ON TOP FOURIER COEFFICIENTS OF CERTAIN AUTOMORPHIC REPRESENTATIONS OF GL_n

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Abstract. We study the top Fourier coefficients of isobaric automorphic representations of $\mathrm{GL}_n(\mathbb{A})$ of the form

$$\Pi_{\underline{s}} = \mathrm{Ind}_{P(\mathbb{A})}^{\mathrm{GL}_n(\mathbb{A})} \Delta(\tau_1, b_1)|\cdot|^{s_1} \otimes \cdots \otimes \Delta(\tau_r, b_r)|\cdot|^{s_r},$$

where $s_i \in \mathbb{C}$, $\Delta(\tau_i, b_i)$'s are Speh representations in the discrete spectrum of $\mathrm{GL}_{a_i b_i}(\mathbb{A})$ with τ_i's being unitary cuspidal representations of $\mathrm{GL}_{a_i}(\mathbb{A})$, and $n = \sum_{i=1}^r a_i b_i$. In particular, we prove a part of a conjecture of Ginzburg, and also a conjecture of Jiang under certain assumptions. The result of this paper will facilitate the study of automorphic forms of classical groups occurring in the discrete spectrum.

1. Introduction

Fourier coefficients are important in the study automorphic forms. For example, Whittaker-Fourier coefficients play an essential role in the theory of constructing automorphic L-functions, either by Rankin-Selberg method or by Langlands-Shahidi method. In general, there is a framework of attaching Fourier coefficients to nilpotent orbits (see [GRS03, G06, J14, GGS17a], and also §2 for details), which has also been used in theory of automorphic descent (see [GRS11]). Let F be a number field and $\mathbb{A}$ be its ring of adeles. Let G be a connected reductive group defined over F. One important topic in the theory of Fourier coefficients is to study all nilpotent orbits providing non-zero Fourier coefficients for a given automorphic representation π of $\mathrm{G}(\mathbb{A})$. We denote the set of all such nilpotent orbits by $\mathfrak{n}(\pi)$. The subset of maximal nilpotent orbits $\mathfrak{n}^m(\pi)$ in $\mathfrak{n}(\pi)$ under the natural ordering of partitions is particularly interesting. For classical groups, nilpotent orbits are parameterized by partitions of certain integers (see [CM93, W01]), and in such cases, a relatively easier question is to characterize the sets of partitions $\mathfrak{p}(\pi)$ and $\mathfrak{p}^m(\pi)$ parameterizing nilpotent orbits in $\mathfrak{n}(\pi)$ and $\mathfrak{n}^m(\pi)$, respectively. A folklore conjecture is that all nilpotent orbits in $\mathfrak{n}^m(\pi)$ belong to the same geometric orbit (namely over the algebraic closure $\overline{F}$), this means that the set $\mathfrak{p}^m(\pi)$ is a singleton in the cases of classical groups. The properties of $\mathfrak{n}(\pi)$, $\mathfrak{n}^m(\pi)$, $\mathfrak{p}(\pi)$, and $\mathfrak{p}^m(\pi)$ have been studied extensively in many papers, for example, [GRS03, G06, J14, JL13, JL15, JL16a, JL16b, JL17, JLS16, C18, Ts17, GGS17a, GGS17b].

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In the case of GL_n, the nilpotent orbits are in one-to-one correspondence with partitions of n (see [CM93]). In the 1970s, Shalika [S74] and Piatetski-Shapiro [PS79] proved independently that any irreducible cuspidal automorphic representation π has a non-zero Whittaker-Fourier coefficient, i.e. $\mathfrak{p}^m(\pi) = \{[n]\}$, corresponding to the largest nilpotent orbit. By the work of Mœglin and Waldspurger [MW89], the discrete spectrum of $\mathrm{GL}_n(\mathbb{A})$ consists of Speh representations $\Delta(\tau, b)$ (see §3.1 for details), where τ runs over irreducible unitary cuspidal automorphic representations of $\mathrm{GL}_a(\mathbb{A})$, and $n = ab$. Ginzburg proved in [G06] that $\mathfrak{p}^m(\Delta(\tau, b)) = \{[a^b]\}$ with a local-global argument, and Jiang and the first-named author proved the same result in [JL13] using a purely global method. Let $n = \sum_{i=1}^r b_i$ and consider the representation

$$\pi = \mathrm{Ind}_{P_{b_1,\cdots,b_r}(\mathbb{A})}^{\mathrm{GL}_n(\mathbb{A})} \delta_{P_{b_1,\cdots,b_r}}^{\underline{s}}$$

with $\underline{s} = (s_1, \cdots, s_r) \in \mathbb{C}^r$ and $\mathrm{Re}(s_i - s_j) \gg 0$ $(1 \le i < j \le r)$, which can be realized as a space of degenerated Eisenstein series. Then it was conjectured by Ginzburg (see [G06, Conjecture 5.1]) and proved recently by Cai in [C18] that

$$\mathfrak{p}^m(\pi) = \{[b_1 b_2 \cdots b_r]^t\} = \{[1^{b_1}] + [1^{b_2}] + \cdots + [1^{b_r}]\}\,.$$

Here the transpose and addition of partitions will be recalled in §2.1.

The purpose of this paper is to generalize the results above and study the top Fourier coefficients of automorphic representations of $\mathrm{GL}_n(\mathbb{A})$ which are induced from Speh representations

$$\Pi_{\underline{s}} = \mathrm{Ind}_{P(\mathbb{A})}^{\mathrm{GL}_n(\mathbb{A})} \Delta(\tau_1, b_1)|\cdot|^{s_1} \otimes \cdots \otimes \Delta(\tau_r, b_r)|\cdot|^{s_r}\,,$$

where $P = MN$ is a parabolic subgroup of GL_n with Levi subgroup M isomorphic to $\mathrm{GL}_{a_1 b_1} \times \cdots \times \mathrm{GL}_{a_r b_r}$, τ_i is an irreducible unitary cuspidal automorphic representation of $\mathrm{GL}_{a_i}(\mathbb{A})$, $n = \sum_{i=1}^r a_i b_i$, and $\underline{s} = (s_1, \ldots, s_r) \in \mathbb{C}^r$.

The first result of this paper is about the top Fourier coefficients of $\Pi_{\underline{s}}$, which verifies a part of a conjecture of Ginzburg (see [G06, Conjecture 5.6]).

Theorem 1.1. *Suppose that* $\mathrm{Re}(s_i - s_j) \gg 0$ *for all* $1 \le i < j \le r$*, then*

$$\mathfrak{p}^m(\Pi_{\underline{s}}) = \{[b_1^{a_1} \cdots b_r^{a_r}]^t\} = \{[a_1^{b_1}] + \cdots + [a_r^{b_r}]\}\,.$$

We are also interested in the case $\underline{s} = (0, \ldots, 0)$, in which case we have the isobaric sum automorphic representation of the form

$$\Pi \;=\; \Delta(\tau_1, b_1) \boxplus \Delta(\tau_2, b_2) \boxplus \cdots \boxplus \Delta(\tau_r, b_r)\,.$$

From the Arthur classification of the discrete spectrum of classical groups (see [A13, M15, KMSW14, Xu14]), endoscopic lifting images of automorphic representations of classical groups occurring in the discrete spectrum form a special class of such isobaric automorphic representations. In [J14], Jiang also conjectured that $\mathfrak{p}^m(\Pi) = \{[a_1^{b_1}] + \cdots + [a_r^{b_r}]\}$.

Our second result in this paper verifies the conjecture of Jiang under the following assumptions:

Assumption 1.2. *We assume the representations* $\tau_1, \ldots, \tau_r$ *satisfy the following conditions.*

(i) *We have* $\tau_i \not\cong \tau_j$ *for all* $1 \le i \ne j \le r$.

(ii) *The local representations* $\tau_{i,v}$ *are tempered for all local places* v *of* F *and* $1 \leq i \leq r$.

(iii) *The* L*-functions* $L(s, \tau_i \times \widetilde{\tau}_j)$ *are non-zero at* $s = \frac{1}{2}$ *for all* $1 \leq i, j \leq r$.

Theorem 1.3. *Under Assumption 1.2, we have*

$$\mathfrak{p}^m(\Pi) = \{[a_1^{b_1}] + \cdots + [a_r^{b_r}]\}.$$

Our proof makes use of some recent results on Fourier coefficients of automorphic forms which pack up some systematical arguments in this topic, and hence can be done in a shorter length. Under our assumptions, we use a result of Gomez, Gourevitch and Sahi in [GGS17a] to show that $\Pi_{\underline{s}}$ (and also Π) has a non-zero *generalized Whittaker-Fourier coefficient* attached to the partition $[a_1^{b_1}] + \cdots + [a_r^{b_r}]$ in §4 (see Proposition 4.2, and see §2 for the definition of such Fourier coefficients). The use of [GGS17a] reduces the argument to the calculation of constant term of Eisenstein series. On the other hand, to show $[a_1^{b_1}] + \cdots + [a_r^{b_r}]$ is exactly the top orbit for $\Pi_{\underline{s}}$, one also needs to show that Π has no non-zero generalized Whittaker-Fourier coefficients attached to any partition bigger than or not related to $[a_1^{b_1}] + \cdots + [a_r^{b_r}]$. For this, we use a local criterion (see Proposition 2.2) which is due to the works of Mœglin-Waldspurger ([MW87]) and Varma ([V14])). This local criterion reduces the proof of vanishing properties of Fourier coefficients to a simpler local vanishing statement, which is proved in §5 (see Proposition 5.1) using Bernstein's localization principle (see [BZ76]) and a combinatorial result of Cai ([C18]). We note that one important feature of the representation $\Pi_{\underline{s}}$ we are considering is that its global top orbit equals to its local top orbit at almost all places, so that this approach works.

We remark that when we were finishing up this paper, we noticed that another proof of the same result for Π is given by Tsiokos in [Ts17], using a different method.

