, \beta}^\sharp}(\pi_{\tau_1 \otimes \sigma})$.

Proposition 3.4. *With the data chosen as above, the following hold:*

- (1) *Suppose that $\beta \in F^\times - (F^\times)^2$. Then any twisted Jacquet module $J_{\psi_{\ell, \beta}^\sharp}(\pi_{\tau_1 \otimes \sigma}) = 0$ for all $n \leq \ell \leq 2n$.*
- (2) *Suppose that $\beta \in (F^\times)^2$. Then any twisted Jacquet module $J_{\psi_{\ell, \beta}^\sharp}(\pi_{\tau_1 \otimes \sigma}) = 0$ for all $n+1 \leq \ell \leq 2n$. When $\ell = n$, any unramified constituent of $J_{\psi_{n+1, \beta}^\sharp}(\pi_{\tau_1 \otimes \sigma})$ is a subquotient of $\text{Ind}_{B_{\text{SO}_{2n}}(F)}^{\text{SO}_{2n}(F)} \mu_1 \otimes \cdots \otimes \mu_{n-1} \otimes \lambda$.*

Proof. The calculation is similar to those in previous cases. By conjugation of some Weyl element, it suffices to consider the unramified constituent of the induced representation $\text{Ind}_{R_{2n-2}(F)}^{\text{SO}_{4n+1}(F)} \tau'_1 \otimes \sigma'$ instead of $\pi_{\tau \otimes \sigma}$, here

$$\tau'_1 = \text{Ind}_{P_{2, \dots, 2}(F)}^{\text{GL}_{2n-2}(F)} \mu_1(\det_{\text{GL}_2}) \otimes \cdots \otimes \mu_{n-1}(\det_{\text{GL}_2}),$$

and $\sigma' = \text{Ind}_{P_{\text{SO}_5(F)}^1}^{\text{SO}_5(F)} \xi \otimes \sigma_\lambda$. As before, we have denoted by σ_λ the unique irreducible quotient of $\text{Ind}_{B_{\text{SO}_3(F)}}^{\text{SO}_3(F)} \lambda \cdot |\cdot|^{1/2}$, which can be viewed as a character of $\text{PGL}_2(F)$ by composing it with the determinant, modulo squares. Note that σ' is non-generic and the derivative $\tau_1^{(\ell)}$ of τ_1' vanishes for $\ell \geq n$. Now applying [10, Theorem 5.1 (i)] with $j = 2n - 2$, we can see that the corresponding twisted Jacquet module vanishes for all $n + 1 \leq \ell \leq 2n$.

When $\ell = n$, by the formula in [10, Theorem 5.1 (i)] we have

$$J_{\psi_{n,\beta}^\#} (\text{Ind}_{R_{2n-2}(F)}^{\text{SO}_{4n+1}(F)} \tau_1' \otimes \sigma') \equiv \text{ind}_{R'_{n-1}(F)}^{\text{SO}_{2n,\beta}(F)} |\cdot|^{2-n} \tau_1'^{(n-1)} \otimes J_{\psi_{1,\beta}^\#} (\sigma').$$

Now we consider $J_{\psi_{1,\beta}^\#} (\sigma')$. Applying [10, Theorem 5.1 (i)] again with $\ell = 1$ and $j = 1$, we get

$$J_{\psi_{1,\beta}^\#} (\sigma') \equiv J_{\psi_{0,\beta}^\#} (\sigma_\lambda),$$

which is zero unless $\beta \in (F^\times)^2$, in which case we have $J_{\psi_{0,\beta}^\#} (\sigma_\lambda) \equiv \lambda$. Then the desired results will follow. \square

4. THE CUSPIDALITY OF THE DESCENT

In the section, we show the cuspidality of the descent representation. We come back to the global settings and start with the following lemma, which is a direct consequence of the local calculations in the previous section (Propositions 3.1 and 3.3). Throughout this section, we let τ and σ be as in §2.2.

Lemma 4.1. *The Bessel-Fourier coefficients f^{ψ_ℓ, w_0} vanish for any anisotropic vector $w_0 \in V^{(\ell)}$ and $f \in \mathcal{E}_{\tau \otimes \sigma}$ while $n + 2 \leq \ell \leq \tilde{m}$ (here we require $n \geq 2$ if $\tilde{m} = 2n$).*

Proof. Write the residual representation as $\mathcal{E}_{\tau \otimes \sigma} = \otimes'_v \Pi_v$. Note that $H(F_v)$ is F_v -split at almost all places v . We fix a finite place v such that $H(F_v) = \text{SO}_{4n+3}(F_v)$ is F_v -split, $\tau_v, \sigma_v, \Pi_v, \psi_v$ are all unramified, and ω_{τ_v} is trivial. By a suitable conjugation we may assume that $w_0 = y_\beta$ for some $\beta \in F^\times$ if $\ell \leq 2n$ (see [14, §2] or [17, Lemma 2.4]), and consider the corresponding twisted Jacquet module $J_{\psi_{\ell,\beta}}(\Pi_v)$. Write

$$(4.1) \quad \tau_v = \mu_{1,v} \times \cdots \times \mu_{n,v} \times \mu_{n,v}^{-1} \times \cdots \times \mu_{1,v}^{-1},$$

where $\mu_{i,v}$'s are unramified characters of F_v^\times . By Propositions 3.1 Part (2) and Proposition 3.3 (for the case $\ell = 2n + 1$), the Jacquet module $J_{\psi_{\ell, w_0}}(\Pi_v)$ vanishes for any $n + 2 \leq \ell \leq 2n + 1$, and hence the Bessel-Fourier coefficient f^{ψ_ℓ, w_0} vanishes for any $n + 2 \leq \ell \leq \tilde{m}$ (we require $n \geq 2$ if $\tilde{m} = 2n$). This completes the proof of the lemma. \square

To prove the cuspidality of the descent, we also need the vanishing property for the depth $\ell = n + 1$.

Proposition 4.2. *Suppose that (τ, σ) satisfies Assumption 1.1. Then the Bessel-Fourier coefficients f^{ψ_{n+1}, w_0} vanish for any anisotropic vector $w_0 \in V^{(\ell)}$ and $f \in \mathcal{E}_{\tau \otimes \sigma}$.*

Proof. For simplicity, we take $w_0 = y_\beta$ ($\beta \in F^\times$) in the proof. This excludes only one case that $n = 1$ and V_0 is anisotropic, where similar arguments can be applied. Assume that the Bessel-Fourier coefficient $f^{\psi_{n+1,\beta}} \neq 0$ for some choice of data. Let $\mathcal{D}_{\psi_{n+1,\beta}}(\mathcal{E}_{\tau \otimes \sigma})$ be the $\mathrm{SO}_{2n,\beta}(\mathbb{A})$ -span of the coefficients $f^{\psi_{n+1,\beta}}|_{\mathrm{SO}_{2n,\beta}(\mathbb{A})}$. Note that $\mathrm{SO}_{2n,\beta}(\mathbb{A})$ is relevant to $\mathrm{SO}(V_0)(\mathbb{A})$. Then by a formula for constant terms in [10, Theorem 7.2] and a similar argument to that in the proof of [10, Theorem 7.6] (see also the remarks in [10, §7.4] and the proof of [17, Proposition 6.3]), the automorphic module $\mathcal{D}_{\psi_{n+1,\beta}}(\mathcal{E}_{\tau \otimes \sigma})$ is cuspidal and has a direct sum decomposition of irreducible cuspidal automorphic representations.

