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THE GENERIC DUAL OF p-ADIC SPLIT SO_{2n} AND LOCAL LANGLANDS PARAMETERS

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ABSTRACT

Let G be the F-rational points of the split group SO_{2n}, where F is a non-Archimedean local field of characteristic 0. In this paper, we give the complete description of the generic dual of G. Based on this result and the local descent result of Jiang and Soudry, we show that the functorial lifting map constructed by Cogdell, Kim, Piatetski-Shapiro and Shahidi is surjective. Along the way, we prove that for any irreducible generic representation σ, if $\sigma \ncong c\sigma$ (c an element of $O_{2n}(F) \setminus SO_{2n}(F)$), then they have the same lifting image and the same twisted local factors, matching Arthur's current results on local Langlands correspondence. Then, for any local Langlands parameter ϕ of G, we construct a representation σ such that ϕ and σ have the same twisted local factors.

As an application, we prove the G-case of a conjecture of Gross–Prasad and Rallis, that is, a local Langlands parameter ϕ of G is generic (i.e., the associated representation σ constructed above is generic) if and only if the adjoint L-function of ϕ is regular at $s = 1$. As another application, we give an alternate proof of a result of Shahidi that for each local Arthur parameter ψ, the representation associated to the corresponding local Langlands parameter ϕ_ψ is generic if and only if ϕ_ψ is tempered.

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1. Introduction

Let **G** be a connected reductive algebraic group split over F, where F is a non-Archimedean local field of characteristic 0. Let $G = \mathbf{G}(F)$ be the F-rational points. Let $\Pi(G)$ be the set of all equivalence classes of irreducible admissible representations of G. Elements in $\Pi(G)$ having nonzero Whittaker models are called generic representations, which have great importance in both theories of representations of p-adic groups and automorphic forms because of the uniqueness of Whittaker models. For classical groups $G = SO_{2n+1}(F)$, $Sp_{2n}(F)$, the generic duals are described by Muić ([M2]). The result has been used extensively to study the local Langlands reciprocity conjecture and the local Langlands functoriality conjecture for these two groups; see [JngS1], [JngS2], [CKPSS], [Liu].

As the first main result of this paper, we give a complete description of the generic dual $\Pi^{(g)}(SO_{2n})$ of the split $SO_{2n}(F)$. To discuss this result, we first briefly recall some notation. A standard parabolic subgroup of $SO_{2n}(F)$ has the form $P = MU$ with $M = GL_{n_1}(F) \times \cdots \times GL_{n_f}(F) \times SO_{2n_0}(F)$, where $n_1+\cdots+n_f+n_0 = n$. If $\tau_1, \ldots, \tau_f$ are representations of $GL_{n_1}(F), \ldots GL_{n_f}(F)$, resp., and τ a representation of $SO_{2n_0}(F)$, we write $\tau_1 \times \cdots \times \tau_f \rtimes \tau$ for the parabolically induced representation $\mathrm{Ind}_P^G(F)$. Note that if $n_0 = 0$, we use $\tau = 1 \otimes e$ and $1 \otimes c$ to disinguish between the two non-conjugate standard parabolic subgroups of this form. (We use $\tau_1 \times \cdots \times \tau_k$ as the corresponding induced representation in $GL_{n_1+\cdots+n_k}(F)$.) See Sections 2 and 3.1 for more details.

The generic dual of $GL_n(F)$ is known from [Jac]; for $SL_n(F)$, it may be obtained by restriction (see [Td1]). The classification for other classical groups—in particular, $Sp(2n, F)$ and $SO(2n+1, F)$—is done in [M2], as mentioned above. Our techniques and results are similar to those in [M2].

By the Langlands classification (cf. [B-W], [Sil1], [Ko], etc.) and the results of [M3], every irreducible generic representation of $SO_{2n}(F)$ may be realized in the form

$$\nu^{x_1}\delta_1 \times \cdots \times \nu^{x_f}\delta_f \rtimes \sigma^{(t)}, \tag{1.1}$$

where $\nu = |det|$, $\delta_1, \ldots, \delta_f$ are irreducible square-integrable representations of general linear groups (automatically generic by [Jac]), $\sigma^{(t)}$ a generic irreducible tempered representation of $SO_{2n_0}(F)$ (possibly $1 \otimes c$), and $x_1 \geq \cdots \geq x_f > 0$. In particular, the induced representation is irreducible. Conversely, an irreducible representation of this form is necessarily generic. Thus, to understand generic representations of $SO_{2n}(F)$, we need to understand when (1.1) is irreducible.

To set up such a result, we first need to discuss generic tempered and generic square-integrable representations of $SO_{2n}(F)$, as characterizations of these are needed to produce explicit reducibility criteria. We use the following notation: if ξ is an irreducible supercuspidal representation of a general linear group, we let $\Sigma = [\nu^c\xi, \nu^d\xi] = \{\nu^c\xi, \nu^{c+1}\xi, \ldots, \nu^d\xi\}$ and $\delta(\Sigma)$ be the unique irreducible quotient of $\nu^c\xi \times \nu^{c+1}\xi \times \cdots \times \nu^d\xi$ (square-integrable up to central character).

Proposition 1.1:

Generic discrete series: *Let $\Delta_i = [\nu^{-a_i}\tau_i, \nu^{b_i}\tau_i]$, $1 \le i \le k$, where τ_i is an irreducible unitary supercuspidal representation if a general linear group. Assume that if $i < j$ has $\tau_i \cong \tau_j$, then $a_i < b_i < a_j < b_j$. Let $\sigma^{(0)}$ be an irreducible supercuspidal generic representation of $SO_{2n'}(F)$ (possibly $1 \otimes c$) and assume that for each i, one of the following holds (necessarily exclusive):*

(1) *$\nu\tau_i \rtimes \sigma^{(0)}$ is reducible, in which case $a_i \in \mathbb{Z} \setminus \{0\}$ and $a_i \ge -1$;*
(2) *$\nu^{\frac{1}{2}}\tau_i \rtimes \sigma^{(0)}$ is reducible, in which case $a_i \in \frac{1}{2} + \mathbb{Z}_{\ge 0}$;*
(3) *$\tau_i \rtimes \sigma^{(0)}$ is reducible, in which case $a_i \in \mathbb{Z}_{\ge 0}$;*
(4) *$\nu^x\tau_i \rtimes \sigma^{(0)}$ is irreducible for all $x \ge 0$ and $\tau_i \cong \tilde{\tau}_i$, in which case $a_i \in \mathbb{Z}_{\ge 0}$.*

Then, if π is the generic subquotient of $\delta(\Delta_1) \times \cdots \times \delta(\Delta_k) \rtimes \sigma^{(0)}$, π is square-integrable. Conversely, any generic irreducible square-integrable π of an even special orthogonal group is of this form (with $\Delta_1, \ldots, \Delta_k$ unique up to permutation), and further

$$\pi \hookrightarrow \delta(\Delta_1) \times \cdots \times \delta(\Delta_k) \rtimes \sigma^{(0)}.$$

Generic tempered representations: *Let $\beta_1, \ldots, \beta_c$ be generic irreducible unitary supercuspidal representations of general linear groups and $\sigma^{(2)}$ a generic irreducible square-integrable representation of an even special orthogonal group. Then the generic component*

$$\sigma^{(t)} \le \delta([\nu^{\frac{-k_1'+1}{2}}\beta_1, \nu^{\frac{k_1'-1}{2}}\beta_1]) \times \cdots \times \delta([\nu^{\frac{-k_c'+1}{2}}\beta_c, \nu^{\frac{k_c'-1}{2}}\beta_c]) \rtimes \sigma^{(2)}$$

is a generic tempered representation. Further, any generic tempered representation may be realized this way (with inducing representation unique up to Weyl conjugation).

Note that the tempered claims above follow directly from a result of Harish-Chandra (cf. Proposition III.4.1 [W]). In the next result, for a segment $\Sigma = [\nu^a\xi, \nu^b\xi]$, we define $\tilde{\Sigma} = [\nu^{-b}\tilde{\xi}, \nu^{-a}\tilde{\xi}]$, so that $\widetilde{\delta(\Sigma)} = \delta(\tilde{\Sigma})$ (where ˜ denotes contragredient).

THEOREM 1.2: *Put*

$$\delta(\Sigma_i) = \nu^{x_i}\delta_i, \quad i = 1, 2, \ldots, f$$

as above (i.e., $x_1 \geq x_2 \geq \cdots \geq x_k > 0$). Then, the representation

$$\delta(\Sigma_1) \times \cdots \times \delta(\Sigma_k) \rtimes \sigma^{(t)}$$

is irreducible if and only if $\{\Sigma_j\}_{j=1}^{f}$ and $\sigma^{(t)}$ satisfy the following properties:

(1) *$\delta(\Sigma_i) \times \delta(\Sigma_j)$ and $\delta(\Sigma_i) \times \delta(\tilde{\Sigma}_j)$ are irreducible for all $1 \leq i \neq j \leq f$; and*
(2) *$\delta(\Sigma_i) \rtimes \sigma^{(t)}$ is irreducible for all $1 \leq i \leq f$.*

The reducibility for (1) is known from [Z]; for (2) we write $\sigma^{(t)}$ as in Proposition 1.1 as above. Then $\delta(\Sigma) \rtimes \sigma^{(t)}$ is irreducible if and only if the following hold:

(1) *$\delta(\Sigma) \times \delta([\nu^{\frac{-k'_j+1}{2}}\beta_j, \nu^{\frac{k'_j-1}{2}}\beta_j])$ and $\delta(\tilde{\Sigma}) \times \delta([\nu^{\frac{-k'_j+1}{2}}\beta_j, \nu^{\frac{k'_j-1}{2}}\beta_j])$ are irreducible for all $1 \leq j \leq c$; and*
(2) *$\delta(\Sigma) \rtimes \sigma^{(2)}$ is irreducible.*

To understand when the second condition above holds, write $\sigma^{(2)}$ as in Proposition 1.1. Then, $\delta(\Sigma) \rtimes \sigma^{(2)}$ is irreducible if and only if the following hold:

(1) *$\delta(\Sigma) \times \delta(\Delta_i)$ and $\delta(\tilde{\Sigma}) \times \delta(\Delta_i)$ are irreducible for all $i = 1, 2, \ldots, k'$; and*
(2) *either (a) $\delta(\Sigma) \rtimes \sigma^{(0)}$ is irreducible, or (b) $\delta(\Sigma) = \delta([\nu\xi, \nu^b\xi])$, with $\xi \rtimes \sigma^{(0)}$ reducible and some i having $\delta(\Delta_i) = \delta([\nu\xi, \nu^{b_i}\xi])$ and $b_i \geq b$.*

Finally, for the second condition above, we have $\delta(\Sigma) \rtimes \sigma^{(0)}$ is irreducible if and only if one of the following hold: for $\Sigma = [\nu^{-a}\xi, \nu^b\xi]$, we have

(1) *$\xi \not\cong \tilde{\xi}$; or*
(2) *$\xi \cong \tilde{\xi}$ and the following: (i) if $\nu^x\xi \rtimes \sigma^{(0)}$ is reducible for some (necessarily unique) $x = \alpha \geq 0$, then $\pm\alpha \notin \{-a, -a+1, \ldots, b\}$; (ii) if $\nu^x\xi \rtimes \sigma^{(0)}$ is irreducible for all $x \geq 0$, then $a \notin \mathbb{Z}_{\geq 0}$.*

Since the Langlands dual group of $SO_{2n}(F)$ is $SO_{2n}(\mathbb{C})$, which has a natural embedding into $GL_{2n}(\mathbb{C})$, by the local Langlands functoriality conjecture, there would have to exist a local functorial lifting $l : \Pi(SO_{2n}) \to \Pi(GL_{2n})$.

In [CKPSS], Cogdell, Kim, Piatetski-Shapiro and Shahidi constructed a local functorial lifting l from $\Pi^{(g)}(SO_{2n})$ to a subset $\Pi^{(go)}(GL_{2n})$ (consists of certain representations of orthogonal type, see Section 4.4 for the definition) of $\Pi(GL_{2n})$, satisfying the following conditions:

$$L(\sigma \times \pi, s) = L(l(\sigma) \times \pi, s),$$

$$\epsilon(\sigma \times \pi, s, \psi) = \epsilon(l(\sigma) \times \pi, s, \psi),$$

for any irreducible generic representation π of $GL_k(F)$, with $k \in \mathbb{Z}_{>0}$, where ψ is a fixed nontrivial character of F. The left-hand sides are the local factors defined by Shahidi [S1], and the right-hand sides are the local factors defined by Jacquet, Piatetski-Shapiro and Shalika [JPSS]; both sides are the Langlands local factors with respect to the standard representations, called standard local factors.

In [JngS3], Jiang and Soudry constructed the descent map from supercuspidal representations of $GL_{2n}(F)$ which are of orthogonal type and have trivial central characters, to irreducible supercuspidal generic representations of $SO_{2n}(F)$, showing that the local Langlands functorial lifting from irreducible supercuspidal generic representations of $SO_{2n}(F)$ is surjective.

As the second main result of this paper, based on the complete description of the generic dual $\Pi^{(g)}(SO_{2n})$ of $SO_{2n}(F)$ and the above result of Jiang and Soudry, we show that the local functorial lifting $l : \Pi^{(g)}(SO_{2n}) \to \Pi^{(go)}(GL_{2n})$ constructed above by Cogdell, Kim, Piatetski-Shapiro and Shahidi is surjective. Note that for SO_{2n+1}, in [JngS2], Jiang and Soudry have already constructed the corresponding local Langlands functorial lifting, and proved that it is actually bijective. In [Liu], Liu proved the surjectivity for Sp_{2n}. The method used here and for the case of Sp_{2n} is the same as the case of SO_{2n+1} in [JngS2]. We remark that, for Sp_{2n} and SO_{2n}, l is expected not to be injective by Jiang's conjecture (see Conjecture 3.7 in [Jng]), which is a refinement of the local converse theorem conjecture.

Let $\Phi(SO_{2n})$ be the set of local Langlands parameters for SO_{2n} (for a definition and discussion of the local Langlands reciprocity conjecture, see the Introduction to [Liu] and the references therein). These are $SO_{2n}(\mathbb{C})$-conjugacy classes of admissible homomorphisms $W_F \times SL_2(\mathbb{C}) \to SO_{2n}(\mathbb{C})$, where

$W_F \times SL_2(\mathbb{C})$ is the Weil–Deligne group. Note that there is a natural embedding $SO_{2n}(\mathbb{C}) \to GL_{2n}(\mathbb{C})$. Given a local Langlands parameter $\phi \in \Phi(GL_{2n})$, $\phi : W_F \times SL_2(\mathbb{C}) \to GL_{2n}(\mathbb{C})$, assume that it factors through $SO_{2n}(\mathbb{C})$ and $\phi \ncong c\phi$ within $SO_{2n}(\mathbb{C})$, where $c\phi$ is the c-conjugate of ϕ. Then ϕ gives two elements in $\Phi(SO_{2n})$ (see Chapter 1 of [A2]), which are denoted by ϕ and $c\phi$. To identify ϕ and $c\phi$ in this situation, let $\widetilde{\Phi}(SO_{2n})$ be the set of c-conjugacy classes of $\phi \in \Phi(SO_{2n})$. For any $\phi \in \Phi(SO_{2n})$, denote its c-conjugacy class by $\widetilde{\phi}$. Note that for any $\phi \in \Phi(SO_{2n})$, if $\phi \ncong c\phi$, then they automatically have the same twisted local factors since they come from the same local Langlands parameter $\phi \in \Phi(GL_{2n})$. Define the twisted local factors of $\widetilde{\phi}$ to be those of ϕ.

The local functorial lifting l enables us to assign a parameter $\phi \in \widetilde{\Phi}(SO_{2n})$ to each $\sigma \in \Pi^{(g)}(SO_{2n})$, which is exactly the parameter corresponding to $l(\sigma)$. That is, there is a map $\iota : \Pi^{(g)}(SO_{2n}) \to \widetilde{\Phi}^{(g)}(SO_{2n})$, where $\widetilde{\Phi}^{(g)}(SO_{2n})$ is the set of parameters corresponding to representations in $\Pi^{(go)}(GL_{2n})$. The surjectivity of l implies that of ι.

Along the way to proving the surjectivity of l and ι, we prove that for any supercuspidal generic representation σ of $SO_{2n}(F)$ with $\sigma \ncong c\sigma$ (c an element of $O_{2n}(F) \setminus SO_{2n}(F)$), both σ and $c\sigma$ have the same lifting image (see Proposition 4.4). This is done by embedding it into a generic cuspidal representation and using Corollary 7.1 of [CKPSS], which says that weak lifting is actually strong. This eventually leads us to the following result:

Theorem 1.3: *For any $\sigma \in \Pi^{(g)}(SO_{2n})$, if $\sigma \ncong c\sigma$, then $l(\sigma) = l(c\sigma)$, and $\iota(\sigma) = \iota(c\sigma)$, that is, they have the same lifting image and the same twisted local factors.*

As the third main result of this paper, for any local Langlands parameter $\phi \in \Phi(SO_{2n})$, by an explicit analysis of its structure, we construct a distinguished irreducible representation σ of $SO_{2n}(F)$ such that $\widetilde{\phi}$ and σ have the same twisted local factors, as in [JngS2] and [Liu]. We now describe this result.

Consider the set of equivalence classes of irreducible admissible representations of $SO_{2n}(F)$. These may be realized as the Langlands quotients of induced representations

$$\delta(\Sigma_1) \times \delta(\Sigma_2) \times \cdots \times \delta(\Sigma_f) \rtimes \sigma^{(t)},$$

where $\sigma^{(t)}$ is an irreducible tempered representation of $SO_{2n^*}(F)$ (possibly $\sigma^{(t)} = 1 \otimes c$—for the definition, see Section 3.1) and $\Sigma_1, \Sigma_2, \ldots, \Sigma_f$ are imbalanced segments whose exponents are positive and in non-increasing order.

We let $\Pi'(SO_{2n})$ be the subset consisting of those irreducible admissible representations having $\sigma^{(t)}$ generic.

THEOREM 1.4: *There is a surjective map ι from $\Pi'(SO_{2n})$ to the set $\widetilde{\Phi}(SO_{2n})$. Moreover, the map ι preserves the local factors:*

$$L(\sigma \times \tau, s) = L(\iota(\sigma) \otimes r^{-1}(\tau), s),$$

$$\epsilon(\sigma \times \tau, s, \psi) = \epsilon(\iota(\sigma) \otimes r^{-1}(\tau), s, \psi),$$

for all $\sigma \in \Pi'(SO_{2n})$ and all irreducible admissible representations τ of $GL_k(F)$, with all $k \in \mathbb{Z}_{>0}$. Here, $r^{-1}(\tau) \in \Phi(GL_k)$ corresponding to τ by the local Langlands reciprocity map for GL_k as in [HT] *and* [H].

In [A2], under an assumption on stabilization of twisted trace formulas for $GL(N)$ and $SO(2n)$, Arthur first classified the set of c-conjugacy classes of $\Pi(SO_{2n})$ up to local packets which are parametrized by $\widetilde{\Phi}(SO_{2n})$. He then refined the classification by classifying the set $\Pi(SO_{2n})$ up to local packets which are parametrized by the set $\Phi(SO_{2n})$. Although a canonical matching between two sets of order 2 (an irreducible representation σ and its c-conjugate, a local Langlands parameter ϕ and its c-conjugate) is still to be determined (see Section 8.4 of [A2]), Arthur's result essentially gives the local Langlands correspondence. Note that by Remark 5.4, for any $\sigma \in \Pi'(SO_{2n})$, if $\sigma \ncong c\sigma$, then σ and $c\sigma$ have the same lifting image and the same twisted local factors, matching Arthur's results.

Given $\widetilde{\phi} \in \widetilde{\Phi}(SO_{2n})$, let σ be the representation attached to $\widetilde{\phi}$ in Theorem 1.4. If $\sigma \ncong c\sigma$, then $\phi \ncong c\phi$. Since a canonical matching between two sets of order 2—$\{\sigma, c\sigma\}$ and $\{\phi, c\phi\}$—is still to be determined (as mentioned above), we say that σ is also attached to ϕ.

There are two applications of the above three main results of this paper. As one application, we prove a conjecture of Gross–Prasad [GP] and Rallis [Ku], saying that a local Langlands parameter ϕ is generic (i.e., there is a generic representation attached to ϕ) if and only if the associated adjoint L-function is regular at $s = 1$. The case of SO_{2n+1} was proved by Jiang and Soudry ([JngS2]), and the case of Sp_{2n} was proved by Liu ([Liu]). We use the same ideas here. Note that for $G = SO_{2n}$, a local Langlands parameter ϕ is generic if the representation attached to ϕ in Theorem 1.4 is generic. There is no ambiguity here, since if σ is generic and attached to $\widetilde{\phi}$, then $c\sigma \ncong \sigma$ is also generic, and they are in different local packets corresponding to ϕ and $c\phi$, respectively ([A2]).

Therefore, at least one generic representation is attached to ϕ. We state the result as follows:

THEOREM 1.5: *For each local Langlands parameter $\phi \in \Phi(SO_{2n})$, the representation σ attached to ϕ in Theorem 1.4 is generic if and only if the local adjoint L-function*

$$L(Ad_{SO_{2n}} \circ \phi, s)$$

is regular at $s = 1$.

In [GR] (see Page 446, Formula (14)), Gross and Reeder proved that for any connected reductive group G with the maximal torus in the center of G anisotropic over F, if ϕ is discrete, then the associated adjoint L-function is regular at $s = 1$. In [AS], Asgari and Schmidt proved this conjecture for GSp_4.

As another application, we give an alternate proof of a result of Shahidi that for each local Arthur parameter ψ (for the definition, see Section 7) with corresponding local Langlands parameter ϕ_ψ, the representation σ attached to ϕ_ψ in Theorem 1.4 is generic if and only if ϕ_ψ is tempered.

Ban proved a similar result for the case of SO_{2n+1} in [Ban2], using the result of Jiang and Soudry in [JngS2]. Liu used the same method later, proving a similar result for the case of Sp_{2n} in [Liu] by generalizing the result of Jiang and Soudry in [JngS2] to the Sp_{2n} case. Since we have given the classification of irreducible generic representations for split SO_{2n}, and generalized the result of Jiang and Soudry in [JngS2] to this case, we are able to use the same idea to prove the above result.

Recently, Shahidi (see Theorem 5.1 of [S4]) proved a similar result for any quasi-split connected reductive group G, with an assumption on the validity of local Langlands conjecture for appropriate Levi subgroups M of G and data. Kim was able to remove this assumption for split GSpin groups, thus fully proving it in this case (see [Kim]). Note that by Lemma 7.2 [CKPSS] and Theorem 1.3 [H1], Theorem 5.1 [S4] implies that the SO_{2n} case of this result is true. We give a different proof here based on our classification. We state the result as follows:

THEOREM 1.6 (Shahidi [S4]): *For each local Arthur parameter ψ, and corresponding local Langlands parameter ϕ_ψ, the representation σ attached to ϕ_ψ in Theorem 1.4 is generic if and only if ϕ_ψ is tempered.*

Note that now for the representation σ attached to the local Langlands ϕ_ψ, we have two criteria to determine its genericity, i.e., those in Theorem 1.5 and Theorem 1.6.

We now discuss the contents by section. The next section introduces notation and background material used in the paper. Section 3 contains an analysis of generic representations which is needed in the later sections. This is broken into four parts. In the first subsection, using a convention regarding the trivial representation of $SO_0(F)$ (the trivial group), we formulate some basic representation theoretic results for $SO_{2n}(F)$ in a fashion which renders them very similar to their counterparts for other classical groups, thereby facilitating proofs which occur in the rest of the section. Sections 3.2 and 3.3 cover Proposition 1.1; Section 3.4 covers Theorem 1.2. The proofs are based on those obtained by Muić ([M2]; also [M1]), though they use subsequent developments to shorten the arguments and are adapted to the case of even-orthogonal groups (facilitated by the results of Section 3.1). In Section 4, we prove the surjectivity of the local functorial lifting l, construct the map ι, prove its surjectivity, and prove Theorem 1.3. Section 5 analyzes the structure of local Langlands parameters and proves Theorem 1.4. In Section 6, we prove the SO_{2n} case of the Gross–Prasad and Rallis conjecture, i.e., Theorem 1.5. In Section 7, we prove Theorem 1.6.

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2. Notation and preliminaries

Let F be a non-archimedean local field of characteristic zero. We fix a non-trivial character ψ of F. Here $SO_{2n}(F)$ denotes the group of F-rational points of the split group SO_{2n}. More precisely, if ${}^{\tau}X$ denotes the transpose of X taken with respect to the second diagonal, then $SO_{2n}(F) = \{X \in SL_{2n}(F) | {}^{\tau}XX = I\}$.

Next, we recall the structures of standard parabolic subgroups of SO_{2n}, which are quite different with those of SO_{2n+1} and Sp_{2n}, and distinguish the classification theory of representations of SO_{2n} from those of SO_{2n+1} and Sp_{2n}.

