ON THE LOCAL CONVERSE THEOREM AND THE DESCENT THEOREM IN FAMILIES

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Abstract. In this paper, we prove an analogue of Jacquet's conjecture on the local converse theorem for ℓ-adic families of co-Whittaker representations of $\mathrm{GL}_n(F)$, where F is a finite extension of $\mathbb{Q}_p$ and $\ell \neq p$. We also prove an analogue of Jacquet's conjecture for a descent theorem, which asks for the smallest collection of gamma factors determining the subring of definition of an ℓ-adic family. These two theorems are closely related to the local Langlands correspondence in ℓ-adic families.

1. Introduction

Let F be a p-adic field whose residue field has order q, and let $G_n := \mathrm{GL}_n(F)$. If A is a commutative ring with unit, denote by $\mathrm{Rep}_A(G_n)$ the category of $A[G_n]$-modules which are smooth: the stabilizer of any element is open. Given irreducible generic representations π_1 and π_2 in $\mathrm{Rep}_{\mathbb{C}}(G_n)$, Jacquet, Piatetski-Shapiro, and Shalika in [JPSS83] defined gamma factors $\gamma(\pi_i \times \tau, s, \psi)$, $i = 1, 2$, for irreducible generic $\tau \in \mathrm{Rep}_{\mathbb{C}}(G_t)$, a non-trivial additive character ψ of F, and a complex variable s. If π_1 is isomorphic to π_2, then

$$\gamma(\pi_1 \times \tau, s, \psi) = \gamma(\pi_2 \times \tau, s, \psi)\,,$$

for all irreducible generic $\tau \in \mathrm{Rep}_{\mathbb{C}}(G_t)$, for all $t \geq 1$. It is a natural problem to identify the smallest collection of representations τ such that the converse statement holds. In [JL70] and [JPSS79], it was shown that when $n = 2$ and 3 respectively, the implication

$$\gamma(\pi_1 \times \tau, s, \psi) = \gamma(\pi_2 \times \tau, s, \psi) \implies \pi_1 \cong \pi_2$$

holds even when τ runs only over characters of G_1. In [H93], it was shown that for general n, the same implication holds when τ runs only over irreducible generic objects in $\mathrm{Rep}_{\mathbb{C}}(G_t)$ for $t = 1, 2, \ldots, n-1$.

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In [Ch96, Ch06, CPS99, HO15], the range of t was improved to be $t = 1, 2, \ldots, n-2$. In general, the converse statement was conjectured by Jacquet to hold when t varies from 1 to $\lfloor \frac{n}{2} \rfloor$. This conjecture was recently proved by Chai in [Ch16], and by Jacquet and the first named author in [JL16], independently, using different methods. Jiang, Nien and Stevens ([JNS15]) also proposed an approach towards Jacquet's conjecture based on the construction of supercuspidal representations in [BK93]. Following this approach, a large part of Jacquet's conjecture is proved in [JNS15], and a combination of the results in [JNS15, ALSX16] proves Jacquet's conjecture for G_n, n prime. The analogue of Jacquet's conjecture for irreducible generic representations of GL_n over Archimedean fields is proved by Adrian and Takeda in [AT17]. There is also an analogue of Jacquet's conjecture for irreducible generic representations of GL_n over finite fields proved by Nien in [N14]. Local converse problems for groups other than GL_n are studied in [B95, B97] ($\mathrm{U}(2,1)$, Sp_4), [JS03] (SO_{2n+1}), [Z15, Z17a, Z17b, Z17c] ($\mathrm{U}(1,1)$, $\mathrm{U}(2,2)$, Sp_{2r}, $\mathrm{U}_{r,r}$, U_{2r+1}).

Let $\ell \neq p$ be a prime number, let k be an algebraically closed field of characteristic ℓ, let $W(k)$ be the ring of Witt vectors of k, and let $\mathcal{K} = W(k)[1/\ell]$ be its fraction field. That is, $W(k)$ is the smallest complete discrete valuation ring of characteristic zero whose residue field is k (for instance, if $k = \overline{\mathbb{F}_\ell}$ then $\mathcal{K}$ is isomorphic to the ℓ-adic completion of the maximal unramified extension of $\mathbb{Q}_\ell$, and $W(k)$ is its ring of integers). Let A be a Noetherian $W(k)$-algebra. An object in $\mathrm{Rep}_A(G_n)$ is an ℓ-adic family of representations in the sense of algebraic geometry: given $\mathfrak{p} \in \mathrm{Spec}(A)$ with residue field $\kappa(\mathfrak{p}) := A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}$, the fiber $V \otimes_A \kappa(\mathfrak{p})$ gives a representation of G_n on a $\kappa(\mathfrak{p})$-vector space. In this paper we follow the method in [JL16] to prove two analogues of Jacquet's conjecture in the setting of ℓ-adic families.

If π is a simple $A[G_n]$-module, then for any ideal I of A, $I \cdot \pi$ is either 0 or all of π, so to have families that encode congruences, we do not use irreducible representations as our basic objects. Ihara's Lemma, and its conjectural generalization beyond GL_2, imply that for representations arising in the cohomology of Shimura varieties, all irreducible subrepresentations of the contragredient are generic after taking the fiber at a maximal ideal of the global Hecke algebra. This motivates us to work with "co-Whittaker" objects, that is, representations in $\mathrm{Rep}_A(G_n)$ that are generic with multiplicity one, admissible, and such that every nonzero quotient is generic (following [EH14, H16b]; see Section 2 for precise definitions).

Since $W(k)$ contains all p-power roots of unity, we may fix a nontrivial character $\psi : F \to W(k)^\times$. If $W(k) \to A$ is any $W(k)$-algebra, we let ψ_A denote the $A[F]$-module A with F acting via the composition $F \overset{\psi}{\to} W(k)^\times \to A^\times$. For any co-Whittaker representation π in $\mathrm{Rep}_A(G_n)$, we define its Whittaker model $\mathcal{W}(\pi,\psi)$ as the image of any nonzero homomorphism $\pi \to \mathrm{Ind}_{U_n}^{G_n}\psi_A$, where ψ_A is extended to the group U_n of upper triangular matrices. Not being irreducible, there may be several non-isomorphic co-Whittaker representations with the same Whittaker model. Two co-Whittaker representations π_1, π_2 are equivalent (in the sense of Section 2) if and only if they have have the same Whittaker model – this amounts to saying their "supercuspidal supports" are the same (see Section 2 and Lemma 2.4). In [M16], the second named author constructed gamma factors $\gamma(\pi \times \tau, X, \psi)$ in $(A \otimes B)((X))$, where $\pi \in \mathrm{Rep}_A(G_n)$ and $\tau \in \mathrm{Rep}_B(G_t)$ are co-Whittaker, A, B are arbitrary Noetherian $W(k)$-algebras, and X is a formal variable, see Section 3 for more details. When $A = B = \mathbb{C}$, let $X = q^{-s+\frac{n-t}{2}}$, then $\gamma(\pi\times\tau, X, \psi)$ is exactly the gamma factor defined in [JPSS83]. The local converse theorem for $t = n-1$ is proven in [M16]. Our first main result is proving an analogue of Jacquet's conjecture for co-Whittaker representations.

Theorem 1.1. *Let A be a reduced, ℓ-torsion free, finite-type $W(k)$-algebra and let π_1, π_2 be co-Whittaker $A[\mathrm{GL}_n(F)]$-modules with the same central character. If*

$$\gamma(\pi_1 \times \tau, X, \psi) = \gamma(\pi_2 \times \tau, X, \psi)\,,$$

for all irreducible generic integral representations $\tau \in \mathrm{Rep}_{\overline{\mathcal{K}}}(G_t)$ with $1 \le t \le \lfloor \frac{n}{2} \rfloor$, then $\mathcal{W}(\pi_1,\psi) = \mathcal{W}(\pi_2,\psi)$ (equivalently, π_1 and π_2 have the same supercuspidal support; see Section 2).

Recall that $\tau \in \mathrm{Rep}_{\overline{\mathcal{K}}}(G_t)$ is *integral* if it contains a stable $\mathcal{O}$-sublattice, where $\mathcal{O}$ is the ring of integers in $\overline{\mathcal{K}}$. Choose an isomorphism $\mathbb{C} \cong \overline{\mathcal{K}}$. Then in the special case of Theorem 1.1 where $A = \mathbb{C}$, we obtain a slightly stronger form of Jacquet's conjecture, where τ need only vary over irreducible generic representations that are integral (c.f. Remark 4.3).

If $n \ge 2$, we use the theory of the universal co-Whittaker family to prove the stability of gamma factors in Proposition 3.4. It follows (3.5) that the condition on the central character is unnecessary, namely, the equalities of GL_1-twist γ-factors imply π_1 and π_2 have the same central character. This is the analogue of [JNS15, Corollary 2.7].

Local converse theorems are especially useful in connection with the local Langlands correspondence. Converse theorems were used in [H02] to show that equalities of twisted local factors uniquely characterize the local Langlands correspondence for GL_n. Motivated by Ihara's Lemma and local-global compatibility results for GL_2, Emerton and Helm conjectured in [EH14] a local Langlands correspondence for GL_n in ℓ-adic families. Their conjecture assigns a co-Whittaker family $\pi(\rho)$ in $\mathrm{Rep}_A(G_n)$ to every ℓ-adically continuous A-representation ρ of the absolute Galois group G_F. The family $\pi(\rho)$ "interpolates local Langlands" in the sense that, at generic points $\mathfrak{p}$ of $\mathrm{Spec}(A)$, the fiber $\pi(\rho)\otimes_A \kappa(\mathfrak{p})$ corresponds to $\rho\otimes_A \kappa(\mathfrak{p})$ under a certain normalization of the classical local Langlands correspondence. They proved that, if $\pi(\rho)$ exists, it is the unique co-Whittaker family satisfying this interpolation property at the generic points.

Recently, Helm and the second author ([Hel, HM16]) proved the existence of $\pi(\rho)$, using local converse theorem and descent techniques in the co-Whittaker setting. Sharpening the techniques of [M16] and [HM16] to the level of Theorem 1.1 will be useful in proving converse theorems and the local Langlands correspondence in families beyond GL_n.

Beyond the complex setting, a new dimension to the local converse problem appears. We say a co-Whittaker $A[G_n]$-module *descends* to a sub-$W(k)$-algebra $A'\subset A$ if there is a co-Whittaker $A'[G_n]$-module π' such that π is equivalent to $\pi'\otimes_{A'} A$. For $\pi\in\mathrm{Rep}_A(G_n)$ and $\tau\in\mathrm{Rep}_B(G_t)$ co-Whittaker, $\gamma(\pi\times\tau, X,\psi)$ defines an element of $(A\otimes B)((X))$, and if π descends to A', then $\gamma(\pi\times\tau,X,\psi)$ must have coefficients in the subring $A'\otimes B$. It is a natural problem to identify collections of representations τ over rings B such that the converse statement holds. In [HM16], converse theorem techniques are used to prove a local gamma factor descent theorem with $t=n-1$, with τ being the compact induction $W_t := \text{c-Ind}_{U_t}^{G_t}\psi$, and with B being $\mathcal{Z}_t$, the center of the category $\mathrm{Rep}_{W(k)}(G_t)$. Our second main result sharpens this theorem to achieve another analogue of Jacquet's conjecture for descent.