Let π be any irreducible summand of $\mathcal{D}_{\psi_{n+1,\beta}}(\mathcal{E}_{\tau \otimes \sigma})$. We claim that π has the generic global Arthur parameter ϕ_τ , and lies in the global Vogan packet $\tilde{\Pi}_{\phi_\tau}[\mathrm{SO}_{2n}^*]$.

We prove the claim case by case. First we assume that $\omega_\tau = 1$. If $\beta \in (F^\times)^2$, then for almost all places v of F , we have $H(F_v)$ is F_v -split, $\tau_v, \sigma_v, \Pi_v, \psi_v$ are all unramified, $\omega_{\tau_v} = 1$, and $\beta \in (F_v^\times)^2$. Fix any such place v , and write

$$\tau_v = \mu_{1,v} \times \cdots \times \mu_{n,v} \times \mu_{n,v}^{-1} \times \cdots \times \mu_{1,v}^{-1},$$

where $\mu_{i,v}$'s are unramified characters of F_v^\times . By Proposition 3.1 Part (2), the unramified constituent of the Jacquet module $J_{\psi_{n+1,\beta}}(\Pi_v)$ is a subquotient of

$$\mathrm{Ind}_{B\mathrm{SO}_{2n}(F_v)}^{\mathrm{SO}_{2n}(F_v)} \mu_{1,v} \otimes \cdots \otimes \mu_{n,v},$$

which shows that the Satake parameter of π_v matches the Satake parameter of τ_v , for almost all places. This shows that the global Arthur parameter $\phi_\pi = \phi_\tau$. If $\beta \notin (F^\times)^2$, then there exists a finite place v such that $H(F_v)$ is F_v -split, $\tau_v, \sigma_v, \Pi_v, \psi_v$ are all unramified, $\omega_{\tau_v} = 1$, and $\beta \notin (F_v^\times)^2$. Then by Proposition 3.1 Part (1), one sees that $\mathcal{D}_{\psi_{n+1,\beta}}(\mathcal{E}_{\tau \otimes \sigma}) = 0$, which is not the case we are considering.

Next we assume that $\omega_\tau \neq 1$. If $\beta \in (F^\times)^2$, then there exists a finite place v such that $H(F_v)$ is F_v -split, $\tau_v, \sigma_v, \Pi_v, \psi_v$ are all unramified, $\omega_{\tau_v} \neq 1$, and $\beta \in (F_v^\times)^2$. Then by Proposition 3.2 Part (1), one sees that $\mathcal{D}_{\psi_{n+1,\beta}}(\mathcal{E}_{\tau \otimes \sigma}) = 0$, which is also not the case we are considering. If $\beta \notin (F^\times)^2$, since we just consider the case that $\mathcal{D}_{\psi_{n+1,\beta}}(\mathcal{E}_{\tau \otimes \sigma}) \neq 0$, then by Proposition 3.1 Part (2) and Proposition 3.2 Part (1), for the finite places v such that $H(F_v)$ is F_v -split, $\tau_v, \sigma_v, \Pi_v, \psi_v$ are all unramified, we have either $\omega_{\tau_v} = 1$ and $\beta \in (F_v^\times)^2$ or $\omega_{\tau_v} \neq 1$ and $\beta \notin (F_v^\times)^2$. Now we consider the parameter of π . For the places such that $H(F_v)$ is F_v -split, $\tau_v, \sigma_v, \Pi_v, \psi_v$ are all unramified, $\omega_{\tau_v} = 1$, and $\beta \in (F_v^\times)^2$, the argument for the first case ($\omega_\tau = 1$ and $\beta \in (F^\times)^2$) already shows that the Satake parameters of τ_v and π_v match each other. And for the places such that $H(F_v)$ is F_v -split, $\tau_v, \sigma_v, \Pi_v, \psi_v$ are all unramified, $\omega_{\tau_v} \neq 1$, and $\beta \notin (F_v^\times)^2$, we write

$$(4.2) \quad \tau_v = \mu_{1,v} \times \cdots \times \mu_{n-1,v} \times 1 \times \lambda_{0,v} \times \mu_{n-1,v}^{-1} \times \cdots \times \mu_{1,v}^{-1},$$

where $\lambda_{0,v}$ is an unramified quadratic character of F_v^\times . Then by Proposition 3.2 Part (1), the unramified constituent of the Jacquet module $J_{\psi_{n+1,\beta}}(\Pi_v)$ is a subquotient of

$$\mathrm{Ind}_{B\mathrm{SO}_{2n}(F_v)}^{\mathrm{SO}_{2n,\beta}(F_v)} \mu_{1,v} \otimes \cdots \otimes \mu_{n-1,v} \otimes 1,$$

which also shows that the Satake parameters of τ_v and π_v match each other. Therefore, the Satake parameter of π_v matches the Satake parameter of τ_v for almost all places, and hence $\phi_\pi = \phi_\tau$ by strong multiplicity one theorem for general linear groups. Moreover, this also shows that here the $\beta \in F^\times$ can only be chosen such that ϕ_τ is $\mathrm{SO}_{2n,\beta}$ -relevant (see §1).

We continue the proof of the proposition. By construction, there exists an automorphic form $\varphi_\pi \in \pi$ (under the automorphic descent realization $\mathcal{D}_{\psi_{n+1,\beta}}(\mathcal{E}_{\tau \otimes \sigma})$) such that the inner product

$$(4.3) \quad \langle \varphi_\pi, \mathrm{Res}_{s=1/2} E^{\psi_{n+1,\beta}}(s, \cdot, \phi_{\tau \otimes \sigma}) \rangle \neq 0$$

for some choice of data. By [17, Corollary 4.4], the pair (π, σ) has a non-zero Bessel period with $\pi \in \tilde{\Pi}_{\phi_\tau}(\mathrm{SO}_{2n,\beta})$. But this contradicts to Assumption 1.1. Hence the proposition is proved. \square

As a corollary, we have the cuspidality of the descent $\mathcal{D}_{\psi_{n,\beta}}(\mathcal{E}_{\tau \otimes \sigma})$.

Proposition 4.3. *Suppose that (τ, σ) satisfies Assumption 1.1. Then the twisted automorphic descent $\mathcal{D}_{\psi_{n,\beta}}(\mathcal{E}_{\tau \otimes \sigma})$ is a cuspidal automorphic $G_{n+1,\beta}(\mathbb{A})$ -module.*

Proof. The proposition can be proved by considering all the constant terms of the Bessel-Fourier coefficient $f^{\psi_{n,\beta}}$ with $f \in \mathcal{E}_{\tau \otimes \sigma}$. The proof is similar to that of [10, Theorem 7.6], see also [14, Proposition 2.6] and [17, Proposition 6.3]. We briefly introduce it as follows.

