ON CORANK 4 UNITARY REPRESENTATIONS OF CLASSICAL GROUPS

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Abstract. In this paper, we explicitly classify the corank 4 unitary representations of symplectic or split odd special orthogonal groups, by classifying Arthur representations of corank 4 and verifying the corresponding unitary dual conjecture recently proposed by Hazeltine-Jiang-Liu-Lo-Zhang in [HJLLZ25].

Contents

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1. Introduction

The unitary dual problem asks for the complete classification of all irreducible unitary representations of a given locally compact group G. Its origins trace back to the early twentieth century with the development of harmonic analysis on groups. For abelian connected Lie groups, Pontryagin duality provided a clean answer – the unitary dual of an abelian group is itself another group (the Pontryagin dual), and Fourier analysis is simply integration over this dual. For compact groups, the Peter–Weyl theorem gave a full decomposition of $L^2(G)$ into finite-dimensional irreducibles. The challenge intensified with non-compact and non-abelian groups, especially real and p-adic reductive Lie groups.

The unitary dual captures all possible symmetries that act unitarily, so it is essential to harmonic analysis, number theory (via automorphic forms and the Langlands program), and quantum physics (where symmetries are represented unitarily). In the Langlands framework, the structure of the unitary dual is deeply related to spectral decompositions L^2-spaces on arithmetic quotients and thus to major conjectures linking representation theory and arithmetic geometry. Despite major advances, a full description of the unitary dual is still only known for certain classes of groups; for general reductive groups, the classification remains highly nontrivial and continues to drive research in modern representation theory.

In [HJLLZ25], Hazeltine, Jiang, the first two named authors, and Zhang proposed a new conjecture describing the structure of the unitary dual in terms of Arthur representations for connected reductive algebraic groups defined over any non-Archimedean local field of characteristic zero. This conjecture provides a candidate set for the unitary dual, constructed from Arthur representations. For classical groups, they developed an explicit algorithm to generate this candidate set. Evidence for its exhaustiveness includes compatibility with the known generic unitary dual, unramified unitary dual, and low-corank representations. As further support of the conjecture, they verified the conjecture for the unitary dual of the exceptional group of type G_2.

More precisely, let F is a non-Archimedean local field of characteristic zero. The celebrated Langlands classification determines the admissible dual $\Pi(G)$ by means of the tempered dual $\Pi_{\mathrm{temp}}(G)$ of G, which is the subset of $\Pi(G)$ consisting of tempered members. However, there is no general approach available to determine the unitary dual $\Pi_u(G)$ based on the Langlands classification of the admissible dual $\Pi(G)$. On the other hand, the far-reaching endoscopic classification conjecture of J. Arthur ([Art89]) produces local Arthur packets (Definition 2.6), which are expected to be finite subsets of the unitary dual $\Pi_u(G)$. Denote by $\Pi_A(G)$ the subset of $\Pi(G)$ consisting of Arthur representations, also referred to as representations of Arthur type, which is the union of all local Arthur packets. It is strongly desirable that the whole unitary dual $\Pi_u(G)$ can be exhausted via various expected constructions from the Arthur representations $\Pi_A(G)$.

In [HJLLZ25], the authors defined a set $\Pi_{\overline{A}}^{\lim}(G)$ from Arthur representations $\Pi_A(G)$, which is constructed from $\Pi_A(G)$ by applying the standard constructions of unitary representations: unitary parabolic induction, complementary series, and limits of complementary series, as explained in Definition 2.22. Then, they conjectured that this set is exactly the unitary dual. Loosely speaking, the idea is that *complimentary series can be obtained via irreducible continuous Hermitian deformations from irreducible unitary inductions* (see [HJLLZ25, Remark 5.3(2)] for more precise statements). Similar ideas have been clearly reflected in the construction of the unitary dual for general linear groups (see [Tad86] and [HJLLZ25, Introduction]). The conjecture in [HJLLZ25] is as follows.