Theorem 1.2. *Let A be any Noetherian $W(k)$-algebra, let $A'\subset A$ be a sub-algebra, and suppose π is a co-Whittaker $A[G_n]$-module whose central character is valued in A'. Assume A is finitely generated as a module over A'. If $\gamma(\pi\times e'W_t,X,\psi)$ has coefficients in $A'\otimes e'\mathcal{Z}_t$ for all primitive idempotents e' of $\mathcal{Z}_t$, and for $t=1,2\dots,\lfloor\frac{n}{2}\rfloor$, then π descends to A' (equivalently, the supercuspidal support of π is valued in A').*

If $n \geq 2$, we prove in Proposition 7.5 (using stability) that the condition on the central character is automatically implied by the gamma factor condition for $t = 1$.

The theory of gamma factors for ℓ-adically continuous families of representations of W_F was developed in [HM15]. We remark that, by applying the local Langlands correspondence in families, as formulated in [HM16], the converse and descent theorems stated here immediately give analogous theorems for gamma factors of ℓ-adically continuous families of representations of the Weil group W_F. We briefly state these results as follows, referring the reader to [HM15, §2] for all notations and definitions. An ℓ-inertial type ν is a finite dimensional representation of the prime-to-ℓ part of inertia. To any n-dimensional ℓ-inertial type ν, there is associated a primitive idempotent e_ν of $\mathcal{Z}_n$ via Vigneras' mod-ℓ semisimple local Langlands correspondence ([V01], [Hel, Prop 10.1]). There is a ring R^ν and a universal ℓ-adically continuous representation $\rho_\nu : W_F \to \mathrm{GL}_n(R^\nu)$ corresponding under local Langlands in families to $e'_\nu W_n$.

Corollary 1.3. (1) *Let A be a reduced, ℓ-torsion free, finite-type $W(k)$-algebra, and let $\rho_1, \rho_2 : W_F \to \mathrm{GL}_n(A)$ be ℓ-adically continuous representations in the sense of* [HM15, §2] *with the same determinant. If*

$$\gamma(\rho_1 \otimes \sigma, X, \psi) = \gamma(\rho_2 \otimes \sigma, X, \psi)\,,$$

for all ℓ-adically continuous representations $\sigma : W_F \to \mathrm{GL}_t(\mathcal{O})$ with $1 \leq t \leq \lfloor \frac{n}{2} \rfloor$, then ρ_1 and ρ_2 have the same semisimplification.

(2) *Let A be any Noetherian $W(k)$-algebra, let $A' \subset A$ be a subalgebra, and suppose $\rho : W_F \to \mathrm{GL}_n(A)$ is an ℓ-adically continuous representation whose determinant character is valued in A'. Assume A is finitely generated as a module over A'. If $\gamma(\rho \otimes \rho_{\nu'}, X, \psi)$ has coefficients in $A' \otimes R_{\nu'}$ for all t-dimensional ℓ-inertial types ν', for $1 \leq t \leq \lfloor \frac{n}{2} \rfloor$, then ρ descends to A', that is, $\rho(W_F)$ lies in $\mathrm{GL}_n(A')$ after conjugation in $GL_n(A)$.*

In proving local converse theorems, there is a key vanishing result (Theorem 4.1 in this paper) that is required to pass from known information on the Rankin–Selberg zeta integrals to desired information about the Whittaker function. This result was originally proven when $A = \mathbb{C}$ in [JPSS81] using harmonic analysis, and when A is reduced, ℓ-torsion free, and finite-type over $W(k)$ in [M16]. The reduced and ℓ-torsion free hypotheses were required in [M16] to make use of certain algebro-geometric techniques, together with the theory of the integral

Bernstein center, which is the reason why they appear in Theorem 1.1. We hope to remove these hypotheses in future work.

In order to use the method of [JL16] we make repeated used of the Fourier inversion formula (Lemma 6.2) and a Fourier "descent formula" (Lemma 7.9), both of which hold over arbitrary $W(k)$-algebras.

In Theorem 1.2, the assumption that A is finitely generated as a module over A' is required to make use of [HM16, Corollary 4.2] (Proposition 7.6 in this paper). This technical result is needed because passing through the functional equation requires the gamma factor to be a rational function, whereas the technique of [HM16], which we exploit here, necessitates working with the gamma factor in its expansion as a power series. We hope to remove this complication in future work.

A natural question following Theorems 1.1, 1.2, and Corollary 1.3, is whether the bound $\lfloor \frac{n}{2} \rfloor$ is sharp. The sharpness of $\lfloor \frac{n}{2} \rfloor$ for the local converse theorem when $A = \mathbb{C}$ is proved in [ALST16] for the case of n being prime and $p \geq \lfloor \frac{n}{2} \rfloor$. Thus, for ℓ-adic families of co-Whittaker representations, we also expect the bound $\lfloor \frac{n}{2} \rfloor$ is sharp. We leave this discussion to future work.

The structure of this paper is as follows. In Section 2, we introduce basic properties of co-Whittaker representations. In Section 3, we briefly recall the theory of gamma factors in [M16] and show that equality of G_1-twisted gamma factors implies equality of central characters. In Section 4, we prove two basic lemmas, which play important roles in later sections. Theorem 1.1 will be proved in Section 5, with a proposition whose proof is deferred to Section 6. Theorem 1.2 will be proved in Section 7.

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2. Co-Whittaker representations

Let F be a finite extension of $\mathbb{Q}_p$ and $G = G_n = \mathrm{GL}_n(F)$. The case where F is a nonarchimedean local field of positive characteristic is excluded only because it is excluded in [H16a, H16b]. This restriction may be unnecessary, but we will not check this here. Recall that in $W(k)$, p is invertible, there are all roots of unity of order prime to ℓ, and we can fix an isomorphism $\overline{W(k)[\frac{1}{\ell}]} \cong \mathbb{C}$, where $\overline{W(k)[\frac{1}{\ell}]}$ is an algebraic closure of the fraction field of $W(k)$. The base rings A for our families will always have the structure of Noetherian $W(k)$-algebras.

This framework is natural when studying congruences mod ℓ. For example, if $q \equiv 1 \bmod \ell$, there exist smooth characters $\chi_1, \chi_2 : F^\times \to \mathcal{O}^\times$ such that χ_1 is unramified but χ_2 is ramified, and such that $\chi_1 \equiv \chi_2 \bmod \mathfrak{m}$, where $\mathfrak{m}$ is the maximal ideal of $\mathcal{O}$. Let A be the $W(k)$-algebra $\{(a,b) \in \mathcal{O} \times \mathcal{O} : a \equiv b \mod \mathfrak{m}\}$. Then the congruence between χ_1 and χ_2 is captured by saying they interpolate in an ℓ-adic family over $\mathrm{Spec}(A) = \{0 \times \mathfrak{m}, \mathfrak{m} \times 0, \mathfrak{m} \cdot A\}$. More precisely, the character $\chi : F^\times \to A^\times : a \mapsto (\chi_1(a), \chi_2(a))$ satisfies $\chi \otimes_A \kappa(0 \times \mathfrak{m}) \cong \chi_1$, $\chi \otimes_A \kappa(\mathfrak{m} \times 0) \cong \chi_2$, and $\chi \otimes_A \kappa(\mathfrak{m}) \cong \chi \mod \mathfrak{m}$.

Let $B_n = T_n U_n$ be the standard Borel subgroup of G_n consisting of upper triangular matrices with unipotent radical U_n and T_n the group of diagonal matrices. We fix a non-trivial additive character $\psi : F \to W(k)^\times$. Define a non-degenerate character ψ_{U_n} on U_n by

$$\psi_{U_n}(u) := \psi\left(\sum_{i=1}^{n-1} u_{i,i+1}\right), u \in U_n .$$

We will drop the subscript and also refer to ψ_{U_n} as ψ. If A is a $W(k)$-algebra, let ψ_A denote the module A with F (or U_n) acting via the composition $F \xrightarrow{\psi} W(k)^\times \to A^\times$.

Define $V^{(n)}$ to be the ψ-coinvariants $V/V(U_n,\psi)$, where $V(U_n,\psi)$ is the submodule generated by $\{\psi(u)v - uv : u \in U_n, v \in V\}$. This functor is exact and, for any A-module M there is a natural isomorphism

$$(V \otimes_A M)^{(n)} \cong V^{(n)} \otimes_A M .$$

Definition 2.1. *A smooth $A[G_n]$-module V is co-Whittaker if the following conditions hold*

1. *V is admissible as an $A[G_n]$-module,*
2. *$V^{(n)}$ is a free A-module of rank one,*
3. *if Q is a quotient of V such that $Q^{(n)} = 0$, then $Q = 0$.*

Note that we do not require V to be free as an A-module. In fact, the definition of "co-Whittaker" arose from efforts to interpolate the local Langlands correspondence, and it is not possible to interpolate the local Langlands correspondence with free modules [EH14, Ex 6.2.14].

For example, when $A = \mathbb{C}$, $n = 2$, and B is the Borel subgroup, the normalized parabolic induction $i_B^G(\chi)$, where $\chi = \chi_1 \otimes \chi_2$ is a character of T, is co-Whittaker if $\chi_1\chi_2^{-1} \neq |\cdot|^{+1}$

For another example, if χ_1 and χ_2 vary over unramified characters, this defines a geometric family over $\mathrm{Spec}(\mathbb{C}[T_1^{\pm 1}, T_2^{\pm 1}])$. More precisely, let $\chi_{univ} : T \to \mathbb{C}[T_1^{\pm 1}, T_2^{\pm 1}]^\times$ send $\mathrm{diag}(a,b)$ to $T_1^{v_F(a)} T_2^{v_F(b)}$, and consider the normalized induction $i_B^G(\chi_{univ})$. On points $\mathfrak{p} = (T_1 - x_1, T_2 - x_2)$, the fiber $i_B^G(\chi_{univ}) \otimes \kappa(\mathfrak{p})$ is irreducible generic on the open subset of points where $x_1 x_2^{-1} \neq q^{\pm 1}$. At points where $x_1 x_2^{-1} = q^{-1}$ the fiber $i_B^G(\chi_{univ}) \otimes \kappa(\mathfrak{p})$ is reducible, but has a unique irreducible generic quotient, which is a twist of the Steinberg representation by an unramified character of $F^\times$. If $A = \mathbb{C}[T_1^{\pm 1}, T_2^{\pm 1}]$ and $f = T_1 T_2^{-1} - q$ and $A_f = A[1/f]$ denotes the localization, it follows that $i_B^G(\chi_{univ}) \otimes_A A_f$ is co-Whittaker as an $A_f[G_2]$-module.