Let $\tilde{m}_{n,\beta}$ be the Witt index of $y_\beta^\perp \cap V^{(n)}$. Using [10, Theorem 7.3], and also the remarks in [10, §7.4], the constant terms $c_p(f^{\psi_{n,\beta}})$ ($1 \leq p \leq \tilde{m}_{n,\beta}$), along maximal parabolic subgroup of $G_{n+1,\beta}$ with Levi subgroup isomorphic to $\mathrm{GL}_p \times G_{n+1-p,\beta}$, can be expressed as a summation whose summands are expressed as integrals of following terms

$$f^{\psi_{n+p,\beta}}, (f^{U_{p-i}})^{\psi_{n+i,\beta}}, 0 \leq i \leq p-1,$$

where $f^{U_{p-i}}$ is the constant term of f along the maximal parabolic subgroup $Q_{p-i} \subset H$ with Levi subgroup isomorphic to $\mathrm{GL}_{p-i} \times \mathrm{SO}(V^{(p-i)})$. Here $(f^{U_{p-i}})^{\psi_{n+i,\beta}}$ denotes the $\psi_{n+i,\beta}$ -Bessel-Fourier coefficient of $f^{U_{p-i}}|_{\mathrm{SO}(V^{(p-i)})}(\mathbb{A})$. By Lemma 4.1 and Proposition 4.2, $f^{\psi_{n+p,\beta}}$ vanishes since $p \geq 1$. By the cuspidal support of $\mathcal{E}_{\tau \otimes \sigma}$, the constant term $f^{U_{p-i}}$ vanishes unless the parabolic Q_{p-i} contains the cuspidal support $\tau_1 \otimes \cdots \otimes \tau_r \otimes \sigma$. It is easy to see that $f^{U_{p-i}}$ belongs to the representation

$$(\tau_{i_1} \boxplus \cdots \boxplus \tau_{i_k}) \otimes \mathcal{E}_{(\tau_{j_1} \boxplus \cdots \boxplus \tau_{j_{r-k}}) \otimes \sigma},$$

where $\{i_1, \dots, i_k\} \cup \{j_1, \dots, j_{r-k}\} = \{1, 2, \dots, r\}$, and $\mathcal{E}_{(\tau_{j_1} \boxplus \cdots \boxplus \tau_{j_{r-k}}) \otimes \sigma}$ is the residual representation of $\mathrm{SO}(V^{(p-i)})$ constructed in the same way as $\mathcal{E}_{\tau \otimes \sigma}$. Note that $f^{U_{p-i}}|_{\mathrm{SO}(V^{(p-i)})}(\mathbb{A}) \in \mathcal{E}_{(\tau_{j_1} \boxplus \cdots \boxplus \tau_{j_{r-k}}) \otimes \sigma}$. Using a similar argument to that in Lemma 4.1 (note that here $n+i$ is large enough by Assumption 1.1, Propositions 3.1 and 3.4), by Propositions 3.1 and 3.4 again, all $\psi_{n+i,\beta}$ -Bessel-Fourier coefficients of $\mathcal{E}_{(\tau_{j_1} \boxplus \cdots \boxplus \tau_{j_{r-k}}) \otimes \sigma}$ vanish, hence $(f^{U_{p-i}})^{\psi_{n+i,\beta}}$ also vanishes for any $0 \leq i \leq p-1$. Therefore, all the constant terms of $f^{\psi_{n,\beta}}$ along various unipotent subgroups of $G_{n+1,\beta}$ vanish, which

means that the Bessel-Fourier coefficient $f^{\psi_{n,\beta}}$ is cuspidal, and the twisted automorphic descent $\mathcal{D}_{\psi_{n,\beta}}(\mathcal{E}_{\tau \otimes \sigma})$ is a cuspidal automorphic $G_{n+1,\beta}(\mathbb{A})$ -module.

This completes the proof of the proposition. \square

In following proposition, we show that in certain cases, to obtain the cuspidality of the descent, we do not need to assume Assumption 1.1.

Proposition 4.4. *If we take $w_0 = y_\beta$ with $\beta \notin (F^\times)^2$ when $\omega_\tau = 1$, or take $w_0 = y_\beta$ with $\beta \in (F^\times)^2$ when $\omega_\tau \neq 1$, then the Bessel-Fourier coefficient $f^{\psi_{\ell,\beta}}$ vanishes for any $f \in \mathcal{E}_{\tau \otimes \sigma}$ and any $n+1 \leq \ell \leq \tilde{m}$, and the descent $\mathcal{D}_{\psi_{n,\beta}}(\mathcal{E}_{\tau \otimes \sigma})$ is cuspidal.*

Proof. We have already known from Lemma 4.1 that $f^{\psi_{\ell,\beta}}$ vanishes for any $n+2 \leq \ell \leq \tilde{m}$. Hence, for the first statement, we only need to consider the case of $\ell = n+1$. As we have seen in the proof of Proposition 4.2, if $\beta \notin (F^\times)^2$ and $\omega_\tau = 1$, there exists a finite place v of F such that $H(F_v)$ is F_v -split, τ_v, σ_v, Π_v and ψ_v are all unramified, $\beta \notin (F_v^\times)^2$ and $\omega_{\tau_v} = 1$. And if $\beta \in (F^\times)^2$ and $\omega_\tau \neq 1$, there exists a finite place v of F such that $H(F_v)$ is F_v -split, τ_v, σ_v, Π_v and ψ_v are all unramified, $\beta \in (F_v^\times)^2$ and $\omega_{\tau_v} \neq 1$. Then by Proposition 3.1 Part (1) and Proposition 3.2 Part (1), the global Fourier coefficient $f^{\psi_{n+1,\beta}}$ vanishes, as desired.

The cuspidality of the descent follows similarly along the same lines of the proof of Proposition 4.3. \square

5. THE NON-VANISHING OF THE DESCENT CONSTRUCTION

In this section, we show that the descent $\mathcal{D}_{\psi_{n,\beta}}(\mathcal{E}_{\tau \otimes \sigma})$ is non-vanishing.

5.1. Generalized and degenerate Whittaker-Fourier coefficients. First, we recall the generalized and degenerate Whittaker-Fourier coefficients attached to nilpotent orbits, following the formulation in [11]. Let G be a reductive group defined over a number field F , and \mathfrak{g} be the Lie algebra of $G(F)$. Given any semi-simple element $s \in \mathfrak{g}$, under the adjoint action, \mathfrak{g} is decomposed into a direct sum of eigenspaces \mathfrak{g}_i^s corresponding to eigenvalues i . The element s is called *rational semi-simple* if all its eigenvalues are in \mathbb{Q} . Given a nilpotent element u , a *Whittaker pair* is a pair (s, u) with $s \in \mathfrak{g}$ being a rational semi-simple element, and $u \in \mathfrak{g}_{-2}^s$. The element s in a Whittaker pair (s, u) is called a *neutral element* for u if there is a nilpotent element $v \in \mathfrak{g}$ such that (v, s, u) is an \mathfrak{sl}_2 -triple. A Whittaker pair (s, u) with s being a neutral element for u is called a *neutral pair*.