Conjecture 1.1 (Unitary Dual, [HJLLZ25, Conjecture 1.1])**.** *Let G be a general connected reductive group defined over F. Assume that the local Arthur conjecture as in* [Art89, Conjecture 6.1] *holds for every Levi subgroup M of G. Then the following two sets are equal:*

$$\Pi_{\overline{A}}^{\lim}(G) = \Pi_u(G).$$

The significance of Conjecture 1.1 may be described as follows. The unitary dual $\Pi_u(G)$ has a natural and conceptually simple definition in representation theory, yet its classification and internal structure remain largely mysterious. In contrast, according to the conjectures of J. Arthur ([Art89, Art13]), the set of Arthur representations $\Pi_A(G)$ is defined via local stability and (twisted) endoscopic transfer, yielding a rich and highly structured framework; however, its intrinsic representation-theoretic meaning is not fully understood.

Conjecture 1.1 serves as a bridge between these two domains by proposing a candidate subset $\Pi_{\overline{A}}^{\lim}(G)$ for the unitary dual, constructed from $\Pi_A(G)$ and designed to retain structural transparency while remaining amenable to explicit computation. A principal motivation of [HJLLZ25] is precisely that $\Pi_{\overline{A}}^{\lim}(G)$ should be computable, thereby opening the possibility of an algorithmic determination of the unitary dual itself. Supporting this perspective, the authors provide an explicit procedure ([HJLLZ25, Algorithm 8.5]) that produces $\Pi_{\overline{A}}^{\lim}(G)$ for symplectic groups and split odd special orthogonal groups.

On the other hand, in [Tad23], Tadic constructed the full unitary dual of classical groups for representations of corank up to 3 (see Definition 2.3). In this paper, we combine the techniques in both [Tad23] and [HJLLZ25], to construct the full unitary dual of symplectic and split odd special orthogonal groups and verify the corresponding Conjecture 1.1 for representations of corank 4 (see Theorem 11.1). In particular, the main result in this paper completes the classification of the unitary dual for Sp_8 and split SO_9.

Theorem 1.2. *For* $\mathrm{G} = \mathrm{Sp}_8$ *or split* SO_9*, Conjecture 1.1 holds, i.e.,* $\Pi_{\overline{A}}^{\lim}(G) = \Pi_u(G)$.

Our main methods may be summarized as follows. First, we compute the full good parity Arthur dual of corank 4 using the classification of tempered representations of good parity, as well as the techniques in [HJLLZ25] to determine Arthur type representations. By [AM25, Theorem 1.1], this is exactly equal to the unitary dual for good parity representations. Using this and the list of all possible subquotients at the critical points of corank 4, we can determine exactly which critical point contains a non-unitarizable subquotient. Then, we will use a similar reduction technique as the one used in [Tad23], in conjunction with the reduction techniques listed in [HJLLZ25, Algorithm 8.5], to classify all possible unitary connected components in corank 4. This will give us all 4 and 3-dimensional unitary connected components for corank 4 representations. Finally, we use an exhaustive process to compute all possible unitary complementary series of dimensions 1 and 2 to give the full corank 4 unitary dual, which we will prove to be equal to the conjectured set $\Pi_{\overline{A}}^{\lim}(G_n)$.

Here is the structure of the paper. In §2, we introduce the basic notation and give some preliminary results needed in later sections, leading up to the definition of the sets $\Pi_{\overline{A}}(G_n)$ and $\Pi_{\overline{A}}^{\lim}(G_n)$. In sections §3 to §7, we will construct the Arthur dual for representations of corank 4 that are of good parity. In particular, in §3, we begin by classifying good parity tempered representations of corank 3. Then, using these results, we will classify all good parity non-tempered representations of corank 4 in §4 to §6, and identify which one of them are of Arthur type and of critical type. Subsequently, in §7, we classify all good-parity tempered representations of corank 4, which gives the full Arthur dual of corank 4 in the good parity case. We summarize our results in Appendix A and B, by giving respectively the full list of representations that are of critical type and of Arthur type, as well as the complementary list of representations of critical type, but not of Arthur type. Using these two lists, we give the full list of unitary open components, as well as their boundaries, for representations of corank 4 in §8. To construct the full unitary dual, we append to it the unitarizable representations inside a one or two-parameter complementary series, which are constructed in §9 and §10 respectively. Finally, in §11, we give the full unitary dual of corank 4 and prove that it is the indeed the same as the closure of the Arthur dual $\Pi_{\overline{A}}^{\lim}(G_n)$, hence proving Conjecture 1.1.