Finally, it is worthwhile to mention that, towards understanding Fourier coefficients of automorphic representations in the discrete spectrum of classical groups, in [J14, §4.4], Jiang made a conjecture on the connection between Fourier coefficients of automorphic representations in an Arthur packet and the structure of the corresponding Arthur parameter (see [JL16a] for the progress on the cases of symplectic groups). The result of this paper will facilitate the study of Fourier coefficients of automorphic representations in the discrete spectrum of classical groups, since the endoscopic lifting image of each Arthur packet is an isobaric automorphic representation of a general linear group.

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2. Generalized and degenerate Whittaker-Fourier coefficients attached to nilpotent orbits

In this section, we recall the generalized and degenerate Whittaker-Fourier coefficients attached to nilpotent orbits, as well as some related basic definitions mentioned in §1, following the formulation in [GGS17a]. Then we introduce a local criterion due to [MW87, V14] on determining the top generalized Whittaker models in the case of GL_n.

2.1. **The generalized and degenerate Whittaker-Fourier coefficients.** Let G be a reductive group defined over a number field F, and $\mathfrak{g}$ be the Lie algebra of $\mathrm{G}(F)$. Given any semi-simple element $s \in \mathfrak{g}$, under the adjoint action, $\mathfrak{g}$ is decomposed into a direct sum of eigenspaces $\mathfrak{g}_i^s$ corresponding to eigenvalues i. The element s is called *rational semi-simple* if all its eigenvalues are in $\mathbb{Q}$. Given a nilpotent element u and a simi-simple element s in $\mathfrak{g}$, the pair (s,u) is called a *Whittaker pair* if s is a rational semi-simple element, and $u \in \mathfrak{g}^s_{-2}$. The element s in a Whittaker pair (s,u) is called a *neutral element* for u if there is a nilpotent element $v \in \mathfrak{g}$ such that (v,s,u) is an $\mathfrak{sl}_2$-triple. A Whittaker pair (s,u) with s being a neutral element is called a *neutral pair*.

Given any Whittaker pair (s,u), define an anti-symmetric form ω_u on $\mathfrak{g}\times\mathfrak{g}$ by

$$\omega_u(X,Y) := \kappa(u,[X,Y])\,,$$

here κ is the Killing form on $\mathfrak{g}$. For any rational number $r \in \mathbb{Q}$, let $\mathfrak{g}^s_{\geq r} = \oplus_{r'\geq r}\mathfrak{g}^s_{r'}$. Let $\mathfrak{u}_s = \mathfrak{g}^s_{\geq 1}$ and let $\mathfrak{n}_{s,u}$ be the radical of $\omega_u|_{\mathfrak{u}_s}$. Then $[\mathfrak{u}_s,\mathfrak{u}_s] \subset \mathfrak{g}^s_{\geq 2} \subset \mathfrak{n}_{s,u}$. For any $X \in \mathfrak{g}$, let $\mathfrak{g}_X$ be the centralizer of X in $\mathfrak{g}$. By [GGS17a, Lemma 3.2.6], one has $\mathfrak{n}_{s,u} = \mathfrak{g}^s_{\geq 2} + \mathfrak{g}^s_1 \cap \mathfrak{g}_u$. Note that if the Whittaker pair (s,u) comes from an $\mathfrak{sl}_2$-triple (v,s,u), then $\mathfrak{n}_{s,u} = \mathfrak{g}^s_{\geq 2}$. Let $N_{s,u} = \exp(\mathfrak{n}_{s,u})$ be the corresponding unipotent subgroup of G. We define a character of $N_{s,u}(\mathbb{A})$ by

$$\psi_u(n) = \psi(\kappa(u,\log(n)))\,,$$

here $\psi : F\backslash\mathbb{A} \to \mathbb{C}^\times$ is a fixed non-trivial additive character, and we extend the killing form κ to $\mathfrak{g}(\mathbb{A})\times\mathfrak{g}(\mathbb{A})$.

Let π be an irreducible automorphic representation of $\mathrm{G}(\mathbb{A})$. For any $\phi \in \pi$, the *degenerate Whittaker-Fourier coefficient* of ϕ attached to a Whittaker pair (s,u) is defined to be

$$\mathcal{F}_{s,u}(\phi)(g) := \int_{[N_{s,u}]} \phi(ng)\psi_u^{-1}(n)\,\mathrm{d}n\,. \tag{2.1}$$

If (s,u) is a neutral pair, then $\mathcal{F}_{s,u}(\phi)$ is also called a *generalized Whittaker-Fourier coefficient* of ϕ. Let

$$\mathcal{F}_{s,u}(\pi) = \{\mathcal{F}_{s,u}(\phi) \mid \phi \in \pi\}\,.$$

The *wave-front set* $\mathfrak{n}(\pi)$ of π is defined to be the set of nilpotent orbits $\mathcal{O}$ such that $\mathcal{F}_{s,u}(\pi)$ is non-zero for some neutral pair (s,u) with $u \in \mathcal{O}$. Note that if $\mathcal{F}_{s,u}(\pi)$ is non-zero for some neutral pair (s,u) with $u \in \mathcal{O}$, then it is non-zero for any such neutral pair (s,u), since the non-vanishing property of such Fourier coefficients does not depend on the choices of representatives of $\mathcal{O}$. Moreover, we denote by $\mathfrak{n}^m(\pi)$ the set of maximal elements in $\mathfrak{n}(\pi)$ under the natural partial ordering of nilpotent orbits (i.e., $\mathcal{O}_1 \leq \mathcal{O}_2$ if $\mathcal{O}_1 \subset \overline{\mathcal{O}_2}$, the Zariski closure of $\mathcal{O}_2$).

We recall [GGS17a, Theorem C] as follows.

Proposition 2.1 (Theorem C, [GGS17a]). *Let π be an automorphic representation of $\mathrm{G}(\mathbb{A})$. Given a neutral pair (s,u) and a Whittaker pair (s',u), if $\mathcal{F}_{s',u}(\pi)$ is non-zero, then $\mathcal{F}_{s,u}(\pi)$ is non-zero.*

In the rest of the paper, we consider the case of $\mathrm{G} = \mathrm{GL}_n$. In this case, nilpotent orbits are in one-to-one correspondence with partitions of n. Here we recall some notions about partitions, and the basic references are [CM93] and [G06]. For convenience, we define a partition of n to be a tuple $\mu = [q_1\, q_2 \cdots q_m]$ of non-negative integers such that $\sum_{i=1}^m q_i = n$. For any partition $\mu = [q_1\, q_2 \cdots q_m]$, we may reorder it as $[q'_1 \cdots q'_m]$ such that $q'_1 \geq q'_2 \geq \cdots \geq q'_m$, which we denote by μ^{N}. We identify two partitions $\mu = [q_1\, q_2 \cdots q_m]$ and $\nu = [p_1\, p_2 \cdots p_l]$ of n if the non-zero parts of μ^{N} and ν^{N} agree. There is a partial ordering structure on the set of partitions of n. Given two partitions $\mu = [q_1\, q_2 \cdots q_m]$ and $\nu = [p_1\, p_2 \cdots p_m]$ of n (adding zeros if necessary), we say that $\mu \geq \nu$ if $\mu^{\mathrm{N}} = [q'_1 \cdots q'_m] \geq \nu^{\mathrm{N}} = [p'_1 \cdots p'_m]$ under the dominance ordering, i.e. $\sum_{i=1}^k q'_i \geq \sum_{i=1}^k p'_i$ for all $1 \leq k \leq m$.

We also define some operations on the set of partitions. For any partition $\mu = [q_1\, q_2 \cdots q_m]$ of n, one defines its transpose $\mu^t = [q_1\, q_2 \cdots q_m]^t$ to be $[q_1^t \cdots q_m^t]$, where $q_i^t = \sharp\{j \mid q_j \geq i\}$. Moreover, for two partitions $\mu = [q_1\, q_2 \cdots q_m]$ and $\nu = [p_1\, p_2 \cdots p_m]$ of n (adding zeros if necessary), we define

$$\mu + \nu = [(q'_1 + p'_1)\,(q'_2 + p'_2) \cdots (q'_m + p'_m)]$$

with $\mu^{\mathrm{N}} = [q'_1 \cdots q'_m]$ and $\nu^{\mathrm{N}} = [p'_1 \cdots p'_m]$.

Given a partition μ of n, by a Fourier coefficient of an automorphic form ϕ attached to μ, we mean a generalized Whittaker-Fourier coefficient of ϕ attached to the corresponding nilpotent orbit. Given an automorphic representation π of $\mathrm{GL}_n(\mathbb{A})$, let $\mathfrak{p}(\pi)$ and $\mathfrak{p}^m(\pi)$ be the set of partitions parametrizing nilpotent orbits in $\mathfrak{n}(\pi)$ and $\mathfrak{n}^m(\pi)$, respectively.

For latter use, we introduce a particular degenerated Whittaker-Fourier coefficients in the case of GL_n. Let $\lambda = [p_1\, p_2 \cdots p_m]$ be a partition of n. Let $E_{i,j}(x)$ be the matrix with (i,j)-entry being x and zeros elsewhere. Let

$$u_\lambda = \frac{1}{2n}\left(\sum_{i=1}^m \sum_{j=1}^{p_i-1} E_{\sum_{k=1}^{i-1} p_k+j+1, \sum_{k=1}^{i-1} p_k+j}(1)\right) \tag{2.2}$$

be a representative of the nilpotent orbit corresponding to λ. Here we follow the convention that the summation $\sum_{k=1}^{i-1} p_k$ vanishes if $i=1$. We also let s_n be the semi-simple element

$$\mathrm{diag}\,(n-1, n-3, \ldots, 1-n)\,.$$

Then (s_n, u_λ) is a Whittaker pair. Here the multiplication by $\frac{1}{2n}$ in u_λ is due to the difference between the Killing form and the trace form for general linear Lie algebras. For an automorphic form ϕ on $\mathrm{GL}_n(\mathbb{A})$, we will consider the degenerate Whittaker-Fourier coefficient

$$\mathcal{F}_{s_n,u_\lambda}(\phi)(g) := \int_{[N_{s_n,u_\lambda}]} \phi(ng)\psi_{u_\lambda}^{-1}(n)\,\mathrm{d}n\,.$$

We note that this is the λ-*semi-Whittaker coefficient of* ϕ defined in [C18].