Given any Whittaker pair (s, u) , define an anti-symmetric form ω_u on \mathfrak{g} by

$$\omega_u(X, Y) := \kappa(u, [X, Y]).$$

Here κ is the Killing form on $\mathfrak{g} \times \mathfrak{g}$. For any rational number $r \in \mathbb{Q}$, let $\mathfrak{g}_{\geq r}^s = \bigoplus_{r' \geq r} \mathfrak{g}_{r'}^s$. Let $\mathfrak{u}_s = \mathfrak{g}_{\geq 1}^s$ and let $\mathfrak{n}_{s,u}$ be the radical of $\omega_u|_{\mathfrak{u}_s}$. Then it is clear that $[\mathfrak{u}_s, \mathfrak{u}_s] \subset \mathfrak{g}_{\geq 2}^s \subset \mathfrak{n}_{s,u}$. For any $X \in \mathfrak{g}$, let \mathfrak{g}_X be the centralizer of X in \mathfrak{g} . By [11, Lemma 3.2.6], $\mathfrak{n}_{s,u} = \mathfrak{g}_{\geq 2}^s + \mathfrak{g}_1^s \cap \mathfrak{g}_u$. Note that if the Whittaker pair (s, u) comes from an \mathfrak{sl}_2 -triple (v, s, u) , then $\mathfrak{n}_{s,u} = \mathfrak{g}_{\geq 2}^s$. Let $U_s = \exp(\mathfrak{u}_s)$ and $N_{s,u} = \exp(\mathfrak{n}_{s,u})$ be the corresponding unipotent subgroups of G . Define a character of $N_{s,u}$ by

$$\psi_u(n) = \psi(\kappa(u, \log(n))),$$

here $\psi : F \backslash \mathbb{A} \rightarrow \mathbb{C}^\times$ is a fixed non-trivial additive character, and we extend the killing form κ to $\mathfrak{g}(\mathbb{A}) \times \mathfrak{g}(\mathbb{A})$. Moreover, let $N'_{s,u} = N_{s,u} \cap \ker(\psi_u)$. Then $U_s/N'_{s,u}$ is a Heisenberg group with center $N_{s,u}/N'_{s,u}$, here $U_s = \exp(\mathfrak{u}_s)$.

Let π be an irreducible automorphic representation of $G(\mathbb{A})$. For any $\phi \in \pi$, the *degenerate Whittaker-Fourier coefficient* of ϕ attached to (s, u) is defined to be

$$(5.1) \quad \mathcal{F}_{s,u}(\phi)(g) := \int_{[N_{s,u}]} \phi(ng) \psi_u^{-1}(n) \, dn.$$

If s is a neutral element for u , then $\mathcal{F}_{s,u}(\phi)$ is also called a *generalized Whittaker-Fourier coefficient* of ϕ . Let $\mathcal{F}_{s,u}(\pi) = \{\mathcal{F}_{s,u}(\phi) | \phi \in \pi\}$. The *wave-front set* $\mathfrak{n}(\pi)$ of π is defined to be the set of nilpotent orbits \mathcal{O} such that $\mathcal{F}_{s,u}(\pi)$ is non-zero, for some neutral pair (s, u) with $u \in \mathcal{O}$. Note that if $\mathcal{F}_{s,u}(\pi)$ is non-zero for some neutral pair (s, u) with $u \in \mathcal{O}$, then it is non-zero for any such neutral pair (s, u) , since the non-vanishing property of such generalized Whittaker-Fourier coefficients does not depend on the choices of representatives of \mathcal{O} . Let $\mathfrak{n}^m(\pi)$ be the set of maximal elements in $\mathfrak{n}(\pi)$ under the natural order of nilpotent orbits. We recall a theorem from [11] in the following.

Theorem 5.1 (Theorem C, [11]). *Let π be an irreducible automorphic representation of $G(\mathbb{A})$. Given a Whittaker pair (s', u) and a neutral pair (s, u) , if $\mathcal{F}_{s',u}(\pi)$ is non-zero, then $\mathcal{F}_{s,u}(\pi)$ is non-zero.*

When G is a quasi-split classical group, it is known that the nilpotent orbits are parametrized by pairs $(\underline{p}, \underline{q})$, where \underline{p} is a partition and \underline{q} is a set of non-degenerate quadratic forms (see [26, §I.6]). When $G = \mathrm{Sp}_{2n}$, \underline{p} is symplectic partition, namely, odd parts occur with even multiplicities. When $G = \mathrm{SO}_{2n}^\alpha, \mathrm{SO}_{2n+1}$, \underline{p} is orthogonal partition, namely, even parts occur with even multiplicities. In these cases, let $\mathfrak{p}^m(\pi)$ be the set of partitions corresponding to nilpotent orbits in $\mathfrak{n}^m(\pi)$. A well-known folklore conjecture is that $\mathfrak{p}^m(\pi)$ is a singleton. In this section, for any symplectic or orthogonal partition \underline{p} , by a generalized Whittaker-Fourier coefficient of π attached to \underline{p} , we mean a generalized Whittaker-Fourier coefficient $\mathcal{F}_{s,u}(\phi)$ attached to a nilpotent orbit \mathcal{O} parametrized by a pair $(\underline{p}, \underline{q})$ for some \underline{q} , with $\phi \in \pi$, $u \in \mathcal{O}$ and (s, u) being a neutral pair. For convenience, sometimes we also write a generalized Whittaker-Fourier coefficient attached to \underline{p} as $\mathcal{F}^{\psi_{\underline{p}}}(\phi)$, without specifying the F -rational nilpotent orbit \mathcal{O} and neutral pairs.

For $G = \mathrm{SO}_{2n+1}$, an orthogonal partition \underline{p} is called *special* if it has an even number of odd parts between two consecutive even parts and an odd number of odd parts greater than the largest even part (see [6, §6.3]). By the main results of [13], any $\underline{p} \in \mathfrak{p}^m(\pi)$ is special. This will play an important role in the following.

5.2. Non-vanishing of the descent. Now we come back to the global situation where the groups and representations are the same as in §2. First we prove the following proposition.

Proposition 5.2. *$\mathcal{E}_{\tau \otimes \sigma}$ has a non-zero generalized Whittaker-Fourier coefficient attached to the partition $[(2n)^2 1^3]$.*

Proof. Let $\alpha_i = e_i - e_{i+1}$ ($1 \leq i \leq 2n - 1$) be a subset of simple roots for $\mathrm{SO}_{4n+3}^{V_0}$. For $1 \leq i \leq 2n$, let x_{α_i} be the one-dimensional root subgroup in \mathfrak{g} corresponding to α_i . By [26, §I.6], there is only one nilpotent orbit \mathcal{O} corresponding to the partition $[(2n)^2 1^3]$.