To start, we fix a Borel subgroup $B = TU$ consisting of the upper triangular matrices in $SO_{2n}(F)$, where T consists of diagonal matrices and U is the unipotent radical. Given an n-tuple $\underline{a} = \{a_1, \ldots, a_n\} \in (F^*)^n$, let $\psi_{\underline{a}}$ be a character of U defined as follows:

$$\psi_{\underline{a}}(u) = \psi(a_1 u_{1,2} + a_2 u_{2,3} + \cdots + a_{n-1} u_{n-1,n} + a_n u_{n-1,n+1}).$$

A representation (σ, V_σ) of $SO_{2n}(F)$ has a nonzero Whittaker model with respect to $\psi_{\underline{a}}$ if there is a nontrivial linear functional l on V_σ such that

$$l(\sigma(u)v) = \psi_{\underline{a}}(u) l(v),$$

for any $u \in U$ and any $v \in V_\sigma$. For any $t \in T$, let $t \circ \psi_{\underline{a}}(u) = \psi_{\underline{a}}(t^{-1}ut)$. Note that having a nonzero Whittaker model for σ depends only on the T-orbit of $\psi_{\underline{a}}$ (see [Jng], Section 3.2), and it is easy to see that for any $\psi_{\underline{a}}$, there is a $t \in T$ and an $a \in F^*$, such that $t \circ \psi_{\underline{a}} = \psi_a$, where

$$\psi_a(u) = \psi(u_{1,2} + u_{2,3} + \cdots + u_{n-1,n} + a u_{n-1,n+1}).$$

Therefore, we simply say σ is generic if it has a nonzero Whittaker model with respect to ψ_a for some $a \in F^*$.

Let

$$c = \begin{pmatrix} I_{n-1} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & I_{n-1} \end{pmatrix} \in O_{2n}(F),$$

which gives rise to the outer conjugation $c(g) = cgc^{-1}$. Assume σ is an irreducible generic representation of $SO_{2n}(F)$ with respect to ψ_a, $a \in F^*$. And assume that $\sigma \not\cong c\sigma$, where $c\sigma(g) = \sigma(cgc^{-1})$. Then $c\sigma$ is also an irreducible generic representation, but with respect to $c\psi_a$, defined by $c\psi_a(u) = \psi_a(cuc^{-1})$. We remark that $c\psi_a = t_a \circ \psi_a$, where $t_a = \mathrm{diag}(I_{n-1}, a, a^{-1}, I_{n-1})$. Therefore, both σ and $c\sigma$ are generic with respect to the same character ψ_a.

Let $\Pi = \{\alpha_1, \ldots, \alpha_{n-1}, \alpha_n\}$ denote the simple roots in the usual ordering, so that $c\alpha_{n-1} = \alpha_n$. The standard parabolic subgroups $P = MU$ are in bijective correspondence with subsets Π_M of Π. In particular, these have the form

$P = MN$ with Levi factor $M \cong GL_{k_1} \times \cdots \times GL_{k_r} \times SO_{2m}$, where $m \geq 0$. When $m = 0$ and $k_r > 1$, we also have standard parabolic subgroups $c(P) = c(M)c(N)$ with Levi factor $c(M) \cong c(GL_{k_1} \times \cdots \times GL_{k_r}) \not\sim GL_{k_1} \times \cdots \times GL_{k_r} \cong M$, where the non-equivalence means that they are not conjugate inside SO_{2n}.

We now take a moment to recall some notation from [BZ], [Td2]. First, for $P = MN$ a standard parabolic subgroup of a p-adic group G, we let $i_{G,M}$ (resp., $r_{M,G}$) denote normalized induction (resp., the normalized Jacquet module) with respect to P. Let $G = GL_k(F)$ and $P = MU$ the standard parabolic subgroup with $M = GL_{k_1}(F) \times \cdots \times GL_{k_r}(F)$. If $\tau_1 \otimes \cdots \otimes \tau_r$ is a representation of M, we let

$$\tau_1 \times \cdots \times \tau_r = i_{G,M}(\tau_1 \otimes \cdots \otimes \tau_r).$$

Similarly, suppose $P = MU$ is a standard parabolic subgroup of $SO_{2n}(F)$ with $M = GL_{k_1}(F) \times \cdots \times GL_{k_r}(F) \times SO_{2m}(F)$. For $\tau_1 \otimes \cdots \otimes \tau_r \otimes \sigma$ a representation of M, we let

$$\tau_1 \times \cdots \times \tau_r \rtimes \sigma = i_{G,M}(\tau \otimes \cdots \otimes \tau_r \otimes \sigma).$$

Note that this allows $\sigma = 1$ (the trivial representation of $SO_0(F)$, the trivial group).

As in [Z],we consider segments of the form

$$[\nu^a\tau, \nu^b, \tau] = \{\nu^a\tau, \nu^{a+1}\tau, \ldots, \nu^{b-1}\tau, \nu^b\tau\}$$

for τ a supercuspidal representation of a general linear group (usually assumed to be unitarizable) and $a \equiv b \bmod 1$. The induced representation $\nu^a\tau \times \cdots \times \nu^b\tau$ has a unique irreducible quotient (resp., subrepresentation) which we denote $\delta([\nu^a\tau, \nu^b\tau])$ (resp., $\zeta([\nu^a\tau, \nu^b\tau])$). The representations $\delta([\nu^a\tau, \nu^b\tau])$ are essentially square-integrable (i.e., square-integrable after twisting by a character), and every irreducible essentially square-integrable representation has this form. In what follows, we also use $St(\tau, 2m+1)$ to denote $\delta([v^{-m}\tau, v^m\tau])$. We remark that discrete series for general linear groups are generic (cf. [Jac]).

3. Classification of generic representations

In this section, we give the classification of irreducible generic representations of $SO_{2n}(F)$. Here, we focus on generic with respect to a fixed ψ_a, noting there are $|F^\times/(F^\times)^2|$ classes of such characters.

3.1. Background on the representation theory of $SO_{2n}(F)$. We begin by introducing some notation and conventions to be used in this section. Following [Jan5], we let both $1 \otimes e$ and $1 \otimes c$ denote the trivial representation of $SO_0(F)$, but with different interpretations when used with parabolic induction. In particular, suppose $P = MU$ is a standard parabolic subgroup with $\alpha_n \notin \Pi_M$. Then $M = GL_{m_1}(F) \times \cdots \times GL_{m_k}(F)$. For representations $\tau_1, \ldots, \tau_k$ of $GL_{m_1}(F), \ldots, GL_{m_k}(F)$, we let $\tau_1 \otimes \cdots \otimes \tau_k \otimes (1 \otimes e)$ denote a representation of M, while $\tau_1 \otimes \cdots \otimes \tau_k \otimes (1 \otimes c)$ denotes a representation of $c(M)$ (the Levi factor of the standard parabolic subgroup $c(P)$). Thus, we write

$$\tau_1 \times \cdots \times \tau_k \rtimes (1 \otimes e) = i_{G,M}(\tau_1 \otimes \cdots \otimes \tau_k),$$

and

$$\tau_1 \times \cdots \times \tau_k \rtimes (1 \otimes c) = c(i_{G,M}(\tau_1 \otimes \cdots \otimes \tau_k)).$$

In terms of the action of c, we take $c(1 \otimes e) = 1 \otimes c$ and $c(1 \otimes c) = 1 \otimes e$. Note that if χ is a character of $F^\times$, then the representations $\chi \rtimes (1 \otimes e)$ and $\chi^{-1} \rtimes (1 \otimes c)$ constitute the same representation of $SO_2(F) \cong F^\times$ (for $M = GL_m(F) \times SO_0(F)$ with $m > 1$, we have M and $c(M)$ nonconjugate and the corresponding induced representations are not in general equivalent). Note that it is a straightforward consequence of induction in stages and $c \circ i_{G,M} \cong i_{G,c(M)} \circ c$ that

$$\tau_1 \rtimes (\tau_2 \rtimes \sigma) \cong (\tau_1 \times \tau_2) \rtimes \sigma \tag{3.1}$$

in this context.

We now discuss the Casselman criterion (cf. [Cas], [W]) for $SO_{2n}(F)$, $n > 1$. Let π_0 be an irreducible representation of $SO_{2n}(F)$. Suppose

$$\nu^{x_1}\rho_1 \otimes \cdots \otimes \nu^{x_k}\rho_k \otimes \sigma_0 \leq r_{M,G}\pi_0$$

has ρ_i an irreducible unitary supercuspidal representation of $GL_{m_i}(F)$ for $i = 1, \ldots, k$ and σ_0 an irreducible supercuspidal representation of $SO_{2m}(F)$. If $m = 1$, use both $\chi \otimes (1 \otimes e)$ and $\chi^{-1} \otimes (1 \otimes c)$ in the inequalities below. The

Casselman criterion tells us that if π_0 is tempered, the following hold:

$$\begin{aligned} m_1x_1 &\geq 0, \\ m_1x_1 + m_2x_2 &\geq 0, \\ &\vdots \\ m_1x_1 + m_2x_2 + \cdots + m_kx_k &\geq 0. \end{aligned}$$

Conversely, if these inequalities hold for all such $\nu^{x_1}\rho_1 \otimes \cdots \otimes \nu^{x_k}\rho_k \otimes \sigma_0 \leq r_{M,G}\pi_0$, then π_0 is tempered. The criterion for square-integrability is the same except that the inequalities are strict. Note that if $n = 1$, the representations $\chi \rtimes (1 \otimes e)$ and $\chi \rtimes (1 \otimes c)$ are considered tempered if χ is unitary, but not square-integrable or supercuspidal (consistent with other cases of irreducible $\rho \rtimes \sigma$ with ρ, σ supercuspidal).

To describe the Langlands classification (the Langlands quotient theorem) for $SO_{2n}(F)$, let $\delta_1, \ldots, \delta_k$ be irreducible square-integrable representations of $GL_{m_1}(F), \ldots, GL_{m_k}(F)$, resp., and τ an irreducible tempered representation of $SO_{2m}(F)$ (possibly $m = 0$ and $\tau = 1 \otimes e$ or $1 \otimes c$). If $x_1 \geq \cdots \geq x_k > 0$, then the induced representation $\nu^{x_1}\tau_1 \times \cdots \times \nu^{x_k}\tau_k \rtimes \tau$ has a unique irreducible quotient which we denote $L(\nu^{x_1}\delta_1 \otimes \cdots \otimes \nu^{x_k}\delta_k \otimes \tau)$. Further, every irreducible admissible representation of $SO_{2n}(F)$ may be written in this form, uniquely up to permutations among those $\nu^{x_i}\delta_i$ having the same exponent. Note that if δ_i is a representation of $GL_{n_i}(F)$,

$$L(\nu^{x_1}\delta_1 \otimes \cdots \otimes \nu^{x_k}\delta_k \otimes \tau) \hookrightarrow \nu^{-x_1}\tilde{\delta}_1 \times \cdots \times \nu^{-x_k}\tilde{\delta}_k \rtimes c^{n_1+\cdots+n_k}\tau$$

(essentially the subrepresentation formulation of the Langlands classification).

We now discuss some structure theory from [Z]. Let

$$R = \bigoplus_{n\geq 0} \mathcal{R}(GL_n(F)) \quad \text{and} \quad R[D] = \bigoplus_{n \geq 0} \mathcal{R}(SO_{2n}(F)),$$

where $\mathcal{R}(G)$ denotes the Grothendieck group of the category of smooth finite-length representations of G. Note that this is slightly different than the $R[D]$ in [Jan4] as we have both $1 \otimes e$ and $1 \otimes c$ for $SO_0(F)$. We define multiplication on R by extending the semisimplification of $\times$ to give the multiplication $\times : R \times R \longrightarrow R$. To describe the comultiplication on R, let $M_{(i)}$ denote the standard Levi factor for $GL_n(F)$ having $M_{(i)} = GL_i(F) \times GL_{n-i}(F)$. For a representation τ of $GL_n(F)$, we define

$$m^*(\tau) = \sum_{i=0}^{n} r_{M_{(i)},G}\tau,$$

the sum of semisimplified Jacquet modules (lying in $R \otimes R$). This extends to a map $m^* : R \longrightarrow R \otimes R$. We note that with this multiplication and comultiplication—and antipode map given by the Zelevinsky involution (a special case of the general duality operator of [Aub], [S-S])—R is a Hopf algebra. Further, if we extend the semisimplification of $\rtimes$ to a map $\mu : R \otimes R[D] \longrightarrow R[D]$ (also using $\rtimes$ for this map), it follows from (3.1) that we have $R[D]$ as a module over R. It remains to construct a comodule structure.

With this convention, one can construct a μ^* structure which closely resembles that of [Td3] and [Ban1] for the other classical groups (more so than what is done in [Jan4]). To this end, we set

$$\Omega_k = \begin{cases} \Pi \setminus \{\alpha_k\} & \text{if } k \leq n-2, \\ \Pi \setminus \{\alpha_{n-1}, \alpha_n\} & \text{if } k = n-1, \\ \Pi \setminus \{\alpha_n\} & \text{if } k = n. \end{cases}$$

Note $c\Omega_n = \Pi \setminus \{\alpha_{n-1}\}$. For π an irreducible representation of $G = SO_{2n}(F)$ with $n \geq 2$, and $0 \leq k \leq n$, write

$$r_{M_{\Omega_k},G}(\pi) = \sum_{i \in I_k} \tau_{i,k} \otimes \theta_{i,k}$$

and

$$r_{M_{c\Omega_n},G}(\pi) = \sum_{j \in J} \tau_j \otimes (1 \otimes c).$$

We then define

$$\mu^*(\pi) = \sum_{k=0}^{n} \sum_{i \in I_k} \tau_{i,k} \otimes \theta_{i,k} + \sum_{j \in J} \tau_j \otimes (1 \otimes c).$$

For $n = 0$, we have only $1 \otimes e$ and $1 \otimes c$; for $d = e$ or c, we define

$$\mu^*(1 \otimes d) = 1 \otimes (1 \otimes d).$$

For $n = 1$, an irreducible representation of $SO_2(F)$ has the form $\chi \rtimes (1 \otimes e) = \chi^{-1} \rtimes (1 \otimes c)$ for χ a (quasi)character of $F^\times$ (noting that under $SO_2(F) \cong F^\times$, this corresponds to the character χ), and we set

$$\mu^*(\chi \rtimes (1 \otimes e)) = 1 \otimes (\chi \rtimes (1 \otimes e)) + \chi \otimes (1 \otimes e) + \chi^{-1} \otimes (1 \otimes c).$$

We then linearly extend μ^* to a map $\mu^* : R[D] \longrightarrow R \otimes R[D]$.

We define $M_D^* : R \longrightarrow R \otimes R \otimes \mathbb{Z}[C]$, where $C = O_{2n}(F)/SO_{2n}(F)$, as in Definition 3.1 [Jan4]. If we define the action of $R \otimes R \otimes \mathbb{Z}[C]$ on $R \otimes R[D]$ by

$$(\tau_1 \otimes \tau_2 \otimes d) \rtimes (\tau \otimes \theta) = (\tau_1 \times \tau) \otimes (\tau_2 \rtimes d\theta),$$

we have the following:

THEOREM 3.1: *With notation as above,*

$$\mu^*(\tau \rtimes \theta) = M_D^*(\tau) \rtimes \mu^*(\theta).$$

Proof. Let $\mathcal{K} = \text{span}_{\mathbb{Z}}\{\theta \otimes c - c\theta \otimes e\}$, where θ runs over irreducible representations of $SO_{2n}(F)$ for all $n > 0$ and set $R_D = (R[D] \otimes \mathbb{Z}[C])/\mathcal{K}$.

Consider the map defined on irreducible representations by

$$\begin{aligned} \psi : R[D] &\longrightarrow R_D \\ \theta &\longmapsto \begin{cases} \theta \otimes e + \mathcal{K} & \text{if } \theta \neq 1 \otimes d, \\ \theta \otimes d + \mathcal{K} & \text{if } \theta = 1 \otimes d. \end{cases} \end{aligned}$$

This map extends to an isomorphism of vectors spaces, with inverse given on irreducible representations by

$$\begin{aligned} \psi^{-1} : \qquad R_D &\longrightarrow R[D] \\ \theta \otimes d + \mathcal{K} &\mapsto \begin{cases} d\theta & \text{if } \theta \neq 1, \\ 1 \otimes d & \text{if } \theta = 1. \end{cases} \end{aligned}$$

It is a straightforward matter to check that these maps respect the module and comodule structures, i.e., we have $\psi \circ \mu = \mu_D \circ (\text{id} \otimes \psi)$ and $(\text{id} \otimes \psi) \circ \mu^* = \mu_D^* \circ \psi$, where μ_D and μ_D^* are defined in Section 3 of [Jan4]. Noting that the former says $\psi(\tau \rtimes \theta) = \tau \rtimes \psi(\theta)$, we have

$$\mu^*(\tau \rtimes \theta) = (\text{id} \otimes \psi^{-1}) \circ \mu_D^* \circ \psi(\tau \rtimes \theta) = (\text{id} \otimes \psi^{-1}) \circ \mu_D^*(\tau \rtimes \psi(\theta)).$$

By Theorem 3.4 [Jan4], we then obtain

$$\begin{aligned} \mu^*(\tau \rtimes \theta) &= (\text{id} \otimes \psi^{-1})[M_D^*(\tau) \rtimes \mu_D^*(\psi(\theta))] \\ &= (\text{id} \otimes \psi^{-1})[M_D^*(\tau) \rtimes (\text{id} \otimes \psi) \circ \mu^*(\theta)]. \end{aligned}$$

It is a straightforward matter to check that $\mathrm{id}\otimes\psi$ respects the $\rtimes$ on the right-hand side above, i.e.,

$$(\tau_1\otimes\tau_2)\rtimes(\mathrm{id}\otimes\psi)(\tau\otimes\theta)=(\mathrm{id}\otimes\psi)\left((\tau_1\otimes\tau_2)\rtimes(\tau\otimes\theta)\right)$$

(which one could write as $\mu_D\circ((\mathrm{id}\otimes id)\otimes(\mathrm{id}\otimes\psi))=(\mathrm{id}\otimes\psi)\circ\mu$ for the corresponding μ_D,μ). That $\mu^*(\tau\rtimes\theta)=M_D^*(\tau)\rtimes\mu^*(\theta)$ now follows. ∎

Remark 3.2: Assume σ is an irreducible supercuspidal representation of SO_{2l}, and ρ is an irreducible unitary representation of GL_k. It follows from [Sil3] (also see [Mœ], [Zh]) that $\nu^x\rho\rtimes\sigma$ has at most one nonnegative x for which it is reducible. If it is reducible for $x=\alpha\geq 0$, we say (ρ,σ) satisfies $(C\alpha)$. Characterizations of α, assuming certain conjectures, are given in [Mœ], [Zh]; Shahidi has shown that if σ is generic, $\alpha\in\{0,\frac{1}{2},1\}$ (see [S1], [S3]). Further, we note that if $\tilde{\rho}\cong\rho$, then (ρ,σ) satisfies $(C\alpha)$ for some $\alpha\geq 0$ unless k is odd and $c\sigma\ncong\sigma$ (e.g., see Lemma 2.2 [Zh]). If this holds (i.e., $\tilde{\rho}\cong\rho$, k is odd and $c\sigma\ncong\sigma$), then $\nu^x\rho\rtimes\sigma$ is irreducible for all $x\in\mathbb{R}$ (e.g., see Lemma 4.3 [Ban4] and Proposition 3.5(b) [S3]) and we say (ρ,σ) satisfies (CN).

By Theorem 8.1 of [S1], a pair $(\rho,\sigma)$with σ generic is $(C1)$ if and only if $L(\rho\times\sigma,s)$ has a pole at $s=0$, and a pair (ρ,σ) is $(C\frac{1}{2})$ if and only if $L(\rho,\wedge^2,s)$ has a pole at $s=0$.

3.2. Square-integrable generic representations. Let P' be a finite set of irreducible supercuspidal self-contragredient (unitary) representations of general linear groups. If τ is such a representation, let $GL_{k_\tau}(F)$ denote the underlying group. Assume that for each $\tau\in P'$, there is a sequence of segments

$$D_i(\tau)=[\nu^{-a_i(\tau)}\tau,\nu^{b_i(\tau)}\tau],\quad i=1,2,\ldots,e_\tau,$$

satisfying

$$2a_i(\tau)\in\mathbb{Z}\quad\text{and}\quad 2b_i(\tau)\in\mathbb{Z}_{\geq 0},\tag{3.2}$$

and

$$a_1(\tau)<b_1(\tau)<a_2(\tau)<b_2(\tau)<\cdots<a_{e_\tau}(\tau)<b_{e_\tau}(\tau).\tag{3.3}$$

Let $\sigma^{(0)}$ be an irreducible supercuspidal ψ_a-generic representation of $SO_{2n'}(F)$, $n'\geq 0$ ($n'\neq 1$). Assume the following hold:

(DS1$'$) if $(\tau,\sigma^{(0)})$ satisfies $(C1)$, then $-1\leq a_i(\tau)\in\mathbb{Z}\setminus\{0\}$ for $1\leq i\leq e_\tau$;
(DS2$'$) if $(\tau,\sigma^{(0)})$ satisfies $(C0)$, then $a_i(\tau)\in\mathbb{Z}_{\geq 0}$ for $1\leq i\leq e_\tau$;
(DS3$'$) if $(\tau,\sigma^{(0)})$ satisfies $(C\frac{1}{2})$, then $a_i(\tau)\in-\frac{1}{2}+\mathbb{Z}_{\geq 0}$ for $1\leq i\leq e_\tau$;

(DS4′) if $(\tau, \sigma^{(0)})$ satisfies (CN), then $a_i(\tau) \in \mathbb{Z}_{\geq 0}$ for $1 \leq i \leq e_\tau$.

Then the unique ψ_a-generic constituent of

$$(\times_{\tau \in P'} \times_{i=1}^{e_\tau} \delta(D_i(\tau))) \rtimes \sigma^{(0)} \tag{3.4}$$

(possibly $\sigma^{(0)} = 1 \otimes c$) is square-integrable (see [Ban3] or Proposition 3.3 below; the proof of Proposition 3.3 is needed as well as the statement). Further, we show below that every irreducible square-integrable ψ_a-generic representation of G is obtained in this way for a unique set of data—consisting of a finite set P', segments $\{D_i(\tau) | 1 \leq i \leq e_\tau, \tau \in P'\}$ and a ψ_a-generic supercuspidal representation $\sigma^{(0)}$—satisfying conditions (3.2), (3.3), (DS1′)–(DS4′) (see [Ban3]). Further, we have an embedding into (3.4) (see Proposition 3.6). Our proof is based on that in [M2], but uses subsequent developments to shorten parts.

Let $\Pi^{(dg)}(SO_{2n})$ be the set of all equivalence classes of irreducible discrete series generic representations of G.

Proposition 3.3: *Let π be the ψ_a-generic subquotient of $\delta(\Delta_1) \times \cdots \times \delta(\Delta_k) \rtimes \sigma$ (possibly $\sigma = 1 \otimes c$), where $\sigma = \sigma^{(0)}$, and Δ_i, $i = 1, \ldots, k$ is an enumeration of $\{D_{i'}(\tau) \,|\, 1 \leq i' \leq e_\tau, \tau \in P'\}$ above. Then π is square-integrable.*

Proof. To set up the proof, let us define μ^*_{GL} to be the sum of everything in μ^* of the form $\tau \otimes \theta$ with θ supercuspidal and $M^*_{D,GL}$ to be the sum of all terms $\tau_1 \otimes \tau_2 \otimes c^j$ in M^*_D having $\tau_2 = 1$. It follows from Theorem 3.1 that $\mu^*_{GL}(\tau \rtimes \theta) = M^*_{D,GL}(\tau) \rtimes \mu^*_{GL}(\theta)$, as for other classical groups. Now, following Lemma 4.6 [Td5], we prove the following by induction on k:

(3.5)

$$\mu^*_{GL}(\pi) \leq d \sum_{i_1=-a_1}^{|a_1|} \cdots \sum_{i_k=-a_k}^{|a_k|} \delta([\nu^{-i_1+1}\tau_1, \nu^{a_1}\tau_1]) \times \delta([\nu^{i_1}\tau_1, \nu^{b_1}\tau_1]) \times \cdots \\ \times \delta([\nu^{-i_k+1}\tau_k, \nu^{a_k}\tau_k]) \times \delta([\nu^{i_k}\tau_k, \nu^{b_k}\tau_k]) \otimes c_{i_1,\ldots,i_k}\sigma$$

for some d depending on σ and the segments and suitable power of $c_{i_1,\ldots,i_k}$ of c (the particular power is not important in what follows). The case $k = 1$ is known (follows from duality and degenerate principal series results, e.g.).