2. Notations and preliminary results

Throughout the rest of this paper, let F denote a non-Archimedean local field of characteristic 0, otherwise known as a p-adic local field. Let $|\cdot|$ be the character of $\mathrm{GL}_n(F)$ obtained by composing the determinant function of $\mathrm{GL}_n(F)$ with the normalized p-adic absolute value of F.

Fix any positive integer n, we consider the representations of the general linear group $\mathrm{GL}_n(F)$, or the symplectic (resp. split odd special orthogonal) groups, denoted by $G_n = \mathrm{G}_n(F)$, where $\mathrm{G}_n = \mathrm{Sp}_{2n}$ (resp. SO_{2n+1}).

2.1. **Local Langlands classification.** In this subsection, we recall the explicit form of the Langlands classification for both the general linear group and classical groups over p-adic local fields ([Sil78, BW00, Kon03]).

In the case of general linear groups $\mathrm{GL}_n(F)$, fix a Borel subgroup B and let $P = MN$ be the standard parabolic subgroup of $\mathrm{GL}_n(F)$ with Levi subgroup $M \cong \mathrm{GL}_{n_1}(F) \times \cdots \times \mathrm{GL}_{n_r}(F)$. Given smooth representations τ_i of $\mathrm{GL}_{n_i}(F)$ for $i = 1, 2, \ldots r$, we denote the normalized parabolic induction by:

$$\tau_1 \times \cdots \times \tau_r := \mathrm{Ind}_P^{\mathrm{GL}_n(F)}(\tau_1 \otimes \cdots \otimes \tau_r).$$

Throughout the rest of this paper, let ρ denote an irreducible supercuspidal representation of $\mathrm{GL}_n(F)$. A (Zelevinsky) segment $[x, y]_\rho$ is the set of supercuspidal representations of the form

$$[x, y]_\rho = \{\rho|\cdot|^x, \rho|\cdot|^{x-1}, \ldots, \rho|\cdot|^y\},$$

where $x, y \in \mathbb{R}$ and $x - y$ is a non-negative integer. The unique irreducible subrepresentation of $\rho|\cdot|^x \times \rho|\cdot|^{x-1} \times \ldots \rho|\cdot|^y$ is called the (essentially) Steinberg representation attached to the segment $[x, y]_\rho$ and is denoted by $\Delta_\rho[x, y]_\rho$. For simplicity, we will treat $\Delta_\rho[x, y]$ as the trivial representation of $\mathrm{GL}_0(F)$ when $x < y$.

Langlands classification provides a way to classify all equivalence classes of smooth admissible representations of $\mathrm{GL}_n(F)$, or the admissible dual $\Pi(\mathrm{GL}_n(F))$. Specifically, it states that any irreducible admissible representation τ of $\mathrm{GL}_n(F)$ can be realized as the unique irreducible subrepresentation of some

$$\Delta_{\rho_1}[x_1, y_1] \times \cdots \times \Delta_{\rho_r}[x_r, y_r],$$

where the ρ_i's are irreducible unitary supercuspidal representations of $\mathrm{GL}_n(F)$, $[x_i, y_i]_{\rho_i}$ are segments, and $x_1 + y_1 \leq \cdots \leq x_r + y_r$. In this case, we write

$$\tau = L(\Delta_{\rho_1}[x_1, y_1], \ldots \Delta_{\rho_r}[x_r, y_r]).$$

Let $(x_{i,j})_{1\leq i\leq s, 1\leq j\leq t}$ be real numbers such that $x_{i,j} = x_{1,1} - i + j$. A *generalized Speh representation* is an irreducible representation of the form

$$\begin{pmatrix} x_{1,1} & \cdots & x_{1,t} \\ \vdots & \ddots & \vdots \\ x_{s,1} & \cdots & x_{s,t} \end{pmatrix}_\rho := L(\Delta_\rho[x_{1,1}, x_{s,1}], \ldots, \Delta_\rho[x_{1,t}, x_{s,t}]). \tag{2.1}$$

These representations will be useful in the classification of unitary dual for $\mathrm{GL}_n(F)$.