A representative of the nilpotent orbit \mathcal{O} can be taken to be $u = \sum_{i=1}^{2n-1} x_{-\alpha_i}(1)$. Let s be the following semi-simple element

$$s = \mathrm{diag}(2n - 1, 2n - 3, \dots, 1 - 2n, 0, 0, 0, 2n - 1, 2n - 3, \dots, 1 - 2n).$$

Then it is clear that (s, u) is a neutral pair.

We want to show that $\mathcal{F}_{s,u}(\mathcal{E}_{\tau \otimes \sigma})$ is non-zero. To this end, we take another semisimple element

$$s' = \mathrm{diag}(4n, 4n - 2, \dots, 2, 0, 0, 0, -2, \dots, -4n).$$

It is clear that (s', u) is a Whittaker pair. We consider $\mathcal{F}_{s',u}(\mathcal{E}_{\tau \otimes \sigma})$. Recall that Q_{2n} is the parabolic subgroup of $\mathrm{SO}_{4n+3}^{V_0}$ with Levi subgroup isomorphic to $\mathrm{GL}_{2n} \times \mathrm{SO}_3(V_0)$ and unipotent radical subgroup U_{2n} . Then, by definition, for any $\phi \in \mathcal{E}_{\tau \otimes \sigma}$, $\mathcal{F}_{s',u}(\phi)$ is the constant term integral over $U_{2n}(F) \backslash U_{2n}(\mathbb{A})$ combined with a non-degenerate Whittaker-Fourier coefficient of τ . Since $\mathcal{E}_{\tau \otimes \sigma}$ is constructed from data $\tau \otimes \sigma$ on the Levi subgroup $\mathrm{GL}_{2n}(\mathbb{A}) \times \mathrm{SO}_3(V_0)(\mathbb{A})$ with τ generic, the constant term integral over $U_{2n}(F) \backslash U_{2n}(\mathbb{A})$ is non-zero and non-degenerate Whittaker-Fourier coefficients of τ are also non-zero. Hence, $\mathcal{F}_{s',u}(\mathcal{E}_{\tau \otimes \sigma})$ is non-zero. Then, by Theorem 5.1, $\mathcal{F}_{s,u}(\mathcal{E}_{\tau \otimes \sigma})$ is also non-zero. This completes the proof of the proposition. \square

Next we prove the following.

Proposition 5.3. *$\mathcal{E}_{\tau \otimes \sigma}$ has a non-zero generalized Whittaker-Fourier coefficient attached to the partition $[(2n + 1)(2n - 1)1^3]$.*

Proof. By Proposition 5.2, we know that $\mathcal{E}_{\tau \otimes \sigma}$ has a non-zero generalized Whittaker-Fourier coefficient attached to the partition $[(2n)^2 1^3]$. It is clear that as an orthogonal partition, $[(2n)^2 1^3]$ is not special, and the smallest special partition which is greater than it is $[(2n + 1)(2n - 1)1^3]$, which is called the special expansion of the partition $[(2n)^2 1^3]$. By [13, Theorem 11.2], we must have that $\mathcal{E}_{\tau \otimes \sigma}$ has a non-zero generalized Whittaker-Fourier coefficient attached to the partition $[(2n + 1)(2n - 1)1^3]$. \square

Now we are ready to prove Part (a) of Theorem 1.2. In the following, given $\beta \in F^\times$, we do not distinguish β with its square class or the quadratic form corresponding to it.

Theorem 5.4. *There exists $\beta \in F^\times$, such that $\mathcal{D}_{\psi_{n,\beta}}(\mathcal{E}_{\tau \otimes \sigma})$ is non-zero.*

Proof. By Proposition 5.3, we know that $\mathcal{E}_{\tau \otimes \sigma}$ has a non-zero generalized Whittaker-Fourier coefficient attached to the partition $[(2n + 1)(2n - 1)1^3]$. By [26, §I.6], nilpotent orbits corresponding to the partition $[(2n + 1)(2n - 1)1^3]$ are parametrized by certain quadratic forms $\{\beta_{2n+1}, \beta_{2n-1}, q_{V_0}\}$, corresponding to the parts $(2n + 1)$, $(2n - 1)$ and 1^3 , where β_{2n+1} and β_{2n-1} are square classes, and q_{V_0} is the quadratic form in 3 variables on V_0 (see §2.1). This parametrization can be refined according to [13, Proposition 8.1], that is, $\mathcal{E}_{\tau \otimes \sigma}$ actually has a non-zero generalized Whittaker-Fourier

coefficient attached to the nilpotent \mathcal{O} , corresponding to the partition $[(2n+1)(2n-1)1^3]$ and parametrized by quadratic forms $\{\beta, -\beta, q_{V_0}\}$ for some $\beta \in F^\times$ (an explicit representative of this nilpotent orbit will be explicated in the next two paragraphs). Note that the normalization of the bilinear form for the irreducible representation of $\mathfrak{sl}_2(\mathbb{C})$ of dimension i in [13] differs from the one in [26] by the factor $(-1)^{\lfloor (i-1)/2 \rfloor}$ (See [13, §7]). In the following, we show that such a β will suffice for the theorem. As in the proof of Proposition 5.2, let $\alpha_i = e_i - e_{i+1}$ ($1 \leq i \leq 2n-1$) be a subset of simple roots for $\mathrm{SO}_{4n+3}^{V_0}$.

For the nilpotent orbit \mathcal{O} above which is parametrized by quadratic forms $\{\beta, -\beta, q_{V_0}\}$, one can take a representative $u = u_1 + u_2$, where

$$u_1 = \sum_{i=1}^{n-1} x_{-\alpha_i}(1) + x_{e_{2n}-e_n}(1) + x_{-e_{2n}-e_n}(\beta/2),$$

u_2 is the embedding of any representative of a certain nilpotent orbit in the Levi part of the stabilizer of u_1 (which is $\mathrm{SO}_{2n+2,\beta}$) into $\mathrm{SO}_{4n+3}^{V_0}$. This nilpotent orbit in $\mathrm{SO}_{2n+2,\beta}$ corresponds to the partition $[(2n-1)1^3]$ and is parametrized by quadratic forms $\{-\beta, q_{V_0}\}$. Let s_1 be the following semisimple element of the Lie algebra of $\mathrm{SO}_{4n+3}^{V_0}$:

$$\mathrm{diag}(2n, 2n-2, \dots, 2, 0, \dots, 0, -2, \dots, 2-2n, -2n).$$

Then it is clear that (s_1, u_1) is a neutral pair.