Now, observe that for any $1 \leq j \leq k$,

$$\pi \leq \delta(\Delta_j) \rtimes (\delta(\Delta_1) \times \cdots \times \delta(\Delta_{j-1}) \times \delta(\Delta_{j+1}) \times \cdots \times \delta(\Delta_k) \rtimes \sigma)$$
$$\Downarrow$$
$$\pi \leq \delta(\Delta_j) \rtimes \lambda$$

for some $\lambda \leq \delta(\Delta_1) \times \cdots \times \delta(\Delta_{j-1}) \times \delta(\Delta_{j+1}) \times \cdots \times \delta(\Delta_k) \rtimes \sigma$. By genericity, it is the ψ_a-generic subquotient, $\delta(\Delta_1, \dots \Delta_{j-1}, \Delta_{j+1}, \dots, \Delta_k; \sigma)_{\psi_a}$; square-integrable by inductive hypothesis. Thus,

$$\begin{aligned}\mu^*_{GL}(\pi) &\leq \mu^*_{GL}(\delta(\Delta_j) \rtimes \delta(\Delta_1, \dots \Delta_{j-1}, \Delta_{j+1}, \dots, \Delta_k; \sigma)_{\psi_a}) \\ &= M^*_{D,GL}(\delta(\Delta_j)) \rtimes \mu^*_{GL}(\delta(\Delta_1, \dots \Delta_{j-1}, \Delta_{j+1}, \dots, \Delta_k; \sigma)_{\psi_a}).\end{aligned}$$

Thus, for each such j, we get

$$\begin{aligned}\mu^*_{GL}(\pi) \leq d_j &\sum_{i_1=-a_1}^{|a_1|} \cdots \sum_{i_{j-1}=-a_{j-1}}^{|a_{j-1}|} \sum_{i_j=-a_j}^{b_j+1} \sum_{i_{j+1}=-a_{j+1}}^{|a_{j+1}|} \cdots \sum_{i_k=-a_k}^{|a_k|} \\ &\delta([\nu^{-i_1+1}\tau_1, \nu^{a_1}\tau_1]) \times \delta([\nu^{i_1}\tau_1, \nu^{b_1}\tau_1]) \times \cdots \\ &\times \delta([\nu^{-i_k+1}\tau_k, \nu^{a_k}\tau_k]) \times \delta([\nu^{i_k}\tau_k, \nu^{b_k}\tau_k]) \otimes c_{i_1,\dots,i_k}\sigma.\end{aligned} \tag{3.6}$$

Without loss of generality, we may assume that if $\tau_i = \tau_j$ for some $i < j$, then $a_i < b_i < a_j < b_j$. Looking at (3.6) for $j = 1$ (noting $k > 1$), we see that $\nu^{-a_k-1}\tau_k, \nu^{-a_k-2}\tau_k, \dots, \nu^{-b_k}\tau_k$ do not appear in the supercuspidal support of π. Therefore, looking at (3.6) for $j = k$, we may refine the bound by removing those terms which contain one of $\nu^{-a_k-1}\tau_k, \nu^{-a_k-2}\tau_k, \dots, \nu^{-b_k}\tau_k$, i.e., if $a_k > 0$, those terms having $i_k > a_k + 1$. (If $a_k \leq 0$—which can happen if $\tau_i \neq \tau_k$ for any $i < k$—all but $i_k = |a_k|$ are removed and we immediately obtain the needed bound.) This gives

$$\begin{aligned}\mu^*_{GL}(\pi) \leq d_k &\sum_{i_1=-a_1}^{|a_1|} \cdots \sum_{i_{k-1}=-a_{k-1}}^{|a_{k-1}|} \sum_{i_k=-a_k}^{a_k+1} \delta([\nu^{-i_1+1}\tau_1, \nu^{a_1}\tau_1]) \\ &\times \delta([\nu^{i_1}\tau_1, \nu^{b_1}\tau_1]) \times \cdots \times \delta([\nu^{-i_k+1}\tau_k, \nu^{a_k}\tau_k]) \\ &\times \delta([\nu^{i_k}\tau_k, \nu^{b_k}\tau_k]) \otimes c_{i_1,\dots,i_k}\sigma.\end{aligned} \tag{3.7}$$

The only terms in (3.7) which are not part of (3.5) are those corresponding to $i_k = a_k + 1$, i.e.,

$$\begin{aligned}d_k &\sum_{i_1=-a_1}^{|a_1|} \cdots \sum_{i_{k-1}=-a_{k-1}}^{|a_{k-1}|} \delta([\nu^{-i_1+1}\tau_1, \nu^{a_1}\tau_1]) \times \delta([\nu^{i_1}\tau_1, \nu^{b_1}\tau_1]) \\ &\times \cdots \times \delta([\nu^{-i_{k-1}+1}\tau_{k-1}, \nu^{a_{k-1}}\tau_{k-1}]) \times \delta([\nu^{i_{k-1}}\tau_{k-1}, \nu^{b_{k-1}}\tau_{k-1}]) \\ &\times \delta([\nu^{-a_k}\tau_k, \nu^{a_k}\tau_k]) \times \delta([\nu^{a_k+1}\tau_k, \nu^{b_k}\tau_k]) \otimes c_{i_1,\dots,i_k}\sigma.\end{aligned}$$

Now, observe that

$$\begin{aligned}&\delta([\nu^{-a_k}\tau_k,\nu^{a_k}\tau_k])\times\delta([\nu^{a_k+1}\tau_k,\nu^{b_k}\tau_k])\\&\qquad=\delta([\nu^{-a_k}\tau_k,\nu^{b_k}\tau_k])+\mathcal{L}_{sub}(\delta([\nu^{-a_k}\tau_k,\nu^{a_k}\tau_k]),\delta([\nu^{a_k+1}\tau_k,\nu^{b_k}\tau_k]))\end{aligned}$$

in the subrepresentation setting of the Langlands classification for general linear groups. We note that in any term in the Jacquet module of

$$\mathcal{L}_{sub}(\delta([\nu^{-a_k}\tau_k,\nu^{a_k}\tau_k]),\delta([\nu^{a_k+1}\tau_k,\nu^{b_k}\tau_k])),$$

the copy of $\nu^{a_k}\tau_k$ always precedes the copy of $\nu^{a_k+1}\tau_k$; the opposite holds for $\delta([\nu^{-a_k}\tau_k,\nu^{b_k}\tau_k])$. Now, looking at (3.6) with $j=1$, we see the only terms there having $\nu^{-a_k}\tau_k$ in their supercuspidal support are those corresponding to $i_j=-a_k$, which all have the form $\cdots\times\delta([\nu^{-a_k}\tau_k,\nu^{b_k}\tau_k])$. In particular, the only copy of $\nu^{a_k+1}\tau_k$ in such a term always precedes the only copy of $\nu^{a_k}\tau_k$. This means that the terms above coming from $\mathcal{L}_{sub}(\delta([\nu^{-a_k}\tau_k,\nu^{a_k}\tau_k]),\delta([\nu^{a_k+1}\tau_k,\nu^{b_k}\tau_k]))\times\cdots$ cannot contribute to $\mu^*_{GL}(\pi)$. Thus, removing those terms from (3.7), we get

$$\begin{aligned}&\mu^*_{GL}(\pi)\\&\quad\le d_k\sum_{i_1=-a_1}^{|a_1|}\cdots\sum_{i_{k-1}=-a_{k-1}}^{|a_{k-1}|}\sum_{i_k=-a_k}^{|a_k|}\delta([\nu^{-i_1+1}\tau_1,\nu^{a_1}\tau_1])\times\delta([\nu^{i_1}\tau_1,\nu^{b_1}\tau_1])\\&\qquad\quad\times\cdots\times\delta([\nu^{-i_k+1}\tau_k,\nu^{a_k}\tau_k])\times\delta([\nu^{i_k}\tau_k,\nu^{b_k}\tau_k])\otimes c_{i_1,\dots,i_k}\sigma\\&\qquad+d_k\sum_{i_1=-a_1}^{|a_1|}\cdots\sum_{i_{k-1}=-a_{k-1}}^{|a_{k-1}|}\delta([\nu^{-i_1+1}\tau_1,\nu^{a_1}\tau_1])\times\delta([\nu^{i_1}\tau_1,\nu^{b_1}\tau_1])\\&\qquad\quad\times\cdots\times\delta([\nu^{-i_{k-1}+1}\tau_{k-1},\nu^{a_{k-1}}\tau_{k-1}])\times\delta([\nu^{i_{k-1}}\tau_{k-1},\nu^{b_{k-1}}\tau_{k-1}])\\&\qquad\quad\times\delta([\nu^{-a_k}\tau_k,\nu^{b_k}\tau_k])\otimes c_{i_1,\dots,i_k}\sigma.\end{aligned}$$

If we take $i_k=-a_k$ in the first set of sums, we obtain the second set of sums. Therefore,

$$\begin{aligned}&\mu^*_{GL}(\pi)\\&\quad\le 2d_k\sum_{i_1=-a_1}^{|a_1|}\cdots\sum_{i_{k-1}=-a_{k-1}}^{|a_{k-1}|}\sum_{i_k=-a_k}^{|a_k|}\delta([\nu^{-i_1+1}\tau_1,\nu^{a_1}\tau_1])\times\delta([\nu^{i_1}\tau_1,\nu^{b_1}\tau_1])\\&\qquad\quad\times\cdots\times\delta([\nu^{-i_k+1}\tau_k,\nu^{a_k}\tau_k])\times\delta([\nu^{i_k}\tau_k,\nu^{b_k}\tau_k])\otimes c_{i_1,\dots,i_k}\sigma\end{aligned}$$

the needed inequality. ∎

Remark 3.4: Observe that the proof shows more: any irreducible subquotient appearing in both

$$\delta(\Delta_1) \rtimes \delta(\Delta_2, \ldots, \Delta_k; \sigma)_{\psi_a}$$

and

$$\delta(\Delta_k) \rtimes \delta(\Delta_2, \ldots, \Delta_{k-1}; \sigma)_{\psi_a}$$

is square-integrable.

The following is Lemma 2.2 in [M2], which was taken from [Sil2]. For G and Z (the F-points of) a reductive F-group and maximal split torus in the center of G, we have the following:

LEMMA 3.5: *Let Π be a tempered representation of G with central character. If π is any square-integrable (mod Z) subquotient of Π, then $Hom(\pi, \Pi) \neq 0$.*

PROPOSITION 3.6: *Let π be an irreducible ψ_a-generic discrete series representation. Then π is of the form $\delta(\Delta_1, \ldots, \Delta_k; \sigma)_{\psi_a}$ for some $\Delta_1, \ldots, \Delta_k$ as in Proposition 3.3 (possibly $\sigma = 1 \otimes c$). That is, if $i < j$ and $\tau_i \cong \tau_j$, then $a_i < b_i < a_j < b_j$. Further,*

$$\pi \hookrightarrow \delta(\Delta_1) \times \cdots \times \delta(\Delta_k) \rtimes \sigma.$$

Proof. First, we show π is of the form $\delta(\Delta_1, \ldots, \Delta_k; \sigma)_{\psi_a}$. If π^* is a component of $\mathrm{Ind}_{SO_{2n}(F)}^{O_{2n}(F)}(\pi)$ (which has at most two components), it follows from [MT] or [Jan3] that

$$\pi^* \leq \nu^{b_1}\tau_1 \times \nu^{b_1-1}\tau_1 \times \cdots \times \nu^{-a_1}\tau_1 \times \cdots \times \nu^{b_k}\tau_k \times \nu^{b_k-1}\tau_k \times \cdots \times \nu^{-a_k}\tau_k \rtimes \sigma^*,$$

for some $\tau_1, \ldots, \tau_k, a_1, \ldots, a_k, b_1, \ldots, b_k$ satisfying (3.2), (3.3), (DS1′)–(DS3′) (note that there is no (CN) case for $O_{2n}(F)$) and some σ^* an irreducible supercuspidal representation of an even orthogonal group (e.g., in the context of [MT], the number of times $\nu^{\pm x}\tau$ appears in the supercuspidal support of π depends only on $Jord_\tau(\pi)$—see Lemma 3.1 [Jan3]). Therefore,

$$\pi \leq \nu^{b_1}\tau_1 \times \nu^{b_1-1}\tau_1 \times \cdots \times \nu^{-a_1}\tau_1 \times \cdots \times \nu^{b_k}\tau_k \times \nu^{b_k-1}\tau_k \times \cdots \times \nu^{-a_k}\tau_k \rtimes \sigma$$

for some $\sigma \leq \mathrm{Res}_{SO_{2n}(F)}^{O_{2n}(F)}(\sigma^*)$. Since this has π as unique irreducible ψ_a-generic subquotient and $\delta(\Delta_1, \ldots, \Delta_k; \sigma)_{\psi_a}$ appears as a subquotient, we must have $\pi = \delta(\Delta_1, \ldots, \Delta_k; \sigma)_{\psi_a}$.

To see that $\pi \hookrightarrow \delta(\Delta_1) \times \cdots \times \delta(\Delta_k) \rtimes \sigma$, we work inductively. For $k = 1$, it follows from [M4], e.g., that the representation $\delta(\Delta) \rtimes \sigma^*$ has a unique nontempered irreducible subquotient, and the remaining irreducible subquotients are square-integrable. It then follows from restriction that the same holds for $\delta(\Delta) \rtimes \sigma$ (noting that the nontempered subquotient has the form $L(\delta(\Delta); \sigma^*)$, hence splits upon restriction if and only if σ^* does—see Lemmas 4.1 and 4.6 [BJ1]). For $k > 1$, let Π denote the intersection of $\delta(\Delta_1) \rtimes \delta(\Delta_2, \ldots, \Delta_k; \sigma)_{\psi_a}$ and $\delta(\Delta_k) \rtimes \delta(\Delta_1, \ldots, \Delta_{k-1}; \sigma)_{\psi_a}$. Note that it follows easily from the inductive assumption that $\Pi \hookrightarrow \delta(\Delta_1) \times \cdots \times \delta(\Delta_k) \rtimes \sigma$. Now, by Remark 3.4, all the subquotients of Π are square-integrable, hence tempered. Therefore, Lemma 3.5 implies $\pi \hookrightarrow \Pi$, from which the desired embedding follows. ∎

By Remark 3.2, (DS1′)–(DS4′) can be described as follows, and it is convenient for us to use this description:

(DS1) In the $(C1)$ case, if $L(\sigma^{(0)} \times \tau, s)$ has a pole at $s = 0$, then

$$-1 \leq a_i(\tau) \in \mathbb{Z} \setminus \{0\}, \quad \text{for } 1 \leq i \leq e_\tau;$$

(DS2) in the $(C0)$ or (CN) cases, if $L(\tau, Sym^2, s)$ has a pole at $s = 0$, but $L(\sigma^{(0)} \times \tau, s)$ is holomorphic at $s = 0$, then $a_i(\tau) \in \mathbb{Z}_{\geq 0}$, for $1 \leq i \leq e_\tau$;

(DS3) in the $(C\frac{1}{2})$ case, if $L(\tau, \wedge^2, s)$ has a pole at $s = 0$, then $a_i(\tau) \in -\frac{1}{2} + \mathbb{Z}_{\geq 0}$, for $1 \leq i \leq e_\tau$.

Remark 3.7: If $L(\sigma^{(0)} \times \tau, s)$ has a pole at $s = 0$ (case $(C1)$), then $L(\tau, Sym^2, s)$ has a pole at $s = 0$. So, we can see that (DS1) and (DS2) cover all possible cases where $L(\tau, Sym^2, s)$ has a pole at $s = 0$.

3.3. Tempered generic representations. Let $\sigma^{(2)}$ be a ψ_a-generic discrete series representation of $SO_{2n''}(F)$. Let $\beta_1, \ldots, \beta_c$ (with possible repetitions) be irreducible unitary supercuspidal representations of $GL_{k_{\beta_1}}(F), \ldots, GL_{k_{\beta_c}}(F)$, respectively. Take a sequence of square-integrable representations

$$\{St(\beta_i, 2e_i + 1)\}_{i=1}^{c}, \quad 2e_i \in \mathbb{Z}_{\geq 0}$$

of $GL_{k_{\beta_i}(2e_i+1)}(F) (i = 1, 2, \ldots, c)$. Then the unique ψ_a-generic constituent σ of

$$St(\beta_1, 2e_1 + 1) \times \cdots \times St(\beta_c, 2e_c + 1) \rtimes \sigma^{(2)} \tag{3.8}$$

(possibly $\sigma^{(2)} = 1 \otimes c$) is a tempered representation of $SO_{2n}(F)$ ($n = n'' + \sum_{i=1}^{c} (2e_i + 1) k_{\beta_i}$).

It follows from a result of Harish-Chandra (cf. Proposition III.4.1 [W]) that all tempered generic representations of G are obtained this way, and the inducing representation is unique up to conjugation. In particular, the data $\{St(\beta_i, 2e_i+1)\}_{i=1}^{c}$ and $\sigma^{(2)}$ are uniquely determined up to replacements of the following form:

$$St(\beta_i, 2e_i+1) \leftrightarrow St(\widetilde{\beta}_i, 2e_i+1), \quad \sigma^{(2)} \leftrightarrow c^{n_i}\sigma^{(2)},$$

where $n_i = (2e_i+1)k_{\beta_i}$ (noting not all such replacements are nontrivial).

Let $\Pi^{(tg)}(SO_{2n})$ be the equivalence classes of irreducible tempered generic representations of G.

3.4. Generic representations. We consider the following representations of general linear groups: $\delta(\Sigma_1), \ldots, \delta(\Sigma_f)$, where

$$\begin{aligned} \Sigma_1 &= [v^{-q_1}\xi_1, v^{-q_1+w_1}\xi_1],\\ \Sigma_2 &= [v^{-q_2}\xi_2, v^{-q_2+w_2}\xi_2],\\ &\vdots\\ \Sigma_f &= [v^{-q_f}\xi_f, v^{-q_f+w_f}\xi_f], \end{aligned} \tag{3.9}$$

where $\xi_1, \xi_2, \ldots, \xi_f$ are irreducible unitary and supercuspidal, with possible repetitions, $q_i \in \mathbb{R}$, $w_i \in \mathbb{Z}_{\geq 0}$ and $q_i \neq w_i/2$. Let $\sigma^{(t)} \in \Pi^{(tg)}(SO_{2n^*})$ as in (3.8).

We are interested in the induced representation $\delta(\Sigma_1) \times \cdots \times \delta(\Sigma_f) \rtimes \sigma^{(t)}$. In the Grothendieck group, we have $\delta(\Sigma_i) \times \delta(\Sigma_j) = \delta(\Sigma_j) \times \delta(\Sigma_i)$ and $\delta(\Sigma_i) \rtimes \sigma^{(t)} = \delta(\tilde{\Sigma}_i) \rtimes c^{n_i}\sigma^{(t)}$ (where $\delta(\Sigma_i)$ is a representation of $GL_{n_i}(F)$). Therefore, replacing $\sigma^{(t)}$ by $c\sigma^{(t)}$ if necessary, we may assume that the exponents of $\delta(\Sigma_1), \delta(\Sigma_2), \ldots, \delta(\Sigma_f)$ are positive and in non-increasing order (the Langlands inducing data), i.e.,

$$\frac{w_1}{2} - q_1 \geq \frac{w_2}{2} - q_2 \geq \cdots \geq \frac{w_f}{2} - q_f > 0.$$

First, we have Theorem 3.8 below. The proof is essentially the same as in [Jan1] (see Theorem 3.3 and Remark 3.4 in [Jan1]), which in turn is based on that in [Td2]. As the only difference for $SO_{2n}(F)$ is that we must keep track of sign changes on $\sigma^{(t)}$—and these do not affect the argument—we omit the details.

Theorem 3.8: *The representation σ of G defined by*

$$\sigma := \delta(\Sigma_1) \times \delta(\Sigma_2) \times \cdots \times \delta(\Sigma_f) \rtimes \sigma^{(t)}$$

is irreducible if and only if $\{\Sigma_j\}_{j=1}^f$ *and* $\sigma^{(t)}$ *satisfy the following properties:*

(G1) $\delta(\Sigma_i) \times \delta(\Sigma_j)$ *and* $\delta(\Sigma_i) \times \delta(\tilde{\Sigma}_j)$ *are irreducible for all* $1 \leq i \neq j \leq f$ *(i.e., the segment* Σ_i *is not linked to either* Σ_j *or* $\widetilde{\Sigma}_j$ *for* $1 \leq i \neq j \leq f$*);*

(G2) $\delta(\Sigma_i) \rtimes \sigma^{(t)}$ *is irreducible for all* $1 \leq i \leq f$.

From Theorem 3.8, assume that $\{\Sigma_j\}_{j=1}^f$ and $\rho^{(t)}$ satisfy the conditions (G1) and (G2). Then, the representation σ of G defined by

$$\sigma := \delta(\Sigma_1) \times \delta(\Sigma_2) \times \cdots \times \delta(\Sigma_f) \rtimes \sigma^{(t)}$$

(possibly $\sigma^{(t)} = 1 \otimes c$) is irreducible and ψ_a-generic. Moreover, it follows from [M3] that all irreducible ψ_a-generic representations of G can be obtained in this way, and the collection $\delta(\Sigma_1), \delta(\Sigma_2), \ldots, \delta(\Sigma_f), \sigma^{(t)}$ is uniquely determined.

Let $\Pi^{(g)}(SO_{2n})$ be the set of equivalence classes of irreducible generic representations of G.

To understand the irreducibility condition in $(G2)$, we give several theorems below, which are SO_{2n}-analogues of Theorems 4.2 and 4.3 of [M2]. Before this, however, we begin with a lemma which is used in some of the proofs (and also helps explain the similarity of the results below with those of [M2]).

Lemma 3.9: *Suppose* $\sigma_{SO}^{(0)} \leq Res_{SO}^{O}(\sigma_O^{(0)})$ *are irreducible supercuspidal representations and* $(\rho; \sigma_{SO}^{(0)})$ *satisfies* $(C\alpha)$ *with* $\alpha \in \{0, \frac{1}{2}, 1\}$. *Let* π *be an irreducible representation of* $O(2n, F)$ *whose supercuspidal support is contained in* $\{\nu^x\rho\}_{x\in\alpha+\mathbb{Z}} \cup \{\sigma_O^{(0)}\}$ *and* $\pi' \leq Res_{SO}^{O}(\pi)$, *also irreducible. If* τ *is an irreducible representation of a general linear group having supercuspidal support in* $\{\nu^x\rho\}_{x\in\alpha+\mathbb{Z}}$, *then*

$$\tau \rtimes \pi \text{ is irreducible } \Leftrightarrow \tau \rtimes \pi' \text{ is irreducible.}$$

Proof. First, suppose $\hat{c}\sigma_O^{(0)} \cong \sigma_O^{(0)}$, where $\hat{c}$ is the character of $O_{2n}(F)$ which is 1 on $SO_{2n}(F)$ and -1 on $O_{2n}(F)\backslash SO_{2n}(F)$. Since $(\rho, \sigma^{(0)}$ is $(C\alpha)$ for some $\alpha \in \{0, \frac{1}{2}, 1\}$, the hypotheses of Theorem 5.3 [Jan5] are satisfied. Now, if $\tau \rtimes \pi$ is irreducible, it follows from Theorem 5.3 [Jan5] that $\text{Res}_{SO}^{O}(\tau \rtimes \pi)$ has two components. However, applying Theorem 5,3 [Jan5] also tells us $\text{Res}_{SO}^{O}(\pi) = \pi' \oplus c\pi'$, with $c\pi' \not\cong \pi'$. We then have (see Lemma 4.2 [BJ1], e.g.)

$$\text{Res}_{SO}^{O}(\tau \rtimes \pi) = (\tau \rtimes \pi') \oplus (\tau \rtimes c\pi'),$$

from which we see that $\tau \rtimes \pi'$ and $\tau \rtimes c\pi'$ must both be irreducible. On the other hand, if $\tau \rtimes \pi = \sum_i \lambda_i$ is reducible, Theorem 5.3 [Jan5] tells us

$\mathrm{Res}^O_{SO}(\lambda_i) = \lambda'_i \oplus c\lambda'_i$ for each i. Therefore,

$$\mathrm{Res}^O_{SO}(\tau \rtimes \pi) = (\tau \rtimes \pi') + (\tau \rtimes c\pi') = \sum_i (\lambda'_i + c\lambda'_i),$$

which implies the reducibility of $\tau \rtimes \pi'$ and $\tau \rtimes c\pi'$.