The Langlands classification of the classical groups G_n can be given similarly as follows. Let $P = MN$ be a standard parabolic subgroup of G_n with Levi subgroup $M \cong \mathrm{GL}_{n_1}(F) \times \cdots \times \mathrm{GL}_{n_r}(F) \times G_m$ (G_m a group of the same type as G_n with $m \leq n$). Given smooth representations τ_i of $\mathrm{GL}_{n_i}(F)$ for $i = 1, 2, \ldots r$, and a smooth representation σ of G_m $(m \leq n)$, we denote the normalized parabolic induction by:

$$\tau_1 \times \ldots \tau_r \rtimes \sigma := \mathrm{Ind}_P^{G_n}(\tau_1 \otimes \cdots \otimes \tau_r \otimes \sigma).$$

Then, Langlands classification for G_n states that every irreducible representation π of G_n is a unique irreducible subrepresentation of

$$\Delta_{\rho_1}[x_1, y_1] \times \cdots \times \Delta_{\rho_r}[x_r, y_r] \rtimes \pi_{temp},$$

where the ρ_i's are irreducible unitary supercuspidal representations of $\mathrm{GL}_n(F)$, $x_1 + y_1 \leq \ldots \leq x_r + y_r < 0$, and π_{temp} is tempered. In this case, we write

$$\pi = L(\Delta_{\rho_1}[x_1, y_1], \ldots \Delta_{\rho_r}[x_r, y_r]; \pi_{temp}).$$

We call the tuple $(\Delta_{\rho_1}[x_1, y_1], \ldots, \Delta_{\rho_r}[x_r, y_r], \pi_{temp})$ the Langlands data, or L-data, of π, and the multi-set $\{\Delta_{\rho_1}[x_1, y_1], \ldots, \Delta_{\rho_r}[x_r, y_r]\}$ the non-tempered portion of the L-data of π.

2.2. **Arthur's parameterization of the tempered spectrum.** In this subsection, for classical groups G_n, we recall the local Langlands correspondence ([Bor79, §8.2] and the Arthur's parametrization of the tempered spectrum ([Art13, Theorem 1.5.1]).

For any split reductive algebraic group G, the Langlands dual group of G is denoted by $\mathrm{G}^\vee(\mathbb{C})$. A local L-parameter of G_n is a $\mathrm{G}_n^\vee(\mathbb{C})$-conjugacy class of an admissible homomorphism $\phi : W_F \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{G}_n^\vee(\mathbb{C})$ (more generally, see [Bor79, §8.2]). In this paper, we do not distinguish ϕ from its conjugacy class. We say that ϕ is tempered if its restriction to W_F has bounded image. The component group of ϕ is defined by

$$\mathcal{S}_\phi := \pi_0(\mathrm{Cent}_{\mathrm{G}_n^\vee(\mathbb{C})}(\mathrm{Im}(\phi))/Z(\mathrm{G}_n^\vee(\mathbb{C}))^\Gamma),$$

where Γ is the absolute Galois group of F.

By the local Langlands correspondence for $\mathrm{GL}_n(F)$ ([Hen00, HT01, Sch13]), we may identify an irreducible supercuspidal representation ρ of $\mathrm{GL}_n(F)$ with its local L-parameter ϕ_ρ, which is an irreducible n-dimensional representation of the Weil group W_F. Explcitly, we can write any tempered local L-parameter ϕ in the following form:

$$\phi = \bigoplus_{i=1}^r \rho_i \otimes S_{a_i},$$

where ρ_i are irreducible unitary supercuspidal representations of $\mathrm{GL}_{n_i}(F)$ and S_k denotes the unique irreducible representation of $\mathrm{SL}_2(\mathbb{C})$ of dimension k. Let $\mathrm{Jord}(\phi)$ denote the multi-set consisting of all the irreducible summands occurring in ϕ (counting multiplicities), i.e. for ϕ as above, we have $\mathrm{Jord}(\phi) = \{\rho_1 \otimes S_{a_1}, \ldots, \rho_r \otimes S_{a_r}\}$.