If V and V' are co-Whittaker, any nonzero G-equivariant map $V \to V'$ is surjective, as otherwise the cokernel would be a nongeneric quotient. In this case V is said to dominate V'. We say V and V' are *equivalent* if there exists a co-Whittaker $A[G_n]$-module V'' dominating both V and V'. This is an equivalence relation on isomorphism classes of co-Whittaker modules. Lemma 2.4 below will show that it is equivalent to define equivalence to mean there exists V'' dominated by both V and V'.

In [H16b], Helm constructs a co-Whittaker module which is "universal" up to this notion of equivalence. The key tool is the integral Bernstein center of G_n, i.e. the center of the category $\mathrm{Rep}_{W(k)}(G_n)$.

The center of an abelian category is the endomorphism ring of the identity functor, in other words the ring of natural transformations from the identity functor to itself. It acts on every object in the category in a way compatible with all morphisms. We denote by $\mathcal{Z}_n$ the center of $\mathrm{Rep}_{W(k)}(G_n)$.

For any co-Whittaker $A[G_n]$-module V, the map $A \to \mathrm{End}_{A[G_n]}(V)$ is an isomorphism (c.f. [H16b, Prop 6.2]), and thus there exists a map $f_V : \mathcal{Z}_n \to A$, which we call the *supercuspidal support* of V. Note that V also admits a central character $\omega_V : F^\times \to A^\times$.

A primitive idempotent element of the ring $\mathcal{Z}_n$ is called a *primitive central integral idempotent*, but we will often refer to it as a "primitive idempotent" for short. Any such primitive idempotent element e in $\mathcal{Z}_n$ gives rise to a direct factor category $e\mathrm{Rep}_{W(k)}(G_n)$, which is the full subcategory of $\mathrm{Rep}_{W(k)}(G_n)$ on which e acts as the identity. As described in [H16a], the primitive idempotents in $\mathcal{Z}_n$ are in bijection with inertial equivalence classes of pairs (L, π), where L is a Levi subgroup of G_n and π is an irreducible supercuspidal k-representation of L. If e is the idempotent corresponding to the pair (L, π), then a representation V in $\mathrm{Rep}_{W(k)}(G_n)$ lies in $e\mathrm{Rep}_{W(k)}(G_n)$ if and only if every simple subquotient of V has mod-ℓ inertial supercuspidal support given by (L, π) in the sense of [H16a, Def 4.12].

Theorem 2.2 ([H16a], Thm 10.8)**.** *Let e be any primitive central idempotent in $\mathcal{Z}_n$. The ring $e\mathcal{Z}_n$ is a finitely generated, reduced, ℓ-torsion free $W(k)$-algebra.*

Now let A be a Noetherian $W(k)$-algebra, and let V be a co-Whittaker $A[G_n]$-module. Suppose further that V lies in $e\mathrm{Rep}_{W(k)}(G_n)$ for some primitive central idempotent e (so the supercuspidal support map f_V factors through the projection $\mathcal{Z}_n \to e\mathcal{Z}_n$). Let W_n be the smooth $W(k)[G_n]$-module c-$\mathrm{Ind}_{U_n}^{G_n}\psi$. For any primitive central idempotent e in $\mathcal{Z}_n$, we have an action of $e\mathcal{Z}_n$ on eW_n.

Theorem 2.3 ([H16b], Theorem 6.3)**.** *Let e be any primitive central idempotent in $\mathcal{Z}_n$. The smooth $e\mathcal{Z}_n[G_n]$-module eW_n is a co-Whittaker $e\mathcal{Z}_n[G_n]$-module. If A is Noetherian and has an $e\mathcal{Z}_n$-algebra structure, the module $eW_n \otimes_{e\mathcal{Z}_n} A$ is a co-Whittaker $A[G_n]$-module. Conversely, V is dominated by $eW_n \otimes_{e\mathcal{Z}_n, f_V} A$.*

We thus say that, up to the equivalence relation induced by dominance, eW_n is the universal co-Whittaker module in $e\mathrm{Rep}_{W(k)}(G_n)$.

If $V^{(n)}$ is a free A-module of rank one, V admits a nonzero Whittaker functional $\iota : V \to \mathrm{Ind}_{U_n}^{G_n}\psi_A$ whose image, denoted by $\mathcal{W}(V, \psi)$ and called the (A-valued) *Whittaker model of V* with respect to ψ, is independent of the choice of Whittaker functional. In particular, this definition applies to co-Whittaker modules. Fixing a nonzero Whittaker functional ι, for $v \in V$ we denote

$$W_v := \iota(v) \in \mathcal{W}(V, \psi)\,.$$

Lemma 2.4. *Let A be a Noetherian $W(k)$-algebra and suppose V_1 and V_2 are two co-Whittaker $A[G_n]$-modules. The following are equivalent:*

1. *There exists W in $\mathcal{W}(V_1, \psi) \cap \mathcal{W}(V_2, \psi)$ such that $W(g) \in A^\times$ for some $g \in G$.*
2. *$\mathcal{W}(V_1, \psi) = \mathcal{W}(V_2, \psi)$.*
3. *$f_{V_1} = f_{V_2}$.*
4. *V_1 and V_2 are equivalent.*

Proof. First the equivalence of (3) and (4). Any co-Whittaker $A[G_n]$-module V is a direct sum of the subrepresentations $e_iV \in e_i\mathrm{Rep}_{W(k)}(G_n)$ for primitive idempotents e_i in $\mathcal{Z}$. Since $A = \mathrm{End}_G(V)$ is Noetherian, there are only finitely many primitive orthogonal idempotents e_i which can act nontrivially, and we may assume A is connected and $V_1 = e_iV_1$, $V_2 = e_iV_2$ have a single component. If $f_{V_1} = f_{V_2}$ then both V_1 and V_2 are dominated by $eW_n \otimes_{e\mathcal{Z}_n, f_{V_i}} A$ (Theorem 2.3), so they are equivalent. Conversely, if V_1 and V_2 are equivalent, each dominated by some V_3, say, then the action of $e\mathcal{Z}_n$ on V_1 and V_2 is given by f_{V_3} since the action of the center is functorial. Next we show (1) $\Longrightarrow$ (2). Let $I = \mathcal{W}(V_1, \psi) \cap \mathcal{W}(V_2, \psi)$. The assumption that W takes values in $A^\times$ guarantees that $I \otimes \kappa(\mathfrak{p})$ is nonzero for all $\mathfrak{p} \in \operatorname{Spec}(A)$, and $(I \otimes \kappa(\mathfrak{p}))^{(n)} = (\mathcal{W}(V_1, \psi) \otimes \kappa(\mathfrak{p}))^{(n)}$. The finitely generated A-module $(\mathcal{W}(V_1, \psi)/I)^{(n)} \otimes \kappa(\mathfrak{p})$ is therefore zero for all $\mathfrak{p}$, and by Nakayama's lemma the localizations $(\mathcal{W}(V_1, \psi)/I)^{(n)}_{\mathfrak{p}}$ are zero for all $\mathfrak{p}$, hence $(\mathcal{W}(V_1, \psi)/I)^{(n)} = 0$. Since $\mathcal{W}(V_1, \psi)$ is co-Whittaker, we get $I = \mathcal{W}(V_1, \psi)$, and the parallel argument gives the same for V_2. For (2) $\Longrightarrow$ (1) we note that upon restriction to P_n, $\mathcal{W}(V_i, \psi)$ contains $\text{c-Ind}_{U_n}^{P_n} \psi_A$, which has functions valued in $A^\times$. The equivalence of (3) and (2) were proven in [M16, Prop 6.2]. $\square$

There is a duality operation on co-Whittaker modules which interpolates the contragredient across a co-Whittaker family. If V is a smooth $W(k)[G_n]$-module, let V^ι denote the $W(k)[G_n]$-module with the same underlying $W(k)$-module structure, and for which the G_n action, which we will denote by $g \cdot v$, is given by $g \cdot v = g^\iota v$, where $g^\iota = {}^t g^{-1}$. Then V^ι is co-Whittaker with respect to ψ^{-1}, and specializes to the contragredient representation at characteristic zero points of the irreducible locus ([HM16, Prop 2.5,2.6]). This duality has a very concrete interpretation in terms of Whittaker functions. Let ω_n be the longest Weyl element of G_n, and for any function W on G_n, let $\widetilde{W}(g) = W(\omega_n g^\iota)$. If W is in $\mathcal{W}(V, \psi)$, then $\widetilde{W}$ is in $\mathcal{W}(V^\iota, \psi^{-1})$.

If $m \leq n$ we define, for use in the next section, the element

$$\omega_{n,m} := \begin{pmatrix} I_{n-m} & 0 \\ 0 & \omega_m \end{pmatrix}.$$

3. Rankin-Selberg theory and gamma factors

Let A and B be Noetherian $W(k)$-algebras and let $R = A \otimes_{W(k)} B$. Let V and V' be co-Whittaker $A[G_n]$- and $B[G_m]$-modules respectively. For $W \in \mathcal{W}(V, \psi)$ and $W' \in \mathcal{W}(V', \psi^{-1})$, and for $0 \leq j \leq n-m-1$, we define (following [M16]) the formal series with coefficients in R (X is a formal variable):

$$\Psi(W, W', X; j)$$
$$:= \sum_{r \in \mathbb{Z}} \int_{M_{j,m}(F)} \int_{U_m \backslash \{g \in G_m : v(\det g) = r\}} \left(W \begin{pmatrix} g & & \\ x & I_j & \\ & & I_{n-m-j} \end{pmatrix} \otimes W'(g) \right) X^r dg dx .$$

Denote $\Psi(W,W',X) := \Psi(W,W',X;0)$.

Proving $\Psi(W,W',X;j)$ is a well-defined element of $R[[X]][X^{-1}]$ entails showing the integral for each coefficient is a finite sum, and there are only finitely many negative-power terms. This follows from the Iwasawa decomposition, and is dealt with in [M16], but for completeness and future reference we add the following.