Now, suppose $\hat{c}\sigma_O^{(0)} \not\cong \sigma_O^{(0)}$. By Theorem 4.1 [Jan5], we then have $\mathrm{Res}^O_{SO}(\pi) = \pi'$ with $c\pi' \cong \pi'$. We then have (cf. Lemma 4.2 [BJ1])

$$\mathrm{Res}^O_{SO}(\tau \rtimes \pi) = \tau \rtimes \pi'.$$

It then follows immediately that $\tau \rtimes \pi'$ irreducible implies $\tau \rtimes \pi$ irreducible. On the other hand, by Theorem 4.1 [Jan5], we have $\tau \rtimes \pi$ irreducible implies $\mathrm{Res}^O_{SO}(\tau \rtimes \pi) = \tau \rtimes \pi'$ irreducible, finishing this case and the lemma. ∎

THEOREM 3.10: *For $\Sigma = [\nu^{-q}\xi, \nu^{-q+w}\xi]$ as above, $\delta(\Sigma) \rtimes \sigma^{(t)}$ is irreducible if and only if the following hold:*

- (G3) *$\delta(\Sigma) \times St(\beta_j, 2e_j + 1)$ and $\delta(\tilde{\Sigma}) \times St(\beta_j, 2e_j + 1)$ are irreducible for all $1 \leq j \leq c$ (i.e., Σ and $\tilde{\Sigma}$ not linked to any $[\nu^{-e_j}\beta_j, \nu^{e_j}\beta_j]$, $1 \leq j \leq c$), and*
- (G4) *$\delta(\Sigma) \rtimes \sigma^{(2)}$ is irreducible.*

Proof. Let $\delta_i = St(\beta_i, 2e_i + 1)$.

First, suppose $(G3)$ and $(G4)$ hold. We must show $\nu^x\delta \rtimes \sigma^{(t)}$ is irreducible. Let $\pi = L(\delta(\Sigma) \otimes \sigma^{(t)})$. Then,

$$\begin{aligned} \pi &\hookrightarrow \delta(\tilde{\Sigma}) \times \delta_1 \times \cdots \times \delta_c \rtimes c^n\sigma^{(2)} \\ &\quad \text{(using (G3))} \\ &\cong \delta_1 \times \cdots \times \delta_c \times \delta(\tilde{\Sigma}) \rtimes c^n\sigma^{(2)} \\ &\quad \text{(using (G4))} \\ &\cong \delta_1 \times \cdots \times \delta_c \rtimes \delta(\Sigma) \rtimes \sigma^{(2)} \\ &\quad \text{(using (G3))} \\ &\cong \delta(\Sigma) \times \delta_1 \times \cdots \times \delta_c \rtimes \sigma^{(2)}. \end{aligned}$$

Therefore, $\pi \hookrightarrow \delta(\Sigma) \rtimes T'$ for some irreducible $T' \leq \delta_1 \times \cdots \times \delta_c \rtimes \sigma^{(2)}$. Now, π appears as the Langlands quotient of $\delta(\Sigma) \rtimes \sigma^{(t)}$, from which it follows that $T' = \sigma^{(t)}$. We now have π appearing as the unique irreducible quotient of the standard module $\delta(\Sigma) \rtimes \sigma^{(t)}$ and as a subrepresentation. This contradicts multiplicity one unless we have irreducibility, as needed.

Now, suppose (at least) one of (G3) or (G4) fails. We must show $\delta(\Sigma) \rtimes \sigma^{(t)}$ is reducible.

First, suppose (G3) fails with some $\delta(\tilde{\Sigma}) \times \delta_i$ reducible; without loss of generality, say $\delta(\tilde{\Sigma}) \times \delta_1$. Write $\sigma^{(t)} \hookrightarrow \delta_1 \rtimes T'$. Now, suppose $\delta(\Sigma) \rtimes \sigma^{(t)}$ were irreducible. Then, $\delta(\Sigma) \rtimes \sigma^{(t)} = \pi$ as above. We have

$$\pi \cong \delta(\tilde{\Sigma}) \rtimes c^{n_\Sigma}\sigma^{(t)} \hookrightarrow \delta(\tilde{\Sigma}) \times \delta_1 \rtimes T'.$$

Since $\delta_1 = St(\beta_1, 2e_1+1)$, reducibility implies $\tilde{\xi} \cong \beta_1$ and $q-w+1 \leq -e_1 \leq q+1 \leq e_1$. Now, since (in the Grothendieck group)

$$\begin{aligned}\delta([\nu^{q-w}\beta_1, \nu^q\beta_1]) \times \delta([\nu^{-e_1}\beta_1, \nu^{e_1}\beta_1]) \\ = & \mathcal{L}_{sub}(\delta([\nu^{q-w}\beta_1, \nu^q\beta_1]) \otimes \delta([\nu^{-e_1}\beta_1, \nu^{e_1}\beta_1])) \\ & + \delta([\nu^{q-w}\beta_1, \nu^{e_1}\beta_1]) \times \delta([\nu^{-e_1}\beta_1, \nu^q\beta_1])\end{aligned}$$

(subrepresentation version of the Langlands classification for general linear groups), Lemma 5.5 [Jan2] tells us

$$\pi \hookrightarrow \mathcal{L}_{sub}(\delta([\nu^{q-w}\beta_1, \nu^q\beta_1]) \otimes \delta([\nu^{-e_1}\beta_1, \nu^{e_1}\beta_1])) \rtimes T'$$

or

$$\delta([\nu^{q-w}\beta_1, \nu^{e_1}\beta_1]) \times \delta([\nu^{-e_1}\beta_1, \nu^q\beta_1]) \rtimes T'.$$

Now, $\delta([\nu^{q-w}\beta_1, \nu^{e_1}\beta_1]) \times \delta([\nu^{-e_1}\beta_1, \nu^q\beta_1])$ is the generic subquotient of $\delta([\nu^{q-w}\beta_1, \nu^q\beta_1]) \times \delta([\nu^{-e_1}\beta_1, \nu^{e_1}\beta_1])$. Therefore, we must have

$$\pi \hookrightarrow \delta([\nu^{q-w}\beta_1, \nu^{e_1}\beta_1]) \times \delta([\nu^{-e_1}\beta_1, \nu^q\beta_1]) \rtimes T'.$$

On the other hand, π appears as a subrepresentation in

$$\mathcal{L}_{sub}\left(\delta([\nu^{q-w}\beta_1, \nu^q\beta_1]) \otimes \delta([\nu^{-e_1}\beta_1, \nu^{e_1}\beta_1])\right) \rtimes T'$$

(a fairly easy consequence of the Langlands classification). Also, π appears with multiplicity one in $\delta([\nu^{q-w}\beta_1, \nu^q\beta_1]) \rtimes \sigma^{(t)}$, hence with multiplicity one in

$$\delta([\nu^{q-w}\beta_1, \nu^q\beta_1]) \times \delta([\nu^{-e_1}\beta_1, \nu^{e_1}\beta_1]) \rtimes T'$$

(noting that if $\delta([\nu^{-e_1}\beta_1, \nu^{e_1}\beta_1]) \rtimes T' = \sigma^{(t)} \oplus T^*$, π does not appear in $\delta([\nu^{q-w}\beta_1, \nu^q\beta_1]) \rtimes T^*$ as $L(\delta([\nu^{-q}\beta_1, \nu^{-q+w}\beta_1]) \otimes T^*)$ is the unique irreducible subquotient of $(\delta([\nu^{q-w}\beta_1, \nu^{-q+w}\beta_1]) \rtimes T^*$ having its central character—see [BJ2], e.g.). In particular, π cannot then appear in

$$\delta([\nu^{q-w}\beta_1, \nu^{e_1}\beta_1]) \times \delta([\nu^{-e_1}\beta_1, \nu^q\beta_1]) \rtimes T',$$

a contradiction. Thus, we must have $\delta(\Sigma) \rtimes \sigma^{(t)}$ reducible, as needed.

Now, suppose it is $\delta(\Sigma) \times \delta_1$ which is reducible. Observe that $\delta(\Sigma) \times \delta_1$ is reducible if and only if $\delta(\tilde{\Sigma}) \times \tilde{\delta}_1$ is reducible. The same argument as above then shows that $\delta(\tilde{\Sigma}) \rtimes \widetilde{\sigma^{(t)}}$ is reducible, implying $\delta(\Sigma) \rtimes \sigma^{(t)}$ reducible, as needed.

The proof for (G4) is similar. Here, since (G3) has been addressed above, we are free to assume (G3) holds but (G4) fails. Then,

$$\pi \hookrightarrow \delta(\Sigma) \times \delta_1 \times \cdots \times \delta_c \rtimes \sigma^{(2)} \cong \delta_1 \times \cdots \times \delta_c \times \delta(\Sigma) \rtimes \sigma^{(2)}.$$

Write $\delta(\Sigma) \rtimes \sigma^{(2)} = L(\delta(\Sigma) \otimes \sigma^{(2)}) + \sum_j Q_j$. Note that by [M3], $L(\delta(\Sigma) \otimes \sigma^{(2)})$ is not generic. Again, by Lemma 5.5 [Jan2],

$$\pi \hookrightarrow \delta_1 \times \cdots \times \delta_c \times \delta(\Sigma) \rtimes \sigma^{(2)}$$

$$\Downarrow$$

$$\pi \hookrightarrow \delta_1 \times \cdots \times \delta_c \rtimes L(\delta(\Sigma) \otimes \sigma^{(2)})$$

or

$$\pi \hookrightarrow \delta_1 \times \cdots \times \delta_c \rtimes Q_i$$

for some i. As above, genericity implies it must be the second, but properties of the Langlands classification require it be the first. We again have π appearing with multiplicity one in $\delta_1 \times \cdots \times \delta_c \times \delta(\Sigma) \rtimes \sigma^{(2)}$, giving a contradiction and finishing the proof. ∎

THEOREM 3.11: *For Σ as above, $\delta(\Sigma) \rtimes \sigma^{(2)}$ is irreducible if and only if the following hold:*

- (G5) *$\delta(\Sigma) \times \delta(D_i(\tau))$ and $\delta(\tilde{\Sigma}) \times \delta(D_i(\tau))$ are irreducible for any $D_i(\tau) = [v^{-a_i(\tau)}\tau, v^{b_i(\tau)}\tau]$, $i = 1, 2, \ldots, e_\tau, \tau \in P'$ (i.e., Σ and $\widetilde{\Sigma}$ are not linked to any segment $[v^{-a_i(\tau)}\tau, v^{b_i(\tau)}\tau]$, $i = 1, 2, \ldots, e_\tau, \tau \in P'$), and*
- (G6) *either* (a) *$\delta(\Sigma) \rtimes \sigma^{(0)}$ is irreducible, or* (b) *$q = -1$ (so $(\xi, \sigma^{(0)})$ satisfies (C1)) , and $\xi \in P'$ with $a_j(\xi) = -1$ and $b_j(\xi) \geq -q + w = 1 + w$ for some $1 \leq j \leq e_\xi$.*

Note that if we let $X' = \{\tau \in P' \mid (\tau, \sigma^{(0)})$ satisfies $(C1)\}$, we may reformulate (G6) *as follows: either $\delta(\Sigma)$ is not linked to any $\tau \in X'$ or there is a $\tau \in X'$ such that $\tau \cong \xi$, $q = -1$, and $a_j(\xi) = -1$, $1 + w \leq b_j(\xi)$ (linked to τ meaning linked to the segment consisting of τ only).*

Proof. We first address ($\Leftarrow$). That is, we assume (G5) and (G6) both hold and show $\delta(\Sigma) \rtimes \sigma^{(2)}$ is irreducible. We assume it is (G6)(a) which holds, and comment on the changes needed for (G6)(b) afterwards. Let $\pi \hookrightarrow \delta(\Sigma) \rtimes \sigma^{(2)}$ be an irreducible subrepresentation. Then,

$$
\begin{aligned}
\pi &\hookrightarrow \delta(\Sigma) \times \delta([\nu^{-a_1}\tau_1, \nu^{b_1}\tau_1]) \times \cdots \times \delta([\nu^{-a_k}\tau_k, \nu^{b_k}\tau_k]) \rtimes \sigma^{(0)} \\
&\quad \text{(using (G5))} \\
&\cong \delta([\nu^{-a_1}\tau_1, \nu^{b_1}\tau_1]) \times \cdots \times \delta([\nu^{-a_k}\tau_k, \nu^{b_k}\tau_k]) \times \delta(\Sigma) \rtimes \sigma^{(0)} \\
&\quad \text{(using (G6)(a))} \\
&\hookrightarrow \delta([\nu^{-a_1}\tau_1, \nu^{b_1}\tau_1]) \times \cdots \times \delta([\nu^{-a_k}\tau_k, \nu^{b_k}\tau_k]) \times \delta(\tilde{\Sigma}) \rtimes c^{n_\Sigma}\sigma^{(0)} \\
&\quad \text{(using (G5))} \\
&\cong \delta(\tilde{\Sigma}) \times \delta([\nu^{-a_1}\tau_1, \nu^{b_1}\tau_1]) \times \cdots \times \delta([\nu^{-a_k}\tau_k, \nu^{b_k}\tau_k]) \rtimes c^{n_\Sigma}\sigma^{(0)}.
\end{aligned} \tag{3.10}
$$

In particular, π has a term of the form $\delta(\tilde{\Sigma}) \otimes \theta$ in its Jacquet module. From properties of the Langlands classification ([BJ2]), the only term of the form $\delta(\tilde{\Sigma}) \otimes \theta$ in the Jacquet module of $\delta(\tilde{\Sigma}) \rtimes c^{n_\Sigma}\sigma^{(2)}$ is $\delta(\tilde{\Sigma}) \otimes c^{n_\Sigma}\sigma^{(2)}$. Thus $\theta = c^{n_\Sigma}\sigma^{(2)}$. Now,

$$
\begin{gathered}
\pi \hookrightarrow \delta(\tilde{\Sigma}) \rtimes c^{n_\Sigma}\sigma^{(2)} \\
\Downarrow \\
\pi \cong L(\delta(\Sigma) \otimes \sigma^{(2)})
\end{gathered}
$$

as $L(\delta(\Sigma) \otimes \sigma^{(2)})$ is the Langlands subrepresentation of $\delta(\tilde{\Sigma}) \rtimes c^{n_\Sigma}\sigma^{(2)}$. Thus, π appears as both irreducible subrepresentation (from above) and unique irreducible quotient in $\delta(\Sigma) \rtimes \sigma^{(2)}$, contradicting multiplicity one in the Langlands classification unless we have irreducibility.

We now discuss the changes needed if it is (G6)(b) which holds. Since $\delta([\nu^{-a_i}\tau_i, \nu^{b_i}\tau_i]) \times \delta([\nu^{-a_j}\tau_j, \nu^{b_j}\tau_j])$ is irreducible for all $i \neq j$ (hence may be commuted while preserving equivalences), we may without loss of generality assume $\tau_k \cong \xi$ with $a_k = -1$. Then, to produce the inversion of $\delta(\Sigma)$ of Equation (3.10), we do the following:

$$
\begin{gathered}
\pi \hookrightarrow \delta(\Sigma) \times \delta([\nu^{-a_1}\tau_1, \nu^{b_1}\tau_1]) \times \cdots \times \delta([\nu^{-a_{k-1}}\tau_{k-1}, \nu^{b_{k-1}}\tau_{k-1}]) \times \delta([\nu\xi, \nu^{b_k}\xi]) \rtimes \sigma^{(0)} \\
\text{(using (G5))} \\
\cong \delta([\nu^{-a_1}\tau_1, \nu^{b_1}\tau_1]) \times \cdots \times \delta([\nu^{-a_{k-1}}\tau_{k-1}, \nu^{b_{k-1}}\tau_{k-1}]) \times \delta(\Sigma) \times \delta([\nu\xi, \nu^{b_k}\xi]) \rtimes \sigma^{(0)} \\
\Downarrow \text{ (Lemma 5.5 [Jan2])} \\
\pi \hookrightarrow \delta([\nu^{-a_1}\tau_1, \nu^{b_1}\tau_1]) \times \cdots \times \delta([\nu^{-a_{k-1}}\tau_{k-1}, \nu^{b_{k-1}}\tau_{k-1}]) \times \delta(\Sigma) \rtimes \theta
\end{gathered}
$$

for some irreducible $\theta \leq \delta([\nu\xi, \nu^{b_k}\xi]) \rtimes \sigma^{(0)}$. By genericity,

$$\theta = \delta([\nu\xi, \nu^{b_k}\xi]; \sigma^{(0)}).$$

We now claim that

$$\delta(\Sigma) \rtimes \delta([\nu\xi, \nu^{b_k}\xi]; \sigma^{(0)}) \cong \delta(\tilde{\Sigma}) \rtimes c^{n_\Sigma}\delta([\nu\xi, \nu^{b_k}\xi]; \sigma^{(0)})$$

by irreducibility. By [M5] the corresponding representation of $O_{2n}(F)$ is irreducible; the claim then follows from Lemma 3.9. The rest of the argument proceeds the same way.

We now turn to ($\Rightarrow$). First, suppose (G5) fails. We first suppose

$$\delta(\tilde{\Sigma}) \times \delta([\nu^{-a_i}\tau_i, \nu^{b_i}\tau_i]) \quad \text{is reducible;}$$

without loss of generality, $\delta(\tilde{\Sigma}) \times \delta([\nu^{-a_1}\tau_1, \nu^{b_1}\tau_1])$ is reducible. Then $\xi \cong \tilde{\tau}_1 \cong \tau_1$ and we may write $\sigma^{(2)} \hookrightarrow \delta([\nu^{-a_1}\tau_1, \nu^{b_1}\tau_1]) \rtimes \delta'_{SO}$. Now, suppose $\delta(\Sigma) \rtimes \sigma^{(2)}$ were irreducible. Then, $\delta(\Sigma) \rtimes \sigma^{(2)} = \pi$ as above. We have

$$\pi \cong \delta(\tilde{\Sigma}) \rtimes c^{n_\Sigma}\sigma^{(2)} \hookrightarrow \delta(\tilde{\Sigma}) \times \delta([\nu^{-a_1}\tau_1, \nu^{b_1}\tau_1]) \rtimes c^{n_\Sigma}\delta'_{SO}.$$

As $\delta(\tilde{\Sigma}) = \delta([\nu^{-w+q}\tau_1, \nu^q\tau_1])$, reducibility implies $-w+q < -a_1 \leq q+1$ and $q < b_1$. Now, since (in the Grothendieck group)

$$\begin{aligned}\delta([\nu^{-w+q}\tau_1, \nu^q\tau_1]) &\times \delta([\nu^{-a_1}\tau_1, \nu^{b_1}\tau_1])\\ &= \mathcal{L}_{sub}(\delta([\nu^{-w+q}\tau_1, \nu^q\tau_1]) \otimes \delta([\nu^{-a_1}\tau_1, \nu^{b_1}\tau_1]))\\ &\quad + \delta([\nu^{-a_1}\tau_1, \nu^q\tau_1]) \times \delta([\nu^{-w+q}\tau_1, \nu^{b_1}\tau_1]),\end{aligned}$$

we have (Lemma 5.5 [Jan2])

$$\pi \hookrightarrow \mathcal{L}_{sub}(\delta([\nu^{-w+q}\tau_1, \nu^q\tau_1]) \times \delta([\nu^{-a_1}\tau_1, \nu^{b_1}\tau_1])) \rtimes c^{n_\Sigma}\delta'_{SO}$$

or

$$\delta([\nu^{-a_1}\tau_1, \nu^q\tau_1]) \times \delta([\nu^{-w+q}\tau_1, \nu^{b_1}\tau_1]) \rtimes c^{n_\Sigma}\delta'_{SO}.$$

As $\delta([\nu^{-a_1}\tau_1, \nu^q\tau_1]) \times \delta([\nu^{-w+q}\tau_1, \nu^{b_1}\tau_1])$ is the generic subquotient of $\delta([\nu^{-w+q}\tau_1, \nu^q\tau_1]) \times \delta([\nu^{-a_1}\tau_1, \nu^{b_1}\tau_1])$, we must have

$$\pi \hookrightarrow \delta([\nu^{-a_1}\tau_1, \nu^q\tau_1]) \times \delta([\nu^{-w+q}\tau_1, \nu^{b_1}\tau_1]) \rtimes c^{n_\Sigma}\delta'_{SO}.$$

However, we claim that this is not the case.

To see this, note that by Frobenius reciprocity,

$$\mu^*(\pi) \geq \delta([\nu^{-w+q}\tau_1, \nu^q\tau_1]) \otimes c^{n_\Sigma}\sigma^{(0)}.$$

Now, in the notation of Section 3.1, we have (in general)

$$(3.11)\quad M_D^*\left(\delta([\nu^{-a}\tau,\nu^b\tau])\right) = \sum_{x=-a}^{b+1}\sum_{s=x}^{b+1}\delta([\nu^{1-x}\tilde{\tau},\nu^a\tilde{\tau}])\times\delta([\nu^s\tau,\nu^b\tau])\otimes\delta([\nu^x\tau,\nu^{s-1}\tau])\otimes c^{n_\tau(x+a)}.$$

If $\mu^*(c^{n_\Sigma}\delta'_{SO})=\sum_i\lambda_i\otimes\theta_i$, looking at

$$\mu^*\left(\delta([\nu^{-a_1}\tau_1,\nu^q\tau_1])\times\delta([\nu^{-w+q}\tau_1,\nu^{b_1}\tau_1])\rtimes c^{n_\Sigma}\delta'_{SO}\right),$$

we see that we must have

$$\delta([\nu^{-w+q}\tau_1,\nu^q\tau_1])\leq\delta([\nu^{1-x}\tau_1,\nu^{-q+w}\tau_1])\times\delta([\nu^s\tau_1,\nu^{b_1}\tau_1])\times\delta([\nu^{1-y}\tau_1,\nu^{a_1}\tau_1])\times\delta([\nu^t\tau_1,\nu^q\tau_1])\times\lambda_i,$$

with $-w+q\leq x\leq s\leq b_1+1$ and $-a_1\leq y\leq t\leq q+1$. Since $b_1,-q+w>q$, we must have $s=b_1+1$ and $1-x=-q+w+1$, eliminating those terms and leaving

$$\delta([\nu^{-w+q}\tau_1,\nu^q\tau_1])\leq\delta([\nu^{1-y}\tau_1,\nu^{a_1}\tau_1])\times\delta([\nu^t\tau_1,\nu^q\tau_1])\times\lambda_i,$$

with $-a_1\leq y\leq t\leq q+1$. Since $t\geq-a_1>-w+q$ and $y\leq q+1<w-q+1\Rightarrow 1-y>-w+q$, we cannot have $\nu^{-w+q}\tau_1$ appearing in the supercuspidal support of $\delta([\nu^{1-y}\tau_1,\nu^{a_1}\tau_1])\times\delta([\nu^t\tau,\nu^q\tau_1])$, hence it must come from λ_i. However, by the Casselman criterion, any contribution $\delta([\nu^{-w+q}\tau_1,\nu^j\tau_1])$ from λ_i must have $j>w-q$—in particular, j too large to allow us to get $\delta([\nu^{-w+q}\tau_1,\nu^q\tau_1])$. Thus there are no terms of the form $\delta([\nu^{-w+q}\tau_1,\nu^q\tau_1])\otimes\cdots$ in the Jacquet module of $\delta([\nu^{-a_1}\tau_1,\nu^q\tau_1])\times\delta([\nu^{-w+q}\tau_1,\nu^{b_1}\tau_1])\rtimes c^{n_\Sigma}\delta'_{SO}$, providing the needed contradiction and finishing the proof for this case.

Next, suppose (G5) fails with $\delta(\Sigma)\times\delta([\nu^{-a_1}\tau_1,\nu^{b_1}\tau_1])$ reducible. Then $\tau_1=\xi$ and either $-q<-a_1\leq-q+w+1$ and $-q+w<b_1$, or $-a_1<-q\leq b_1+1$ and $b_1<-q+w$. The former has $\delta(\tilde{\Sigma})\times\delta([\nu^{-a_1}\tau_1,\nu^{b_1}\tau_1])$ reducible, hence is covered by the previous discussion. Thus we may assume $-a_1<-q\leq b_1+1, b_1<-q+w\Rightarrow-q+w>b_1>a_1>q$. We have

$$\pi\leq\delta([\nu^{-q}\tau,\nu^{-q+w}\tau])\times\delta([\nu^{-a_1}\tau,\nu^{b_1}\tau])\rtimes\delta'_{SO}$$

$$\Downarrow$$

$$\pi\leq\delta([\nu^{-a_1}\tau,\nu^{-q+w}\tau])\times\delta([\nu^{-q}\tau,\nu^{b_1}\tau])\rtimes\delta'_{SO}$$

by genericity. Again, if $\delta(\Sigma) \rtimes \sigma^{(2)}$ were irreducible, we would have to have at least one term of the form $\delta([\nu^{-w+q}\tau, \nu^{q}\tau]) \otimes \dots$ in

$$\mu^*(\pi) \leq \mu^*(\delta([\nu^{-a_1}\tau, \nu^{-q+w}\tau]) \times \delta([\nu^{-q}\tau, \nu^{b_1}\tau]) \rtimes \delta'_{SO}).$$

The proof that this does not happen is similar, but significantly easier, than that in the previous case.

It remains to deal with (G6). From above, we may assume (G5) holds.

Suppose $\delta([\nu^{-q}\xi, \nu^{-q+w}\xi]) \rtimes \sigma^{(0)}$ were reducible. By [M5] (applied to the corresponding representation of $O(2n, F)$) and restriction, this has unique irreducible quotient $L(\delta([\nu^{-q}\xi, \nu^{-q+w}\xi]) \otimes \sigma^{(0)})$ (the Langlands quotient) and remaining subquotients tempered subrepresentations (almost always discrete series). By [M3], the ψ_a-generic subquotient is one of the tempered subrepresentations. Let

$$\delta_{\psi_a}([\nu^{-q}\xi, \nu^{-q+w}\xi]); \sigma^{(0)})$$

denote the ψ_a-generic subrepresention.