We make u_2 explicit as follows. First, we give a representative of the nilpotent orbit in $\mathrm{SO}_{2n+2,\beta}$ corresponding to the partition $[(2n-1)1^3]$ and parametrized by the quadratic forms $\{-\beta, q_{V_0}\}$. Let s_2 be the following semisimple element of the Lie algebra of $\mathrm{SO}_{2n+2,\beta}$:

$$\mathrm{diag}(2n-2, 2n-4, \dots, 2, 0, 0, 0, 0, -2, \dots, 4-2n, 2-2n).$$

Let $N_{s_2} = \exp(\mathfrak{g}_{\geq 2}^{s_2})$, which is the unipotent radical of the parabolic subgroup of $\mathrm{SO}_{2n+2,\beta}$ with Levi subgroup isomorphic to $\mathrm{GL}_1^{n-1} \times \mathrm{SO}_{4,\beta}$, here $\mathrm{SO}_{4,\beta} = \mathrm{SO}(V_{0,\beta})$ (see

§2.3). Write elements of N_{s_2} as $n = \begin{pmatrix} z & zx & y \\ 0 & I_4 & x' \\ 0 & 0 & z^* \end{pmatrix}$, where z is an upper triangular matrix in GL_{n-1} , $x \in \mathrm{Mat}_{(n-1) \times 4}$, and x' is defined in §2.3. Let

$$u_2 = \sum_{i=1}^{n-2} x_{-\alpha_i}(1) + x_{e_n-e_{n-1}}(a) + x_{e_{n+1}-e_{n-1}}(b) + x_{-e_{n+1}-e_{n-1}}(c) + x_{-e_n-e_{n-1}}(d),$$

such that $(a, b, c, d) \in (F^\times)^4$ is an anisotropic vector with respect the quadratic form of $\mathrm{SO}_{4,\beta}$. Then (s_2, u_2) is a neutral pair and u_2 is a representative of the nilpotent orbit in $\mathrm{SO}_{2n+2,\beta}$ corresponding to the partition $[(2n-1)1^3]$ and parametrized by quadratic forms $\{-\beta, q_{V_0}\}$, and $N_{s_2} = N_{s_2, u_2} = U_{s_2}$ (see §5.1, note that now $\mathfrak{g}_1^{s_2} = \emptyset$). Then, we embed s_2 into the Lie algebra of $\mathrm{SO}_{4n+3}^{V_0}$ as follows:

$$\mathrm{diag}(0, \dots, 0, 2n-2, 2n-4, \dots, 2, 0, 0, 0, 0, -2, \dots, 4-2n, 2-2n, 0, \dots, 0),$$

which is still denoted by s_2 . We embed elements of N_{s_2, u_2} into $\mathrm{SO}_{4n+3}^{V_0}$ as follows:

$$n = \begin{pmatrix} z & zx & y \\ 0 & I_4 & x' \\ 0 & 0 & z^* \end{pmatrix} \mapsto \mathrm{diag} \left(I_n, \begin{pmatrix} z & z\bar{x} & y \\ 0 & I_5 & \bar{x}' \\ 0 & 0 & z^* \end{pmatrix}, I_n \right),$$

where $\bar{x} = \{x_4, x^{(4)}, -\frac{2x_4}{\beta}\}$ when we write $x = \{x^{(4)}, 2x_4\}$ with $2x_4$ being the last column of x . Still denote the image subgroup by N_{s_2, u_2} . Similarly, we can embed u_2 into the Lie algebra of $\mathrm{SO}_{4n+3}^{V_0}$, and still denote the image by u_2 .

Let $s = s_1 + s_2$. Then it is clear that (s, u) is a neutral pair. From the above discussion, there is a $\varphi \in \mathcal{E}_{\tau \otimes \sigma}$ such that the generalized Whittaker-Fourier coefficient $\mathcal{F}_{s, u}(\varphi) \neq 0$, for some choice of u_2 . We fix such a choice.

Recall that

$$(5.2) \quad \mathcal{F}_{s, u}(\varphi)(g) := \int_{[N_{s, u}]} \varphi(ng) \psi_u^{-1}(n) \, dn.$$

Note that the elements of $N_{s, u}$ have the form:

$$(5.3) \quad n(x, y, z, w, \tilde{n}) := \begin{pmatrix} I_n & 0 & 0 & 0 & 0 \\ w' & I_{n-1} & 0 & 0 & 0 \\ 0 & 0 & I_5 & 0 & 0 \\ 0 & 0 & 0 & I_{n-1} & 0 \\ 0 & 0 & 0 & w & I_n \end{pmatrix} \begin{pmatrix} z & zx & y \\ 0 & \tilde{n} & \tilde{n}x' \\ 0 & 0 & z^* \end{pmatrix},$$

where z is an upper triangular matrix in GL_n , $\mathrm{diag}(I_n, \tilde{n}, I_n)$ is in N_{s_2, u_2} , $x \in \mathrm{Mat}_{n \times (2n+3)}$, and $w \in \mathrm{Mat}_{n \times (n-1)}$ with some entries being zero specified in the next paragraph. Note that the subgroup of $N_{s, u}$ consisting of elements of the form $n(0, 0, I_n, w, I_{2n+3})$ as above is abelian.

To proceed, we define some unipotent subgroups. For $k = 2, 3, \dots, n-1$, let R_k be the unipotent subgroup consists of elements of the form $\begin{pmatrix} I_n & r & 0 \\ 0 & I_{2n+3} & r' \\ 0 & 0 & I_n \end{pmatrix}$, such

that $r_{i, j}$ are all zero except possibly when $i = k$ and $1 \leq j \leq k-1$. Similarly, for $k = 2, 3, \dots, n-1$, let C_k be the unipotent subgroup consists of elements of the form $\begin{pmatrix} I_n & 0 & 0 \\ c' & I_{2n+3} & 0 \\ 0 & c & I_n \end{pmatrix}$, such that $c_{i, j}$ are all zero except possibly when $j = k+1$ and

$1 \leq i \leq k-1$. One can see that $\prod_{k=2}^{n-1} R_k \cap N_{s, u} = \{I_{4n+3}\}$ and $\prod_{k=2}^{n-1} C_k$ is exactly the subgroup of $N_{s, u}$ consisting of elements of the form $n(0, 0, I_n, w, I_{2n+3})$ as in (5.3). Note that both R_k and C_k are abelian, for $2 \leq k \leq n-1$. Let $N'_{s, u}$ be the subgroup of $N_{s, u}$ consisting of all the elements of the form $n(x, y, z, 0, \tilde{n})$.