For the groups G_n, the Pontrayagin dual $\widehat{\mathcal{S}}_\phi$ of $\mathcal{S}_\phi$ is a finite abelian group consisting of characters which may be identified with functions $\varepsilon : \mathrm{Jord}(\phi) \to \{\pm 1\}$ such that $\varepsilon(\rho_i \otimes S_{a_i}) = \varepsilon(\rho_j \otimes S_{a_j})$ whenever $\rho_i \otimes S_{a_i} \cong \rho_j \otimes S_{a_j}$, and

$$\prod_{\rho \otimes S_a \in \mathrm{Jord}(\phi)} \varepsilon(\rho \otimes S_a)^{m_{\rho,a}} = 1,$$

where $m_{\rho,a}$ denotes the multiplicity of $\rho \otimes S_a$ in ϕ.

From the conjectural local Langlands correspondence, we associate to each local L-parameter ϕ a finite set of irreducible admissible representation of G satisfying certain properties (see [Bor79]), which is called the local L-packet attached to ϕ and is denoted by Π_ϕ. For G_n, the following theorem of Arthur shows that Π_ϕ is in bijection with $\widehat{\mathcal{S}}_\phi$ for tempered parameters ϕ.

Theorem 2.1 ([Art13, Theorem 1.5.1])**.** *Fix a choice of Whittaker datum for G_n and let ϕ be a tempered local L-parameter. Then there is a bijective map between the tempered local L-packet Π_ϕ and $\widehat{\mathcal{S}}_\phi$.*

Henceforth, we fix a choice of Whittaker datum for G_n. When ϕ is tempered, we write $\pi(\phi, \varepsilon)$ for the element of Π_ϕ corresponding to $\varepsilon \in \widehat{\mathcal{S}}_\phi$ via the bijection in Theorem 2.1.

2.3. **Supercuspidal representations and reducibility points.** In this subsection, we recall Mœglin's characterization of local L-parameters for irreducible supercuspidal representations of G_n. Let $\mathcal{C}$ (resp. $\mathcal{C}_{cl}$) be the set of supercuspidal representations of general linear groups (resp. classical groups), $\mathcal{C}^u := \{\rho \in \mathcal{C} \mid \rho \text{ is unitary}\}$, and $\mathcal{C}^{sd} := \{\rho \in \mathcal{C}^u \mid \rho \text{ is self-dual}\}$.

For any tempered L-parameter $\phi = \bigoplus_i \rho_i \otimes S_{a_i}$, we write $\rho \otimes S_a \subset \phi$ if $\rho \otimes S_a$ appears as a direct summand in ϕ. A tempered L-parameter $\phi = \bigoplus_i \rho_i \otimes S_{a_i}$ is called *discrete* if the $\rho_i \otimes S_{a_i}$'s are pairwise non-equivalent. A discrete L-parameter $\phi = \bigoplus_i \rho_i \otimes S_{a_i}$ is said to be *without gaps* if for $a \geq 1$,

$$\rho \otimes S_{a+2} \subset \phi \Rightarrow \rho \otimes S_a \subset \phi.$$

Applying Theorem 2.1, Mœglin's parametrization of irreducible supercuspidal representations of G_n is as follows.

Theorem 2.2 ([Mœ11b, Theorem 1.5.1], [Xu17a, Theorem 3.3])**.** *An irreducible tempered representation $\pi(\phi, \varepsilon)$ of G_n is supercuspidal if and only if the following hold.*

- *As a tempered local L-parameter, ϕ is discrete and without gaps.*
- *If both $\rho \otimes S_a \subset \phi$ and $\rho \otimes S_{a+2} \subset \phi$, then $\varepsilon(\rho \otimes S_a)\varepsilon(\rho \otimes S_{a+2}) = -1$.*
- *If $\rho \otimes S_2 \subset \phi$, then $\varepsilon(\rho \otimes S_2) = -1$.*