Lemma 3.1. *Let V be a co-Whittaker $A[G_n]$-module, and W an element of $\mathcal{W}(V,\psi)$.*

1. *Let $g \in G_m$. Then $W\begin{pmatrix} g & & \\ x & I_j & \\ & & I_{n-m-j}\end{pmatrix} \neq 0$ implies that x belongs to a compact set C independent of g.*
2. *There exists an integer r_0 such that if $v_F(\det g) < r_0$ then $W\begin{pmatrix} g & & \\ x & I_j & \\ & & I_{n-m-j}\end{pmatrix} = 0$.*

Proof. Rather than prove it directly, we deduce it formally by extension of scalars from $\mathcal{Z}_n$ (the argument is very similar to [HM16, Lemma 4.5]). Without loss of generality, suppose $V = eV$ for a primitive central idempotent of $\mathcal{Z}_n$, and let $f : e\mathcal{Z}_n \to A$ denote the action of the Bernstein center. Choose $W' \in \mathcal{W}(eW_n,\psi)$ an $e\mathcal{Z}_n$-valued Whittaker function such that $f \circ W' = W$. If $W'\begin{pmatrix} g & & \\ x & I_j & \\ & & I_{n-m-j}\end{pmatrix} = 0$, then the same is true for W, and so it suffices to prove the lemma with $A = e\mathcal{Z}_n$ and $V = eW_n$. Since $e\mathcal{Z}_n$ is reduced and ℓ-torsion free, the residue field $\kappa(\mathfrak{p})$ of each minimal prime $\mathfrak{p}$ has characteristic zero, so we may choose an isomorphism $\overline{\kappa(\mathfrak{p})} \cong \mathbb{C}$, and apply [JPSS83, (2.6) Lemma] (respectively, [JPSS79, Prop 2.3.6]) to obtain a compact set $C_{\mathfrak{p}}$ (respectively, an integer $r_{0,\mathfrak{p}}$) satisfying the conclusion of part (1) (respectively, part (2)), for the Whittaker model of $V \otimes_{e\mathcal{Z}_n} \kappa(\mathfrak{p})$. Since $e\mathcal{Z}_n$ is reduced, if $W'\begin{pmatrix} g & & \\ x & I_j & \\ & & I_{n-m-j}\end{pmatrix}$ is nonzero it remains nonzero in some $\kappa(\mathfrak{p})$, and so x is in $C_{\mathfrak{p}}$ for some $\mathfrak{p}$. Therefore we can take $C = \bigcup_{\mathfrak{p}} C_{\mathfrak{p}}$. Similarly, we can take $r_0 = \min_{\mathfrak{p}}\{r_{0,\mathfrak{p}}\}$. $\square$

The well-definedness of $\Psi(W,W',X;j)$ now follows from Lemma 3.1 and

Lemma 3.2 ([HM16], Lemma 4.5)**.** *If V is a co-Whittaker $A[G_n]$-module, $W \in \mathcal{W}(V,\psi)$, and $m < n$, then for each integer r there exists a compact subset C of G_m such that if $g \in G_m$ satisfies $v_F(\det g) = r$ and $W\begin{pmatrix} g & \\ & I_{n-m}\end{pmatrix} \neq 0$, then $g \in U_{n-1}C$.*

Proof. The proof in [HM16, Lemma 4.5] for $m = 1$ works verbatim for $m > 1$, referring to [JPSS79, Prop 2.3.6] after passing to characteristic zero. □

Let S be the multiplicative system in $R[X, X^{-1}]$ consisting of all polynomials whose leading and trailing coefficients are units.

Theorem 3.3 ([M16]). *Suppose V and V' are co-Whittaker*

(1) *For all $0 \leq j \leq n-m-1$, the power series $\Psi(W, W', X; j)$ is in the image of the natural inclusion $S^{-1}(R[X, X^{-1}]) \hookrightarrow R[[X]][X^{-1}]$.*

(2) *There exists a unique element $\gamma(V \times V', X, \psi) \in S^{-1}(R[X, X^{-1}])$ such that*
$$\begin{aligned}&\Psi(W, W', X; j)\gamma(V \times V', X, \psi)\omega_{V'}(-1)^{n-1}\\ &= \Psi(\omega_{n,m}\widetilde{W}, \widetilde{W'}, \frac{q^{n-m-1}}{X}; n-m-1-j)\,,\end{aligned}$$
for any $W \in \mathcal{W}(V, \psi)$, $W' \in \mathcal{W}(V', \psi)$ and for any $0 \leq j \leq n-m-1$.

(3) *Given ring homomorphisms $f_1 : A \to \tilde{A}$ and $f_2 : B \to \tilde{B}$, the gamma factor is compatible with base change in the sense that*
$$(f_1 \otimes f_2)\left(\gamma(V \times V', X, \psi)\right) = \gamma(V \otimes_A \tilde{A} \times V' \otimes_B \tilde{B}, X, \psi),$$
where $f_1 \otimes f_2$ is extended to a map on $S^{-1}(R[X, X^{-1}])$ in the natural way.

Proposition 3.4. *Let A be a Noetherian $W(k)$-algebra, let π be a co-Whittaker $A[G_n]$-module with $n \geq 2$. Then there exists an integer m_π such that for any character $\chi : F^\times \to W(k)^\times$ of conductor $m \geq m_\pi$, and any $c \in \mathfrak{p}^{-m}$ satisfying $\chi(1+x) = \psi(cx)$ for any $x \in \mathfrak{p}^{[m/2]+1}$, we have*
$$\gamma(\pi \times \chi, X, \psi) = \omega_\pi(c)^{-1}\gamma(1_A \times \chi, X, \psi)^n\,,$$
where 1_A denotes the trivial character of $F^\times$.

Proof. To begin, assume that A is reduced and ℓ-torsion free. There are finitely many minimal prime ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ of A. If $\kappa(\mathfrak{p}_i)$ denotes the residue field $\mathrm{Frac}(A/\mathfrak{p}_i)$, then $\kappa(\mathfrak{p}_i)$ has characteristic zero and contains $W(k)$. As π is defined over a subalgebra which is finite type over $W(k)$, we may assume without loss of generality that A is finite-type over $W(k)$ and choose an isomorphism $\overline{\kappa(\mathfrak{p}_i)} \cong \mathbb{C}$. Let $\pi_{i,0}$ denote the cosocle of $\pi \otimes_A \kappa(\mathfrak{p}_i)$. Then $\pi_{i,0}$ is semisimple, but by Definition 2.1(2), only one of its simple summands is generic, and by Definition 2.1(3), all the other summands must be zero. Since $\mathrm{End}_G(\pi_{i,0}) = \kappa(\mathfrak{p}_i)$, $\pi_{i,0}$ remains irreducible upon extending to $\overline{\kappa(\mathfrak{p}_i)}$.

In [JNS15, Proposition 2.6], they prove the analogous proposition for irreducible generic representations of G_n over $\mathbb{C}$. Therefore, for all i, there exists $m_{\pi_{i,0}}$ such that

$$\gamma(\pi_{i,0} \times \chi, X, \psi) = \omega_{\pi_{i,0}}(c)^{-1}\gamma(1_{\overline{\kappa(\mathfrak{p}_i)}} \times \chi, X, \psi)^n \,,$$

for any character $\chi : F^\times \to W(k)^\times$ of conductor $m \geq m_{\pi_{i,0}}$, and any $c \in \mathfrak{p}^{-m}$ satisfying $\chi(1+x) = \psi(cx)$ for any $x \in \mathfrak{p}^{[m/2]+1}$. On the other hand,

$$\gamma(\pi, X, \psi) \equiv \gamma(\pi_{i,0} \times \chi, X, \psi) \mod \mathfrak{p}_i \,,$$

by the compatibility of the gamma factor with homomorphisms of the base ring. For the same reason, we also have

$$\omega_\pi(c)^{-1}\gamma(1_A \times \chi, X, \psi)^n \equiv \omega_{\pi_{i,0}}(c)^{-1}\gamma(1_{\overline{\kappa(\mathfrak{p}_i)}} \times \chi, X, \psi)^n \mod \mathfrak{p}_i \,.$$

Therefore, for m larger than the maximum of all $m_{\pi_{i,0}}$, the difference

$$\gamma(\pi, X, \psi) - \omega_\pi(c)^{-1}\gamma(1_A \times \chi, X, \psi)^n$$

lies in $\mathfrak{p}_i$ for all i. Since A is reduced, $\bigcap_i \mathfrak{p}_i = 0$, hence this difference is zero.

To go beyond the case where A is reduced and ℓ-torsion free we recall that each component $e\mathcal{Z}_n$ of the integral Bernstein center is reduced and ℓ-torsion free. Without loss of generality, we may assume that $e\pi = \pi$ for a primitive idempotent e of $\mathcal{Z}_n$. By Theorem 2.3, π is equivalent to the base change, from $e\mathcal{Z}_n$ to A, of the co-Whittaker $e\mathcal{Z}_n[G_n]$-module W_n. We now apply the preceding paragraph to the co-Whittaker $e\mathcal{Z}_n[G_n]$-module W_n to conclude that

$$\gamma(W_n \times \chi, X, \psi) = \omega_{W_n}(c)^{-1}\gamma(1_A \times \chi, X, \psi)^n \,,$$

where $\omega_{W_n} : F^\times \to e\mathcal{Z}_n$ denotes the central character of W_n. In general, there is a homomorphism $f_\pi : e\mathcal{Z}_n \to A$ such that

$$\gamma(\pi \times \chi, X, \psi) = f_\pi(\gamma(W_n \times \chi, X, \psi) \,,$$

and similarly for ω_π and $\gamma(1_A \times \chi, X, \psi)^n$. The equal quantities over $e\mathcal{Z}_n$ map to equal quantities over A, which proves the theorem. □

The following result is an analogue of [JNS15, Corollary 2.7].

Proposition 3.5. *Let A be a Noetherian $W(k)$-algebra, let π_1, π_2 be co-Whittaker $A[G_n]$-modules with $n \geq 2$. Assume that*

$$\gamma(\pi_1 \times \chi, X, \psi) = \gamma(\pi_2 \times \chi, X, \psi) \,,$$

for any character $\chi : F^\times \to W(k)^\times$. Then $\omega_{\pi_1} = \omega_{\pi_2}$.

Proof. Let $m_{\pi_{1,i,0}}, m_{\pi_{2,i,0}}$ be the numbers given in the proof of Proposition 3.4 for π_1 and π_2, respectively, $i = 1, \ldots, r$. Let m_0 the max of $\{m_{\pi_{1,i,0}}, m_{\pi_{2,i,0}}, i = 1, \ldots, r\}$.

For any $c \in \mathfrak{p}^{-m} \backslash \mathfrak{p}^{1-m}$, with $m \geq m_0$, there exists a character χ_c of conductor m such that $\chi_c(1+x) = \psi(cx)$ for $x \in \mathfrak{p}^{[m/2]+1}$; thus Proposition 3.4 implies $\omega_{\pi_1}(c) = \omega_{\pi_2}(c)$. Since any element of $F^\times$ can be expressed as the quotient of two elements of valuation at most $-m$, we deduce that $\omega_{\pi_1} = \omega_{\pi_2}$. □

4. Two lemmas

In this section, we prove two lemmas, which will play important roles in later sections. For two $W(k)$-algebras A, B, and $\phi_1 \in \text{c-Ind}_{U_m}^{G_m} \psi_A$, $\phi_2 \in \text{Ind}_{U_m}^{G_m} \psi_B^{-1}$, we denote by $\langle \phi_1, \phi_2 \rangle$ the element

$$\int_{U_m \backslash G_m} \phi_1(g) \otimes \phi_2(g) dg \in A \otimes_{W(k)} B\,.$$

In practice, ϕ_1 will be a product $W\left(\begin{smallmatrix} g & \\ & I_{n-m}\end{smallmatrix}\right) \Phi_r(g)$, where W is a Whittaker function on G_n for $n > m$ and Φ_r is the characteristic function of the set of elements in $U_m \backslash G_m$ whose determinant has valuation r, for a fixed integer r. Such an object is compactly supported by Lemma 3.2.