2.2. **A criterion on determining local top orbits.** The generalized and degenerate Whittaker-Fourier coefficients also have their local analogues, which are certain local models. Let k be a local field. For an irreducible smooth admissible representation π of $\mathrm{GL}_n(k)$, we say that π has a non-zero *degenerate Whittaker model* attached to a Whittaker pair (s,u) if

$$\mathrm{Hom}_{N_{s,u}(k)}(\pi,\psi_u)\neq 0\,. \tag{2.3}$$

Here $N_{s,u}$ and ψ_u have the same definitions as in the global setting in §2.1, and we use the same convention for admissible representations as in [GGS17a, §1.1]. Moreover, if (s,u) is a neutral pair, then we say that π has a non-zero *generalized Whittaker model* attached to (s,u) in case that (2.3) holds. We also have the analogous definitions for $\mathfrak{n}(\pi)$, $\mathfrak{n}^{\mathfrak{m}}(\pi)$, $\mathfrak{p}(\pi)$, and $\mathfrak{p}^{\mathfrak{m}}(\pi)$, respectively.

We have the following criterion for $\mathfrak{p}^{\mathfrak{m}}(\pi)$:

Proposition 2.2. *Let $\mu=[p_1p_2\cdots p_m]$ be a partition of n. Let π be an irreducible admissible representation of $\mathrm{GL}_n(k)$, then the following are equivalent:*

(1) $\mathfrak{p}^{\mathfrak{m}}(\pi)=\{\mu\}$;
(2) *the representation π has a non-zero degenerate Whittaker model attached to the Whittaker pair (s_n,u_μ), and has no non-zero degenerate Whittaker model attached to the Whittaker pair (s_n,u_λ) for any partition λ of n which is bigger than or not related to μ.*

Proof. The criterion is a special case of the general results in [MW87] and [V14]. □

Remark 2.3. *The more recent works of Gomez, Gourevitch and Sahi* ([GGS17a, GGS17b]) *generalize the works in* [MW87] *and* [V14]*, and hence also give the above local criterion. In* [C18]*, Cai also suggested a global criterion (see* [C18, Proposition 5.3]*). However, we found that there is a gap in the argument for* [C18, Lemma 5.7]*, where the non-trivial orbit in the expansion of the inner integral can not always give the Fourier coefficient for the claimed larger partition. As pointed out to us by Cai, this global criterion can be deduced from the global results* [GGS17a, Theorem C] *and* [GGS17b, Theorem 8.0.3].

3. Certain automorphic representations of GL_n

3.1. **Structure of discrete spectrum for GL_n.** It was a conjecture of Jacquet ([J84]) and then a theorem of Mœglin and Waldspurger ([MW89]) that an irreducible automorphic representation π of $\mathrm{GL}_n(\mathbb{A})$ occurring in the discrete spectrum of the space of all square-integrable automorphic forms on $\mathrm{GL}_n(\mathbb{A})$ is parameterized by a pair (τ,b) with τ being an irreducible unitary cuspidal automorphic representation of $\mathrm{GL}_a(\mathbb{A})$ such that $n=ab$. In particular, we have $b=1$ if π is cuspidal.

For any partition $[p_1p_2\cdots p_m]$ of n, we denote by $P_{p_1,\ldots,p_m}$ the standard parabolic subgroup of GL_n with Levi part isomorphic to $\mathrm{GL}_{p_1}\times\cdots\times\mathrm{GL}_{p_m}$. To describe the discrete spectrum more precisely, we take $n=ab$ with $b>1$, then the standard parabolic subgroup $P_{a^b}=M_{a^b}N_{a^b}$ of GL_{ab} has Levi part M_{a^b} isomorphic to $\mathrm{GL}_a^{\times b}=\mathrm{GL}_a\times\cdots\times\mathrm{GL}_a$ (b copies). Following the theory of Langlands (see [L76] and [MW95]), there is an Eisenstein series $E(\phi_{\tau^{\otimes b}},\overline{s},g)$ attached to the cuspidal datum $(P_{a^b},\tau^{\otimes b})$ of $\mathrm{GL}_{ab}(\mathbb{A})$, where $\overline{s}=(s_1,\cdots,s_b)\in\mathbb{C}^b$. This Eisenstein series converges absolutely for the real part of $\overline{s}$ belonging to a certain cone and has a meromorphic continuation to the whole complex space $\mathbb{C}^b$. Moreover, it has an iterated residue at

$$\Lambda_b := (\frac{b-1}{2}, \frac{b-3}{2}, \ldots, \frac{1-b}{2}),$$

which can be written as

$$E_{-1}(\phi_{\tau^{\otimes b}}, g) = \lim_{\overline{s}\to\Lambda_b} \prod_{i=1}^{b-1}(s_i - s_{i+1} - 1)E(\phi_{\tau^{\otimes b}}, \overline{s}, g)\,. \tag{3.1}$$

It is square-integrable, and hence belongs to the discrete spectrum of the space of all square-integrable automorphic forms of $\mathrm{GL}_{ab}(\mathbb{A})$. Denote by $\Delta(\tau, b)$ the automorphic representation generated by all the residues $E_{-1}(\phi_{\tau^{\otimes b}}, g)$. Mœglin and Waldspurger (see [MW89]) proved that $\Delta(\tau, b)$ is irreducible, and any irreducible non-cuspidal automorphic representation occurring in the discrete spectrum of the general linear group $\mathrm{GL}_n(\mathbb{A})$ is of this form for some $a \geq 1$ and $b > 1$ such that $n = ab$, and has multiplicity one. Moreover, the representation $\Delta(\tau, b)$ can be regarded as the *unique* irreducible quotient of the induced representation

$$\pi_{\tau,b} := \mathrm{Ind}_{P_{a^b}(\mathbb{A})}^{\mathrm{GL}_n(\mathbb{A})} \tau|\cdot|^{\frac{b-1}{2}} \otimes \tau|\cdot|^{\frac{b-3}{2}} \otimes \cdots \otimes \tau|\cdot|^{\frac{1-b}{2}}\,.$$

Here the notation $|\cdot|$ stands for $|\det(\cdot)|$ for short.

Let $\ell_b = \lceil \frac{b}{2} \rceil$ and $k_b = \lfloor \frac{b}{2} \rfloor$. Define $\iota_{\tau,b}$ to be the evaluation map

$$\begin{aligned} t_b^{(\ell_b)} &\mapsto \frac{b-1}{2}\,, \\ t_b^{(\ell_b-1)} &\mapsto \frac{b-3}{2}\,, \\ &\cdots \\ h_b^{(k_b-1)} &\mapsto \frac{3-b}{2}\,, \\ h_b^{(k_b)} &\mapsto \frac{1-b}{2}\,, \end{aligned}$$

on a set of parameters $\{t_b^{(\ell_b)}, \ldots, t_b^{(1)}, h_b^{(1)}, \ldots, h_b^{(k_b)}\}$ with b entries. Let

$$\pi'_{\tau,b} := \mathrm{Ind}_{P_{a^b}(\mathbb{A})}^{\mathrm{GL}_n(\mathbb{A})} \tau|\cdot|^{t_b^{(\ell_b)}} \otimes \cdots \otimes \tau|\cdot|^{t_b^{(1)}} \otimes \tau|\cdot|^{h_b^{(1)}} \otimes \cdots \otimes \tau|\cdot|^{h_b^{(k_b)}}\,. \tag{3.2}$$

Then the map $\iota_{\tau,b}$ naturally induces a map from $\pi'_{\tau,b}$ to $\pi_{\tau,b}$, which we still denote by $\iota_{\tau,b}$.

3.2. **Certain automorphic representations of** GL_n**.** Write $n = \sum_{i=1}^r a_i b_i$, where a_i and b_i are both positive integers. For $1 \leq i \leq r$, let τ_i be an irreducible unitary cuspidal automorphic representation of $\mathrm{GL}_{a_i}(\mathbb{A})$, and $\Delta(\tau_i, b_i)$ be the corresponding representation in the discrete spectrum of $\mathrm{GL}_{a_i b_i}(\mathbb{A})$. Let $P = MN$ be a parabolic subgroup of GL_n with Levi subgroup M isomorphic to $\mathrm{GL}_{a_1b_1} \times \cdots \times \mathrm{GL}_{a_rb_r}$. Consider the induced representation

$$\Pi_{\underline{s}} = \mathrm{Ind}_{P(\mathbb{A})}^{\mathrm{GL}_n(\mathbb{A})} \Delta(\tau_1, b_1)|\cdot|^{s_1} \otimes \cdots \otimes \Delta(\tau_r, b_r)|\cdot|^{s_r},$$

where $\underline{s} = (s_1, \cdots, s_r) \in \mathbb{C}^r$. Sometimes we will also denote such induced representations by

$$\Delta(\tau_1, b_1)|\cdot|^{s_1} \times \cdots \times \Delta(\tau_r, b_r)|\cdot|^{s_r}$$

for short.