Now we apply [12, Lemma 6.4] (it is clear that all the assumptions there hold) to the quadruple

$$(N'_{s, u}, \psi_u, \{R_k\}_{k=2}^{n-1}, \{C_k\}_{k=2}^{n-1}),$$

which is equivalent to applying [10, Corollary 7.1] (it is also clear that all the assumptions there hold) repeatedly to the following sequence of quadruples

$$\begin{aligned} & \left(\prod_{k=3}^{n-1} C_k N'_{s,u}, \psi_u, R_2, C_2 \right), \\ & \left(\prod_{k=4}^{n-1} C_k N'_{s,u} R_2, \psi_u, R_3, C_3 \right), \\ & \quad \vdots \\ & \left(N'_{s,u} \prod_{k=2}^{n-2} R_k, \psi_u, R_{n-1}, C_{n-1} \right). \end{aligned}$$

Then we obtain that the following integral

$$(5.4) \quad \int_{[N'_{s,u} \prod_{k=2}^{n-1} R_k]} \varphi(n g) \psi_u^{-1}(n) \, dn \neq 0,$$

where elements in $N'_{s,u} \prod_{k=2}^{n-1} R_k$ have the form $n(x, y, z, \tilde{n}) := \begin{pmatrix} z & zx & y \\ 0 & \tilde{n} & \tilde{n}x' \\ 0 & 0 & z^* \end{pmatrix}$, here z

is an upper triangular matrix in GL_n , $\mathrm{diag}(I_n, \tilde{n}, I_n)$ is in N_{s_2, u_2} , and $x \in \mathrm{Mat}_{n \times (2n+3)}$ with only $x_{n,j} = 0$, for $1 \leq j \leq n-1$.

Then one finds that, as an inner integral of (5.4), the integral

$$(5.5) \quad \int_{[N_0]} \varphi(n g) \psi_{u_1}^{-1}(n) \, dn$$

is non-vanishing, where N_0 consists of all elements in $N'_{s,u} \prod_{k=2}^{n-1} R_k$ having the form $n(x, y, z, I_{2n+3})$. Note that here $x \in \mathrm{Mat}_{n \times (2n+3)}$ with only $x_{n,j} = 0$, for $1 \leq j \leq n-1$.

Let X be the unipotent subgroup consisting of elements of the form

$$n = \begin{pmatrix} I_n & x & 0 \\ 0 & I_{2n+3} & x' \\ 0 & 0 & I_n \end{pmatrix},$$

such that $x_{i,j}$ are all zero except possibly when $i = n$ and $1 \leq j \leq n-1$. Then X is an abelian subgroup, preserving N_0 and the character ψ_{u_1} . Taking Fourier expansion of the integral (5.5) along $[X]$, there exists a non-zero Fourier coefficient. Assume that one such non-zero Fourier coefficient is given by a Lie algebra element

$u'_1 = \begin{pmatrix} 0_n & 0 & 0 \\ p' & 0_{2n+3} & 0 \\ 0 & p & 0_n \end{pmatrix}$, where $p'_{i,j}$ are all zero except possibly when $j = n$ and

$1 \leq i \leq n-1$. Then we can rewrite this Fourier coefficient as

$$(5.6) \quad \int_{[N_{s_1, u_1 + u'_1}]} \varphi(n g) \psi_{u_1 + u'_1}^{-1}(n) \, dn.$$

We claim that u_1 and $u_1 + u'_1$ are in the same nilpotent orbit, that is, one can find a element $g \in \mathrm{SO}_{4n+3}^{V_0}$ such that $u_1 = g(u_1 + u'_1)g^{-1}$. Indeed, write $u_1 = \begin{pmatrix} x & 0 & 0 \\ y' & 0_{2n+3} & 0 \\ 0 & y & x^* \end{pmatrix}$

and $u_1 + u'_1 = \begin{pmatrix} w & 0 & 0 \\ z' & 0_{2n+3} & 0 \\ 0 & z & w^* \end{pmatrix}$, then $x = w \in \mathrm{Mat}_{n \times n}$ and all columns of y' and z' are zero except the last columns y'_n and z'_n . From the form of u'_1 above, the last column p'_n of p' is isotropic. Hence, regarding our choice of basis, one can see that as two vectors in an orthogonal space with action of $\mathrm{SO}_{2n+3}^{V_0}$, y'_n and z'_n have the same length, which equals $\beta \in F^\times$. So, one can find $h \in \mathrm{SO}_{2n+3}^{V_0}$ such that $y'_n = z'_n h^{-1}$.

Let $g = \begin{pmatrix} I_n & 0 & 0 \\ 0 & h & 0 \\ 0 & 0 & I_n \end{pmatrix}$, then we have $u_1 = g(u_1 + u'_1)g^{-1}$. Therefore, $u_1 + u'_1$ is also

a representative of the nilpotent orbit corresponding to the partition $[(2n+1)1^{2n+2}]$ and parametrized by quadratic forms $\{\beta, q\}$, where q is a quadratic form in $(2n+2)$ -variables such that $\beta \oplus q$ is isomorphic to q_{V_0} composing with n hyperplanes. It follows that $(s_1, u_1 + u'_1)$ is a neutral pair. Therefore, we obtain that the generalized Whittaker-Fourier coefficient \mathcal{F}_{s_1, u_1} is non-zero, that is, $\mathcal{D}_{\psi_{n, \beta}}(\mathcal{E}_{\tau \otimes \sigma})$ is non-zero.

This completes the proof of the theorem. \square

Remark 5.5. *It is clear that the argument above can be easily modified and applied to the case that σ is a cuspidal representation of $\mathrm{SO}(V_0)(\mathbb{A})$ with $\dim_F(V_0) = 2r + 1$, $r \in \mathbb{Z}_{>0}$, and the residual representation $\mathcal{E}_{\tau \otimes \sigma}$ for $\mathrm{SO}_{4n+2r+1}^{V_0}(\mathbb{A})$.*

6. ON THE RECIPROCAL BRANCHING PROBLEM

In this section, we consider the reciprocal branching problem and prove Parts (b) and (c) of Theorem 1.2.

For simplicity, write $\pi_\beta = \mathcal{D}_{\psi_{n, \beta}}(\mathcal{E}_{\tau \otimes \sigma})$. By Theorem 5.4, there exists $\beta \in F^\times$ such that $\pi_\beta \neq 0$. Moreover, if the representation σ satisfies Assumption 1.1, or the $\beta \in F^\times$ in Theorem 5.4 and ω_τ satisfy the condition of Proposition 4.4, then π_β is cuspidal by Propositions 4.3, 4.4. By the uniqueness of local Bessel models (see [1, 7, 16, 24]), one has a multiplicity free direct sum decomposition:

$$\pi_\beta = \bigoplus_i \pi_\beta^{(i)},$$

where $\pi_\beta^{(i)}$'s are non-zero mutual-inequivalent irreducible cuspidal automorphic representations of the group $G_{n+1, \beta}(\mathbb{A}) = \mathrm{SO}_{2n+2, \beta}(\mathbb{A})$.