Under certain hypotheses on A, we can detect the vanishing of ϕ_1 by letting ϕ_2 run over the collection of all Whittaker functions valued in $B = \mathcal{O}$. We recall the following theorem from [M16].

Theorem 4.1 ([M16] Thm 6.4)**.** *Suppose A is a finite-type, reduced, ℓ-torsion free $W(k)$-algebra. Suppose $H \neq 0$ is an element of* $\text{c-Ind}_{U_n}^{G_n} \psi_A$. *Then there exists an irreducible generic integral $\overline{\mathcal{K}}$-representation V' with $\mathcal{O}$-integral structure V, such that there is a Whittaker function $W \in \mathcal{W}(V^\iota, \psi_{\mathcal{O}}^{-1})$ satisfying $\langle H, W \rangle \neq 0$ in $A \otimes_{W(k)} \mathcal{O}$.*

The following lemma is an analogue of [JL16, Lemma 2.4].

Lemma 4.2. *Let A be as in Thm 4.1 and let π_1 and π_2 be two co-Whittaker $A[G_n]$-modules. Suppose $t \leq n-2$ and j with $0 \leq j \leq t$. Suppose that W^1 and W^2 are elements in the Whittaker models of π_1 and π_2, respectively. Suppose further that for all irreducible generic integral representations τ in $\text{Rep}_{\overline{\mathcal{K}}}(G_{n-t-1})$ we have*

$$\Psi(X, W^1, W'; j) = \Psi(X, W^2, W'; j)$$

for all $W' \in \mathcal{W}(\tau, \psi_{\overline{\mathcal{K}}}^{-1})$. Then

$$\int W^1 \begin{pmatrix} I_{n-t-1} & 0 & 0 \\ X & I_j & 0 \\ 0 & 0 & I_{t+1-j} \end{pmatrix} dX = \int W^2 \begin{pmatrix} I_{n-t-1} & 0 & 0 \\ X & I_j & 0 \\ 0 & 0 & I_{t+1-j} \end{pmatrix} dX\,,$$

where the integrals are over $X \in M_{j\times(n-t-1)}(F)$.

Proof. For $j=0$, the assumption is that

$$\begin{aligned}(4.1)\qquad &\sum_r \int_{(U_{n-t-1}\backslash G_{n-t-1})^r} W^1\begin{pmatrix} g & 0 \\ 0 & I_{t+1}\end{pmatrix} \otimes W'(g)X^r dg \\ &= \sum_r \int_{(U_{n-t-1}\backslash G_{n-t-1})^r} W^2\begin{pmatrix} g & 0 \\ 0 & I_{t+1}\end{pmatrix} \otimes W'(g)X^r dg\,,\end{aligned}$$

for all W', where $(U_{n-t-1}\backslash G_{n-t-1})^r$ is the subset of $U_{n-t-1}\backslash G_{n-t-1}$ consisting of elements whose determinant has valuation r. The conclusion is that $W^1(I_n) = W^2(I_n)$. Indeed, recall that given $r\in\mathbb{Z}$ the relations

$$|\det g| = r\,,\ W^i\left(\begin{array}{cc} g & 0 \\ 0 & I_{t+1}\end{array}\right) \neq 0$$

imply that g is in a set compact modulo U_{n-t-1}. Taking H_r to be

$$\left(W^1\left(\begin{array}{cc} g & 0 \\ 0 & I_{t+1}\end{array}\right) - W^2\left(\begin{array}{cc} g & 0 \\ 0 & I_{t+1}\end{array}\right)\right)\Phi_r(g)\,,$$

where Φ_r is the characteristic function of $(U_{n-t-1}\backslash G_{n-t-1})^r$, we have, by assumption, that $\langle H_r, W'\rangle = 0$ for all Whittaker functions W' of irreducible generic integral representations τ, and hence of the contragredient representations τ^ι. On the other hand, by Theorem 4.1, if H_r were nonzero, there would be some V', V as in Theorem 4.1 such that $\langle H_r, W'\rangle \neq 0$ in $A\otimes_{W(k)}\mathcal{O}$ for some $W'\in\mathcal{W}(V^\iota,\psi_{\mathcal{O}}^{-1})$. Then taking $W'\otimes 1$ in $\mathcal{W}((V')^\iota,\psi_{\overline{\mathcal{K}}}^{-1})$, we have $\langle H_r, W'\otimes 1\rangle = \langle H_r, W'\rangle\otimes 1 \neq 0$, a contradiction, so H_r must equal zero. Since $H_r = 0$ for all r, we then have $W^1 - W^2 = 0$ on G_{n-t-1}.

For $0 < j \leq t$, one observes that there is a compact subset Ω of $M_{j\times(n-t-1)}(F)$ such that for all $g\in G_{n-t-1}$ and $i=1,2$,

$$W^i\begin{pmatrix} g & 0 & 0 \\ X & I_j & 0 \\ 0 & 0 & I_{t+1-j}\end{pmatrix} \neq 0$$

implies that $X\in\Omega$. Thus, for $i=1,2$, there is an element $W_0^i\in\mathcal{W}(\pi_i,\psi)$ such that for all $g\in G_{n-t-1}$

$$\int_{M_{j\times(n-t-1)}(F)} W^i\begin{pmatrix} g & 0 & 0 \\ X & I_j & 0 \\ 0 & 0 & I_{t+1-j}\end{pmatrix} dX = W_0^i\begin{pmatrix} g & 0 & 0 \\ 0 & I_j & 0 \\ 0 & 0 & I_{t+1-j}\end{pmatrix}.$$

We are therefore reduced to the case $j=0$. $\square$

Remark 4.3. *We point out that if V' is an irreducible* integral *object in* $\mathrm{Rep}_{\overline{\mathcal{K}}}(G_n)$ *(such as in Theorems 1.1, 4.1, 4.2), then it contains an $\mathcal{O}_E$-lattice V_E, where E is a finite extension of $\mathcal{K}$, V_E is finitely generated as an $\mathcal{O}_E[G_n]$-module, and free as an $\mathcal{O}_E$-module (this is* [V96, II.4.13]*, taking $R = \mathcal{K}$ in II.4.9 of loc. cit.). In fact this statement is true for E a finite extension of $\mathbb{Q}_\ell$ (loc. cit. with $R = \mathbb{Q}_\ell$), however the theory of gamma factors and co-Whittaker modules is only written in the literature for $W(k)$-algebras, so we descend no further than extensions of $\mathcal{K}$.*

Remark 4.4. *There is a gap in the proof of* [M16, Theorem 1.1]*. A* priori*, infinitely many different finite extensions $\mathcal{O}'$ of $W(k)$ could be necessary as W_i and m vary in* [M16, §6.1]*. Thus, the argument in that paper only proves the following slightly weaker result: "if $\gamma(V_1 \times V', X, \psi) = \gamma(V_2 \times V', X, \psi)$ for all irreducible generic integral representations V' of G_{n-1} over $\overline{\mathcal{K}}$, then V_1 and V_2 have the same supercuspidal support." The authors believe* [M16, Theorem 1.1] *is correct as stated; this will be addressed in future work.*

At the cost of taking larger rings $B = e'\mathcal{Z}_n$, for primitive idempotents e' of $\mathcal{Z}_n$, but without any hypotheses on A, the collection of Whittaker functions over B can detect a subring in which ϕ_1 takes values, essentially by duality. We recall the following result in [HM16].

Theorem 4.5 (Cor 3.6, [HM16])**.** *Let A' be a $W(k)$-subalgebra of A, and suppose $H \neq 0$ is an element of* $\text{c-Ind}_{U_n}^{G_n} \psi_A$*. If $\langle H, W' \rangle$ lies in $A' \otimes e'\mathcal{Z}_n$ for all primitive idempotents e' of $\mathcal{Z}_n$ and all $W' \in \mathcal{W}(e'W_n, \psi^{-1})$, then H lies in* $\text{c-Ind}_{U_n}^{G_n} \psi_{A'}$*.*

Lemma 4.6. *Let A be any Noetherian $W(k)$-algebra and $A' \subset A$ a subalgebra, and let π be a co-Whittaker $A[G_n]$-module. Suppose $t \leq n-2$ and $0 \leq j \leq t$. Suppose that for W in $\mathcal{W}(\pi, \psi)$, the power series $\Psi(W, W', X; j)$ takes coefficients in A' for all primitive idempotents e' of $\mathcal{Z}_{n-t-1}$, and all $W' \in \mathcal{W}(e'W_{n-t-1}, \psi^{-1})$. Then*

$$\int W \begin{pmatrix} I_{n-t-1} & 0 & 0 \\ X & I_j & 0 \\ 0 & 0 & I_{t+1-j} \end{pmatrix} dX \text{ is in } A',$$

where the integrals are over $X \in M_{j \times (n-t-1)}(F)$.

Proof. For $j = 0$, the assumption is that

$$\sum_r \int_{(U_{n-t-1} \backslash G_{n-t-1})^r} W \begin{pmatrix} g & 0 \\ 0 & I_{t+1} \end{pmatrix} \otimes W'(g) X^r dg \tag{4.2}$$

is in $A'[[X]][X^{-1}]$ for all W', where $(U_{n-t-1}\backslash G_{n-t-1})^r$ is the subset of $U_{n-t-1}\backslash G_{n-t-1}$ consisting of elements whose determinant has valuation r. The conclusion is that $W(I_n)$ is in A'. Indeed, recall that given $r \in \mathbb{Z}$ the relations

$$|\det g| = r\,,\; W\begin{pmatrix} g & 0 \\ 0 & I_{t+1} \end{pmatrix} \neq 0$$

imply that g is in a set compact modulo U_{n-t-1}. Taking

$$H_r = W\left(\begin{smallmatrix} g & 0 \\ 0 & I_{t+1} \end{smallmatrix}\right)\Phi_r(g)\,,$$

where Φ_r is the characteristic function of $(U_{n-t-1}\backslash G_{n-t-1})^r$, we have, by assumption, that $\langle H_r, W'\rangle \in A' \otimes e'\mathcal{Z}_{n-t-1}$ for all Whittaker functions W' of $e'W_{n-t-1}$, and for all e'. By Theorem 4.5 H_r must take values in A'. Since this is true for all r, we then have $W\left(\begin{smallmatrix} g & 0 \\ 0 & I_{t+1} \end{smallmatrix}\right)$ is in A'.

For $0 < j \leq t$, one observes that there is a compact subset Ω of $M_{j\times(n-t-1)}(F)$ such that for all $g \in G_{n-t-1}$,

$$W\begin{pmatrix} g & 0 & 0 \\ X & I_j & 0 \\ 0 & 0 & I_{t+1-j} \end{pmatrix} \neq 0$$

implies that $X \in \Omega$. Thus, there is an element $W_0 \in \mathcal{W}(\pi,\psi)$ such that for all $g \in G_{n-t-1}$

$$\int_{M_{j\times(n-t-1)}(F)} W\begin{pmatrix} g & 0 & 0 \\ X & I_j & 0 \\ 0 & 0 & I_{t+1-j} \end{pmatrix} dX = W_0\begin{pmatrix} g & 0 & 0 \\ 0 & I_j & 0 \\ 0 & 0 & I_{t+1-j} \end{pmatrix}.$$

We are therefore reduced to the case $j = 0$. □

5. Proof of Theorem 1.1

In this section, we prove Theorem 1.1, following the proof of the [JL16, Theorem 1.3].