For

$$\overline{s} = (s_1^{(1)}, \ldots, s_{b_1}^{(1)}, s_1^{(2)}, \ldots, s_{b_2}^{(2)}, \ldots, s_1^{(r)}, \ldots, s_{b_r}^{(r)}) \in \mathbb{C}^{b_1+\cdots+b_r},$$

we define the Eisenstein series attached to the cuspidal datum

$$(P_{a_1^{b_1},\ldots,a_r^{b_r}}, \tau_1^{\otimes b_1} \otimes \cdots \otimes \tau_r^{\otimes b_r})$$

to be the meromorphic continuation of the series

$$E(\phi_{\tau_1^{\otimes b_1},\ldots,\tau_r^{\otimes b_r}}, \overline{s}, g) = \sum_{\gamma \in P_{a_1^{b_1},\ldots,a_r^{b_r}}(F)\backslash \mathrm{GL}_n(F)} \phi_{\tau_1^{\otimes b_1},\ldots,\tau_r^{\otimes b_r},\overline{s}}(\gamma g), \tag{3.3}$$

where $\phi_{\tau_1^{\otimes b_1},\ldots,\tau_r^{\otimes b_r},\overline{s}}$ is a holomorphic section in

$$\mathrm{Ind}_{P_{a_1^{b_1},\ldots,a_r^{b_r}}(\mathbb{A})}^{\mathrm{GL}_n(\mathbb{A})} \tau_1|\cdot|^{s_1^{(1)}} \otimes \cdots \otimes \tau_1|\cdot|^{s_{b_1}^{(1)}} \otimes \tau_2|\cdot|^{s_1^{(2)}} \otimes \cdots \otimes \tau_2|\cdot|^{s_{b_2}^{(2)}} \otimes \cdots \otimes \tau_r|\cdot|^{s_1^{(r)}} \otimes \cdots \otimes \tau_r|\cdot|^{s_{b_r}^{(r)}}.$$

In the latter parts, we always denote vectors in $\mathbb{C}^r$ by $\underline{s}$, and vectors in $\mathbb{C}^{b_1+\cdots+b_r}$ by $\overline{s}$. Recall that we have defined the notation Λ_b in the definition of $\Delta(\tau, b)$. By Langlands' theory of Eisenstein series (see [L76, L79a]), for $\mathrm{Re}(s_i - s_j) \gg 0$ $(1 \leq i < j \leq r)$, the automorphic representation $\Pi_{\underline{s}}$ can be realized via the residues of the Eisenstein series $E(\phi_{\tau_1^{\otimes b_1},\ldots,\tau_r^{\otimes b_r}}, \overline{s}, g)$ at

$$\Lambda_{\underline{s}} = (\Lambda_{b_1} + s_1, \ldots, \Lambda_{b_r} + s_r) \in \mathbb{C}^{b_1+\cdots+b_r}.$$

If $\underline{s} = (0, \ldots, 0)$, the automorphic representation Π can be realized via the residues of the Eisenstein series $E(\phi_{\tau_1^{\otimes b_1},\ldots,\tau_r^{\otimes b_r}}, \overline{s}, g)$ at

$$\Lambda = (\Lambda_{b_1}, \ldots, \Lambda_{b_r}) \in \mathbb{C}^{b_1+\cdots+b_r}.$$

In sense of [L79b, Section 2] (see also [A13, Section 1.3]), the representation Π is *isobaric*.

4. Non-vanishing for the top orbit

In this section, we show that the representations Π and $\Pi_{\underline{s}}$ have non-zero generalized Whittaker-Fourier coefficients attached to the partition

$$[a_1^{b_1}] + \cdots + [a_r^{b_r}],$$

under the assumptions in Theorem 1.3 and Theorem 1.1, respectively.

4.1. The case $\underline{s} = (0, \ldots, 0)$. We consider the isobaric automorphic representation Π at first. This case includes all main ingredients of our approach, and also some additional treatments which can be avoided for $\Pi_{\underline{s}}$ by the assumption $\mathrm{Re}(s_i - s_j) \gg 0$ $(1 \leq i < j \leq r)$.

The main idea is to show that Π is a subquotient of a representation induced from certain parabolic subgroup and generic data. Before carrying out the argument, we first explain the steps using an explicit example.

Example 4.1. *We consider the representation*

$$\Pi = \Delta(\tau_1, 3) \times \Delta(\tau_2, 4) \times \Delta(\tau_3, 5)\,,$$

where τ_i is a unitary cuspidal representation of $\mathrm{GL}_{a_i}(\mathbb{A})$, $1 \le i \le 3$. Note that $\Delta(\tau_1, 3)$ is the unique irreducible quotient of

$$\pi_{\tau_1,3} = \tau_1|\cdot|^1 \times \tau_1|\cdot|^0 \times \tau_1|\cdot|^{-1}\,;$$

$\Delta(\tau_2, 4)$ is the unique irreducible quotient of

$$\pi_{\tau_2,4} = \tau_2|\cdot|^{\frac{3}{2}} \times \tau_2|\cdot|^{\frac{1}{2}} \times \tau_2|\cdot|^{-\frac{1}{2}} \times \tau_2|\cdot|^{-\frac{3}{2}}\,;$$

and $\Delta(\tau_3, 5)$ is the unique irreducible quotient of

$$\pi_{\tau_3,5} = \tau_3|\cdot|^2 \times \tau_3|\cdot|^1 \times \tau_3|\cdot|^0 \times \tau_3|\cdot|^{-1} \times \tau_3|\cdot|^{-2}\,.$$

Then Π is also an irreducible quotient of

$$\pi_{\tau_3,5} \times \pi_{\tau_2,4} \times \pi_{\tau_1,3}\,.$$

We put the above inducing data in a table as follows:

$\tau_3\|\cdot\|^{-2}$	$\tau_3\|\cdot\|^{-1}$	$\tau_3\|\cdot\|^{0}$	$\tau_3\|\cdot\|^{1}$	$\tau_3\|\cdot\|^{2}$
$\tau_2\|\cdot\|^{-\frac{3}{2}}$	$\tau_2\|\cdot\|^{-\frac{1}{2}}$	$\tau_2\|\cdot\|^{\frac{1}{2}}$	$\tau_2\|\cdot\|^{\frac{3}{2}}$	
	$\tau_1\|\cdot\|^{-1}$	$\tau_1\|\cdot\|^{0}$	$\tau_1\|\cdot\|^{1}$	

Here the rules of the placement are:

(1) *We put the inducing data from $\Delta(\tau_i, b_i)$ with largest b_i into the first row (here it is $\Delta(\tau_3, 5)$), and so on with b_i's in non-increasing order.*
(2) *For $\Delta(\tau_i, b_i)$'s with b_i being odd, (here they are $\Delta(\tau_1, 3)$ and $\Delta(\tau_3, 5)$), we put the inducing components $\tau_i|\cdot|^0$ into the same column, and for $\Delta(\tau_i, b_i)$'s with b_i being even, (here it is $\Delta(\tau_2, 4)$), we also put $\tau_i|\cdot|^{\frac{1}{2}}$ into the same column as those $\tau_j|\cdot|^0$.*
(3) *Then we put the other inducing components belonging to the same $\Delta(\tau_i, b_i)$, with exponents increasing, into the corresponding rows.*

The placement of the data is not unique, for example, for the inducing data of $\Delta(\tau_2, 4)$, we can also put $\tau_2|\cdot|^{-\frac{1}{2}}$ into the center column of the above table. Now we rearrange all the inducing data by columns of the above table from the left to the right, i.e., let

$$\begin{aligned}
\eta_1 &= \tau_3|\cdot|^{-2} \times \tau_2|\cdot|^{-\frac{3}{2}}\,,\\
\eta_2 &= \tau_3|\cdot|^{-1} \times \tau_2|\cdot|^{-\frac{1}{2}} \times \tau_1|\cdot|^{-1}\,,\\
\eta_3 &= \tau_3|\cdot|^{0} \times \tau_2|\cdot|^{\frac{1}{2}} \times \tau_1|\cdot|^{0}\,,\\
\eta_4 &= \tau_3|\cdot|^{1} \times \tau_2|\cdot|^{\frac{3}{2}} \times \tau_1|\cdot|^{1}\,,\\
\eta_5 &= \tau_3|\cdot|^{2}\,,
\end{aligned}$$

then η_i's are irreducible generic representations of certain general linear groups. It follows that, the representation Π, realizing as residues of the Eisenstein series (3.3) at $\overline{s} = \Lambda$, has a non-zero constant term with respect to the parabolic subgroup whose Levi subgroup is

$$\mathrm{GL}_{a_2+a_3}(\mathbb{A}) \times \mathrm{GL}^3_{a_1+a_2+a_3}(\mathbb{A}) \times \mathrm{GL}_{a_3}(\mathbb{A})\,.$$

Moreover, the constant term gives a non-zero vector in the representation

$$\eta_1 \times \eta_2 \times \eta_3 \times \eta_4 \times \eta_5 .$$

One can see the proof of Proposition 4.2 for more details, note that we need to use Assumption 1.2. It follows that Π *has a non-zero degenerate Whittaker-Fourier coefficient attached to the partition* $[(a_2+a_3)(a_1+a_2+a_3)^3a_3] = [(a_1+a_2+a_3)^3(a_2+a_3)a_3]$ *(see §2.1 for the definition). By Proposition 2.1,* Π *also has a non-zero generalized Whittaker-Fourier coefficient attached to the partition* $[(a_1+a_2+a_3)^3(a_2+a_3)a_3]$*, which is exactly* $[a_1^3]+[a_2^4]+[a_3^5]$.

Now we carry out the general argument and prove the following proposition.

Proposition 4.2. *Under Assumption 1.2, the representation* Π *has a non-zero generalized Whittaker-Fourier coefficient attached to the partition* $\mu = [a_1]^{b_1} + \cdots + [a_r]^{b_r}$.

Proof. Without loss of generality (note that Π is irreducible), we may reorder $\Delta(\tau_1, b_1) \times \cdots \times \Delta(\tau_r, b_r)$ such that $b_1 \geq \cdots \geq b_r$.