Let $\rho \in \mathcal{C}^u$ and $\sigma \in \mathcal{C}_{cl}$. If ρ is not self-dual, then $\rho|\cdot|^x \rtimes \sigma$ is irreducible for all $x \in \mathbb{R}$. If ρ is self-dual, then there exists a unique $\alpha_{\rho,\sigma} \in \mathbb{R}_{\geq 0}$, such that $\rho|\cdot|^{\alpha_{\rho,\sigma}} \rtimes \sigma$ is reducible ([Sil80]). The number $\alpha_{\rho,\sigma}$ is known as the reducibility point, and it is known that $\alpha_{\rho,\sigma} \in \frac{1}{2}\mathbb{Z}_{\geq 0}$. In fact, there is an explicit description of $\alpha_{\rho,\sigma}$ based on the local L-parameter ϕ_σ of σ as follows, according to [MW06, Remark 4.5.2].

By Theorem 2.2, we may write

$$\phi_\sigma = \bigoplus_{\rho \in R} \bigoplus_{i=0}^{a_\rho} \rho \otimes S_{2(i+\epsilon_\rho)+1}, \tag{2.2}$$

where R is a finite set of $\mathcal{C}^{sd}$ and $a_\rho \in \mathbb{Z}_{\geq 0}$. Here $\epsilon_\rho \in \{0, \frac{1}{2}\}$ based on the parity of ρ. More explicitly, if G_n is an orthogonal (resp. symplectic) group, then $\epsilon_\rho = 0$ (resp. $\epsilon_\rho = \frac{1}{2}$) if the image of the homomorphism $\rho : W_F \to \mathrm{GL}(V)$ preserves a symplectic bilinear form on V, and $\epsilon_\rho = \frac{1}{2}$ (resp. $\epsilon_\rho = 0$) if the image of ρ preserves a symmetric bilinear form on V. Set $a_\rho = -1$ if ρ is not in the finite set R. Then the reducibility point $\alpha_{\rho,\sigma}$ is given by $a_\rho + \epsilon_\rho + 1$ and the decomposition (2.2) can be rewritten as

$$\phi_\sigma = \bigoplus_{\rho \in R} (\rho \otimes S_{2\epsilon_\rho+1} + \rho \otimes S_{2\epsilon_\rho+3} + \cdots + \rho \otimes S_{2(\alpha_{\rho,\sigma}-1)+1}). \tag{2.3}$$

Let us now recall the definition of corank for representations, which gives the notion of dimension in our construction of the unitary dual.

Definition 2.3. *A representation* $\pi \in \Pi(G_n)$ *is said to be of corank* r *if there exists an injection* $\pi \hookrightarrow \rho_1 \times \cdots \times \rho_r \times \pi_{sc}$*, where* $\rho_i \in \mathcal{C}$ *and* $\pi_{sc} \in \mathcal{C}_{cl}$*.*

2.4. **Representations of good parity and critical type.** In this subsection, we recall the definition of a representation being *null parity, good parity, bad parity* and *critical type.* We often use the unitarizability of a representation of good parity to determine whether a continuous family of representations are unitary.

Definition 2.4. *Let* π *be an irreducible admissible representation of* G_n*, which is a subquotient of* $\rho_1|\cdot|^{x_1} \times \cdots \times \rho_r|\cdot|^{x_r} \rtimes \pi_{sc}$*, where* $\rho_i \in \mathcal{C}^u$*,* $x_i \in \mathbb{R}_{\geq 0}$ *and* $\pi_{sc} \in \mathcal{C}_{cl}$*.*

1. *Define* π *to be of* null parity (*named as "ugly" in* [AM23]) *if there exists* $1 \leq i \leq r$*, such that either* $\rho_i \notin \mathcal{C}^{sd}$ *or* $x_i \notin \frac{1}{2}\mathbb{Z}$*.*
2. *Define* π *to be of* good parity *if for any* $1 \leq i \leq r$*,* $\rho_i \in \mathcal{C}^{sd}$ *and* $x_i \in \alpha_{\rho_i,\pi_{sc}} + \mathbb{Z}$*, where* $\alpha_{\rho_i,\pi_{sc}} \in \mathbb{R}_{\geq 0}$ *is the reducibility point of the pair* (ρ_i, π_{sc})*.*
3. *Define* π *to be of* bad parity *if* π *is of neither null-parity nor good parity.*
4. *Define* π *to be of* critical type *if it is of good parity and for each* $\rho \in \mathcal{C}^{sd}$*, the set (not multi-set)* $\{x_i \mid \rho_i \cong \rho\}$ *is either empty or forms a segment containing* $\alpha_{\rho,\pi_{sc}}$*.*