Now, since $\delta([\nu^{-q}\xi, \nu^{-q+w}\xi]) \rtimes \sigma^{(2)}$ is irreducible, we have

$$\begin{aligned}\pi &\hookrightarrow \delta([\nu^{-q}\xi, \nu^{-q+w}\xi]) \rtimes \sigma^{(2)}\\ &\hookrightarrow \delta([\nu^{-q}\xi, \nu^{-q+w}\xi]) \times \delta([\nu^{-a_1}\tau_1, \nu^{b_1}\tau_1]) \times \cdots \times \delta([\nu^{-a_k}\tau_k, \nu^{b_k}\tau_k]) \rtimes \sigma^{(0)}\\ &\cong \delta([\nu^{-a_1}\tau_1, \nu^{b_1}\tau_1]) \times \cdots \times \delta([\nu^{-a_k}\tau_k, \nu^{b_k}\tau_k]) \times \delta([\nu^{-q}\xi, \nu^{-q+w}\xi) \rtimes \sigma^{(0)}\end{aligned}$$

using the irreducibility for $(G5)$ already proved. By Lemma 5.5 [Jan2], this implies

$$\pi \hookrightarrow \delta([\nu^{-a_1}\tau_1, \nu^{b_1}\tau_1]) \times \cdots \times \delta([\nu^{-a_k}\tau_k, \nu^{b_k}\tau_k]) \rtimes \theta$$

for some $\theta \leq \delta([\nu^{-q}\xi, \nu^{-q+w}\xi]) \rtimes \sigma^{(0)}$. We must have θ ψ_a-generic, so

$$\pi \hookrightarrow \delta([\nu^{-a_1}\tau_1, \nu^{b_1}\tau_1]) \times \cdots \times \delta([\nu^{-a_k}\tau_k, \nu^{b_k}\tau_k]) \rtimes \delta_{\psi_a}([\nu^{-q}\xi, \nu^{-q+w}\xi]; \sigma^{(0)}).$$

Since $\pi \cong \delta([\nu^{-w+q}\xi, \nu^{q}\xi]) \rtimes c^n\sigma^{(2)}$ by irreducibility, we must have a term of the form $\delta([\nu^{-w+q}\xi, \nu^{q}\xi]) \otimes \cdots$ in

$$\begin{aligned}&\mu^*(\delta([\nu^{-a_1}\tau_1, \nu^{b_1}\tau_1]) \times \cdots \times \delta([\nu^{-a_k}\tau_k, \nu^{b_k}\tau_k]) \rtimes (\delta_{\psi_a}([\nu^{-q}\xi, \nu^{-q+w}\xi]; \sigma^{(0)}))\\ =&M^*_D(\delta([\nu^{-a_1}\tau_1, \nu^{b_1}\tau_1])\times\cdots\times\delta([\nu^{-a_k}\tau_k, \nu^{b_k}\tau_k]))\rtimes\mu^*(\delta_{\psi_a}([\nu^{-q}\xi, \nu^{-q+w}\xi]; \sigma^{(0)}))\end{aligned}$$

(see Theorem 3.1). If

$$\mu^*(\delta_{\psi_a}([\nu^{-q}\xi, \nu^{-q+w}\xi]; \sigma^{(0)})) = \sum_i \lambda_i \otimes \theta_i,$$

this requires (using Equation (3.11))

$$\delta([\nu^{-w+q}\xi,\nu^{q}\xi]) \leq \delta([\nu^{1-x_1}\tilde{\tau}_1,\nu^{a_1}\tilde{\tau}_1]) \times \delta([\nu^{s_1}\tau_1,\nu^{b_1}\tau_1]) \times \cdots \times \delta([\nu^{1-x_k}\tilde{\tau}_k,\nu^{a_k}\tilde{\tau}_k]) \times \delta([\nu^{s_k}\tau_k,\nu^{b_k}\tau_k]) \times \lambda_i.$$

We show this cannot happen.

To show this, we focus on where $\nu^{-w+q}\xi$ can appear above. First, it cannot appear in λ_i as this would require (by Equation (3.11) applied to $\delta_{\psi_a}([\nu^{-q}\xi,\nu^{-q+w}\xi];\sigma^{(0)}) \leq \delta([\nu^{-q}\xi,\nu^{-q+w}\xi]) \rtimes \sigma^{(0)}$)

$$\mu^*(\delta_{\psi_a}([\nu^{-q}\xi,\nu^{-q+w}\xi];\sigma^{(0)})) \geq \delta([\nu^{-w+q}\xi,\nu^{q}\xi]) \otimes \cdots ,$$

contradicting the Casselman criterion. It clearly cannot appear in $\delta([\nu^{1-x_i}\tilde{\tau}_i,\nu^{a_i}\tilde{\tau}_i]) \times \delta([\nu^{s_i}\tau_i,\nu^{b_i}\tau_i])$ if $\tau_i \not\cong \xi$, so assume $\tau_i \cong \xi$. Now, if $\nu^{-w+q}\xi$ appears in $\delta([\nu^{s_i}\xi,\nu^{b_i}\xi])$, we must have $s_i = -w+q$ and $b_i \leq q$ to contribute to a $\delta([\nu^{-w+q}\xi,\nu^{q}\xi])$. However, $-a \leq -w+q \leq s_i$ and $b_i \leq q$ imply $q \geq b_i > a_i \geq w-q$, contradicting $w > 2q$. Thus $\nu^{-w+q}\xi$ cannot appear in $\delta([\nu^{s_i}\xi,\nu^{b_i}\xi])$.

The only possibility remaining is that $\nu^{-w+q}\xi$ appears in $\delta([\nu^{1-x_i}\xi,\nu^{a_i}\xi])$. In this case, we must have $1-x_i = -w+q$ and $a_i \leq q$ to contribute to a $\delta([\nu^{-w+q}\xi,\nu^{q}\xi])$. Now,

$$-a_i \leq x = 1+w-q \leq b_i+1$$

implies $\delta([\nu^{-w+q}\xi,\nu^{q}\xi]) \times \delta([\nu^{-a_i}\xi,\nu^{b_i}\xi])$ is reducible—hence (G5) fails—unless either $-a_i \leq -q+w$ or $q < -a_i-1$. Since $-a_i \leq -q+w$ implies $a_i \geq w-q > q$—a contradiction—we must have $q < -a_i-1$. Combining this with $a_i \geq q$, we have that $a_i = -1$. Further, combining $-1 = a \leq q$ and $q < -a-1 = 0$, we see that we must also have $q=-1$. Since $b_i \geq w-q = w+1$, we see that (G6)(b) holds. Therefore, we have proved that if (G6)(a) does not hold, then either there is a contradiction, or (G6)(b) holds. This completes the proof of this theorem. ∎

Remark 3.12: See Remark 4.9 for further discussion of the condition $1+w \leq b_j(\tau)$ in (G6).

THEOREM 3.13: *For Σ as above, $\delta(\Sigma) \rtimes \sigma^{(0)}$ is irreducible if and only if the following hold:*

(G7) $\xi \not\cong \widetilde{\xi}$; *or*

(G8) $\xi \cong \tilde{\xi}$ *and the following:* (i) *if* $(\xi, \sigma^{(0)})$ *is* $(C\alpha)$, $\alpha \in \{0, \frac{1}{2}, 1\}$, *then* $\pm\alpha \notin \{-q, -q+1, \ldots, -q+w\}$; (ii) *if* $(\xi, \sigma^{(0)})$ *is* (CN), *then* $q \notin \mathbb{Z}_{\geq 0}$.

Proof. First, if (G7) holds, let $\pi = L(\delta([\nu^{-q}\xi, \nu^{-q+w}\xi]) \otimes \sigma^{(0)})$. Then, noting that $\nu^x\xi \times \nu^y\tilde{\xi}$ and $\nu^x\xi \rtimes \sigma^{(0)}$ are irreducible for all $x, y \in \mathbb{R}$, we may argue as in the proof of Theorem 3.8 to get

$$\begin{aligned}\pi &\hookrightarrow \delta([\nu^{-w+q}\tilde{\xi}, \nu^{q}\tilde{\xi}]) \rtimes c^{n_\Sigma}\sigma^{(0)} \\ &\hookrightarrow \nu^{-w+q}\tilde{\xi} \times \nu^{-w+q+1}\tilde{\xi} \times \cdots \times \nu^{q}\tilde{\xi} \rtimes c^{n_\Sigma}\sigma^{(0)} \\ &\ \ \vdots \\ &\cong \nu^{-q}\xi \times \nu^{-q+1}\xi \times \cdots \times \nu^{-q+w}\xi \rtimes \sigma^{(0)}.\end{aligned}$$

By Lemma 5.5 [Jan2],

$$\pi \hookrightarrow \lambda \rtimes \sigma^{(0)}$$

for some irreducible $\lambda \leq \nu^{-q}\xi \times \nu^{-q+1}\xi \times \cdots \times \nu^{-q+w}\xi$; necessarily $\lambda \cong \delta([\nu^{-q}\xi, \nu^{-q+w}\xi])$ (as it is the only $\lambda \leq \nu^{-q}\xi \times \nu^{-q+1}\xi \times \cdots \times \nu^{-q+w}\xi$ which has $\nu^{-q}\xi \otimes \cdots \otimes \nu^{-q+w}\xi \otimes \sigma^{(0)}$ in the Jacquet module of $\lambda \rtimes \sigma^{(0)}$). We now have π as a subrepresentation of $\delta([\nu^{-q}\xi, \nu^{-q+w}\xi]) \rtimes \sigma^{(0)}$ and the unique irreducible quotient (the Langlands quotient). As π must appear with multiplicity one, this implies the needed irreducibility.

We now turn to (G8). Let us write $\sigma^{(0)}_{SO}$ for the irreducible supercuspidal representation of $SO_{2n}(F)$ and $\sigma^{(0)}_{O}$ for a component of $\mathrm{Ind}^{O}_{SO}(\sigma^{(0)}_{SO})$ (so that $\sigma_{SO} \leq \mathrm{Res}^{O}_{SO}(\sigma_O)$). In the cases (C0),(C1/2),(C1), Lemma 3.9 tells us the reducibility points of $\delta(\Sigma) \rtimes \sigma^{(0)}_{SO}$ match those of $\delta(\Sigma) \rtimes \sigma^{(0)}_{O}$. Those reducibility points are given in Theorem 9.1 of [Td4], and match those given in the statement of the theorem above. We note that [Td4] mentions only symplectic and odd-orthogonal groups. This is because it predates results such as [Ban1]—necessary extensions of certain key results to the non-connected case. The combinatorics is the same, and the results hold for (non-connected) orthogonal groups as well.

We now turn our attention to the (CN) case. The dual to $\delta(\Sigma) \rtimes \sigma^{(0)}_{SO}$ is $\widetilde{\zeta(\Sigma)} \rtimes c^n\sigma^{(0)}_{SO} = \zeta(\Sigma) \rtimes \sigma^{(0)}_{SO}$, equality in the Grothendieck group. These have the same reducibility points. In the (CN) case, these duals have been analyzed in [BJ1]—see Proposition 5.2 of that paper. Again, the reducibility points match those in the statement of this theorem. ∎

From the discussion above and based on the three theorems above, we make the following definition:

Definition 3.14: Let $\{\Sigma_i\}_{i=1}^f$ and $\sigma^{(t)}$ be given as above. Then $\{\Sigma_i\}_{i=1}^f$ is called an SO_{2n}**-generic sequence of segments with respect to** $\sigma^{(t)}$ if it satisfies the following conditions:

(1) the segment Σ_i is not linked to either Σ_j or $\widetilde{\Sigma}_j$ for $1 \leq i \neq j \leq f$;
(2) for $1 \leq i \leq f$, Σ_i and $\widetilde{\Sigma}_i$ are not linked to any segment, which corresponds to a representation in any of the families

$$\delta(D_i(\tau)), \quad i = 1, 2, \ldots, e_\tau, \quad \tau \in P',$$

$$\{St(\beta_j, 2e_j + 1)\}_{j=1}^c, \quad \{St(\widetilde{\beta}_j, 2e_j + 1) | \beta_j \ncong \widetilde{\beta}_j, 1 \leq j \leq c\};$$

(3) one of the following three conditions holds:
  (3a) $\xi_i \ncong \widetilde{\xi}_i$; or
  (3b) there exists $\tau \in X'$, such that $\tau \cong \xi_i$, $q_i = -1$, and there is some $1 \leq j \leq e_\tau$, with $a_j(\tau) = -1$ and $1 + w_i \leq b_j(\tau)$; or
  (3c) $(\xi_i, \sigma^{(0)})$ is $(C\alpha)$ $(\alpha = 0, \frac{1}{2}, 1)$, but $\pm\alpha \notin \{-q_i, -q_i+1, \ldots, -q_i+w_i\}$; $(\xi_i, \sigma^{(0)})$ is (CN), but $q_i \notin \mathbb{Z}_{\geq 0}$.

4. Surjectivity of local Langlands functorial lifting maps

In this section, first we summarize the results on the local Langlands functorial lifting from $\Pi^{(sg)}(SO_{2n})$ to $\Pi^{(sgo)}(GL_{2n})$, then using the same descent method as in [JngS2], we prove that the rest of local Langlands functorial lifting given by Cogdell, Kim, Piatetski-Shapiro and Shahidi [CKPSS] is also surjective. In each case, we write down the corresponding local Langlands parameters. Due to the similarity between the SO_{2n} case here and the cases of SO_{2n+1} in [JngS2], Sp_{2n} in [Liu], most of the proofs are omitted.

4.1. Supercuspidal generic representations. Let $\Pi^{(sg)}(SO_{2n})$ be the set of all equivalence classes of irreducible supercuspidal generic representations of G. Let $\Pi^{(sgo)}(GL_{2n})$ be the set of all equivalence classes of irreducible tempered representations of $GL_{2n}(F)$ with trivial central characters which have the following form:

$$\tau_1 \times \tau_2 \times \cdots \times \tau_r, \tag{4.1}$$

where for each $1 \le i \le r$, τ_i is an irreducible supercuspidal self-contragredient representation of $GL_{n_i}(F)$ such that $L(\tau_i, Sym^2, s)$ has a pole at $s = 0$ and for $i \neq j$, $\tau_i \ncong \tau_j$.

From the definition, we can see that any $\rho \in \Pi^{(go)}(GL_{2n})$ is of orthogonal type. For properties of symmetric square and exterior square L-functions, see [S3].

Cogdell, Kim, Piatetski-Shapiro and Shahidi [CKPSS] gave the following local Langlands functorial lifting map:

THEOREM 4.1 (Cogdell–Kim–Piatetski-Shapiro–Shahidi): *There is a map l from $\Pi^{(sg)}(SO_{2n})$ to $\Pi^{(sgo)}(GL_{2n})$. Moreover, the map l preserves local L and ϵ factors with GL twists, namely,*

$$L(\sigma \times \pi, s) = L(l(\sigma) \times \pi, s),$$

$$\epsilon(\sigma \times \pi, s, \psi) = \epsilon(l(\sigma) \times \pi, s, \psi),$$

for any $\sigma \in \Pi^{(sg)}(SO_{2n})$ and any irreducible generic representation π of $GL_k(F)$ (k any positive integer).

Jiang and Soudry [JngS3] constructed the local descent map from supercuspidal representations of GL_{2n} to irreducible supercuspidal representations of SO_{2n}, and proved the following theorem, which is one of the main ingredients of this paper.

THEOREM 4.2 (Jiang–Soudry): *The map l in Theorem 4.1 is surjective.*

Remark 4.3: For $\sigma \in \Pi^{(sg)}(SO_{2n})$, assume $\tau_1 \times \tau_2 \times \cdots \times \tau_r \in \Pi^{(sgo)}(GL_{2n})$ is the lifting of σ. Then it is clear that $L(\tau_i \times \sigma, s)$ has a pole at $s = 0$, $1 \le i \le r$. Therefore, by Remark 3.2, each pair (τ_i, σ) must be $(C1)$.

We have the following proposition about lifting images of σ and $c\sigma$ when $\sigma \ncong c\sigma \in \Pi^{(sg)}(SO_{2n})$.

PROPOSITION 4.4: *If $\sigma \ncong c\sigma \in \Pi^{(sg)}(SO_{2n})$, then $l(\sigma) = l(c\sigma) \in \Pi^{(sgo)}(GL_{2n})$. In particular,*

$$L(\sigma \times \tau, s) = L(c\sigma \times \tau, s),$$

$$\epsilon(\sigma \times \tau, s, \psi) = \epsilon(c\sigma \times \tau, s, \psi),$$

for any irreducible generic representation τ of $GL_m(F)$, where m is any positive integer.

Proof. By Proposition 5.1 of [S1], we can embed σ as a local component of a generic cuspidal representation Σ of $SO_{2n}(\mathbb{A})$, with $\Sigma_v = \sigma$ for some local place v. Assume that Π is the lifting of Σ to $GL_{2n}(\mathbb{A})$. Consider $c\Sigma$. Since for unramified places v, Σ_v and $c\Sigma_v$ lift to the same representation, and by [CKPSS] the weak lifting is actually strong, $c\Sigma$ also lifts to Π. Therefore, σ and $c\sigma$ lift to the same representation, and hence

$$L(\sigma \times \tau, s) = L(c\sigma \times \tau, s),$$
$$\epsilon(\sigma \times \tau, s, \psi) = \epsilon(c\sigma \times \tau, s, \psi),$$

for any irreducible generic representation τ of $GL_m(F)$, where m is any positive integer, by Lemma 7.2 of [CKPSS]. ∎

Next, we will figure out the corresponding parameters of irreducible supercuspidal generic representations of G. For this, we need to recall the following, Proposition 4.4 of [Liu]:

PROPOSITION 4.5 (Liu): (1) *Assume τ is an irreducible supercuspidal self-contragredient representation of $GL_m(F)$ having local Langlands parameter ϕ (which is an irreducible admissible m-dimensional complex representation of W_F) and whose local symmetric square L-function $L(\tau, Sym^2, s)$ has a pole at $s = 0$. Then ϕ is orthogonal.*

(2) *Let $\phi = \phi_1 \oplus \phi_2$ be an admissible completely reducible complex representation of W_F with the property that*

$$\mathrm{Hom}_{W_F}(\phi_1 \otimes \phi_2, 1) = 0.$$

Then ϕ is orthogonal if and only if ϕ_1 and ϕ_2 are both orthogonal.

Let $\Phi^{(sg)}(SO_{2n})$ be the subset of $\Phi(SO_{2n})$ consisting of all parameters of type

$$\phi = \bigoplus_i \phi_i$$

with the properties that:

(1) $\phi_i \ncong \phi_j$ if $i \neq j$;

(2) for each i, ϕ_i is orthogonal.

Let $\widetilde{\Phi}^{(sg)}(SO_{2n})$ be the image of $\Phi^{(sg)}(SO_{2n})$ in $\widetilde{\Phi}(SO_{2n})$ (for the definition and related discussion, see the Introduction). As a consequence of Theorem 4.2 and Proposition 4.5, we have the following result for irreducible generic supercuspidal representations of G:

THEOREM 4.6: *There is a surjective map ι from $\Pi^{(sg)}(SO_{2n})$ to the set $\widetilde{\Phi}^{(sg)}(SO_{2n})$. The map ι preserves the local factors as follows:*

$$L(\sigma \times \tau, s) = L(\iota(\sigma) \otimes r^{-1}(\tau), s),$$

$$\epsilon(\sigma \times \tau, s, \psi) = \epsilon(\iota(\sigma) \otimes r^{-1}(\tau), s, \psi),$$

for any $\sigma \in \Pi^{(sg)}(SO_{2n})$ and any irreducible generic representations τ of $GL_{k_\tau}(F)$, with all $k_\tau \in \mathbb{Z}_{>0}$. Here, $r^{-1}(\tau)$ is the irreducible admissible representation of $W_F \times SL_2(\mathbb{C})$ of dimension k_τ which corresponds to τ under the local Langlands reciprocity map for GL_{k_τ}.

Remark 4.7: Note that as mentioned in the Introduction, for any $\widetilde{\phi} \in \widetilde{\Phi}(SO_{2n})$, its twisted local factors are defined to be those of ϕ (a representative of $\widetilde{\phi}$).

Also note that by Proposition 4.4 and Theorem 4.6, if $\sigma \not\cong c\sigma \in \Pi^{(sg)}(SO_{2n})$, then they have the same lifting image and the same twisted local factors.

4.2. DISCRETE SERIES GENERIC REPRESENTATIONS. Let $\Pi^{(dgo)}(GL_{2n})$ be the set of all equivalence classes of irreducible tempered representations of $GL_{2n}(F)$ with trivial central characters which have the following form:

$$St(\tau_1, 2m_1 + 1) \times St(\tau_2, 2m_2 + 1) \times \cdots \times St(\tau_r, 2m_r + 1), \tag{4.2}$$

where the balanced segments $[v^{-m_i}\tau_i, v^{m_i}\tau_i]$ are pairwise distinct, self-contragredient (i.e., $\tau_i \cong \widetilde{\tau_i}$), and satisfy the following properties for each i:

(1) if $L(\tau_i, \wedge^2, s)$ has a pole at $s = 0$, then $m_i \in \frac{1}{2} + \mathbb{Z}_{\geq 0}$;
   or,
(2) if $L(\tau_i, Sym^2, s)$ has a pole at $s = 0$, then $m_i \in \mathbb{Z}_{\geq 0}$.

From the definition, we can see that any $\pi \in \Pi^{(dgo)}(GL_{2n})$ is of orthogonal type.

Then we have the following theorem:

THEOREM 4.8: *There is a surjective map l (which extends the one in Theorem 4.2) from $\Pi^{(dg)}(SO_{2n})$ to $\Pi^{(dgo)}(GL_{2n})$. Moreover, l preserves local factors:*

$$L(\sigma \times \pi, s) = L(l(\sigma) \times \pi, s), \tag{4.3}$$

$$\epsilon(\sigma \times \pi, s, \psi) = \epsilon(l(\sigma) \times \pi, s, \psi), \tag{4.4}$$

for any $\sigma \in \Pi^{(dg)}(SO_{2n})$ and any irreducible generic representation π of any $GL_k(F)$, $k \in \mathbb{Z}_{>0}$.

Proof. The idea of the proof is the same as in [JngS2].

Let $\Pi^{(ss)}(GL_k)$ be the set of equivalence classes of irreducible self-dual supercuspidal representations of $GL_k(F)$ ($k \in \mathbb{Z}_{>0}$). Given a $\rho \in \Pi^{(dgo)}(GL_{2n})$, let

$$P(\rho) := \{\tau \in \Pi^{(ss)}(GL_k) | L(\rho \times \tau, s) \text{ has a pole in } \mathbb{R},\ k \in \mathbb{Z}_{>0}\}.$$

Then $P(\rho)$ is finite. For $\tau \in P(\rho)$, we list the real poles of $L(\rho \times \tau, s)$ as follows:

$$-m_{d_\tau}(\tau) < \cdots < -m_2(\tau) < -m_1(\tau) \leq 0. \tag{4.5}$$

Put $d_\tau = 0$ if $L(\rho \times \tau, s)$ is holomorphic for τ irreducible supercuspidal (self-contragredient or not). We consider the following subset of $P(\rho)$:

$$\begin{aligned} A(\rho) =& \{\tau \in P(\rho) | L(\tau_i, Sym^2, s) \text{ has a pole at } s = 0, \text{ and } d_\tau \text{ is odd}\}, \\ B(\rho) =& \{\tau \in P(\rho) | L(\tau_i, Sym^2, s) \text{ has a pole at } s = 0, \text{ and } d_\tau \text{ is even}\}, \\ C(\rho) =& \{\tau \in P(\rho) | L(\tau_i, \wedge^2, s) \text{ has a pole at } s = 0\}. \end{aligned}$$

Then

$$P(\rho) = A(\rho) \cup B(\rho) \cup C(\rho).$$

Further, if $\tau \in A(\rho) \cup B(\rho)$, then $\{m_i(\tau)\}_{i=1}^{d_\tau} \subset \mathbb{Z}_{\geq 0}$; if $\tau \in C(\rho)$, then $\{m_i(\tau)\}_{i=1}^{d_\tau} \subset \frac{1}{2} + \mathbb{Z}_{\geq 0}$.