Before proceeding, let us recall the definition of Bessel period in a situation related to our construction. As in §1, let $G_n \times H_m$ be a relevant pure inner form of $G_n^* \times H_m^*$ over F with $n > m$. Note that in both G_n and H_m , we can define unipotent subgroups $N_\kappa^{(n)}$ and $N_\kappa^{(m)}$ analogous to N_ℓ in §2.3, and also characters $\psi_{\kappa, \alpha}^{(n)}$ and $\psi_{\kappa, \alpha}^{(m)}$ ($\alpha \in F^\times$) analogous to $\psi_{\ell, \alpha}$ in §2.3. Let π be an irreducible cuspidal automorphic representation of $G_n(\mathbb{A})$, and take $\varphi_\pi \in \pi$. Since G_n and H_m are relevant, there exists an $\alpha \in F^\times$

such that the Bessel-Fourier coefficient $\varphi_\pi^{\psi_{n-m-1,\alpha}^{(n)}}(\cdot)$ is an automorphic function on $H_m(\mathbb{A})$. Here the Bessel-Fourier coefficient has an analogous definition as in (2.2). Then for an irreducible cuspidal automorphic representation σ of $H_m(\mathbb{A})$ and $\varphi_\sigma \in \sigma$, one defines the *Bessel period* $\mathcal{P}(\varphi_\pi, \varphi_\sigma)$ by

$$(6.1) \quad \mathcal{P}(\varphi_\pi, \varphi_\sigma) = \int_{[H_m(\mathbb{A})]} \varphi_\sigma(h) \varphi_\pi^{\psi_{n-m-1,\alpha}^{(n)}}(h) dh.$$

The integral (6.1) converges absolutely by the cuspidality of σ . For a fixed automorphic realization of π , we say the pair of cuspidal representations (π, σ) has a non-zero Bessel period if $\mathcal{P}(\varphi_\pi, \varphi_\sigma) \neq 0$ for some choice of data.

Now let $\pi = \pi_\beta^{(i)}$ be any irreducible summand of π_β . By construction, there exists an automorphic form $\varphi_\pi \in \pi$ such that the inner product

$$(6.2) \quad \langle \varphi_\pi, \text{Res}_{s=1/2} E^{\psi_{n,\beta}}(s, \cdot, \phi_{\tau \otimes \sigma}) \rangle \neq 0,$$

for some choice of data. By Corollary 4.4 of [17], the pair (π, σ) has a non-zero Bessel period. This finishes the proof of Part (b) of Theorem 1.2. We note here that by construction, the groups $G_{n+1,\beta}$ and $H_1^{V_0}$ are relevant. In particular, we have

Proposition 6.1. *The automorphic descent $\pi_\beta = \mathcal{D}_{\psi_{n,\beta}}(\mathcal{E}_{\tau \otimes \sigma})$ has a non-zero Fourier coefficient attached to the nilpotent orbit corresponding to the partition $[(2n-1)1^3]$ and parameterized by quadratic forms $\{-\beta, q_{V_0}\}$.*

To get a connection with the reciprocal branching problem we have introduced in §1, we need to study the Arthur parameter ϕ_π of each irreducible summand π of π_β . In particular, we hope that ϕ_π is generic. Combining with the local results we have obtained in §3, we have the following proposition, which is Part (c) of Theorem 1.2.

Proposition 6.2. *Assume that $\omega_\tau = 1$, and the $\beta \in F^\times$ in Theorem 5.4 is not a square. Then each irreducible summand π of the descent π_β belongs to a global Arthur packet corresponding to a generic global Arthur parameter ϕ_π . The parameter ϕ_π has the central character $\eta_{V_0,\beta}$, and satisfies the property that $L(\frac{1}{2}, \phi_\pi \times \phi_{\tau_0}) \neq 0$ and (π, σ) has a non-zero Bessel period.*

Proof. By Part (b) of Theorem 1.2, under the assumption of the proposition, π_β is cuspidal, so is π . Write $\pi = \otimes'_v \pi_v$, and consider all the finite places v (infinitely many) such that both $\mathcal{E}_{\tau \otimes \sigma, v}$ and ψ_v are unramified and $\beta \in F_v^\times - (F_v^\times)^2$. Note that τ_v is a self-dual irreducible generic unitary unramified representation, which has the following form (see [25])

$$\tau_v = \chi_1 |\cdot|^{s_1} \times \cdots \times \chi_n |\cdot|^{s_n} \times \chi_n^{-1} |\cdot|^{-s_n} \times \cdots \times \chi_1^{-1} |\cdot|^{-s_1},$$

where χ_i is a unitary unramified character of F_v^\times and $0 \leq s_i < \frac{1}{2}$, for each $1 \leq i \leq n$. By Proposition 3.1, Part (1), π_v is the fully-induced irreducible generic unitary unramified representation

$$\pi_v = \chi_1 |\cdot|^{s_1} \times \cdots \times \chi_n |\cdot|^{s_n} \rtimes 1.$$

By the endoscopic classification theory of Arthur [2], if π has a non-generic global Arthur parameter, then π_v is non-generic for almost all finite places. Therefore, π must have a generic global Arthur parameter ϕ_π . This proves the first statement.

The central character of the parameter ϕ_π is determined by the form of the group $G_{n+1,\beta}$, and hence is $\eta_{V_0,\beta}$. From the discussion right before Proposition 6.1, the pair (π, σ) has a non-zero Bessel period, and hence by Theorem 5.7 of [17], one has that $L(\frac{1}{2}, \phi_\pi \times \phi_{\tau_0}) \neq 0$. \square

To end this section, we state a theorem on the reciprocal branching problem, which completes the proof of Theorem 1.2.

Theorem 6.3. *Let $\tau = \tau_1 \boxplus \cdots \boxplus \tau_r$ be an irreducible isobaric sum automorphic representation of $\mathrm{GL}_{2n}(\mathbb{A})$ with trivial central character, such that each τ_i is an irreducible unitary cuspidal representation of $\mathrm{GL}_{n_i}(\mathbb{A})$ of orthogonal type, and $\tau_i \not\cong \tau_j$ if $i \neq j$. Let V_0 be a quadratic space of dimension 3 over F , and σ be an irreducible cuspidal automorphic representation of $\mathrm{SO}(V_0)(\mathbb{A})$ which lies in the global Vogan packet $\tilde{\Pi}_{\phi_{\tau_0}}[H_1^*]$ (here $H_1^* = \mathrm{SO}_3^*$, F -split) for some irreducible cuspidal automorphic representation τ_0 of $\mathrm{GL}_2(\mathbb{A})$ of symplectic type. Assume that*

- (i) $L(\frac{1}{2}, \tau \times \tau_0) \neq 0$,
- (ii) *the representation σ satisfies Assumption 1.1.*

Then there exists a $\beta \in F^\times$, such that the twisted descent $\pi_\beta = \mathcal{D}_{\psi_{n,\beta}}(\mathcal{E}_{\tau \otimes \sigma})$ has all of its irreducible summands $\pi_\beta^{(i)}$, as cuspidal automorphic representations of $G_{n+1,\beta}(\mathbb{A})$, enjoying the property that each $(\pi_\beta^{(i)}, \sigma)$ has a non-trivial Bessel period. Moreover, if the $\beta \in F^\times$ taken in Theorem 5.4 is not a square, then each $\pi_\beta^{(i)}$ has a generic global Arthur parameter $\phi^{(i)}$, and gives an answer to the reciprocal branching problem with respect to $\sigma \in \tilde{\Pi}_{\phi_{\tau_0}}[H_1^]$.*

Remark 6.4. *A more precise description of those parameters $\phi^{(i)}$ can be deduced from the refined local theory of the global zeta integrals in [15], we will not discuss them here.*

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