Let A be a finite-type $W(k)$-algebra which is reduced and ℓ-torsion free. Let π_1 and π_2 be co-Whittaker $A[G_n]$-modules with the same central character ω. Let P be the maximal parabolic subgroup of G_n with Levi subgroup $G_{n-1} \times G_1$. Let $Z = Z_n$ and $U = U_n$, and V_0 be the canonical sub-module of π_i which is isomorphic to $\text{c-Ind}_{ZU}^{P}\omega\psi_A$ (see [HM16, Lemma 3.1]). We have

$$W_v^1(p) = W_v^2(p)\,,\ \forall p \in P\,,\ \forall v \in V_0\,, \tag{5.1}$$

$$W_v^i(gp) = W_{pv}^i(g)\,,\ \forall g \in G_n\,,\ \forall p \in P\,,\ \forall v \in V_0\,,\ i = 1,2\,. \tag{5.2}$$

We recall the decomposition of G_n into double cosets of U and P as in [Ch06]:

$$G_n = \dot{\bigcup}_{i=0}^{n-1} U\alpha^i P\,, \text{ where } \alpha = \begin{pmatrix} 0 & I_{n-1} \\ 1 & 0 \end{pmatrix}.$$

Note that $\alpha^i = \left(\begin{smallmatrix} 0 & I_{n-i} \\ I_i & 0 \end{smallmatrix}\right)$, in particular, $\alpha^0 = \alpha^n = I_n$.

Definition 5.1. *For each double coset $U\alpha^i P$, $0 \leq i \leq n-1$, we call i the height of the double coset. We say that π_1 and π_2 agree at height i if*

$$W_v^1(g) = W_v^2(g)\,,\ \forall g \in U\alpha^i P\,,\ \forall v \in V_0\,.$$

By (5.1), π_1 and π_2 agree at height 0. The following lemma, which is the analogue of [Ch06, Lemma 3.1] with the same proof, gives a characterization of π_1 and π_2 agreeing at height i.

Lemma 5.2. *π_1 and π_2 agree at height i if and only if*

$$W_v^1(\alpha^i) = W_v^2(\alpha^i)\,,\ \forall v \in V_0\,.$$

Definition 5.3. *For $1 \leq t \leq n-1$, we say π_1 and π_2 satisfy hypothesis $\mathcal{H}_t$ if $\gamma(\pi_1\times\tau, X, \psi) = \gamma(\pi_2\times\tau, X, \psi)$ for all irreducible generic integral representations $\tau \in \mathrm{Rep}_{\overline{\mathcal{K}}}(G_t)$. We say π_1 and π_2 satisfy hypothesis $\mathcal{H}_{\leq s}$ if they satisfy $\mathcal{H}_t$ for all $t \leq s$.*

The following lemma is the analogue of [Ch06, Proposition 3.1] with same proof.

Lemma 5.4. *Let t with $1 \leq t \leq n-1$. If π_1 and π_2 satisfy hypothesis $\mathcal{H}_t$, then they agree at height t.*

The following proposition is an analogue of [JL16, Proposition 3.6], which allows us to prove Theorem 1.1 inductively.

Proposition 5.5. *Assume that π_1 and π_2 satisfy hypothesis $\mathcal{H}_{\leq[\frac{n}{2}]}$. Let t with $[\frac{n}{2}] \leq t \leq n-2$. Suppose that for any s with $0 \leq s \leq t$, the representations π_1 and π_2 agree at height s. Then they agree at height $t+1$.*

Before proving the proposition, we apply it to the proof of our main result as follows.

Proof of Theorem 1.1. Assume that π_1 and π_2 satisfy hypothesis $\mathcal{H}_{\leq[\frac{n}{2}]}$. By Lemma 5.4, π_1 and π_2 agree at heights $1, 2, \ldots, [\frac{n}{2}]$. Note that by (5.1), π_1 and π_2 already agree at height 0. Applying Proposition 5.5 repeatedly for t from $[\frac{n}{2}]$ to $n-2$, we obtain that π_1 and π_2 also agree at heights $[\frac{n}{2}]+1, \ldots, n-1$. Hence, π_1 and π_2 agree at all the heights $0, 1, \ldots, n-1$, that is, $W_v^1(g) = W_v^2(g)$, for all $g \in G_n$ and for all $v \in V_0$. Since there is some $v \in V_0$ such that $W_v^i(g) \in A^\times$ for some g, we have $\mathcal{W}(\pi_1, \psi) = \mathcal{W}(\pi_2, \psi)$ by Lemma 2.4. This completes the proof of Theorem 1.1. $\square$

Therefore, we only need to prove Proposition 5.5, which will be done in Section 6.

6. Proof of Proposition 5.5

First, we recall [JL16, Lemma 3.5], which characterizes certain supports of Whittaker functions W_v^1, W_v^2, for $v \in V_0$.

Lemma 6.1 ([JL16], Lemma 3.5). *Let t with $[\frac{n}{2}] \leq t \leq n-2$. Suppose that for any s with $0 \leq s \leq t$ the representations π_1 and π_2 agree at height s. Then the following equality holds for all $X \in M_{(n-t-1)\times(2t+2-n)}(F)$, all $g \in G_{n-t-1}$, and all $v \in V_0$:*

$$W_v^1 \begin{pmatrix} I_{n-t-1} & 0 & 0 \\ 0 & I_{2t+2-n} & 0 \\ 0 & X & g \end{pmatrix} = W_v^2 \begin{pmatrix} I_{n-t-1} & 0 & 0 \\ 0 & I_{2t+2-n} & 0 \\ 0 & X & g \end{pmatrix}. \tag{6.1}$$

The proof of Proposition 5.5 is similar to that of [JL16, Proposition 3.6], with [JL16, Lemma 4.1] being replaced by Lemma 6.3 below. We omit it here.

Lemma 6.3 below is an analogue of [JL16, Lemma 4.1]. One of the basic formulas we will repeatedly use is the Fourier inversion formula. For completeness, we record the well-known fact that the Fourier inversion formula still holds in this setting.

Fix $\psi : F \to W(k)^\times$, let $\mathfrak{Y} = F^l$ for some integer l, and let

$$\langle , \rangle : \mathfrak{Y}^2 \to F$$

be a non-degenerate bilinear pairing on $\mathfrak{Y}$. We define the Fourier transform $\widehat{\Phi}$ of $\Phi \in C_c^\infty(\mathfrak{Y}, A)$ by

$$\widehat{\Phi}(Y) := \int_{\mathfrak{Y}} \Phi(X)\psi(\langle Y, X\rangle)dX\,,$$

where $dX = d\mu(X)$ for a $W(k)$-valued Haar measure μ on $\mathfrak{Y}$.

Lemma 6.2 (Fourier inversion formula). *Let A be any $W(k)$-algebra. Given $\Phi \in C_c^\infty(\mathfrak{Y}, A)$, then $\widehat{\Phi}$ is also in $C_c^\infty(\mathfrak{Y}, A)$ and there is a Haar measure μ on $\mathfrak{Y}$ for which the Fourier inversion formula $\widehat{\widehat{\Phi}}(X) = \Phi(-X)$, holds for all $\Phi \in C_c^\infty(\mathfrak{Y}, A)$. If $\ell \neq 2$, this Haar measure is unique.*

Proof. The A-module $C_c^\infty(\mathfrak{Y}, A)$ is spanned by the characteristic functions Φ_j of $a + \varpi^j\mathfrak{Y}$, $a \in \mathfrak{Y}$, $j \in \mathbb{Z}$. In addition, the map $X \mapsto$

$\psi(\langle -, X\rangle)$ gives an isomorphism from $\mathfrak{Y}$ to the set of characters $\mathfrak{Y} \to W(k)^\times$ in the usual way (cf. [BH06, 1.7]). The inversion formula follows from the identity $\int_{\mathfrak{Y}} \psi(\langle A, B\rangle)dA = 0$ unless $B = 0$, in which case it is a unit u in $W(k)$. One can choose a Haar measure, depending on ψ, such that $u = 1$. If $\ell \neq 2$, we may proceed exactly as in [BH06, 23.1], choosing a square root of q in $W(k)^\times$ and letting $\mu(\mathcal{O}_F^k) = q^{kl/2}$, where l is the level of ψ. □

The proof of Lemma 6.3 below goes exactly as that of [JL16, Lemma 4.1], applying the Fourier inversion formula in Lemma 6.2. We omit it here.

Lemma 6.3. *Recall that $X_v^i = \rho(\alpha^{t+1})W_v^i$, $i = 1, 2$. If*

$$\begin{aligned}
&\int_{M_{(n-t-2)\times(n-t-1)}(F)} X_v^1 \begin{pmatrix} I_{n-t-1} & 0 & 0 \\ X & I_{n-t-2} & 0 \\ 0 & 0 & I_{2t+3-n} \end{pmatrix} dX \\
&= \int_{M_{(n-t-2)\times(n-t-1)}(F)} X_v^2 \begin{pmatrix} I_{n-t-1} & 0 & 0 \\ X & I_{n-t-2} & 0 \\ 0 & 0 & I_{2t+3-n} \end{pmatrix} dX\,,
\end{aligned} \tag{6.2}$$

for all $v \in V_0$, then $X_v^1(I_n) = X_v^2(I_n)$, for all $v \in V_0$.

7. A Descent Theorem

Let A be a Noetherian $W(k)$-algebra and let (π, V) be a co-Whittaker $A[G_n]$-module. Let A' be a sub-$W(k)$-algebra of A.

Definition 7.1. (1) *π satisfies hypothesis $\mathcal{H}_0(A')$ if its central character $\omega_\pi : Z \to A^\times$ factors through the inclusion $(A')^\times \subset A^\times$.*
(2) *For $t \geq 1$, we say that π satisfies hypothesis $\mathcal{H}_t(A')$ if $\gamma(\pi \times e'W_t, X, \psi)$ has coefficients in $A' \otimes e'\mathcal{Z}_t$ for all primitive idempotents e' of $\mathcal{Z}_t$. It satisfies $\mathcal{H}_{\leq s}(A')$ if it satisfies $\mathcal{H}_t(A')$ for all $t \leq s$.*

In this section we prove the following version of Theorem 1.2.

Theorem 7.2. *Assume that A is a finite extension of A' and π satisfies hypothesis $\mathcal{H}_{\leq [n/2]}(A')$. Then the supercuspidal support map*

$$f_V : \mathcal{Z}_n \to A$$

factors through the inclusion $A' \subset A$.

In [HM16], following version of descent theorem has been proved.