Observe that for $\tau \in A(\rho)$, d_τ is odd and the central character ω_τ is quadratic; for $\tau \in B(\rho)$, d_τ is even and the central character ω_τ is quadratic; for $\tau \in C(\rho)$, $L(\tau, \wedge^2, s)$ has a pole at $s = 0$ which implies that the central character ω_τ is trivial. Hence, the following character is trivial:

$$\prod_{\tau \in A(\rho)} \omega_\tau^{d_\tau - 1} \prod_{\tau \in B(\rho)} \omega_\tau^{d_\tau} \prod_{\tau \in C(\rho)} \omega_\tau^{d_\tau}.$$

Therefore, the representation $\times_{\tau \in A(\rho)} \tau$ is a representation of $GL_{2k}(F)$ with trivial central character, k is an integer, $2k = \sum_{\tau \in A(\rho)} k_\tau$, where k_τ is so defined that τ is a representation of $GL_{k_\tau}(F)$. Since for $\tau \in A(\rho)$, $L(\tau, Sym^2, s)$ has a pole at $s = 0$, it follows from Theorem 4.2 that there exists an irreducible supercuspidal generic representation $\sigma^{(0)}$ (not necessarily unique up to equivalence) of $SO_{2k}(F)$, such that

$$l(\sigma^{(0)}) = \times_{\tau \in A(\rho)} \tau \tag{4.6}$$

on $GL_{2k}(F)$.

Let

$$\begin{aligned}A_0(\rho) =&\{\tau \in A(\rho) | d_\tau = 1 \text{ and } m_1(\tau) = 0\},\\ A_1(\rho) =&\{\tau \in A(\rho) | d_\tau \geq 3 \text{ and } m_1(\tau) = 0\},\\ A_2(\rho) =&\{\tau \in A(\rho) | m_1(\tau) \geq 1\}.\end{aligned}$$

Then they form a partition of $A(\rho)$. For $\tau \in A_1(\rho)$, let

$$\Delta_i(\tau) = \delta([\nu^{-m_{2i}(\tau)}\tau, \nu^{m_{2i+1}(\tau)}\tau]), \quad i = 1, 2, \dots, \frac{d_\tau - 1}{2}; \tag{4.7}$$

for $\tau \in A_2(\rho)$, let

$$\Delta_0(\tau) = \delta([\nu\tau, \nu^{m_1(\tau)}\tau]), \quad \Delta_i(\tau) = \delta([\nu^{-m_{2i}(\tau)}\tau, \nu^{m_{2i+1}(\tau)}\tau]), \quad i = 1, 2, \dots, \frac{d_\tau - 1}{2}. \tag{4.8}$$

For $\tau \in B(\rho)$, let

$$\Delta_i(\tau) = \delta([\nu^{-m_{2i-1}(\tau)}\tau, \nu^{m_{2i}(\tau)}\tau]), \quad i = 1, 2, \dots, \frac{d_\tau}{2}. \tag{4.9}$$

Similarly, for $\tau \in C(\rho)$, if d_τ is odd, let

$$\Delta_0(\tau) = \delta([\nu^{\frac{1}{2}}\tau, \nu^{m_1(\tau)}\tau]), \quad \Delta_i(\tau) = \delta([\nu^{-m_{2i}(\tau)}\tau, \nu^{m_{2i+1}(\tau)}\tau]), \quad i = 1, 2, \dots, \frac{d_\tau - 1}{2}. \tag{4.10}$$

Finally, for $\tau \in C(\rho)$, if d_τ is even, let

$$\Delta_i(\tau) = \delta([\nu^{-m_{2i-1}(\tau)}\tau, \nu^{m_{2i}(\tau)}\tau]), \quad i = 1, 2, \dots, \frac{d_\tau}{2}. \tag{4.11}$$

We now define

$$J_\tau = \begin{cases} \{1, 2, \dots, \frac{d_\tau - 1}{2}\}, & \text{in case (4.7)},\\ \{0, 1, 2, \dots, \frac{d_\tau - 1}{2}\}, & \text{in cases (4.8), (4.10)},\\ \{1, 2, \dots, \frac{d_\tau}{2}\}, & \text{in cases (4.9), (4.11)},\end{cases}$$

and let σ_ρ be the unique irreducible generic constituent of

$$(\times_{\tau \in P(\rho) \smallsetminus A_0(\rho)} \times_{j \in J_\tau} \Delta_j(\tau)) \rtimes \sigma^{(0)}, \tag{4.12}$$

where possibly $\sigma^{(0)} = 1 \otimes c$. Observe that σ_ρ is a representation of G. It is now easy to see that the sequence of segments in (4.7)–(4.11), together with $\sigma^{(0)}$ satisfy (3.2), (3.3), (DS1)–(DS3), hence σ_ρ is square-integrable.

The proof that the local factors are preserved is similar to that in [JngS2] or [CKPSS]. ∎

Remark 4.9: Note that if $\rho^{(2)} = l(\sigma^{(2)})$, then the X' in Theorem 3.11 is equal to $A_2(\rho^{(2)})$.

In this case, by Part (2), Part (3b) and Part (3c) of Definition 3.14, if $\{\Sigma_i\}_{i=1}^f$ is an SO_{2n}-generic sequence of segments with respect to $\sigma^{(t)}$, then for $1 \leq i \leq f$, Σ_i and $\widetilde{\Sigma_i}$ are not linked to any segment associated to $\rho^{(2)}$. That is, as a representation of $GL_*(F)$, $\delta(\Sigma_1) \times \cdots \times \delta(\Sigma_f) \times \rho^{(2)}$ is irreducible and generic, where $\sigma^{(2)}$ is the irreducible discrete series generic representation occurring in $\sigma^{(t)}$, and $\rho^{(2)} = l(\sigma^{(2)})$.

Note that if in Part (3b) of Definition 3.14, $1 + w_i > b_j(\tau)$, then Σ_i is linked to $[\nu^{-b_j(\tau)}\tau, \nu^{b_j(\tau)}\tau]$, which is a segment associated to $\rho^{(2)}$ (see the proof of Theorem 4.8). Then, $\delta(\Sigma_1) \times \cdots \times \delta(\Sigma_f) \times \rho^{(2)}$ is no longer irreducible and generic.

In the case of irreducible generic representations of $SO_{2n+1}(F)$ and $Sp_{2n}(F)$ (see [M2]), Part (2) of Definition 5.1 of [JngS2] and Definition 4.14 of [Liu] implies that $1 + w_i \leq b_j(\tau)$ holds in Part (3b).

Next we generalize Theorem 4.6 to $\Pi^{(dg)}(SO_{2n})$.

Let $\Phi^{(d)}(SO_{2n})$ be the subset of $\Phi(SO_{2n})$ consisting of all the local Langlands parameters of type

$$\phi = \bigoplus_i \phi_i \otimes S_{2m_i+1},$$

where the ϕ_i's are irreducible self-contragredient representations of W_F of dimension k_{ϕ_i}, the S_{2m_i+1}'s are irreducible representations of $SL_2(\mathbb{C})$ of dimension $2m_i + 1$, and they satisfy the following:

(1) the tensor products $\phi_i \otimes S_{2m_i+1}$ are irreducible and orthogonal;
(2) $\phi_i \otimes S_{2m_i+1}$ and $\phi_j \otimes S_{2m_j+1}$ are not equivalent if $i \neq j$;
(3) the image $\phi(W_F \times SL_2(\mathbb{C}))$ is not contained in any proper Levi subgroup of $SO_{2n}(\mathbb{C})$.

The local Langlands parameters in $\Phi^{(d)}(SO_{2n})$ are called **discrete**. Let $\widetilde{\Phi}^{(d)}(SO_{2n})$ be the image of $\Phi^{(d)}(SO_{2n})$ in $\widetilde{\Phi}(SO_{2n})$. The following theorem is parallel to Theorem 2.2 [JngS2] and Theorem 4.9 [Liu].

THEOREM 4.10: *There is a surjective map ι (which extends the one in Theorem 4.6) from $\Pi^{(dg)}(SO_{2n})$ to the set $\widetilde{\Phi}^{(d)}(SO_{2n})$. The map ι preserves the local* *factors:*

$$L(\sigma \times \tau, s) = L(\iota(\sigma) \otimes r^{-1}(\tau), s), \tag{4.13}$$

$$\epsilon(\sigma \times \tau, s, \psi) = \epsilon(\iota(\sigma) \otimes r^{-1}(\tau), s, \psi), \tag{4.14}$$

for all $\sigma \in \Pi^{(dg)}(SO_{2n})$ and all irreducible generic representations τ of any $GL_{k_\tau}(F)$, $k_\tau \in \mathbb{Z}_{>0}$. Here, $r^{-1}(\tau)$ is the irreducible admissible representation of $W_F \times SL_2(\mathbb{C})$ of dimension k_τ corresponding to τ by the local Langlands reciprocity map for GL_{k_τ}.

Remark 4.11: Suppose $\sigma^{(2)} \in \Pi^{(dg)}(SO_{2n})$ is the unique generic constituent of $(\times_{\tau \in P'} \times_{i=1}^{e_\tau} \delta(D_i(\tau))) \rtimes \sigma^{(0)}$ (possibly $\sigma^{(0)} = 1 \otimes c$) and $\sigma^{(0)} \ncong c\sigma^{(0)}$. Then, by Section 3.2, $\sigma^{(2)} \ncong c\sigma^{(2)}$ are both in $\Pi^{(dg)}(SO_{2n})$, and $c\sigma^{(2)}$ is the unique generic constituent of $(\times_{\tau \in P'} \times_{i=1}^{e_\tau} \delta(D_i(\tau))) \rtimes c\sigma^{(0)}$. Note that if $(\tau, \sigma^{(0)})$ satisfies $(C\alpha)$ or (CN), then so does $(\tau, c\sigma^{(0)})$.

By Remark 4.7, Theorem 4.8 and Theorem 4.10, and by the multiplicativity of local factors (see [S2], [JngS2] and [CKPSS]), in the above situation, $\sigma^{(2)}$ and $c\sigma^{(2)}$ have the same lifting image and the same twisted local factors.

4.3. Tempered generic representations. Let $\Pi^{(tgo)}(GL_{2n})$ be the set of equivalence classes of tempered representations of $GL_{2n}(F)$ with trivial central characters which have the following form:

$$St(\lambda_1, 2h_1 + 1) \times St(\lambda_2, 2h_2 + 1) \times \cdots \times St(\lambda_f, 2h_f + 1), \tag{4.15}$$

where $\lambda_1, \lambda_2, \ldots, \lambda_f$ are unitary supercuspidal representations, and $2h_i \in \mathbb{Z}_{\geq 0}$, such that for $1 \leq i \leq f$:

(1) if $\lambda_i \ncong \widetilde{\lambda}_i$, then $St(\lambda_i, 2h_i + 1)$ occurs in (4.15) as many times as $St(\widetilde{\lambda}_i, 2h_i + 1)$ does;
(2) if $L(\lambda_i, \wedge^2, s)$ has a pole at $s = 0$, and $h_i \in \mathbb{Z}_{\geq 0}$, then $St(\lambda_i, 2h_i + 1)$ occurs an even number of times in (4.15);
(3) if $L(\lambda_i, Sym^2, s)$ has a pole at $s = 0$, and $h_i \in \frac{1}{2} + \mathbb{Z}_{\geq 0}$, then $St(\lambda_i, 2h_i + 1)$ occurs an even number of times in (4.15).

From the definition, we can see that any $\pi \in \Pi^{(tgo)}(GL_{2n})$ is of orthogonal type.

The following theorem is parallel to Theorem 4.1 [JngS2] and Theorem 4.12 [Liu].

THEOREM 4.12: *There is a surjective map l (which extends the one in Theorem 4.8) from $\Pi^{(tg)}(SO_{2n})$ to $\Pi^{(tgo)}(GL_{2n})$. Moreover, l preserves local factors:*

$$L(\sigma \times \pi, s) = L(l(\sigma) \times \pi, s), \tag{4.16}$$

$$\epsilon(\sigma \times \pi, s, \psi) = \epsilon(l(\sigma) \times \pi, s, \psi), \tag{4.17}$$

for any $\sigma \in \Pi^{(tg)}(SO_{2n})$ and any irreducible generic representation π of any $GL_k(F)$, $k \in \mathbb{Z}_{>0}$.

Remark 4.13: By Theorem 4.12, for each $\sigma^{(t)} \in \Pi^{(tg)}(SO_{2n})$,

$$\begin{aligned} \rho^{(t)} = l(\sigma^{(t)}) =& St(\beta_1, 2e_1+1) \times \cdots \times St(\beta_c, 2e_c+1) \times l(\sigma^{(2)}) \\ & \times \widetilde{St}(\beta_c, 2e_c+1) \times \cdots \times \widetilde{St}(\beta_1, 2e_1+1). \end{aligned} \tag{4.18}$$

Therefore, by Remark 4.9, if $\{\Sigma_i\}_{i=1}^{f}$ is an SO_{2n}-generic sequence of segments with respect to $\sigma^{(t)}$, then for $1 \leq i \leq f$, Σ_i and $\widetilde{\Sigma_i}$ are not linked to any segment associated to $\rho^{(t)}$. That is, as a representation of $GL_*(F)$,

$$\delta(\Sigma_1) \times \cdots \times \delta(\Sigma_f) \times \rho^{(t)}$$

is irreducible and generic.

Next, we write down the parameters for representations in $\Pi^{(tg)}(SO_{2n})$. From (4.18), we can see that the local Langlands parameter of σ is

$$\phi_{\sigma^{(2)}} \oplus \bigoplus_{i=1}^{c} [\phi_{\beta_i} \times S_{2e_i+1} \oplus \widetilde{\phi}_{\beta_i} \times S_{2e_i+1}].$$

Let $\Phi^{(t)}(SO_{2n})$ be the subset of $\Phi(SO_{2n})$ consisting of the local Langlands parameters ϕ with the property that $\phi(W_F)$ is bounded in $SO_{2n}(\mathbb{C})$. The parameters in $\Phi^{(t)}(SO_{2n})$ are called **tempered**. Then we have the following result that the local Langlands parameters corresponding to representations in $\Pi^{(tg)}(SO_{2n})$ are exactly the tempered parameters. Let $\widetilde{\Phi}^{(t)}(SO_{2n})$ be the image of $\Phi^{(t)}(SO_{2n})$ in $\widetilde{\Phi}(SO_{2n})$. The following theorem is parallel to Theorem 4.2 [JngS2] and Theorem 4.13 [Liu].

THEOREM 4.14: *There is a surjective map ι (which extends the one in Theorem 4.10) from $\Pi^{(tg)}(SO_{2n})$ to the set $\widetilde{\Phi}^{(t)}(SO_{2n})$. It preserves the local factors:*

$$L(\sigma \times \tau, s) = L(\iota(\sigma) \otimes r^{-1}(\tau), s),$$

$$\epsilon(\sigma \times \tau, s, \psi) = \epsilon(\iota(\sigma) \otimes r^{-1}(\tau), s, \psi),$$

for all $\sigma \in \Pi^{(tg)}(SO_{2n})$ *and all irreducible generic representations* τ *of any* $GL_{k_\tau}(F)$, $k_\tau \in \mathbb{Z}_{>0}$. *Here,* $r^{-1}(\tau)$ *is the irreducible admissible representation of* $W_F \times SL_2(\mathbb{C})$ *of dimension* k_τ *corresponding to* τ *by the local Langlands reciprocity map for* GL_{k_τ}.

Remark 4.15: By Section 3.3, if $\sigma^{(t)} \in \Pi^{(tg)}(SO_{2n})$ is the unique generic constituent of $St(\beta_1, 2e_1+1) \times \cdots \times St(\beta_c, 2e_c+1) \rtimes \sigma^{(2)}$ (possibly $\sigma^{(2)} = 1 \otimes c$) and $\sigma^{(2)} \not\cong c\sigma^{(2)}$, then $\sigma^{(t)} \not\cong c\sigma^{(t)}$, both are in $\Pi^{(tg)}(SO_{2n})$, and $c\sigma^{(t)}$ is the unique generic constituent of $St(\beta_1, 2e_1+1) \times \cdots \times St(\beta_c, 2e_c+1) \rtimes c\sigma^{(2)}$.

By Remark 4.11, Theorem 4.12, Theorem 4.14, and the multiplicativity of local factors (see [S2], [JngS2] and [CKPSS]), in the above situation, $\sigma^{(t)}$ and $c\sigma^{(t)}$ have the same lifting image and the same twisted local factors.

4.4. Generic representations. Let $\Pi^{(go)}(GL_{2n})$ be the set of equivalence classes of irreducible self-contragredient representations of $GL_{2n}(F)$ with trivial central characters which are Langlands quotients of representations

$$\delta(\Sigma_1) \times \cdots \times \delta(\Sigma_f) \times \rho^{(t)} \times \delta(\widetilde{\Sigma}_f) \times \cdots \times \delta(\widetilde{\Sigma}_1), \tag{4.19}$$

where $\{\Sigma_j\}_{j=1}^{f}$ are of the form (3.9), $\xi_1, \xi_2, \ldots, \xi_f$ are irreducible unitary and supercuspidal, with possible repetitions, $q_i \in \mathbb{R}$, $w_i \in \mathbb{Z}_{\geq 0}$, and $\rho^{(t)} \in \Pi^{(tg)}(GL_{2n*})$, such that the following hold:

1. $\frac{w_1}{2} - q_1 \geq \frac{w_2}{2} - q_2 \geq \cdots \geq \frac{w_f}{2} - q_f > 0$.
2. The segment Σ_i is not linked to either Σ_j or $\widetilde{\Sigma}_j$ for $1 \leq i \neq j \leq f$.
3. The representations $\delta(\Sigma_i) \times \rho^{(t)}$ and $\delta(\widetilde{\Sigma}_i) \times \rho^{(t)}$ are irreducible for all $1 \leq i \leq f$.
4. Assume ξ_i is self-contragredient and $2q_i \in \mathbb{Z}$, such that if $L(\xi_i, \wedge^2, s)$ has a pole at $s=0$, then $q_i \in \frac{1}{2} + \mathbb{Z}$, and if $L(\xi_i, Sym^2, s)$ has a pole at $s=0$, then $q_i \in \mathbb{Z}$. Then Σ_i is not linked to $\widetilde{\Sigma}_i$. Moreover, if $L(\rho^{(0)} \times \xi_i, s)$ has a pole at $s=0$ and $q_i \in \mathbb{Z}$, then either (a) $-q_i \geq 2$ or (b) $q_i = -1$, $\xi_i = \tau \in A_2(\rho^{(2)})$ and there is some $1 \leq j \leq e_\tau$ such that $a_j(\tau) = -1$ and $1 + w_i \leq b_j(\tau)$.

From the definition, we can see that any $\pi \in \Pi^{(go)}(GL_{2n})$ is of orthogonal type.

Note that by Remark 4.9, $A_2(\rho^{(2)})$ here is actually X' in Theorem 3.11. The following theorem is parallel to Theorem 5.1 [JngS2] and Theorem 4.15 [Liu].

THEOREM 4.16: *There is a surjective map l (which extends the one in Theorem 4.12) from $\Pi^{(g)}(SO_{2n})$ to $\Pi^{(go)}(GL_{2n})$. It preserves the local factors:*

$$L(\sigma \times \pi, s) = L(l(\sigma) \times \pi, s), \tag{4.20}$$

$$\epsilon(\sigma \times \pi, s, \psi) = \epsilon(l(\sigma) \times \pi, s, \psi), \tag{4.21}$$

for any $\sigma \in \Pi^{(g)}(SO_{2n})$ and any irreducible generic representation π of any $GL_k(F)$, $k \in \mathbb{Z}_{>0}$.

At last, we write down the corresponding parameters.

Let $\Phi^{(g)}(SO_{2n})$ be the subset of $\Phi(SO_{2n})$ consisting of elements of the following form:

$$\phi_\sigma = \phi^{(t)} \oplus \bigoplus_{i=1}^{f} \left[|\cdot|^{-q_i + \frac{w_i}{2}} r^{-1}(\xi_i) \otimes S_{w_i+1} \oplus |\cdot|^{q_i - \frac{w_i}{2}} r^{-1}(\widetilde{\xi_i}) \otimes S_{w_i+1} \right],$$

where $\phi^{(t)}$ is a representative of $\iota(\sigma^{(t)})$, and the sequence

$$\{\Sigma_j = [v^{-q_j}\xi_j, v^{-q_j+w_j}\xi_j]\}_{j=1}^{f}$$

is an SO_{2n}-generic sequence of segments with respect to $\sigma^{(t)}$ (see Definition 3.14), ι is the reciprocity map given in Theorem 4.14 for irreducible tempered generic representations in $\Pi^{(tg)}(SO_{2n})$, r is the reciprocity map for $GL_*(F)$, and $|\cdot|^s$ is the character of W_F normalized as in [T] via local class field theory. Let $\widetilde{\Phi}^{(g)}(SO_{2n})$ be the image of $\Phi^{(g)}(SO_{2n})$ in $\widetilde{\Phi}(SO_{2n})$.

The following theorem is parallel to the result in the last paragraph of Section 5 of [JngS2] and Theorem 4.17 [Liu].

THEOREM 4.17: *There is a surjective map ι (which extends the one in Theorem 4.14) from $\Pi^{(g)}(SO_{2n})$ to $\widetilde{\Phi}^{(g)}(SO_{2n})$. The map ι preserves the local factors:*

$$L(\sigma \times \tau, s) = L(\iota(\sigma) \otimes r^{-1}(\tau), s),$$

$$\epsilon(\sigma \times \tau, s, \psi) = \epsilon(\iota(\sigma) \otimes r^{-1}(\tau), s, \psi),$$

for all $\sigma \in \Pi^{(g)}(SO_{2n})$ and all irreducible generic representations τ of any $GL_{k_\tau}(F)$, $k_\tau \in \mathbb{Z}_{>0}$. Here, $r^{-1}(\tau)$ is the irreducible admissible representation of $W_F \times SL_2(\mathbb{C})$ of dimension k_τ corresponding to τ by the local Langlands reciprocity map for GL_{k_τ}.

Proof. Given any $\sigma \in \Pi^{(g)}(SO_{2n})$, by the classification of generic representations of $SO_{2n}(F)$ in Section 3.4, there exists an irreducible tampered generic representation $\sigma^{(t)}$ of $SO_{2n^*}(F)$ and a sequence of segments

$$\{\Sigma_j = [v^{-q_j}\xi_j, v^{-q_j+w_j}\xi_j]\}_{j=1}^f$$

which is an SO_{2n}-generic sequence of segments with respect to $\sigma^{(t)}$ (see Definition 3.14), such that

$$\sigma = \delta(\Sigma_1) \times \delta(\Sigma_2) \times \cdots \times \delta(\Sigma_f) \rtimes \sigma^{(t)}.$$

Let $\phi^{(t)}$ be a representative of $\iota(\sigma^{(t)})$, and let

$$\phi = \phi^{(t)} \oplus \bigoplus_{i=1}^{f} \left[|\cdot|^{-q_i+\frac{w_i}{2}} r^{-1}(\xi_i) \otimes S_{w_i+1} \oplus |\cdot|^{q_i-\frac{w_i}{2}} r^{-1}(\widetilde{\xi_i}) \otimes S_{w_i+1} \right].$$

It is easy to see that $\phi \in \Phi^{(g)}(SO_{2n})$. Define $\iota(\sigma) = \widetilde{\phi}$, the image of ϕ in $\widetilde{\Phi}(SO_{2n})$. Therefore, we have constructed a map ι from $\Pi^{(g)}(SO_{2n})$ to $\widetilde{\Phi}^{(g)}(SO_{2n})$, which naturally extends the one in Theorem 4.14. Using multiplicativity of local factors, it is easy to check that ι preserves local factors.

To show that this map is surjective, take any $\widetilde{\phi} \in \widetilde{\Phi}^{(g)}(SO_{2n})$, and let

$$\phi = \phi^{(t)} \oplus \bigoplus_{i=1}^{f} \left[|\cdot|^{-q_i+\frac{w_i}{2}} r^{-1}(\xi_i) \otimes S_{w_i+1} \oplus |\cdot|^{q_i-\frac{w_i}{2}} r^{-1}(\widetilde{\xi_i}) \otimes S_{w_i+1} \right]$$

be a representative, where $\sigma^{(t)}$ is an irreducible tampered generic representation of $SO_{2n^*}(F)$ and the sequence of segments

$$\{\Sigma_j = [v^{-q_j}\xi_j, v^{-q_j+w_j}\xi_j]\}_{j=1}^f$$

is an SO_{2n}-generic sequence of segments with respect to $\sigma^{(t)}$.

Let

$$\sigma = \delta(\Sigma_1) \times \delta(\Sigma_2) \times \cdots \times \delta(\Sigma_f) \rtimes \sigma^{(t)}.$$

Since $\sigma^{(t)}$ is an irreducible tampered generic representation of $SO_{2n^*}(F)$ and the sequence of segments

$$\{\Sigma_j = [v^{-q_j}\xi_j, v^{-q_j+w_j}\xi_j]\}_{j=1}^f$$

is an SO_{2n}-generic sequence of segments with respect to $\sigma^{(t)}$, by the classification of irreducible generic representations in Section 3.4, we can see that σ is irreducible and generic. Hence $\sigma \in \Pi^{(g)}(SO_{2n})$. And we can also easily see that $\iota(\sigma)$ is actually equal to $\widetilde{\phi}$. Therefore, ι is indeed surjective.