We associate the notion of being good parity for representations to that of L-parameters as follows. A local L-parameter ϕ of G_n is said to be of good parity if the local L-packet Π_ϕ contains any representation of good parity. In fact, this condition is equivalent to the one that all representations of Π_ϕ being good parity.

According to the following theorem proven by Atobe and Mínguez, for good parity representations, being unitary is equivalent to be of Arthur type (see §2.5).

Theorem 2.5. [AM25, Theorem 1.1] *Let* π *be an irreducible unitary representation of* G_n *of good parity. Then* π *is unitary if and only if it is of Arthur type.*

This result gives a full conjectural description of the unitary dual for good parity representations, and greatly simplifies our classification of the unitary dual of corank 4 in this paper.

2.5. **Local Arthur packets and reduction to good parity.** In this subsection, we recall the theory of local Arthur packets, as established in [Art13], and the reduction of the Arthur dual to the good parity case. Recall that F is a p-adic local field, and $G_n = \mathrm{G}_n(F)$, where $\mathrm{G}_n = Sp_{2n}$ or SO_{2n+1}. Let W_F be the Weil group of F and $G_n^\vee(\mathbb{C})$ be the Langlands dual group of G_n.

Definition 2.6. *A* local Arthur parameter *is a homomorphism*

$$\psi : W_F \times \mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C}) \longrightarrow \mathrm{G}_n^\vee(\mathbb{C}),$$

$$\psi = \bigoplus_{i=1}^{r} \phi_i \otimes S_{a_i} \otimes S_{b_i}, \tag{2.4}$$

satisfying the following conditions:

1. $\phi_i(W_F)$ *is bounded and consists of semi-simple elements, and* $\dim(\phi_i) = d_i$*;*
2. *the restrictions of* ψ *to the two copies of* $\mathrm{SL}_2(\mathbb{C})$ *are algebraic,* S_k *is the* k*-dimensional irreducible representation of* $\mathrm{SL}_2(\mathbb{C})$*, and*

$$\sum_{i=1}^{r} d_i a_i b_i = N := \begin{cases} 2n+1 & \text{when } G_n = Sp_{2n}, \\ 2n & \text{when } G_n = SO_{2n+1}. \end{cases}$$

The first copy of $\mathrm{SL}_2(\mathbb{C})$ *is called the Deligne-*$\mathrm{SL}_2(\mathbb{C})$ *and is denoted by* $\mathrm{SL}_2^D(\mathbb{C})$*. The second copy of* $\mathrm{SL}_2(\mathbb{C})$ *is called the Arthur-*$\mathrm{SL}_2(\mathbb{C})$ *and is denoted by* $\mathrm{SL}_2^A(\mathbb{C})$*. A local Arthur parameter* ψ *given in* (2.4) *is called* generic *if* $b_i = 1$ *for* $i = 1, \dots, r$*.*

Given a local Arthur parameter as in (2.4)*, Arthur defined a packet* Π_ψ *in* [Art13, Theorem 2.2.1]*, called a* local Arthur packet*, which is a finite subset of* $\Pi_u(G_n)$*, satisfying certain twisted endoscopic character identities. Let* $\Psi(G_n)$ *be the subset of local Arthur parameters. We say that a representation* π *is of* **Arthur type** *if* $\pi \in \Pi_\psi$ *for some local Arthur parameter* $\psi \in \Psi(G_n)$*. Let*

$$\Pi_A(G_n) = \{\pi \in \Pi_\psi \mid \psi \in \Psi(G_n)\}.$$

Representations in $\Pi_A(G_n)$ *are called* **Arthur representations***. For* $\pi \in \Pi_A(G_n)$*, we let*