Theorem 7.3 ([HM16], Theorem 3.2)**.** *Assume that A is a finite extension of A' and π satisfies hypothesis $\mathcal{H}_{\leq n-1}(A')$. Then f_V factors through the inclusion $A' \subset A$.*

Recall that P is the maximal parabolic subgroup of G_n with Levi subgroup $G_{n-1} \times G_1$, $Z = Z_n$ is the center of G_n and $U = U_n$. Also recall that V_0 is the canonical sub-module of V which is isomorphic to c-$\mathrm{Ind}_{ZU}^{P}(\omega_\pi \psi_A)$. Let V_0' be the canonical sub-$W(k)$-module of V_0 which is isomorphic to c-$\mathrm{Ind}_{ZU}^{P}(\omega_\pi \psi_{A'})$. We recall the following decomposition of G_n from Section 5:

$$G_n = \dot{\bigcup}_{i=0}^{n-1} U\alpha^i P\,, \text{ where } \alpha = \begin{pmatrix} 0 & I_{n-1} \\ 1 & 0 \end{pmatrix}.$$

Definition 7.4. *We say π is A'-valued at height t if $W_v(g)$ lies in A', for all $g \in U\alpha^t P$ and all $v \in V_0'$. Note that if $\mathcal{H}_0(A')$ is satisfied, π is A'-valued at height t if and only if $W_v(\alpha^i) \in A'$, for all $v \in V_0'$.*

Note that if $\mathcal{H}_0(A')$ is satisfied, π is A'-valued at height 0.

Proposition 7.5. *Suppose $n \geq 2$ and π satisfies $\mathcal{H}_1(A')$. Then π satisfies $\mathcal{H}_0(A')$.*

Proof. If $\chi : F^\times \to W(k)^\times$ is any smooth character, then it is trivially co-Whittaker and has a supercuspidal support map $f_\chi : \mathcal{Z}_1 \to W(k)$. By compatibility of the gamma factor with base change, we thus have

$$(id \otimes f_\chi)(\gamma(\pi \times W_1, X, \psi)) = \gamma(\pi \times \chi, X, \psi)$$

has coefficients in $A' \otimes_{W(k)} W(k)$ by hypothesis $\mathcal{H}_1$. Taking χ to have sufficiently large conductor m, Proposition 3.4 allows us to conclude $\omega_\pi(c)^{-1}\gamma(1_A \times \chi, X, \psi)^n$ has coefficients in $A' \otimes W(k)$ for any $c \in \mathfrak{p}^{-m}$ satisfying $\chi(1+x) = \psi(cx)$ for $x \in \mathfrak{p}^{[m/2]+1}$. But since the supercuspidal support of 1_A factors through A' (in fact, $W(k)$), $\gamma(1_A \times \chi, X, \psi)$ is valued in $S^{-1}(A' \otimes W(k)[X, X^{-1}])$. Since $\gamma(1_A \times \chi, X, \psi)$ is a unit in this ring ([M16, Cor 5.6]) we have $\omega_\pi(c)^{-1} \otimes 1$ is in $A' \otimes W(k)$, showing $\omega_\pi(c)^{-1}$ is in A'. For any $c \in \mathfrak{p}^{-m} \backslash \mathfrak{p}^{1-m}$, there exists χ_c of conductor m such that $\chi_c(1 + x) = \psi(cx)$ for $x \in \mathfrak{p}^{[m/2]+1}$. Since any element of $F^\times$ can be expressed as the quotient of two elements of valuation at most $-m$, we deduce that ω_π is valued in A'. $\square$

The functional equation requires that the zeta integrals lie in the ring $S^{-1}R[X, X^{-1}]$ of "rational functions", however, we only know this rationality is preserved by *finite* descent, in the following sense.

Proposition 7.6 ([HM16], Corollary 4.2)**.** *Suppose R' is a Noetherian $W(k)$-subalgebra of R such that R is finitely generated as an R'-module. Let S' be the subset of $R'[X, X^{-1}]$ consisting of polynomials whose first and last nonzero coefficients are units in R'. Then $(S')^{-1}R'[X, X^{-1}]$ is the intersection, in $R[[X]][X^{-1}]$, of the subrings $R'[[X]][X^{-1}]$ and $S^{-1}R[X, X^{-1}]$.*

In what follows, we will refer to Proposition 7.6 when $R = A \otimes e'\mathcal{Z}_t$ and R' is the subring $A' \otimes e'\mathcal{Z}_t$, for some e'. Note that this is indeed a subring since $e'\mathcal{Z}_t$ is flat over $W(k)$.

The following proposition is the analogue of [Ch06, Proposition 3.1].

Proposition 7.7. *Suppose A is a finite extension of A', and suppose π satisfies $\mathcal{H}_0(A')$. Let t be an integer with $1 \leq t \leq n-1$. If π satisfies hypothesis $\mathcal{H}_t(A')$, then π is A'-valued at height t.*

Proof. Since π satisfies $\mathcal{H}_0$, by definition of V_0', $W_v(p) \in A'$, $\forall p \in P$, $\forall v \in V_0'$. Hence,

$$W_v \begin{pmatrix} g_t & & \\ x & I_{n-t-1} & \\ & & 1 \end{pmatrix} \in A' \,, \forall v \in V_0' \,,$$

where g_t is any element in G_t, $x \in M_{(n-t-1)\times t}$. Then, for any primitive idempotents e' of $\mathcal{Z}_t$, for any $W' \in \mathcal{W}(e'W_t, \overline{\psi})$,

$$\begin{aligned} &\Psi(W, W', X; n-t-1) \\ = &\sum_{r\in\mathbb{Z}} \int_{M_{n-t-1,t}(F)} \int_{N_t \backslash \{g \in G_t : v(\det g) = r\}} \left(W \left(\begin{smallmatrix} g & & \\ x & I_{n-t-1} & \\ & & 1 \end{smallmatrix}\right) \otimes W'(g) \right) X^r dg dx \\ \in &(A' \otimes e'\mathcal{Z}_t)[[X]][X^{-1}] \,. \end{aligned}$$

Since $A' \subset A$ is finite, by Proposition 7.6, $\Psi(W, W', X; n-t-1)$ lies in $(S')^{-1}(A' \otimes e'\mathcal{Z}_t)[X, X^{-1}]$. Applying the involution $X \mapsto \frac{q^{n-t-1}}{X}$ gives

$$\Psi(W, W', \frac{q^{n-t-1}}{X}; n-t-1) \in (S')^{-1}(A' \otimes e'\mathcal{Z}_t)[X, X^{-1}] \,.$$

By assumption, π satisfies hypothesis $\mathcal{H}_t$, that is, $\gamma(\pi \times e'W_t, \frac{q^{n-t-1}}{X}, \psi)$ has coefficients in $A' \otimes e'\mathcal{Z}_t$ for all primitive idempotents e' of $\mathcal{Z}_t$. Hence,

$$\Psi(\omega_{n,t}\widetilde{W}, \widetilde{W'}, X; 0) \in (A' \otimes e'\mathcal{Z}_t)[[X]][X^{-1}] \,.$$

Therefore,

$$\widetilde{W}\left(\begin{pmatrix} g_t & \\ & I_{n-t} \end{pmatrix} \begin{pmatrix} I_t & \\ & \omega_{n-t} \end{pmatrix} \right) \in A' \otimes e'\mathcal{Z}_t \,, \forall g_t \in G_t \,, \forall v \in V_0' \,.$$

Take $g_t = I_t$, we get that

$$\widetilde{W}\begin{pmatrix} I_t & \\ & \omega_{n-t} \end{pmatrix} \in A' \otimes e'\mathcal{Z}_t \,, \forall v \in V_0' \,,$$

that is,

$$W\left(\omega_n \begin{pmatrix} I_t & \\ & \omega_{n-t} \end{pmatrix} \right) \in A' \otimes e'\mathcal{Z}_t \,, \forall v \in V_0' \,,$$

which is exactly

$$W(\alpha^t) \in A' \otimes e'\mathcal{Z}_t \,,\ \forall v \in V_0' \,.$$

Therefore, π is A'-valued at height t. This completes the proof of the proposition. □

The following proposition is an analogue of [JL16, Proposition 3.6].

Proposition 7.8. *Assume A is a finite extension of A' and π satisfies hypothesis $\mathcal{H}_{\leq[n/2]}(A')$. Let t be such that $[n/2] \leq t \leq n-2$. Suppose that for any s with $0 \leq s \leq t$, π is A'-valued at height s. Then π is A'-valued at height $t+1$.*

Before proving Proposition 7.8, we use it to deduce Theorem 7.2.

Proof of Theorem 7.2. Assume π satisfies hypothesis $\mathcal{H}_{\leq[n/2]}$. By Proposition 7.7, π is A'-valued at heights $1, 2, \ldots, [n/2]$. Note that π is already A'-valued at height 0 because it satisfies $\mathcal{H}_1$. Applying Proposition 7.8 repeatedly for t ranging from $[n/2]$ to $n-2$, we find that π is A'-valued at heights $[n/2]+1, \ldots, n-1$. Hence π is A'-valued at all heights $0, 1, \ldots, n-1$, and Lemma 7.3 finishes the proof. □

We now prove Proposition 7.8.

Proof of Proposition 7.8. It is shown in the course of proving [JL16, Lemma 3.5] that, for all $X \in M_{(n-t-1)\times(2t+2-n)}(F)$, all $g \in G_{n-t-1}$, and all $v \in V_0'$, the matrices $\begin{pmatrix} I_{n-t-1} & 0 & 0 \\ 0 & I_{2t+2-n} & 0 \\ 0 & X & g \end{pmatrix}$ are in $U\alpha^{n-i}P$ with $n-i \leq t$. Therefore, by assumption,

$$W_v \begin{pmatrix} I_{n-t-1} & 0 & 0 \\ 0 & I_{2t+2-n} & 0 \\ 0 & X & g \end{pmatrix} \in A' \,.$$

Then,

$$W_v \left(\omega_n \omega_n \begin{pmatrix} I_{n-t-1} & 0 & 0 \\ 0 & I_{2t+2-n} & 0 \\ 0 & X & g \end{pmatrix} \right) \in A' \,,$$

that is,

$$W_v \left(\omega_n \begin{pmatrix} g_1 & X_1 & 0 \\ 0 & I_{2t+2-n} & 0 \\ 0 & 0 & I_{n-t-1} \end{pmatrix} \omega_n \right) \in A' \,,$$

where $g_1 = \omega_{n-t-1} g \omega_{n-t-1}$, $X_1 = \omega_{n-t-1} X \omega_{2t+2-n}$.

Note that

$$\omega_n = \begin{pmatrix} \omega_{n-t-1} & 0 \\ 0 & I_{t+1} \end{pmatrix} \omega_{n,n-t-1} \alpha^{t+1} \,.$$

Recall that
$$\omega_{n,n-t-1} = \begin{pmatrix} I_{n-t-1} & 0 \\ 0 & \omega_{t+1} \end{pmatrix} .$$
Hence,
$$W_v \left(\omega_n \begin{pmatrix} g_2 & X_1 & 0 \\ 0 & I_{2t+2-n} & 0 \\ 0 & 0 & I_{n-t-1} \end{pmatrix} \omega_{n,n-t-1} \alpha^{t+1} \right) \in A' ,$$
where $g_2 = \omega_{n-t-1} g$, $X_1 = \omega_{n-t-1} X \omega_{2t+2-n}$.