This completes the proof of the theorem. ∎

Remark 4.18: Given any $\sigma \in \Pi^{(g)}(SO_{2n})$, let $\iota(\sigma) = \widetilde{\phi} \in \widetilde{\Phi}^{(g)}(SO_{2n})$, and let ϕ be a representative of $\widetilde{\phi}$. Let $i : SO_{2n}(\mathbb{C}) \to GL_{2n}(\mathbb{C})$ be the canonical embedding. From Theorem 4.16 and Theorem 4.17, we can see that the composition $i \circ \phi$ is actually the local Langlands parameter corresponding to the lifing $l(\sigma)$ of σ under the local Langlands reciprocity map of $GL_{2n}(F)$.

A local Langlands parameter $\phi \in \Phi(SO_{2n})$ is called **generic** if there is a generic representation in the corresponding local L-packet. From Theorem 4.17, we can see that $\Phi^{(g)}(SO_{2n})$ is actually the set of all generic local Langlands parameters of $SO_{2n}(F)$.

Remark 4.19: By Section 3.4, if $\sigma^{(g)} \in \Pi^{(g)}(SO_{2n})$ is the irreducible generic representation $\pi_1 \times \pi_2 \times \cdots \times \pi_f \rtimes \sigma^{(t)}$ (possibly $\sigma^{(t)} = 1 \otimes c$) and $\sigma^{(t)} \ncong c\sigma^{(t)}$, then $\sigma^{(g)} \ncong c\sigma^{(g)}$, both are in $\Pi^{(g)}(SO_{2n})$, and $c\sigma^{(g)}$ is the irreducible generic representation $\pi_1 \times \pi_2 \times \cdots \times \pi_f \rtimes c\sigma^{(t)}$.

By Remark 4.15, Theorem 4.16 and Theorem 4.17, and by the multiplicativity of local factors (see [S2], [JngS2] and [CKPSS]), in the above situation, $\sigma^{(g)}$ and $c\sigma^{(g)}$ have the same lifting image and the same twisted local factors. We record this result as the following theorem.

THEOREM 4.20: *For any $\sigma \in \Pi^{(g)}(SO_{2n})$, if $\sigma \ncong c\sigma$, then $l(\sigma) = l(c\sigma)$, and $\iota(\sigma) = \iota(c\sigma)$. That is, they have the same lifting image and the same twisted local factors.*

5. Representations attached to parameters

In this section, as in [JngS2] and [Liu], we associate one irreducible representation of G to each local Langlands parameter $\phi \in \Phi(SO_{2n})$. The key idea is to analyze the structure of each local Langlands parameter.

PROPOSITION 5.1: *Let $\phi \in \Phi(SO_{2n})$. Then either $\phi \in \Phi^{(t)}(SO_{2n})$, or*

$$\phi = \phi^{(t)} \oplus \phi^{(n)}, \tag{5.1}$$

where $\phi^{(t)} \in \Phi^{(t)}(SO_{2n^*})$ $(n^* < n)$ and $\phi^{(n)} \in \Phi(SO_{2(n-n^*)})$ is of the form

$$\phi^{(n)} = \bigoplus_{i=1}^{f} \left[|\cdot|^{-q_i + \frac{w_i}{2}} \phi_i \otimes S_{w_i+1} \oplus |\cdot|^{q_i - \frac{w_i}{2}} \widetilde{\phi}_i \otimes S_{w_i+1} \right]. \tag{5.2}$$

Here, $f \in \mathbb{Z}_{>0}$, $w_1, w_2, \ldots, w_f \in \mathbb{Z}_{\geq 0}$, $q_1, q_2, \ldots, q_f \in \mathbb{R}$, such that ϕ_i is an irreducible bounded representation of W_F for $1 \leq i \leq f$, and

$$\frac{w_1}{2} - q_1 \geq \frac{w_2}{2} - q_2 \geq \cdots \geq \frac{w_f}{2} - q_f > 0$$

(with $|\cdot|$ the character of W_F normalized as in [T] *via local class field theory).*

Proof. Given a parameter $\phi \in \Phi(SO_{2n})$, assume $V = \mathbb{C}^{2n}$ is the corresponding non-degenerate orthogonal space of dimension $2n$, with an orthogonal form $\langle \, , \rangle$.

Let V_1 be the direct sum of all irreducible subspaces which are stable under the action of $W_F \times SL_2(\mathbb{C})$ and in which $\phi(W_F)$ is bounded. Let V_2 be the direct sum of all irreducible subspaces which are stable under the action of $W_F \times SL_2(\mathbb{C})$ and in which $\phi(W_F)$ is unbounded. Then

$$V = V_1 \oplus V_2.$$

In a similar manner to [JngS2], one can see that both subspaces V_1 and V_2 are non-degenerate with respect to the restriction of the non-degenerate orthogonal form $\langle \, , \rangle$.

Denote by $\phi^{(t)}$ the subrepresentation of $W_F \times SL_2(\mathbb{C})$ on V_1, and by $\phi^{(n)}$ the subrepresentation of $W_F \times SL_2(\mathbb{C})$ on V_2. Then there are two cases:

(a) $\phi^{(t)} \in \Phi^{(t)}(SO_{2n^*})$ and $\phi^{(n)} \in \Phi(SO_{2(n-n^*)})$;
(b) $\phi^{(t)} \in \Phi^{(t)}(Sp_{2n^*})$ and $\phi^{(n)} \in \Phi(Sp_{2(n-n^*-1)})$.

Using an argument similar to that in the proof of Proposition 5.1 of [Liu], we can prove that case (b) cannot occur. Also, we can see that $\phi^{(t)} \in \Phi^{(t)}(SO_{2n^*})$, and $\phi^{(n)}$ is of the form (5.2). This completes the proof. ∎

Let $\Pi'(SO_{2n})$ be the set of (equivalence classes of) all Langlands quotients $L(\nu^{x_1}\delta_1 \otimes \cdots \otimes \nu^{x_k}\delta_k \otimes \sigma^{(t)})$, where $\sigma^{(t)}$ is an irreducible tempered generic representation of $SO_{2n^*}(F)$ (possibly $\sigma^{(t)} = 1 \otimes c$—for the definition, see Section 3.1), $x_1 \geq x_2 \geq \cdots \geq x_k > 0$, and δ_i is a square-integrable representation of $GL_{n_i}(F)$, for $i = 1, 2, \ldots, k$. Then, we have the following result:

Theorem 5.2: *There is a surjective map ι (which extends the one in Theorem 4.14) from $\Pi'(SO_{2n})$ to the set $\widetilde{\Phi}(SO_{2n})$. It preserves the local factors:*

$$L(\sigma \times \tau, s) = L(\iota(\sigma) \otimes r^{-1}(\tau), s),$$

$$\epsilon(\sigma \times \tau, s, \psi) = \epsilon(\iota(\sigma) \otimes r^{-1}(\tau), s, \psi),$$

for all $\sigma \in \Pi'(SO_{2n})$ and all irreducible admissible representations τ of any $GL_{k_\tau}(F)$, $k_\tau \in \mathbb{Z}_{>0}$. Here, $r^{-1}(\tau)$ is the irreducible admissible representation of $W_F \times SL_2(\mathbb{C})$ of dimension k_τ corresponding to τ by the local Langlands reciprocity map for GL_{k_τ}.

Proof. Given any $\sigma \in \Pi'(SO(2n))$ which is the Langlands quotient $L(\nu^{x_1}\delta_1 \otimes \cdots \otimes \nu^{x_k}\delta_k \otimes \sigma^{(t)})$, where $\sigma^{(t)}$ is an irreducible tempered generic representation of $SO_{2n^*}(F)$ (possibly $\sigma^{(t)} = 1 \otimes c$—for the definition, see Section 3.1), $x_1 \geq x_2 \geq \cdots \geq x_k > 0$, and δ_i is a square-integrable representation of $GL_{n_i}(F)$, for $i = 1, 2, \ldots, k$.

Using the surjective map ι in Theorem 4.14, let $\widetilde{\phi}^{(t)} = \iota(\sigma^{(t)}) \in \widetilde{\Phi}^t(SO(2n))$, and let $\phi^{(t)}$ be a representative. Assume that ϕ_i is the corresponding Langlands parameter for δ_i under the local Langlands raciprocity map for $GL_{n_i}(F)$, for $i = 1, 2, \ldots, k$. Then let

$$\phi = \bigoplus_{i=1}^{k} [|\cdot|^{x_i}\phi_i \oplus |\cdot|^{-x_i}\widetilde{\phi_i}] \bigoplus \phi^{(t)},$$

where $\widetilde{\phi_i}$ is the contragredient of ϕ_i. Define $\iota(\sigma) = \widetilde{\phi}$, the image of ϕ in $\widetilde{\Phi}(SO_{2n})$. Then using multiplicativity of local factors, it is easy to see that the local factors are preserved. In this way, we construct a map ι from $\Pi'(SO_{2n})$ to the set $\widetilde{\Phi}(SO_{2n})$ which preserves local factors and naturally extends the one in Theorem 4.14.

To prove that this map ι is surjective, take any $\widetilde{\phi} \in \widetilde{\Phi}(SO_{2n})$, and let $\phi \in \Phi(SO_{2n})$ be a representative. By Proposition 5.1, it can be written as

$$\phi = \phi^{(t)} \oplus \phi^{(n)},$$

where $\phi^{(t)} \in \Phi^{(t)}(SO_{2n^*})$ $(n^* < n)$ and $\phi^{(n)} \in \Phi(SO_{2(n-n^*)})$ is of the form

$$\phi^{(n)} = \bigoplus_{i=1}^{f} \left[|\cdot|^{-q_i+\frac{w_i}{2}}\phi_i \otimes S_{w_i+1} \oplus |\cdot|^{q_i-\frac{w_i}{2}}\widetilde{\phi_i} \otimes S_{w_i+1}\right].$$

Here, $f \in \mathbb{Z}_{>0}$, $w_1, w_2, \ldots, w_f \in \mathbb{Z}_{\geq 0}$, $q_1, q_2, \ldots, q_f \in \mathbb{R}$, such that ϕ_i is an irreducible bounded representation of W_F for $1 \leq i \leq f$, and

$$\frac{w_1}{2} - q_1 \geq \frac{w_2}{2} - q_2 \geq \cdots \geq \frac{w_f}{2} - q_f > 0.$$

By Theorem 4.14, there exists $\sigma^{(t)} \in \Pi^{(tg)}(SO_{2n^*})$ such that

$$\iota(\sigma^{(t)}) = \widetilde{\phi}^{(t)} \in \widetilde{\Phi}^{(t)}(SO_{2n}). \tag{5.3}$$

Using the local Langlands reciprocity map r for $GL_k(F)$, define

$$\Sigma_i = [v^{-q_i} r(\phi_i), v^{-q_i+w_i} r(\phi_i)], \quad 1 \leq i \leq f. \tag{5.4}$$

Let σ be the Langlands quotient of the induced representation

$$\delta(\Sigma_1) \times \delta(\Sigma_2) \times \ldots \delta(\Sigma_f) \rtimes \sigma^{(t)}$$

(possibly $\sigma^{(t)} = 1 \otimes c$). Then, it is easy to see that $\sigma \in \Pi'(SO_{2n})$. And we can easily see that $\iota(\sigma)$ is actually equal to $\widetilde{\phi}$. Therefore ι is indeed surjective.

This completes the proof of the theorem. ∎

Remark 5.3: When $\widetilde{\phi} \in \widetilde{\Phi}^{(g)}(SO_{2n})$, the subset of $\widetilde{\Phi}(SO_{2n})$, let $\phi \in \Phi^{(g)}(SO_{2n})$ be a representative, which is a generic local Langlands parameter. Then, by definition, ϕ is of the form

$$\phi^{(t)} \oplus \bigoplus_{i=1}^{f} \left[|\cdot|^{-q_i+\frac{w_i}{2}} r^{-1}(\xi_i) \otimes S_{w_i+1} \oplus |\cdot|^{q_i-\frac{w_i}{2}} r^{-1}(\widetilde{\xi_i}) \otimes S_{w_i+1} \right],$$

where $\sigma^{(t)}$ is an irreducible tampered generic representation of $SO_{2n^*}(F)$ and the sequence of segments

$$\{\Sigma_j = [v^{-q_j}\xi_j, v^{-q_j+w_j}\xi_j]\}_{j=1}^{f}$$

is an SO_{2n}-generic sequence of segments with respect to $\sigma^{(t)}$.

Then by the classification of generic representations in Section 3.4,

$$\delta(\Sigma_1) \times \delta(\Sigma_2) \times \cdots \times \delta(\Sigma_f) \rtimes \sigma^{(t)}$$

is irreducible and generic. And the σ constructed in Theorem 5.2 is actually equal to $\delta(\Sigma_1) \times \delta(\Sigma_2) \times \cdots \times \delta(\Sigma_f) \rtimes \sigma^{(t)}$, hence generic. And, from the construction in Theorem 4.17, we can see that this σ matches the one constructed in Theorem 4.17 for this generic local Langlands parameter ϕ. Hence, the map ι constructed in Theorem 5.2 is a natural extension of the one constructed in Theorem 4.17.

Therefore, we can conclude that $\phi \in \Phi(SO_{2n})$ is a generic local Langlands parameter if and only if the representation σ attached to ϕ in Theorem 5.2 is generic.

Remark 5.4: If $\sigma \in \Pi'(SO_{2n})$ is the Langlands quotient of the induced representation $\delta(\Sigma_1) \times \delta(\Sigma_2) \times \cdots \times \delta(\Sigma_f) \rtimes \sigma^{(t)}$ (possibly $\sigma^{(t)} = 1 \otimes c$) and $\sigma^{(t)} \not\cong c\sigma^{(t)}$, then $\sigma \not\cong c\sigma$ and $c\sigma$ is the Langlands quotient of the induced representation

$\delta(\Sigma_1) \times \delta(\Sigma_2) \times \cdots \times \delta(\Sigma_f) \rtimes c\sigma^{(t)}$. This matches the local Langlands classification for $SO_{2n}(F)$—see Proposition 6.3 and Section 2 of [BJ1].

By Remark 4.15, Theorem 5.2, and the multiplicativity of local factors (see [S2], [JngS2] and [CKPSS]), in the above situation, σ and $c\sigma$ have the same twisted local factors.

As we discussed in the Introduction, when σ is attached to $\widetilde{\phi}$ as in Theorem 5.2, we also say it is attached to ϕ.

6. A conjecture of Gross–Prasad and Rallis

In this section, we give an application of the results in previous sections to a conjecture of Gross–Prasad [GP] and Rallis [Ku], which says that a local Langlands parameter ϕ is generic (i.e., the corresponding local L-packet has a generic member) if and only if the associated adjoint L-function is regular at $s = 1$.

We prove the SO_{2n} case of this conjecture; the method is the same as in the case of SO_{2n+1} ([JngS2]) or the case of Sp_{2n} ([Liu]). For a general formulation and discussion of this conjecture, see [JngS2]. Note that in [GR] (see page 446, formula (14)), Gross and Reeder proved that for any connected reductive group G with maximal torus in the center of G anisotropic over F, if ϕ is discrete, then the associated adjoint L-function is regular at $s = 1$. In [AS], Asgari and Schmidt proved this conjecture for GSp_4.

Note that for $\mathbf{G} = SO_{2n}$, by the discussion in Remark 5.3, ϕ is generic if and only if the representation σ attached to ϕ in Theorem 5.2 is generic. Note that, by the discussion in the Introduction, there is no ambiguity here. By the classification of irreducible generic representations of $G = SO_{2n}(F)$ in Section 3, we have the following characterization of the genericity of the local Langlands parameters of SO_{2n}:

Proposition 6.1: *For any local Langlands parameter*

$$\phi : W_F \times SL_2(\mathbb{C}) \to SO_{2n}(\mathbb{C}),$$

the representation σ attached to ϕ in Theorem 5.2 is generic if and only if $\sigma^{(t)}$ and Σ_i $(i = 1, 2, \ldots, f)$ defined in (5.3) *and* (5.4) *satisfy the conditions of Definition 3.14.*

The following theorem is the SO_{2n} case of the conjecture. It gives a criterion for determining the genericity of the representation attached to each ϕ in Section 5. Note that the proof is similar to that for SO_{2n+1} in [JngS2] and for Sp_{2n} in [Liu]. To be complete, we give full details here.

THEOREM 6.2: *For any local Langlands parameter*

$$\phi : W_F \times SL_2(\mathbb{C}) \to SO_{2n}(\mathbb{C}),$$

the representation σ attached to ϕ in Theorem 5.2 is generic if and only if the associated adjoint L-function $L(\mathrm{Ad}_{SO_{2n}} \circ \phi, s)$ is regular at $s = 1$.

Proof. STEP (1) ($\Rightarrow$). Assume that σ is generic. Write $\phi = \phi^{(t)} \oplus \phi^{(n)}$ as in (5.1) and (5.2). Put

$$\theta = \bigoplus_{i=1}^{f} |\cdot|^{\frac{w_i}{2}-q_i}\phi_i \otimes S_{w_i+1}.$$

Then $\phi^{(n)} = \theta \oplus \widetilde{\theta}$, and we have the following decomposition of $L(\mathrm{Ad}_{SO_{2n}} \circ \phi, s)$:

$$\begin{aligned} L(\mathrm{Ad}_{SO_{2n}} \circ \phi, s) =& L(\theta \otimes \widetilde{\theta}, s) L(\theta \otimes \phi^{(t)}, s) L(\widetilde{\theta} \otimes \phi^{(t)}, s) \\ & \cdot L(\mathrm{Ad}_{SO_{2n^*}} \circ \phi^{(t)}, s) L(\wedge^2 \circ \theta, s) L(\wedge^2 \circ \widetilde{\theta}, s). \end{aligned} \tag{6.1}$$

In what follows, we show that each factor in the above product is holomorphic at $s = 1$. Hence, the adjoint L-function $L(\mathrm{Ad}_{SO_{2n}} \circ \phi, s)$ is regular at $s = 1$.

By conditions (1) and (2) of Definition 3.14, the representation $\pi = r(\theta) \times \rho^{(t)}$ is irreducible and generic, with $r^{-1}(\pi) = \theta \otimes \phi^{(t)}$, where r is the local Langlands reciprocity map for GL. Further, $\rho^{(t)} = l(\sigma^{(t)})$ by Theorem 4.12, $\phi^{(t)} = \iota(\sigma^{(t)})$ by Theorem 4.14, and $r(\theta) = \delta(\Sigma_1) \times \cdots \times \delta(\Sigma_f)$. Assume that π is a representation of $GL_{n_1}(F)$. Then by Proposition 7.1 of [JngS2], we know that

$$L(r^{-1}(\pi) \times \widetilde{r^{-1}(\pi)}, s) = L(\mathrm{Ad}_{GL_{n_1}} \circ r^{-1}(\pi), s)$$

is holomorphic at $s = 1$.

On the other hand, $L(r^{-1}(\pi) \times \widetilde{r^{-1}(\pi)}, s)$ has the following decomposition:

$$\begin{aligned} L(r^{-1}(\pi) \times \widetilde{r^{-1}(\pi)}, s) =& L((\theta \otimes \phi^{(t)}) \times (\widetilde{\theta} \otimes \phi^{(t)}), s) \\ =& L(\theta \times \widetilde{\theta}, s) L(\theta \times \phi^{(t)}, s) L(\widetilde{\theta} \times \phi^{(t)}, s) L(\phi^{(t)} \times \phi^{(t)}, s). \end{aligned} \tag{6.2}$$

By [HT] and [H], $L(\phi^{(t)} \times \phi^{(t)}, s) = L(\rho^{(t)} \times \rho^{(t)}, s)$, which is $P(q^{-s})^{-1}$ by definition, with $P(q^{-s})$ a polynomial in q^{-s}. Thus, it does not vanish at $s = 1$.

Therefore,

$$L(\theta \times \widetilde{\theta}, s)L(\theta \times \phi^{(t)}, s)L(\widetilde{\theta} \times \phi^{(t)}, s)$$

is holomorphic at $s = 1$.

Since $r(\phi^{(t)}) = \rho^{(t)}$, which is irreducible generic and self-contragredient, it follows from Proposition 7.1 of [JngS2] that

$$L(\phi^{(t)} \times \phi^{(t)}, s) = L(\phi^{(t)} \times \widetilde{\phi^{(t)}}, s) = L(\mathrm{Ad}_{GL_{2n^*}} \circ \phi^{(t)}, s) \tag{6.3}$$

is holomorphic at $s = 1$. Since as polynomials of q^{-s}, $L(\mathrm{Ad}_{SO_{2n^*}} \circ \phi^{(t)}, s)^{-1}$ divides $L(\mathrm{Ad}_{GL_{2n^*}} \circ \phi^{(t)}, s)^{-1}$, $L(\mathrm{Ad}_{SO_{2n^*}} \circ \phi^{(t)}, s)$ is holomorphic at $s = 1$.

From its definition above, we know that θ has positive exponents, so the L-function $L(\theta \otimes \theta, s)$ is holomorphic at $s = 1$. Since $L(\theta \otimes \theta, s) = L(Sym^2 \circ \theta, s)L(\wedge^2 \circ \theta, s)$, and $L(Sym^2 \circ \theta, s)$ does not vanish at $s = 1$, $L(\wedge^2 \circ \theta, s)$ must be holomorphic at $s = 1$.

It remains to show that $L(\wedge^2 \circ \widetilde{\theta}, s)$ is holomorphic at $s = 1$. Let $\theta_i = \phi_i \otimes S_{w_i+1}$. Then, we have the following decomposition:

$$\begin{aligned} L(\wedge^2 \circ \widetilde{\theta}, s) = &\prod_{i=1}^{f} L(\wedge^2 \circ \widetilde{\theta}_i, s - w_i + 2q_i) \\ &\cdot \prod_{1 \le i < j \le f} L(\widetilde{\theta}_i \otimes \widetilde{\theta}_j, s - \frac{w_i + w_j}{2} + q_i + q_j). \end{aligned}$$

For $1 \le i < j \le f$, it follows from [HT], [H], and (0.17) in [JngS2] that

$$\begin{aligned} L(\widetilde{\theta}_i \otimes \widetilde{\theta}_j, s - \frac{w_i + w_j}{2} + q_i + q_j) &= L(\widetilde{St}_i \otimes \widetilde{St}_j, s - \frac{w_i + w_j}{2} + q_i + q_j) \\ &= L(\delta(\widetilde{\Sigma_i}) \times \delta(\widetilde{\Sigma_j}), s), \end{aligned}$$

where $\delta(\widetilde{\Sigma_i}) = \nu^{q_i - \frac{w_i}{2}} \widetilde{St}_i$.

By Proposition 7.1 in [JngS2], we have the following fact: for $i < j$, Σ_i and Σ_j are linked if and only if

$$L(\delta(\Sigma_i) \times \delta(\widetilde{\Sigma_j}), s)L(\delta(\widetilde{\Sigma_i}) \times \delta(\Sigma_j), s)$$

has a pole at $s = 1$.

Since there is no linkage between $\widetilde{\Sigma_i}$ and Σ_j (by condition (1) of Definition 3.14), by the fact above, we have $L(\delta(\widetilde{\Sigma_i}) \times \delta(\widetilde{\Sigma_j}), s)$ is holomorphic at $s = 1$. That is,

$$L(\widetilde{\theta}_i \otimes \widetilde{\theta}_j, s - \frac{w_i + w_j}{2} + q_i + q_j)$$

is holomorphic at $s = 1$.