Recall from §3.1 that given an integer b, we have defined $\ell_b = \lceil \frac{b}{2} \rceil$ and $k_b = \lfloor \frac{b}{2} \rfloor$. For $1 \leq i \leq r$, we form parameters

$$\left(t_{b_i}^{(\ell_{b_1})}, \ldots, t_{b_i}^{(1)}, h_{b_i}^{(1)}, \ldots, h_{b_i}^{(k_{b_1})}\right)$$

with b_1 entries by adding zeros from the front if $\ell_{b_i} < \ell_{b_1}$, and adding zeros from the end if $k_{b_i} < k_{b_1}$ (note that $b_1 \geq b_j$ for $2 \leq j \leq r$). In other words, one has $t_{b_i}^{(j)} = 0$ if $j > \ell_{b_i}$, and $h_{b_i}^{(j)} = 0$ if $j > k_{b_i}$. Then, for $1 \leq j \leq k_{b_1}$, we construct representations

$$\sigma_j^1 = \tau_1|\cdot|^{h_{b_1}^{(j)}} \times \cdots \times \tau_r|\cdot|^{h_{b_r}^{(j)}} ,$$

where we omit the $\tau_i|\cdot|^{h_{b_i}^{(j)}}$-term if $j > k_{b_i}$ $(1 \leq i \leq r)$. Similarly, for $1 \leq q \leq \ell_{b_1}$, we construct representations

$$\rho_q^1 = \tau_1|\cdot|^{t_{b_1}^{(q)}} \times \cdots \times \tau_r|\cdot|^{t_{b_r}^{(q)}} ,$$

where we omit the $\tau_i|\cdot|^{t_{b_i}^{(q)}}$-term if $q > \ell_{b_i}$ $(1 \leq i \leq r)$.

For $1 \leq j \leq k_{b_1}$, we assume that σ_j^1 is a representation of $\mathrm{GL}_{n_j}(\mathbb{A})$, and for $1 \leq q \leq \ell_{b_1}$, we assume that ρ_q^1 is a representation of $\mathrm{GL}_{m_q}(\mathbb{A})$. Note that the n_j's and m_q's are among the integers

$$\left\{ \sum_{i=1}^{r} \delta_i a_i \mid \delta_i = 0 \text{ or } 1 \right\} .$$

Let $t_1 = n_{k_{b_1}}$, $t_2 = n_{k_{b_1}-1}$, …, $t_{k_{b_1}} = n_1$, $t_{k_{b_1}+1} = m_1$, $t_{k_{b_1}+2} = m_2$, …, $t_{b_1} = m_{\ell_{b_1}}$, then the partition $[t_1 t_2 \cdots t_{b_1}]$ is just the partition $[a_1^{b_1}] + \cdots + [a_r^{b_r}]$ (see the definitions and conventions in §2.1). We rename the representations $\{\sigma_j^1, 1 \leq j \leq k_{b_1}, \rho_q^1, 1 \leq q \leq \ell_{b_1}\}$ as $\{\varepsilon_1, \ldots, \varepsilon_{b_1}\}$, where $\varepsilon_1 = \sigma_{k_{b_1}}^1$, $\varepsilon_2 = \sigma_{k_{b_1}-1}^1$, …, $\varepsilon_{k_{b_1}} = \sigma_1^1$, $\varepsilon_{k_{b_1}+1} = \rho_1^1$, $\varepsilon_{k_{b_1}+2} = \rho_2^1$, …, $\varepsilon_{b_1} = \rho_{\ell_{b_1}}^1$. Here each ε_i is a representation of $\mathrm{GL}_{t_i}(\mathbb{A})$ $(1 \leq i \leq b_1)$.

Consider the induced representation

$$\mathrm{Ind}_{Q(\mathbb{A})}^{\mathrm{GL}_n(\mathbb{A})} \varepsilon_1 \otimes \cdots \otimes \varepsilon_{b_1} , \tag{4.1}$$

where $Q = LV$ is the parabolic subgroup of GL_n with Levi subgroup $L \subset Q$ isomorphic to $\mathrm{GL}_{t_1} \times \cdots \times \mathrm{GL}_{t_{b_1}}$. Recall that we have defined maps ι_{τ_i,b_i} in §3.1. We denote by

$$\mathrm{Ind}_{Q(\mathbb{A})}^{\mathrm{GL}_n(\mathbb{A})} \eta_1 \otimes \cdots \otimes \eta_{b_1}$$

the image of (4.1) under the set of maps

$$\{\iota_{\tau_1,b_1}, \ldots, \iota_{\tau_r,b_r}\}.$$

We suppose Assumption 1.2 in the rest of the proof. We need two lemmas to finish the proof. The first one is:

Lemma 4.3. *Each representation η_i is an irreducible generic representation of* $\mathrm{GL}_{t_i}(\mathbb{A})$.

Proof of Lemma 4.3. In fact, by the construction above, each η_i is of the form

$$\tau_{\kappa_1}|\cdot|^{e_1} \times \cdots \times \tau_{\kappa_\alpha}|\cdot|^{e_\alpha} \quad (1 \leq \alpha \leq r),$$

with integers or half-integers e_i's such that $e_i - e_j = 0, \pm\frac{1}{2}$ for $1 \leq i < j \leq \alpha$. Consider the Eisenstein series corresponding to an induced representation

$$\rho_1|\cdot|^{\nu_1} \times \cdots \times \rho_k|\cdot|^{\nu_k} \quad (\nu_i \in \mathbb{C}),$$

with ρ_i's being irreducible unitary cuspidal automorphic representations. The calculation of constant term (see, for example, [Sh10, Chapter 6]) implies that the poles of the Eisenstein series are given by the ratio of Rankin-Selberg L-functions

$$\prod_{1 \leq i < j \leq k} \frac{L(\nu_i - \nu_j, \rho_i \times \widetilde{\rho}_j)}{L(1 + \nu_i - \nu_j, \rho_i \times \widetilde{\rho}_j)}.$$

By [MW89, Appendice, Proposition and Corollaire], $L(\nu_i - \nu_j, \rho_i \times \widetilde{\rho}_j)$ has only simple poles at $\nu_i - \nu_j = 0, 1$, and by [JS76, JS81, Sh80, Sh81] (see also [Cog07, Theorem 4.3]), $L(\nu_i - \nu_j, \rho_i \times \widetilde{\rho}_j)$ is non-vanishing for $\mathrm{Re}(\nu_i - \nu_j) \geq 1$ or $\mathrm{Re}(\nu_i - \nu_j) \leq 0$. It follows that the Eisenstein series is non-zero holomorphic at the point $(e_1, \ldots, e_\alpha)$ (recall that we have $L(\frac{1}{2}, \tau_i \times \widetilde{\tau}_j) \neq 0$ by Assumption 1.2), and hence each η_i is irreducible generic. This proves Lemma 4.3. $\square$

As in (2.2), let

$$u_\mu = \frac{1}{2n} \sum_{i=1}^{b_1} \sum_{j=1}^{t_i - 1} E_{\sum_{k=1}^{i-1} t_k + j + 1, \sum_{k=1}^{i-1} t_k + j}(1)$$

be a representative of the nilpotent orbit $\mathcal{O}$ corresponding to the partition $\mu = [t_1 t_2 \ldots t_{b_1}]$, and let s be the semi-simple element

$$\mathrm{diag}\,(t_1 - 1, \ldots, 1 - t_1, t_2 - 1, \ldots, 1 - t_2, \ldots, t_{b_1} - 1, \ldots, 1 - t_{b_1}).$$

It is easy to see that s is a neutral element for u, and hence (s, u) is a neutral pair. Recall that we have defined another semi-simple element

$$s_n = \mathrm{diag}\,(n - 1, n - 3, \ldots, 1 - n)$$

in §2.1, and (s_n, u_μ) is also a Whittaker pair.

Take $0 \neq f \in \Pi$, and consider the degenerate Fourier coefficient $\mathcal{F}_{s_n,u_\mu}(f)$ attached to the Whittaker pair (s_n, u_μ). It is easy to see that $\mathcal{F}_{s_n,u_\mu}(f)$ is the constant term integral along the parabolic subgroup Q composed with the Whittaker-Fourier coefficient along the Levi subgroup L. The following is the second lemma we need:

Lemma 4.4. *The constant term of f along Q gives us a non-zero vector in the irreducible generic representation $\eta_1 \otimes \cdots \otimes \eta_{b_1}$ of $L(\mathbb{A})$.*

Proof of Lemma 4.4. Using the notation in §3.2, we may write

$$f(g) = \lim_{\overline{s}\to\Lambda} \prod_{j=1}^{r} \prod_{i=1}^{b_i-1} (s_i^{(j)} - s_{i+1}^{(j)} - 1) E(\phi_{\tau_1^{\otimes b_1},\ldots,\tau_r^{\otimes b_r}}, \overline{s}, g)\,. \tag{4.2}$$

Since the multi-residue operator $\lim_{\overline{s}\to\Lambda}\prod_{j=1}^{r}\prod_{i=1}^{b_i-1}(s_i^{(j)} - s_{i+1}^{(j)} - 1)$ and the constant term operator are interchangeable, we consider the constant term $E_Q(\phi_{\tau_1^{\otimes b_1},\ldots,\tau_r^{\otimes b_r}}, \overline{s}, g)$ of the Eisenstein series (3.3) first. For simplicity we denote $P_0 = P_{a_1^{b_1},\ldots,a_r^{b_r}}$, and denote its Levi subgroup by M_0. By [MW95, Proposition II.1.7 (ii)], the constant term is given by

$$E_Q(\phi_{\tau_1^{\otimes b_1},\ldots,\tau_r^{\otimes b_r}}, \overline{s}, g) = \sum_{\omega^{-1}\in W^c(P_0,Q)} E^Q(M(\omega,\overline{s})\phi_{\tau_1^{\otimes b_1},\ldots,\tau_r^{\otimes b_r}}, \omega\overline{s}, g)\,. \tag{4.3}$$

Recall that

$$\overline{s} = (s_1^{(1)},\ldots,s_{b_1}^{(1)}, s_1^{(2)},\ldots,s_{b_2}^{(2)},\ldots,s_1^{(r)},\ldots,s_{b_r}^{(r)}) \in \mathbb{C}^{b_1+\cdots+b_r}\,,$$