Let $X_v = \rho(\alpha^{t+1}) W_v$. Then
$$X_v \left(\omega_n \begin{pmatrix} g_2 & X_1 & 0 \\ 0 & I_{2t+2-n} & 0 \\ 0 & 0 & I_{n-t-1} \end{pmatrix} \omega_{n,n-t-1} \right) \in A' .$$
Recall that $\widetilde{X_v}(g) = X_v(\omega_n {}^t g^{-1})$. Then,
$$\widetilde{X_v} \left(\begin{pmatrix} g_3 & 0 & 0 \\ X_2 & I_{2t+2-n} & 0 \\ 0 & 0 & I_{n-t-1} \end{pmatrix} \omega_{n,n-t-1} \right) \in A' ,$$
where $g_3 = \omega_{n-t-1} {}^t g^{-1}$, $X_2 = -\omega_{2t+2-n} {}^t X {}^t g^{-1}$.

Therefore,
$$\widetilde{X_v} \left(\begin{pmatrix} g & 0 & 0 \\ X & I_{2t+2-n} & 0 \\ 0 & 0 & I_{n-t-1} \end{pmatrix} \omega_{n,n-t-1} \right) \in A' ,$$
for all $X \in M_{(2t+2-n)\times(n-t-1)}(F)$, all $g \in G_{n-t-1}$, and all $v \in V_0'$. Then, by the definition of the zeta integral Ψ, and since A is finite over A' we have (Proposition 7.6):
$$\Psi(\rho(\omega_{n,n-t-1})(\widetilde{X_v}), \widetilde{W'}, \frac{q^t}{X}; 2t+2-n) \text{ is in } (S')^{-1} R'[X, X^{-1}] ,$$
for all idempotents e' of $\mathcal{Z}_{n-t-1}$ and all $W' \in \mathcal{W}(e' W_{n-t-1}, \psi^{-1})$, and all $v \in V_0'$.

Since π satisfies hypothesis $\mathcal{H}_{\leq [\frac{n}{2}]}$, and $n-t-1 \leq [\frac{n}{2}]$, the functional equation (Theorem 3.3) gives
$$\Psi(X_v, W', X; n-t-2) \in R'[[X]][X^{-1}] ,$$
for all irreducible generic representations τ of G_{n-t-1}, all Whittaker functions $W' \in \mathcal{W}(e' W_{n-1}, \psi^{-1})$, and all $v \in V_0'$. Hence, by Lemma 4.6
$$\int_{M_{(n-t-2)\times(n-t-1)}(F)} X_v \begin{pmatrix} I_{n-t-1} & 0 & 0 \\ X & I_{n-t-2} & 0 \\ 0 & 0 & I_{2t+3-n} \end{pmatrix} dX \in A' ,$$

for all $v \in V_0'$. We claim (Lemma 7.10 below) that this identity implies in fact

$$X_v(I_n) \in A' \,,\, \forall v \in V_0' \,,$$

showing that

$$W_v(\alpha^{t+1}) \in A' \,,\, \forall v \in V_0' \,,$$

i.e. π is A'-valued at height $t+1$. This concludes the proof of Proposition 7.8. $\square$

Lemma 7.10 below is an analogue of [JL16, Lemma 4.1]. One of the basic formulas we will repeatedly need is the following Fourier descent formula.

Lemma 7.9. *(Fourier descent formula) Suppose A is an arbitrary $W(k)$-algebra. Suppose $\mathfrak{Y} = F^l$ for some integer l, and $\Phi \in C_c^\infty(\mathfrak{Y}, A)$. If $\widehat{\Phi}(X) \in A'$ for all $X \in \mathfrak{Y}$, then $\Phi(X)$ is in A' for all $X \in \mathfrak{Y}$.*

Proof. This follows immediately from Lemma 6.2 after noting that $\widehat{\widehat{\Phi}}$ is also valued in A', since the integral is a finite sum. $\square$

Lemma 7.10. *Let $X_v := \rho(\alpha^{t+1})W_v$. If*

$$\int_{M_{(n-t-2)\times(n-t-1)}(F)} X_v \begin{pmatrix} I_{n-t-1} & 0 & 0 \\ X & I_{n-t-2} & 0 \\ 0 & 0 & I_{2t+3-n} \end{pmatrix} dX$$

is in A' for all $v \in V_0'$, then $X_v(I_n)$ is in A' for all $v \in V_0'$.

The proof of this lemma is similar to that of [JL16, Lemma 4.1], applying the Fourier descent formula in Lemma 7.9. We omit it here.

References

[AT17] M. Adrian and S. Takeda, *A local converse theorem for* GL(n) *(archimedean case)*. Preprint 2017. arXiv:1702.07632.

[ALSX16] M. Adrian, B. Liu, S. Shaun and P. Xu, *On the Jacquet Conjecture on the local converse problem for p-adic* GL$_N$. Representation Theory **20** (2016), 1–13.

[ALST16] M. Adrian, B. Liu, S. Shaun and K.-F. Tam, *On sharpness of the bound for the local converse theorem of p-adic* GL$_{prime}$*, tame case.* Preprint. 2016.

[B95] E. M. Baruch, *Local factors attached to representations of p-adic groups and strong multiplicity one.* PhD thesis, Yale University, 1995.

[B97] E. M. Baruch, *On the gamma factors attached to representations of* U$(2,1)$ *over a p-adic field.* Israel J. Math. **102** (1997), 317–345.

[BH06] C. Bushnell and G. Henniart, *The local Langlands conjecture for* GL(2). Spring-Verlag Berlin Heidelberg, A Series of Comprehensive Studies in Mathematics **335** (2006).

[BK93] C. Bushnell and P. Kutzko, *The admissible dual of* GL_N *via restriction to compact open subgroups.* Annals of Mathematics Studies, **129**. Princeton University Press, Princeton, NJ, 1993.

[Ch16] J. Chai, *Bessel functions and local converse conjecture of Jacquet.* To appear in JEMS. arXiv:1601.05450.

[Ch96] J.-P. Chen, *Local factors, central characters, and representations of the general linear group over non-Archimedean local fields.* Thesis, Yale University, 1996.

[Ch06] J.-P. Chen, *The* $n \times (n-2)$ *local converse theorem for* $\mathrm{GL}(n)$ *over a* p*-adic field.* J. Number Theory **120** (2006), no. 2, 193–205.

[CPS99] J. Cogdell and I. Piatetski-Shapiro, *Converse theorems for* GL_n*. II*, J. Reine Angew. Math. **507** (1999), 165–188.

[EH14] M. Emerton and D. Helm, *The local Langlands correspondence for* GL_n *in families*, Ann. Sci. Éc. Norm. Supér. (4) **47** (2014), no. 4, 655–722.

[H16a] D. Helm, *The Bernstein center of the category of smooth* $W(k)[\mathrm{GL}_n(F)]$*-modules*, Forum of Math, Sigma (2016), Vol. **4**, e11.

[H16b] D. Helm, *Whittaker models and the integral Bernstein center for* GL_n, Duke. Math. J., Volume **165**, Number 9 (2016), 1597–1628

[Hel] D. Helm, *Curtis homomorphisms and the local Langlands correspondence in families*, preprint, **arXiv:1605.00487**.

[HM15] D. Helm and G. Moss, *Deligne–Langlands gamma factors in families*, submitted, **arXiv:1510.08743v3**.

[HM16] D. Helm and G. Moss, *Converse theorems and the local Langlands correspondence in families*, to appear in Invent. Math., **arXiv:1610.03277**.

[HO15] J. Hakim and O. Offen, *Distinguished representations of* $\mathrm{GL}(n)$ *and local converse theorems.* Manuscripta Math. **148**, 1–27 (2015).

[H93] G. Henniart, *Caractérisation de la correspondance de Langlands locale par les facteurs* ϵ *de paires.* Invent. Math. **113** (1993), no. 2, 339–350.

[H02] G. Henniart, *Une caractérisation de la correspondance de Langlands locale pour* $\mathrm{GL}(n)$*.* Bull. Soc. Math. France **130** (2002), no. 4, 587–602.

[JL70] H. Jacquet and R. Langlands, *Automorphic forms on* $\mathrm{GL}(2)$. Lecture Notes in Mathematics, Vol. **114**. Springer-Verlag, Berlin-New York, 1970. vii+548 pp.

[JL16] H. Jacquet and B. Liu, *On the local converse theorem for* p*-adic* GL_n. To appear in Amer. J. Math. 2016. arXiv:1601.03656.

[JPSS79] H. Jacquet, I. Piatetski-Shapiro and J. Shalika, *Automorphic forms on* $\mathrm{GL}(3)$. Ann. of Math. (2) **109** (1979), no. 1–2, 169–258.

[JPSS83] H. Jacquet, I. Piatetski-Shapiro and J. Shalika, *Rankin–Selberg convolutions.* Amer. J. Math. **105** (1983), 367–464.

[JPSS81] H. Jacquet, I. Piatetski-Shapiro and J. Shalika, *Conducteur des representations due groupe lineaire.* Math. Annalen **256** (1981), 199–214.

[JNS15] D. Jiang, C. Nien and S. Stevens, *Towards the Jacquet conjecture on the Local Converse Problem for* p*-adic* GL_n. J. Eur. Math. Soc. **17** (2015), no. 4, 991–1007.

[JS03] D. Jiang and D. Soudry, *The local converse theorem for* $\mathrm{SO}(2n+1)$ *and applications.* Ann. of Math. (2) **157** (2003), no. 3, 743–806.

[M16] G. Moss, *Gamma factors of pairs and a local converse theorem in families.* Int Math Res Notices (2016) **2016** (16): 4903–4936.

[N14] C. Nien, *A proof of the finite field analogue of Jacquet's conjecture.* Amer. J. Math. **136** (3) (2014), 653–674.

[V01] M.-F. Vignéras, *Correspondance de Langlands semi-simple pour* $\mathrm{GL}(n, F)$ *modulo* $\ell \neq p$, Invent. Math. **144** (2001), no. 1, 177–223.

[V96] M.-F. Vignéras, *Representations* ℓ*-modulaires d'un groupe reductif* p*-adique avec* ℓ *different de* p, Boston: Birkhauser, 1996.

[Z15] Q. Zhang, *A local converse theorem for* $\mathrm{U}(2,2)$. Preprint. 2015. arXiv:1508.07062.

[Z17a] Q. Zhang, *A local converse theorem for* Sp_{2r}. Preprint. 2017. arXiv: 1705.01692.

[Z17b] Q. Zhang, *A local converse theorem for* U_{2r+1}. Preprint. 2017. arXiv: 1705.09410.

[Z17c] Q. Zhang, *A local converse theorem for* $\mathrm{U}(1,1)$. Int. J. Number Theory, Vol. **13**, No. 8 (2017) 1931–1981.

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