What is left are the L-factors $L(\wedge^2 \circ \widetilde{\theta}_i, z_i)$, for $i = 1, 2, \ldots, f$ and $z_i = s - w_i + 2q_i$. First, we have

$$\begin{aligned}\wedge^2 \circ \widetilde{\theta}_i &= \wedge^2 \circ (\widetilde{\phi}_i \otimes S_{w_i+1}) \\ &= (\wedge^2 \circ \widetilde{\phi}_i) \otimes (Sym^2 \circ S_{w_i+1}) \oplus (Sym^2 \circ \widetilde{\phi}_i) \otimes (\wedge^2 \circ S_{w_i+1}) \\ &= (\wedge^2 \circ \widetilde{\phi}_i) \otimes (Sym^2 \circ Sym^{w_i}\mathbb{C}^2) \oplus (Sym^2 \circ \widetilde{\phi}_i) \otimes (\wedge^2 \circ Sym^{w_i}\mathbb{C}^2),\end{aligned}$$

where $S_{w_i+1} = Sym^{w_i}\mathbb{C}^2$ is the $(w_i + 1)$-dimensional irreducible representation of $SL_2(\mathbb{C})$. Then, by Sections 11.2 and 11.3 of [FH],

$$Sym^2(Sym^m\mathbb{C}^2) = \bigoplus_{k=0}^{[\frac{m}{2}]} Sym^{2m-4k}\mathbb{C}^2,$$
$$\wedge^2(Sym^m\mathbb{C}^2) = \bigoplus_{k=0}^{[\frac{m-1}{2}]} Sym^{2(m-1)-4k}\mathbb{C}^2.$$

Therefore,

$$\begin{aligned}&\wedge^2 \circ \widetilde{\theta}_i \\ &= \left[\bigoplus_{k=0}^{[\frac{w_i}{2}]} (\wedge^2 \circ \widetilde{\phi}_i) \otimes S_{2w_i-4k+1}\right] \otimes \left[\bigoplus_{k=0}^{[\frac{w_i-1}{2}]} (Sym^2 \circ \widetilde{\phi}_i) \otimes S_{2(w_i-1)-4k+1}\right].\end{aligned}$$

Hence, by [HT], [H], and (0.17) in [JngS2],

$$\begin{aligned}&L(\wedge^2 \circ \widetilde{\theta}_i, z_i) \\ &= \prod_{k=0}^{[\frac{w_i}{2}]} L(\wedge^2 \circ \widetilde{\phi}_i, z_i + w_i - 2k) \cdot \prod_{k=0}^{[\frac{w_i-1}{2}]} L(Sym^2 \circ \widetilde{\phi}_i, z_i + w_i - 1 - 2k) \qquad (6.4)\\ &= \prod_{k=0}^{[\frac{w_i}{2}]} L(\wedge^2 \circ \widetilde{\phi}_i, s + 2q_i - 2k) \cdot \prod_{k=0}^{[\frac{w_i-1}{2}]} L(Sym^2 \circ \widetilde{\phi}_i, s + 2q_i - 1 - 2k).\end{aligned}$$

Since ϕ_i is an irreducible representation of W_F, by [HT] and [H], $\xi_i = r(\phi_i)$ is an irreducible supercuspidal representation of $GL_*(F)$. If ϕ_i is not self-contragredient, then by Theorem 4.3 of [JngS2],

$$L(\widetilde{\phi}_i \times \widetilde{\phi}_i, s) = L(\wedge^2 \circ \widetilde{\phi}_i, s) L(Sym^2 \circ \widetilde{\phi}_i, s) = L(\widetilde{\xi}_i, \wedge^2, s) L(\widetilde{\xi}_i, Sym^2, s)$$

is holomorphic on the real line. Therefore, we only have to deal with the case of ϕ_i self-contragredient, that is, ξ_i self-contragredient. Then

$$L(\wedge^2 \circ \widetilde{\phi}_i, s)L(Sym^2 \circ \widetilde{\phi}_i, s) = L(\widetilde{\xi}_i, \wedge^2, s)L(\widetilde{\xi}_i, Sym^2, s)$$

has a pole at $s = 0$.

If $L(\xi_i, \wedge^2, s + 2q_i - 2k)$ has a pole at $s = 1$, for some $0 \leq k \leq [\frac{w_i}{2}]$, then $1 + 2q_i - 2k = 0$, that is

$$-q_i = \frac{1}{2} - k \in \frac{1}{2} + \mathbb{Z}_{\leq 0}. \tag{6.5}$$

Since $-q_i + w_i \geq -q_i + \frac{w_i}{2} > 0$, we know from (6.5) that $-q_i + w_i \in \frac{1}{2} + \mathbb{Z}_{\geq 0}$. Also by (6.5), $-q_i \leq \frac{1}{2}$, we can see that

$$\frac{1}{2} \in \{-q_i, -q_i + 1, \ldots, -q_i + w_i\}.$$

This contradicts condition (3c) in Definition 3.14. On the other hand, the conditions (3a) and (3b) in Definition 3.14 do not hold. So, we can see that $L(\xi_i, \wedge^2, s + 2q_i - 2k)$ is holomorphic at $s = 1$ for all $0 \leq k \leq [\frac{w_i}{2}]$. That is, $L(\wedge^2 \circ \widetilde{\phi}_i, s + 2q_i - 2k)$ is holomorphic at $s = 1$ for all $0 \leq k \leq [\frac{w_i}{2}]$.

Similarly, if $L(\xi_i, Sym^2, s + 2q_i - 1 - 2k)$ has a pole at $s = 1$, for some $0 \leq k \leq [\frac{w_i - 1}{2}]$, then $-q_i = -k \in \mathbb{Z}_{\leq 0}$. So, $-q_i + w_i \geq 1$. Therefore, we can see that

$$0, 1 \in \{-q_i, -q_i + 1, \ldots, -q_i + w_i\}, \quad \text{and} \quad q_i \in \mathbb{Z}_{\geq 0}.$$

This contradicts conditions (3b) and (3c) in Definition 3.14. On the other hand, the condition (3a) in Definition 3.14 does not hold. So, we can see that $L(\xi_i, Sym^2, s+2q_i-2k-1)$ is holomorphic at $s = 1$, for all $0 \leq k \leq [\frac{w_i-1}{2}]$. That is, $L(Sym^2 \circ \widetilde{\phi}_i, s + 2q_i - 1 - 2k)$ is holomorphic at $s = 1$ for all $0 \leq k \leq [\frac{w_i-1}{2}]$.

Based on the above discussion, we can see that the adjoint L-function $L(\mathrm{Ad}_{SO_{2n}} \circ \phi, s)$ is holomorphic at $s = 1$ when $\phi \in \Phi(SO_{2n})$ is a generic parameter. That is, the representation σ attached to ϕ in Theorem 5.2 is generic. Hence, Step (1) has been proved.

Step (2) ($\Leftarrow$). We assume that the adjoint L-function $L(\mathrm{Ad}_{SO_{2n}} \circ \phi, s)$ is regular at $s = 1$. We then show that the representation σ attached to ϕ in Theorem 5.2 is generic, that is, $\{\Sigma_i\}_{i=1}^f$ defined in (5.4) is an SO_{2n}-generic sequence of segments with respect to $\sigma^{(t)}$ (see Definition 3.14). We do this by contradiction. Assume that σ is not generic. Then $\{\Sigma_i\}_{i=1}^f$ and $\sigma^{(t)}$ do not satisfy all the conditions in Definition 3.14.

If Part (1) of Definition 3.14 is not satisfied, then there exist $1 \leq i \neq j \leq f$, such that Σ_i is linked to Σ_j or $\widetilde{\Sigma_j}$. Again using the fact above that Σ_i and Σ_j $(i < j)$ are linked if and only if $L(\delta(\Sigma_i) \times \delta(\widetilde{\Sigma_j}), s)L(\delta(\widetilde{\Sigma_i}) \times \delta(\Sigma_j), s)$ has a pole at $s = 0$, we know that the following product

(6.6)

$$L(\delta(\Sigma_i) \times \delta(\widetilde{\Sigma_j}), s)L(\delta(\widetilde{\Sigma_i}) \times \delta(\Sigma_j), s)L(\delta(\Sigma_i) \times \delta(\Sigma_j), s)L(\delta(\widetilde{\Sigma_i}) \times \delta(\widetilde{\Sigma_j}), s)$$

has a pole at $s = 1$. Let $\theta_i' = |\cdot|^{\frac{w_i}{2} - q_i}\phi_i \otimes S_{w_i+1}$, for $1 \leq i \leq f$. Then $r(\theta_i') = \Sigma_i$. By (6.6), [HT] and [H], the following product

$$L(\theta_i' \times \widetilde{\theta_j'}), s)L(\widetilde{\theta_i'} \times \theta_j', s)L(\theta_i' \times \theta_j', s)L(\widetilde{\theta_i'} \times \widetilde{\theta_j'}, s) \tag{6.7}$$

has a pole at $s = 1$. By the proof for Step (1), $L(\theta_i' \times \widetilde{\theta_j'}), s)L(\widetilde{\theta_i'} \times \theta_j', s)$ occurs in $L(\theta \times \widetilde{\theta}, s)$, $L(\theta_i' \times \theta_j', s)$ occurs in $L(\wedge^2 \circ \theta, s)$, and $L(\widetilde{\theta_i'} \times \widetilde{\theta_j'}, s)$ occurs in $L(\wedge^2 \circ \widetilde{\theta}, s)$. Therefore, $L(\mathrm{Ad}_{SO_{2n}} \circ \phi, s)$ has a pole at $s = 1$—contradiction!

If Part (2) of Definition 3.14 is not satisfied, then the representation $r(\theta) \times \rho^{(t)}$ is reducible and its Langlands quotient π is non-generic. Then by Proposition 7.1 of [JngS2], we know that the product in (6.2) has a pole at $s = 1$. By (6.3), the last factor is holomorphic at $s = 1$; the pole at $s = 1$ must occur in the product of the first three factors, which, on the other hand, occurs in $L(\mathrm{Ad}_{SO_{2n}} \circ \phi, s)$ by (6.1). Therefore, we can see that $L(\mathrm{Ad}_{SO_{2n}} \circ \phi, s)$ has a pole at $s = 1$—contradiction!

If there is an integer $1 \leq i \leq f$ such that Part (3) of Definition 3.14 is not satisfied, then ϕ_i is self-contragredient; Σ_i is not linked to an element of X' or there exists $\tau \in X'$ such that $\tau \cong \xi_i$, $q_i = -1$, and there is some $1 \leq j \leq e_\tau$ with $a_j(\tau) = -1$ but $1 + w_i > b_j(\tau)$; and $r(\phi_i)$ does not satisfy condition (3c). Let $\sigma^{(2)}$ be the irreducible discrete series generic representation occurring in $\sigma^{(t)}$, $\rho^{(t)} = l(\sigma^{(t)})$, and $\sigma^{(0)}$ the irreducible supercuspidal generic representation occurring in $\sigma^{(2)}$. Put $\xi_i = r(\phi_i)$. Then ξ_i is self-contragredient.

Assume that $(\xi_i, \sigma^{(0)})$ is $(C1)$, but one of $\pm 1 \in \{-q_i, -q_i + 1, \ldots, -q_i + w_i\}$. Then $q_i \in \mathbb{Z}$. If $q_i < 0$, then $q_i \leq -1$, and so $-q_i \geq 1$, which implies $-q_i = 1$. Since $L(\rho^{(0)} \times \xi_i, s)$ has a pole at $s = 1$, we have Σ_i linked to an element of $A(\rho^{(2)})$. Then Σ_i is linked to an element of $A_0(\rho^{(2)}) \cup A_1(\rho^{(2)})$, or there exists $\tau \in X' = A_2(\rho^{(2)})$ such that $\tau \cong \xi_i$, $q_i = -1$, and there is some $1 \leq j \leq e_\tau$ with $a_j(\tau) = -1$ but $1 + w_i > b_j(\tau)$. In any case, Σ_i is linked to a segment associated to $\rho^{(t)}$, which means $r(\theta) \times \rho^{(t)}$ is reducible and its Langlands quotient π is non-generic. In this case, for the same reason as above, $L(\mathrm{Ad}_{SO_{2n}} \circ \phi, s)$ has a pole at $s=1$—contradiction! So we must have $q_i \geq 0$. Since $q_i < w_i/2$, we have

$$0 \leq q_i \leq \left[\frac{w_i - 1}{2}\right], \quad q_i \in \mathbb{Z}.$$

Now, as $L(\xi_i, Sym^2, s)$ has a pole at $s=0$, the second product in (6.4) must have a pole at $s=1$ if there exists a $0 \leq k' \leq [\frac{w_i-1}{2}]$ such that

$$1 + 2q_i - 2k' - 1 = 0,$$

that is, $k' = q_i$. So, the second product in (6.4) has a pole at $s=1$, as its factor corresponding to $k' = q_i$ has a pole at $s=1$—contradiction!

Suppose $(\xi_i, \sigma^{(0)})$ is $(C0)$, but $0 \in \{-q_i, -q_i+1, \ldots, -q_i+w_i\}$. Then we can see that $q_i \in \mathbb{Z}$. So $-q_i \leq 0$, that is, $q_i \geq 0$. On the other hand, $q_i < w_i/2$, so we have

$$0 \leq q_i \leq \left[\frac{w_i - 1}{2}\right], \quad q_i \in \mathbb{Z}.$$

Then, as in the last case, we can see that the second product in (6.4) has a pole at $s=1$—contradiction! If $(\xi_i, \sigma^{(0)})$ is (CN), but $q \in \mathbb{Z}_{\geq 0}$, then we have a similar contradiction!

Suppose $(\xi_i, \sigma^{(0)})$ is $(C\frac{1}{2})$, but one of $\pm\frac{1}{2} \in \{-q_i, -q_i+1, \ldots, -q_i+w_i\}$. Then $q_i \in \frac{1}{2}+\mathbb{Z}$ and $-q_i \leq \frac{1}{2}$, that is, $q_i + \frac{1}{2} \geq 0$. Since $L(\xi_i, \wedge^2, s)$ has a pole at $s=0$, the first product in (6.4) must have a pole at $s=1$ if there exists a $0 \leq k' \leq [\frac{w_i}{2}]$ such that

$$1 + 2q_i - 2k' = 0,$$

that is, $k' = q_i + \frac{1}{2}$. On the other hand, $q_i < w_i/2$, so we have

$$0 \leq q_i + \frac{1}{2} \leq \left[\frac{w_i}{2}\right], \quad q_i \in \frac{1}{2} + \mathbb{Z}.$$

Thus, the first product in (6.4) has a pole at $s=1$—contradiction!

Therefore $\{\Sigma_i\}_{i=1}^f$ and $\sigma^{(t)}$ do satisfy all the conditions in Definition 3.14, that is, σ is not generic. This completes the proof of Step (2), hence proves the theorem. ∎

7. Genericity and Arthur parameters

In this section, we give another application of the results in Sections 3–5. Given a local Arthur parameter ψ, and the corresponding local Langlands parameter ϕ_ψ, we give an alternate proof of a result of Shahidi that the representations attached to ϕ_ψ in Section 5 are generic if and only if ϕ_ψ is tempered, that is, ψ is trivial on the second SL_2.

Ban proved a similar result for the case of SO_{2n+1} in [Ban2], using the result of Jiang and Soudry in [JngS2]. Liu used the same method later, proving a similar result for the case of Sp_{2n} by generalizing the result of Jiang and Soudry in [JngS2] to the Sp_{2n} case. In previous sections, we gave the classification of irreducible generic representations for split SO_{2n} and generalized the result of Jiang and Soudry in [JngS2] to this case, so we can use the same strategy to prove the above result.

Recently, Shahidi (see Theorem 5.1 of [S4]) proved a similar result for any quasi-split connected reductive group G, with an assumption on the validity of the local Langlands conjecture for appropriate Levi subgroups M of G and data. Kim was able to remove this assumption for split GSpin groups, thus fully proving it in this case (see [Kim]). Note that by Lemma 7.2 [CKPSS] and Theorem 1.3 [H1], Theorem 5.1 [S4] implies that the SO_{2n} case of this result is true. We give a different proof here based on our classification.

First, let us recall the definition of local Arthur parameters (see [A1] and [A2]). The local Arthur parameter (A-parameter) for G is of the following form, a direct sum of irreducible representations:

$$\psi : W_F \times SL_2(\mathbb{C}) \times SL_2(\mathbb{C}) \to SO_{2n}(\mathbb{C})$$

$$\psi = \bigoplus_{i=1}^{t} \phi_i \otimes S_{m_i} \otimes S_{n_i},$$

satisfying the following conditions:

(1) $\phi_i(W_F)$ is bounded and consists of semi-simple elements;
(2) the restrictions of ψ to the two copies of $SL_2(\mathbb{C})$ are analytic.

For each A-parameter ψ, Arthur associated a local Langlands parameter (L-parameter) ϕ_ψ as follows:

$$\phi_\psi(w,x) = \psi\left(w, x, \begin{pmatrix} |w|^{\frac{1}{2}} & 0 \\ 0 & |w|^{-\frac{1}{2}} \end{pmatrix}\right). \tag{7.1}$$

Note that for any L-parameter ϕ,

$$\phi(w) \otimes S_m(x) \otimes S_n\left(\begin{pmatrix} |w|^{\frac{1}{2}} & 0 \\ 0 & |w|^{-\frac{1}{2}} \end{pmatrix}\right) = \bigoplus_{j=-\frac{n-1}{2}}^{\frac{n-1}{2}} |w|^j \phi(w) \otimes S_m(x).$$

Arthur also showed that $\psi \mapsto \phi_\psi$ is injective.

The following is the main theorem in this section:

Theorem 7.1 (Shahidi [S4]): *For each A-parameter ψ, and corresponding L-parameter ϕ_ψ, the representations attached to ϕ_ψ in Section 5 are generic if and only if ϕ_ψ is tempered.*

Proof. Assume $\psi = \bigoplus_{i=1}^t \phi_i \otimes S_{m_i} \otimes S_{n_i}$. Then, by (7.1),

$$\begin{aligned}
\phi_\psi &= \bigoplus_{i=1}^{t} \bigoplus_{j_i=-\frac{n_i-1}{2}}^{\frac{n_i-1}{2}} |\cdot|^{j_i}\phi_i \otimes S_{m_i} \\
&= \bigoplus_{\{i \mid n_i \text{ even}\}} \bigoplus_{j_i=\frac{1}{2}}^{\frac{n_i-1}{2}} |\cdot|^{j_i}\phi_i \otimes S_{m_i} \oplus |\cdot|^{-j_i}\widetilde{\phi_i} \otimes S_{m_i} \\
&\quad \oplus \bigoplus_{\{i \mid n_i \text{ odd}\}} \left(\bigoplus_{j_i=1}^{\frac{n_i-1}{2}} |\cdot|^{j_i}\phi_i \otimes S_{m_i} \oplus |\cdot|^{-j_i}\widetilde{\phi_i} \otimes S_{m_i} \right) \oplus \phi_i \otimes S_{m_i}.
\end{aligned} \tag{7.2}$$

Assume that σ is a representation attached to ϕ_ψ in Section 5. If ϕ_ψ is tempered, then σ is obviously generic by Theorem 5.2. Therefore, it suffices to assume that σ is generic and to show $n_i = 1$, for $1 \leq i \leq t$. To do this, we use Definition 3.14 to rule out the possibility of $n_i > 1$ for $1 \leq i \leq t$.

First, from (7.2), we can write down all the segments associated to σ as follows:

$$\begin{aligned}
&\bigcup_{n_i \text{ even}} \{[|\cdot|^{j_i+\frac{1-m_i}{2}}\tau_i, |\cdot|^{j_i+\frac{m_i-1}{2}}\tau_i], [|\cdot|^{-j_i+\frac{1-m_i}{2}}\widetilde{\tau}_i, |\cdot|^{-j_i+\frac{m_i-1}{2}}\widetilde{\tau}_i]\}_{j_i=\frac{1}{2}}^{\frac{n_i-1}{2}}, \\
&\bigcup_{n_i \text{ odd}} \{[|\cdot|^{j_i+\frac{1-m_i}{2}}\tau_i, |\cdot|^{j_i+\frac{m_i-1}{2}}\tau_i], [|\cdot|^{-j_i+\frac{1-m_i}{2}}\widetilde{\tau}_i, |\cdot|^{-j_i+\frac{m_i-1}{2}}\widetilde{\tau}_i]\}_{j_i=1}^{\frac{n_i-1}{2}}, \\
&\bigcup_{n_i \text{ odd}} \{[|\cdot|^{\frac{1-m_i}{2}}\tau_i, |\cdot|^{\frac{m_i-1}{2}}\tau_i]\},
\end{aligned} \tag{7.3}$$

where $\tau_i = r(\phi_i)$, an irreducible supercuspidal representation under the local Langlands reciprocity map r for GL.

For the case of n_i odd, we claim that n_i must be 1. Indeed, otherwise

$$\{[|\cdot|^{1+\frac{1-m_i}{2}}\tau_i, |\cdot|^{1+\frac{m_i-1}{2}}\tau_i], [|\cdot|^{-1+\frac{1-m_i}{2}}\widetilde{\tau}_i, |\cdot|^{-1+\frac{m_i-1}{2}}\widetilde{\tau}_i], [|\cdot|^{\frac{1-m_i}{2}}\tau_i, |\cdot|^{\frac{m_i-1}{2}}\tau_i]\}$$

is a subset of (7.3), which contradicts Part (2) of Definition 3.14 since $[|\cdot|^{1+\frac{1-m_i}{2}}\tau_i, |\cdot|^{1+\frac{m_i-1}{2}}\tau_i]$ is linked with $[|\cdot|^{\frac{1-m_i}{2}}\tau_i, |\cdot|^{\frac{m_i-1}{2}}\tau_i]$. Therefore, if n_i is odd, then $n_i = 1$.

For the case of n_i even, we claim that for any $1 \leq i \leq t$, $\phi_i \otimes S_{m_i} \otimes S_{n_i}$ is of orthogonal type. Indeed, otherwise, $\widetilde{\phi_i} \otimes S_{m_i} \otimes S_{n_i}$ is also included in ϕ_ψ. Then

$$\{[|\cdot|^{\frac{1}{2}+\frac{1-m_i}{2}}\tau_i, |\cdot|^{\frac{1}{2}+\frac{m_i-1}{2}}\tau_i], [|\cdot|^{-\frac{1}{2}+\frac{1-m_i}{2}}\widetilde{\tau_i}, |\cdot|^{-\frac{1}{2}+\frac{m_i-1}{2}}\widetilde{\tau_i}]\}$$
$$\cup \{[|\cdot|^{\frac{1}{2}+\frac{1-m_i}{2}}\widetilde{\tau_i}, |\cdot|^{\frac{1}{2}+\frac{m_i-1}{2}}\widetilde{\tau_i}], [|\cdot|^{-\frac{1}{2}+\frac{1-m_i}{2}}\tau_i, |\cdot|^{-\frac{1}{2}+\frac{m_i-1}{2}}\tau_i]\}$$

is a subset of (7.3), which contradicts Part (1) of Definition 3.14, since $[|\cdot|^{\frac{1}{2}+\frac{1-m_i}{2}}\tau_i, |\cdot|^{\frac{1}{2}+\frac{m_i-1}{2}}\tau_i]$ is linked with $[|\cdot|^{-\frac{1}{2}+\frac{1-m_i}{2}}\tau_i, |\cdot|^{-\frac{1}{2}+\frac{m_i-1}{2}}\tau_i]$. Therefore, for any $1 \leq i \leq t$, if n_i is even, then $\phi_i \otimes S_{m_i} \otimes S_{n_i}$ is of orthogonal type.

If there exists i, such that $n_i \geq 4$, even, then we can see that

$$\{[|\cdot|^{\frac{1}{2}+\frac{1-m_i}{2}}\tau_i, |\cdot|^{\frac{1}{2}+\frac{m_i-1}{2}}\tau_i], [|\cdot|^{-\frac{1}{2}+\frac{1-m_i}{2}}\widetilde{\tau_i}, |\cdot|^{-\frac{1}{2}+\frac{m_i-1}{2}}\widetilde{\tau_i}]\}$$
$$\cup \{[|\cdot|^{\frac{3}{2}+\frac{1-m_i}{2}}\tau_i, |\cdot|^{\frac{3}{2}+\frac{m_i-1}{2}}\tau_i], [|\cdot|^{-\frac{3}{2}+\frac{1-m_i}{2}}\widetilde{\tau_i}, |\cdot|^{-\frac{3}{2}+\frac{m_i-1}{2}}\widetilde{\tau_i}]\}$$

is a subset of (7.3), which also contradicts Part (1) of Definition 3.14, since $[|\cdot|^{\frac{1}{2}+\frac{1-m_i}{2}}\tau_i, |\cdot|^{\frac{1}{2}+\frac{m_i-1}{2}}\tau_i]$ is linked with $[|\cdot|^{\frac{3}{2}+\frac{1-m_i}{2}}\tau_i, |\cdot|^{\frac{3}{2}+\frac{m_i-1}{2}}\tau_i]$.

So, it remains to consider the case of $n_i = 2$. By the discussion above, $\phi_i \otimes S_{m_i} \otimes S_2$ is of orthogonal type. In particular, $\tau_i = r(\phi_i)$ is self-contragredient, and $\phi_i \otimes S_{m_i}$ is of symplectic type. Since σ is generic, the segment $\Sigma = [|\cdot|^{\frac{1}{2}+\frac{1-m_i}{2}}\tau_i, |\cdot|^{\frac{1}{2}+\frac{m_i-1}{2}}\tau_i]$ must satisfy Part (3b) or Part (3c) of Definition 3.14. It is easy to see that Σ does not satisfy Part (3b), since $\frac{1}{2} + \frac{1-m_i}{2} = 1$ implies $m_i = 0$. For Part (3c), if m_i is even, then τ_i is of orthogonal type, that is, $L(\tau_i, Sym^2, s)$ has a pole at $s = 0$. But now, $0, 1 \in \{-\frac{m_i}{2} + 1, \ldots, \frac{m_i}{2}\}$, and $\frac{m_i}{2} - 1 \in \mathbb{Z}_{\geq 0}$. If m_i is odd, then τ_i is of symplectic type, that is, $L(\tau_i, \wedge^2, s)$ has a pole at $s = 0$. But now, $\frac{1}{2} \in \{-\frac{m_i}{2} + 1, \ldots, \frac{m_i}{2}\}$. Therefore, Σ does not satisfy Part (3c) either. Contradiction!

This completes the proof of the theorem. ∎

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