Here

(1) the set $W^c(P_0,Q)$ consists of Weyl elements $\omega^{-1} \in W(\mathrm{GL}_n)$ (the Weyl group of GL_n) with the properties that $\omega(\alpha) > 0$ for any $\alpha \in \Phi^+(M_0)$ (the positive roots in M_0), $\omega^{-1}(\beta) > 0$ for any $\beta \in \Phi^+(L)$ (the positive roots in L), and $\omega M_0 \omega^{-1} \subset L$;

(2) the Eisenstein series $E^Q(\phi_{\tau_1^{\otimes b_1},\ldots,\tau_r^{\otimes b_r}}, \overline{s}, g)$ is defined as the meromorphic continuation of the series

$$E^Q(\phi_{\tau_1^{\otimes b_1},\ldots,\tau_r^{\otimes b_r}}, \overline{s}, g) = \sum_{\gamma\in(P_0\cap L)(F)\backslash L(F)} \phi_{\tau_1^{\otimes b_1},\ldots,\tau_r^{\otimes b_r},\overline{s}}(\gamma g)\,.$$

Note that by construction, elements in $W^c(P_0,Q)$ can be viewed as certain elements in the permutation group $\mathcal{S}_{b_1+\cdots+b_r}$, which permute the cuspidal support

$$\tau_1|\cdot|^{\frac{b_1-1}{2}} \otimes\cdots\otimes \tau_1|\cdot|^{\frac{1-b_1}{2}} \otimes \tau_2|\cdot|^{\frac{b_2-1}{2}} \otimes\cdots\otimes \tau_2|\cdot|^{\frac{1-b_2}{2}} \otimes\cdots\otimes \tau_r|\cdot|^{\frac{b_r-1}{2}} \otimes\cdots\otimes \tau_r|\cdot|^{\frac{1-b_r}{2}}\,. \tag{4.4}$$

Let ω_0 be an element in $\mathcal{S}_{b_1+\cdots+b_r}$ which permutes the blocks in (4.4) to those corresponding to $\eta_1 \otimes\cdots\otimes \eta_{b_1}$. It is clear that $\omega_0^{-1} \in W^c(P_0,Q)$. We consider the summand

$$E^Q(M(\omega_0,\overline{s})\phi_{\tau_1^{\otimes b_1},\ldots,\tau_r^{\otimes b_r}}, \omega_0\overline{s}, g)$$

in the constant term (4.3). The occurrence of this summand (which can not be canceled) is guaranteed by Part (i) of Assumption 1.2. We are going to show that

$$\lim_{\overline{s}\to\Lambda} \prod_{j=1}^{r} \prod_{i=1}^{b_i-1} (s_i^{(j)} - s_{i+1}^{(j)} - 1) M(\omega_0,\overline{s})$$

is non-zero holomorphic, and hence the Eisenstein series

$$E^Q\left(\lim_{\overline{s}\to\Lambda}\prod_{j=1}^{r}\prod_{i=1}^{b_i-1}(s_i^{(j)}-s_{i+1}^{(j)}-1)M(\omega_0,\overline{s})\phi_{\tau_1^{\otimes b_1},\ldots,\tau_r^{\otimes b_r}},\omega_0\overline{s},g\right)$$
$$=\lim_{\overline{s}\to\Lambda}\prod_{j=1}^{r}\prod_{i=1}^{b_i-1}(s_i^{(j)}-s_{i+1}^{(j)}-1)E^Q(M(\omega_0,\overline{s})\phi_{\tau_1^{\otimes b_1},\ldots,\tau_r^{\otimes b_r}},\omega_0\overline{s},g)$$

on $L(\mathbb{A})$ gives a non-zero vector in $\eta_1\otimes\cdots\otimes\eta_{b_1}$.

For simplicity, we denote

$$\tau_1|\cdot|^{s_1^{(1)}}\otimes\cdots\otimes\tau_1|\cdot|^{s_{b_1}^{(1)}}\otimes\tau_2|\cdot|^{s_1^{(2)}}\otimes\cdots\otimes\tau_2|\cdot|^{s_{b_2}^{(2)}}\otimes\cdots\otimes\tau_r|\cdot|^{s_1^{(r)}}\otimes\cdots\otimes\tau_r|\cdot|^{s_{b_r}^{(r)}} \tag{4.5}$$

by

$$\rho_1|\cdot|^{s_1}\otimes\rho_2|\cdot|^{s_2}\otimes\cdots\otimes\rho_{b_1+\cdots+b_r}|\cdot|^{s_{b_1+\cdots+b_r}}\,. \tag{4.6}$$

Define

$$\mathrm{inv}(\omega_0)=\{(i,j)\mid 1\leq i<j\leq b_1+\cdots+b_r,\ \omega_0(i)>\omega_0(j)\}\,.$$

For each local place v of F, we define

$$r_v(\omega_0,\overline{s})=\prod_{(i,j)\in\mathrm{inv}(\omega_0)}\frac{L(s_i-s_j,\rho_{i,v}\times\widetilde{\rho}_{j,v})}{L(1+s_i-s_j,\rho_{i,v}\times\widetilde{\rho}_{j,v})\varepsilon(s_i-s_j,\rho_{i,v}\times\widetilde{\rho}_{j,v},\psi_v)}\,,$$

then we define

$$r(\omega_0,\overline{s})=\prod_v r_v(\omega_0,\overline{s})=\prod_{(i,j)\in\mathrm{inv}(\omega_0)}\frac{L(s_i-s_j,\rho_i\times\widetilde{\rho}_j)}{L(1+s_i-s_j,\rho_i\times\widetilde{\rho}_j)\varepsilon(s_i-s_j,\rho_i\times\widetilde{\rho}_j,\psi)}\,.$$

Here $\varepsilon(s_i-s_j,\rho_{i,v}\times\widetilde{\rho}_{j,v},\psi_v)$ is the local ε-factor defined in [JPSS83]. We normalize the intertwining operator $M(\omega_0,\overline{s})$ by

$$N(\omega_0,\overline{s})=r(\omega_0,\overline{s})^{-1}M(\omega_0,\overline{s})\,.$$

Recall the following features between the orders of cuspidal blocks in (4.4) and those in $\eta_1\otimes\cdots\otimes\eta_{b_1}$:

- we have $b_1\geq b_2\geq\cdots\geq b_r$;
- in $\eta_1\otimes\cdots\otimes\eta_{b_1}$, the order of the cuspidal blocks in every single $\Delta(\tau_i,b_i)$ are totally reversed.

Then if s_i in (4.6) corresponds to $s_{\mathfrak{l}}^{(l)}$ in (4.5), then for $(i,j)\in\mathrm{inv}(\omega_0)$, we must have $s_j=s_{\mathfrak{k}}^{(k)}$ with $\mathfrak{k}\geq\mathfrak{l}$. It follows that for $\overline{s}=\Lambda$, one has $s_i-s_j>0$ for all $(i,j)\in\mathrm{inv}(\omega_0)$. Recall from Part (ii) of Assumption 1.2 that all τ_i are locally tempered, hence, by [MW89, §I.1], the normalized intertwining operator $N(\omega_0,\underline{s})$ is holomorphic non-zero at $\overline{s}=\Lambda$. Then it suffices to show that the limit

$$\lim_{\overline{s}\to\Lambda}\prod_{j=1}^{r}\prod_{i=1}^{b_i-1}(s_i^{(j)}-s_{i+1}^{(j)}-1)r(\omega_0,\overline{s}) \tag{4.7}$$

exists and is non-zero. Again regarding to the order of the cuspidal blocks in $\eta_1\otimes\cdots\otimes\eta_{b_1}$, all the pairs $(i,i+1)$ such that (s_i,s_{i+1}) in (4.6) corresponding to $(s_{\mathfrak{l}}^{(\mathfrak{k})},s_{\mathfrak{l}+1}^{(\mathfrak{k})})$ $(1\leq\mathfrak{k}\leq r, 1\leq\mathfrak{l}\leq b_{\mathfrak{k}}-1)$ in (4.5) lie in $\mathrm{inv}(\omega_0)$. Taking limit $\overline{s}\to\Lambda$, we have $s_{\mathfrak{l}}^{(\mathfrak{k})}-s_{\mathfrak{l}+1}^{(\mathfrak{k})}=1$

for $1 \leq \mathfrak{k} \leq r$ and $1 \leq \mathfrak{l} \leq b_{\mathfrak{k}} - 1$, which give poles in $r(\omega_0, \overline{s})$, and are canceled by the limit in (4.7). On the other hand, for other pairs $(i,j) \in \mathrm{inv}(\omega_0)$, when taking limit $\overline{s} \to \Lambda$, we have $s_i - s_j \geq \frac{1}{2}$, and if (s_i, s_j) corresponds to $(s_{\mathfrak{l}}^{(\mathfrak{k})}, s_{\mathfrak{t}}^{(\mathfrak{k})})$ for the $1 \leq \mathfrak{k} \leq r$ and any $1 \leq \mathfrak{l}, \mathfrak{t} \leq b_{\mathfrak{k}} - 1$, we also have $s_i - s_j \neq 1$. Then by Assumption 1.2 and the results on Rankin-Selberg L-functions we have used in the proof of Lemma 4.3, the other terms in $r(\omega_0, \overline{s})$ is non-zero holomorphic at $\overline{s} = \Lambda$. This shows that (4.7) is non-zero holomorphic, and hence finishes the proof of Lemma 4.4. $\square$

Granting Lemma 4.4, and recall that η_i is an irreducible generic representation of $\mathrm{GL}_{t_i}(\mathbb{A})$ for $1 \leq i \leq b_1$ by Lemma 4.3, we have $\mathcal{F}_{s_n,u_\mu}(f) \neq 0$. Moreover, we also have $\mathcal{F}_{s,u_\mu}(f) \neq 0$ by Proposition 2.1, this means that Π has a non-zero generalized Whittaker-Fourier coefficient attached to the partition μ.

This completes the proof of the proposition. $\square$

Remark 4.5. *We apply the general result in* [GGS17a] (*Proposition 2.1*) *to show that* Π *has a non-zero generalized Whittaker-Fourier coefficient attached to the partition* $\mu = [a_1^{b_1}] + \cdots + [a_r^{b_r}]$, *from the result that* $\mathcal{F}_{s_n,u_\mu}(f) \neq 0$ *for some* $f \in \Pi$. *Similar arguments have been used in* [JLX19] *in the study of certain twisted automorphic descent constructions and a reciprocal branching problem related to the global Gan-Gross-Prasad conjecture, and are expected to be applied in more general situations.*

4.2. **The case** $\mathrm{Re}(s_i - s_j) \gg 0$ $(1 \leq i < j \leq r)$**.** Now we consider $\Pi_{\underline{s}}$ with $\mathrm{Re}(s_i - s_j) \gg 0$ for all $1 \leq i < j \leq r$. The proof follows the same path as the case $\underline{s} = (0, \ldots, 0)$, with some small modifications.

Proposition 4.6. *For* $\mathrm{Re}(s_j - s_j) \gg 0$ $(1 \leq i < j \leq r)$, *the representation* $\Pi_{\underline{s}}$ *has a non-zero generalized Whittaker-Fourier coefficient attached to the partition* $\mu = [a_1]^{b_1} + \cdots + [a_r]^{b_r}$.

Proof. Let $b_0 = \max_{1 \leq i \leq r}\{b_i\}$, and recall that we denote $\underline{s} = (s_1, \ldots, s_r) \in \mathbb{C}^r$. As in §4.1, for $1 \leq i \leq r$, we form parameters

$$\left(t_{b_i}^{(\ell_{b_0})}, \ldots, t_{b_i}^{(1)}, h_{b_i}^{(1)}, \ldots, h_{b_i}^{(k_{b_0})}\right)$$

with b_0 entries by adding zeros from the front if $\ell_{b_i} < \ell_{b_0}$, and adding zeros from the end if $k_{b_i} < k_{b_0}$ (note that $b_0 \geq b_j$ for $1 \leq j \leq r$). In other words, one has $t_{b_i}^{(j)} = 0$ if $j > \ell_{b_i}$, and $h_{b_i}^{(j)} = 0$ if $j > k_{b_i}$. For $1 \leq j \leq k_{b_0}$, we construct representations

$$\sigma_{j,\underline{s}}^1 = \tau_1 |\cdot|^{h_{b_1}^{(j)} + s_1} \times \cdots \times \tau_r |\cdot|^{h_{b_r}^{(j)} + s_r},$$

where we omit the $\tau_i |\cdot|^{h_{b_i}^{(j)} + s_i}$-term if $j > k_{b_i}$ $(1 \leq i \leq r)$. Similarly, for $1 \leq q \leq \ell_{b_0}$, we construct representations

$$\rho_{q,\underline{s}}^1 = \tau_1 |\cdot|^{t_{b_1}^{(q)} + s_1} \times \cdots \times \tau_r |\cdot|^{t_{b_r}^{(q)} + s_r},$$

where we omit the $\tau_i |\cdot|^{t_{b_i}^{(q)} + s_i}$-term if $q > \ell_{b_i}$ $(1 \leq i \leq r)$. Then we get representations

$$\varepsilon_{1,\underline{s}}, \ldots, \varepsilon_{b_0,\underline{s}}$$

analogous to ε_i's in §4.1, where $\varepsilon_{1,\underline{s}} = \sigma^1_{k_{b_0},\underline{s}}$, $\varepsilon_{2,\underline{s}} = \sigma^1_{k_{b_0}-1,\underline{s}}$, $\ldots$, $\varepsilon_{k_{b_0},\underline{s}} = \sigma^1_{1,\underline{s}}$, $\varepsilon_{k_{b_0}+1,\underline{s}} = \rho^1_{1,\underline{s}}$, $\varepsilon_{k_{b_0}+2,\underline{s}} = \rho^1_{2,\underline{s}}$, $\ldots$, $\varepsilon_{b_0,\underline{s}} = \rho^1_{\ell_{b_0},\underline{s}}$. We also suppose that each ε_i is a representation of $\mathrm{GL}_{t_i}(\mathbb{A})$, $1 \le i \le b_0$, and hence $\{t_1, \ldots, t_{b_0}\}$ are among the parts in the partition $[a_1]^{b_1} + \cdots + [a_r]^{b_r}$.

We also denote by

$$\mathrm{Ind}_{Q(\mathbb{A})}^{\mathrm{GL}_n(\mathbb{A})} \eta_{1,\underline{s}} \otimes \cdots \otimes \eta_{b_0,\underline{s}}$$

the image of the induced representation

$$\mathrm{Ind}_{Q(\mathbb{A})}^{\mathrm{GL}_n(\mathbb{A})} \varepsilon_{1,\underline{s}} \otimes \cdots \otimes \varepsilon_{b_0,\underline{s}}$$

under the set of maps $\{\iota_{\tau_1,b_1}, \ldots, \iota_{\tau_r,b_r}\}$, where $Q = LV$ is the parabolic subgroup of GL_n with Levi subgroup $L \subset Q$ isomorphic to $\mathrm{GL}_{t_1} \times \cdots \times \mathrm{GL}_{t_{b_0}}$.

Following the approach in §4.1, Proposition 4.6 will follow from the lemma below:

Lemma 4.7. *For $\mathrm{Re}(s_i - s_j) \gg 0$ $(1 \le i < j \le r)$, we have*

1. *Each representation $\eta_{i,\underline{s}}$ is an irreducible generic representation of $\mathrm{GL}_{t_i}(\mathbb{A})$.*
2. *For any $f \in \Pi_{\underline{s}}$, the constant term of f along Q gives us a non-zero vector in the irreducible generic representation $\eta_1 \otimes \cdots \otimes \eta_{b_0}$ of $L(\mathbb{A})$.*

Proof of Lemma 4.7. The proof is similar to the proof of Lemma 4.3 and Lemma 4.4 in §4.1. We just give a sketch and indicate the use of the assumption $\mathrm{Re}(s_i - s_j) \gg 0$ $(1 \le i < j \le r)$.

Note that each $\eta_{i,\underline{s}}$ is of the form

$$\tau_{\kappa_1}|\cdot|^{e_1+s_{\kappa_1}} \times \cdots \times \tau_{\kappa_\alpha}|\cdot|^{e_\alpha+s_{\kappa_\alpha}} \quad (s_{\kappa_i} \in \mathbb{C}, \quad 1 \le \alpha \le r),$$

with integers or half-integers e_i's such that $e_i - e_j = 0, \pm\frac{1}{2}$ for $1 \le i < j \le \alpha$. By assumption, we can take $\mathrm{Re}(s_{\kappa_i} - s_{\kappa_j})$ $(1 \le i < j \le \alpha)$ large enough such that the ratio of Rankin-Selberg L-functions

$$\prod_{1 \le i < j \le \alpha} \frac{L(e_i + s_{\kappa_i} - e_j - s_{\kappa_j}, \tau_{\kappa_i} \times \widetilde{\tau}_{\kappa_j})}{L(1 + e_i + s_{\kappa_i} - e_j - s_{\kappa_j}, \tau_{\kappa_i} \times \widetilde{\tau}_{\kappa_j})}$$

is non-zero and holomorphic. Then by the same arguments as in the proof of Lemma 4.3 we get Part (1) of the lemma.

Now we prove Part (2) of the lemma. We also write

$$f(g) = \lim_{\overline{s} \to \Lambda_{\underline{s}}} \prod_{j=1}^{r} \prod_{i=1}^{b_i-1} (s_i^{(j)} - s_{i+1}^{(j)} - 1) E(\phi_{\tau_1^{\otimes b_1}, \ldots, \tau_r^{\otimes b_r}}, \overline{s}, g), \tag{4.8}$$

and consider the constant term of $E(\phi_{\tau_1^{\otimes b_1}, \ldots, \tau_r^{\otimes b_r}}, \overline{s}, g)$ along Q. In particular, as in the proof of Lemma 4.4, we denote ω_0 to be the permutation which permutes the blocks in

$$\tau_1|\cdot|^{\frac{b_1-1}{2}+s_1} \otimes \cdots \otimes \tau_1|\cdot|^{\frac{1-b_1}{2}+s_1} \otimes \cdots \otimes \tau_r|\cdot|^{\frac{b_r-1}{2}+s_r} \otimes \cdots \otimes \tau_r|\cdot|^{\frac{1-b_r}{2}+s_r}$$

to those corresponding to $\eta_{1,\underline{s}} \otimes \cdots \otimes \eta_{b_0,\underline{s}}$, and consider summand

$$E^Q(M(\omega_0, \overline{s})\phi_{\tau_1^{\otimes b_1}, \ldots, \tau_r^{\otimes b_r}}, \omega_0\overline{s}, g)$$

in the constant term. We want to show that

$$\lim_{\overline{s}\to\Lambda_{\underline{s}}}\prod_{j=1}^{r}\prod_{i=1}^{b_i-1}(s_i^{(j)}-s_{i+1}^{(j)}-1)M(\omega_0,\overline{s})$$

is non-zero holomorphic provided $\mathrm{Re}(s_i-s_j)\gg 0$ $(1\le i<j\le r)$, and hence Part (2) follows.

Let

$$r(\omega_0,\overline{s})=\prod_v r_v(\omega_0,\overline{s})$$

be the normalizing factor we have defined in the proof of Lemma 4.4, and let

$$N(\omega_0,\overline{s})=r(\omega_0,\overline{s})^{-1}M(\omega_0,\overline{s})$$

be the normalized intertwining operator. By [MW89, Proposition I.10], the normalized intertwining operator is non-zero holomorphic at $\overline{s}=\Lambda_{\underline{s}}$ if $\mathrm{Re}(s_i-s_j)\gg 0$ for all $1\le i<j\le r$. Then it suffices to show that

$$\lim_{\overline{s}\to\Lambda_{\underline{s}}}\prod_{j=1}^{r}\prod_{i=1}^{b_i-1}(s_i^{(j)}-s_{i+1}^{(j)}-1)r(\omega_0,\overline{s}) \tag{4.9}$$

exists and is non-zero, under the same assumption. But this is clear since the limit $\lim_{\overline{s}\to\Lambda_{\underline{s}}}\prod_{j=1}^{r}\prod_{i=1}^{b_i-1}(s_i^{(j)}-s_{i+1}^{(j)}-1)$ cancels the poles of $r(\omega_0,\overline{s})$ coming from the pairs $(i,i+1)$ such that (s_i,s_{i+1}) in (4.6) corresponds to $(s_{\mathfrak{l}}^{(\mathfrak{k})},s_{\mathfrak{l}+1}^{(\mathfrak{k})})$ $(1\le\mathfrak{k}\le r, 1\le\mathfrak{l}\le b_{\mathfrak{k}}-1)$ in (4.5), and the other terms in $r(\omega_0,\overline{s})$ (ratios of L-functions) are holomorphic non-zero provided $\mathrm{Re}(s_i-s_j)\gg 0$ for all $1\le i<j\le r$. □

□

5. Vanishing for bigger and not related orbits

In this section, we show that for any s_i $(1\le i\le r)$, the induced representation

$$\Pi_{\underline{s}}=\mathrm{Ind}_{P(\mathbb{A})}^{\mathrm{GL}_n(\mathbb{A})}\Delta(\tau_1,b_1)|\cdot|^{s_1}\otimes\cdots\otimes\Delta(\tau_r,b_r)|\cdot|^{s_r}$$

has no non-zero generalized Whittaker-Fourier coefficients attached to any partition either bigger than or not related to the partition

$$\mu=[a_1^{b_1}]+\cdots+[a_r^{b_r}].$$

Combining with Proposition 4.2 and Proposition 4.6, this completes the proof of Theorem 1.1 and Theorem 1.3. Note that $\mu^t=[b_1^{a_1}\cdots b_r^{a_r}]$ from the definitions related to partitions in §2.1. For any partition $\nu=[d_1\, d_2\cdots d_l]$ of n, let P_ν be the standard parabolic subgroup of GL_N whose Levi subgroup is $M_\nu\simeq\mathrm{GL}_{d_1}\times\cdots\times\mathrm{GL}_{d_l}$, and denote the corresponding unipotent subgroup by N_ν. The main result in this section is the following.

Proposition 5.1. *The representation $\Pi_{\underline{s}}$ has no non-zero generalized Whittaker-Fourier coefficients attached to any partition either bigger than or not related to the partition $\mu=[a_1^{b_1}]+\cdots+[a_r^{b_r}]$.*

Proof. Let $b_0 = \max_{1\leq i\leq r}\{b_i\}$ as before. We may write $\mu = [t_1 t_2 \cdots t_{b_0}]$ with $t_1 \geq \cdots \geq t_{b_0}$. Let Π' be any constituent of $\Pi_{\underline{s}}$. By Proposition 4.2 and Proposition 4.6, $\Pi_{\underline{s}}$ has a non-zero degenerate Whittaker-Fourier coefficient attached to the Whittaker pair (s_n, u_μ), and hence for any finite place v, Π'_v has a non-zero degenerate Whittaker model attached to this same Whittaker pair. We claim that at some finite place v, Π'_v has no non-zero *degenerate Whittaker model* attached to the Whittaker pair (s_n, u_λ), for any partition $\lambda = [p_1 p_2 \cdots p_m]$ $(p_1 \geq \cdots \geq p_m)$ of n which is bigger than or not related to μ. With this claim, by Proposition 2.2, $\mathfrak{p}^m(\Pi'_v) = \{\mu\}$, in particular, Π'_v has no non-zero *generalized Whittaker model* attached to any partition either bigger than or not related to μ. Therefore, $\Pi_{\underline{s}}$ has no non-zero generalized Whittaker-Fourier coefficients attached to any partition either bigger than or not related to μ.

In the following, we will prove the above claim. Note that for any such partition λ, there exists $1 \leq i \leq m$ such that $p_1 + \cdots + p_i > t_1 + \cdots + t_i$.

By [L79a, Lemma 1], constituents of $\Pi_{\underline{s}}$ are pairwise nearly equivalent. We consider the local unramified components of any constituent Π'. Let v be a finite place such that Π'_v is unramified. For $1 \leq i \leq r$, write $\tau_{i,v} = \chi_1^{(i)} \times \cdots \times \chi_{a_i}^{(i)}$ with $\chi_j^{(i)}$'s being unramified characters of $F_v^\times$, then Π'_v is the unique irreducible unramified constituent of the following induced representation

$$\mathrm{Ind}_{P_{\mu^t}(F_v)}^{\mathrm{GL}_n(F_v)} \sigma_{1,v}|\cdot|^{s_1} \otimes \cdots \otimes \sigma_{r,v}|\cdot|^{s_r},$$

where

$$\sigma_{i,v} = \chi_1^{(i)}(\det{}_{\mathrm{GL}_{b_i}}) \times \cdots \times \chi_{a_i}^{(i)}(\det{}_{\mathrm{GL}_{b_i}})$$

is a representation of $\mathrm{GL}_{a_i b_i}(F_v)$.

Write $\varrho_v = \sigma_{1,v}|\cdot|^{s_1} \otimes \cdots \otimes \sigma_{r,v}|\cdot|^{s_r}$. Let $U = N_{s_n,u_\lambda}$ for simplicity. We claim that

$$\mathrm{Hom}_{U(F_v)}(\mathrm{Ind}_{P_{\mu^t}(F_v)}^{\mathrm{GL}_n(F_v)} \varrho_v, \psi_{u_\lambda,v}) = 0 \tag{5.1}$$

provided that there exists $1 \leq i \leq m$ such that $p_1 + \cdots + p_i > t_1 + \cdots + t_i$. This implies that Π'_v has no non-zero degenerate Whittaker model attached to the Whittaker pair (s_n, u_λ), for any partition $\lambda = [p_1 p_2 \cdots p_m]$ $(p_1 \geq \cdots \geq p_m)$ of n which is bigger than or not related to μ. Recall that $\psi_{u_\lambda,v}$ is defined in §2.2.

We use Bernstein's localization principle (see [BZ76, §6]) to study the Hom-space

$$\mathrm{Hom}_{U(F_v)}(\mathrm{Ind}_{P_{\mu^t}(F_v)}^{\mathrm{GL}_n(F_v)} \varrho_v, \psi_{u_\lambda,v})\,. \tag{5.2}$$

For $h \in \mathrm{GL}_n(F_v)$, define two actions on the space $C_c^\infty(\mathrm{GL}_n(F_v))$, the compactly supported smooth functions on $\mathrm{GL}_n(F_v)$, by

$$(l_h \cdot f)(g) = f(h^{-1}g) \text{ and } (r_h \cdot f)(g) = f(gh)\,,$$

here $f \in C_c^\infty(\mathrm{GL}_n(F_v))$. Let

$$C_c^\infty(\mathrm{GL}_n(F_v)/U(F_v), \psi_{u_\lambda,v}) \tag{5.3}$$

be the subspace of $C_c^\infty(\mathrm{GL}_n(F_v))$ such that

$$(r_u \cdot f)(g) = \psi_{u_\lambda,v}(u) f(g)$$

for all $u \in U(F_v)$. The parabolic subgroup $P_{\mu^t}(F_v)$ has a left action on the space (5.3) given by $f \mapsto l_p \cdot f$ ($p \in P_{\mu^t}(F_v)$), which is constructive ([BZ76, §6.1]) by [BZ76, Theorem 6.15]. Let $\mathcal{D}$ be the space of complex linear functionals T on (5.3) such that

$$T(l_p \cdot f) = \delta_{P_{\mu^t}}^{\frac{1}{2}}(p)\varrho_v(p)^{-1}T(f)$$

for all $p \in P_{\mu^t}(F_v)$. Consider the restriction of $T \in \mathcal{D}$ to the double coset $P_{\mu^t}(F_v)wU(F_v)$ with $w \in P_{\mu^t}(F_v)\backslash \mathrm{GL}_n(F_v)/U(F_v)$. It is associated to the Hom-space

$$\mathrm{Hom}_{U(F_v)}(\mathrm{ind}_{U(F_v)\cap(w^{-1}P_{\mu^t}(F_v)w)}^{U(F_v)} \varrho_v^w, \psi_{u_\lambda,v})\,, \tag{5.4}$$

where

$$\mathrm{ind}_{U(F_v)\cap(w^{-1}P_{\mu^t}(F_v)w)}^{U(F_v)} \varrho_v^w$$

is the compact induced representation with ϱ_v^w being defined by $\varrho_v^w(g) = \varrho_v(wgw^{-1})$ for $g \in U(F_v) \cap w^{-1}P_{\mu^t}(F_v)w$. Note that by construction, we have $\varrho_v^w|_{U(F_v)\cap(w^{-1}P_{\mu^t}(F_v)w)} \equiv 1$. Moreover, (5.4) is isomorphic to

$$\mathrm{Hom}_{U(F_v)\cap(w^{-1}P_{\mu^t}(F_v)w)}(\varrho_v^w, \psi_{u_\lambda,v}) \tag{5.5}$$

by Frobenius reciprocity. By a root-theoretic result [C18, Theorem 1.3], if there exists $1 \le i \le m$ such that $p_1 + \cdots + p_i > t_1 + \cdots + t_i$, then for any representative $w \in P_{\mu^t}(F_v)\backslash \mathrm{GL}_n(F_v)/U(F_v)$, there exists $u \in U(F_v)$ such that $\psi_{u_\lambda,v}(u) \neq 1$ and $wuw^{-1} \in P_{\mu^t}(F_v)$. Therefore, the space (5.5), and hence the space (5.4), is zero for all $w \in P_{\mu^t}(F_v)\backslash \mathrm{GL}_n(F_v)/U(F_v)$. Then by Bernstein's localization principle (see [BZ76, Theorem 6.9]), we see that the Hom-space (5.2) is zero. This proves the claim above and completes the proof of the proposition. □

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