$$\Psi(\pi) := \{\psi \in \Psi(G_n) \mid \pi \in \Pi_\psi\}.$$

Now we recall the decomposition of local Arthur parameters and the reduction of the construction of local Arthur packets to the good parity case. By the Local Langlands Correspondence for $\mathrm{GL}_{d_i}(F)$, a bounded representation ϕ of W_F can be identified with an irreducible unitary supercuspidal representation ρ of $\mathrm{GL}_{d_i}(F)$ ([Hen00, HT01, Sch13]). Consequently, we may write (2.4) as

$$\psi = \bigoplus_{i \in I} \rho_i |\cdot|^{x_i} \otimes S_{a_i} \otimes S_{b_i}, \tag{2.5}$$

where ρ_i's are irreducible unitary supercuspidal representations of $\mathrm{GL}_{d_i}(F)$. With this decomposition, we say that ψ is of *good parity* if the following holds. Let σ be any irreducible supercuspidal representation of G_n. Then $x_i = 0$ and $\frac{a_i+b_i}{2} \in \alpha_{\rho_i,\sigma} + \mathbb{Z}$ for any $i \in I$. Equivalently, ψ is of good parity if and only if any representation in the local L-packet Π_{ϕ_ψ} is of good parity (Definition 2.4(2)). We remark that by Theorem 2.7 below and the construction of good parity local Arthur packets (see the next subsection), ψ is of good parity if and only if any representation in the local Arthur packet Π_ψ is of good parity.

Let $\psi \in \Psi(G_n)$. Since ψ factors through $\mathrm{G}_n^\vee(\mathbb{C})$, we may rewrite the decomposition (2.5) as

$$\psi = \bigoplus_{i \in I_{ngp}} (\rho_i \otimes S_{a_i} \otimes S_{b_i} + \rho_i^\vee \otimes S_{a_i} \otimes S_{b_i}) \oplus \bigoplus_{i \in I_{gp}} \rho_i \otimes S_{a_i} \otimes S_{b_i},$$

where

- For any $i \in I_{ngp}$, either $\rho_i \not\cong \rho_i^\vee$, or, $\rho_i \cong \rho_i^\vee$ and $\frac{a_i+b_i}{2} \notin \alpha_{\rho_i,\sigma} + \mathbb{Z}$;
- For any $i \in I_{gp}$, $\rho_i \cong \rho_i^\vee$ and $\frac{a_i+b_i}{2} \in \alpha_{\rho_i,\sigma} + \mathbb{Z}$.

For $* \in \{ngp,\ gp\}$, define subrepresentations ψ_* of ψ by

$$\psi_* := \bigoplus_{i \in I_*} \rho_i \otimes S_{a_i} \otimes S_{b_i}.$$

Thus ψ_{gp} is of good parity and

$$\psi = (\psi_{ngp} + \psi_{ngp}^\vee) + \psi_{gp}. \tag{2.6}$$

For a unitary supercuspidal representation ρ of $\mathrm{GL}_d(F)$ and $a, b \in \mathbb{Z}_{>0}$, we let

$$u_\rho(a,b) := \begin{pmatrix} \frac{a-b}{2} & \cdots & \frac{a+b}{2} - 1 \\ \vdots & \ddots & \vdots \\ \frac{-a-b}{2} + 1 & \cdots & \frac{b-a}{2} \end{pmatrix}_\rho, \tag{2.7}$$

be the corresponding unitary generalized Speh representation (see (2.1)). This is the unique member of the local Arthur packet $\Pi_{\rho\otimes S_a\otimes S_b}(\mathrm{GL}_{abd}(F))$. For each $i\in I_{ngp}$, define τ_i to be the generalized Speh representation $u_{\rho_i}(a_i,b_i)$. Then we set

$$\tau_{\psi_{ngp}} := \bigtimes_{i\in I_{ngp}} \tau_i,$$

which is irreducible.

Mœglin showed that the construction of local Arthur packets can be reduced to good parity ones as follows.

Theorem 2.7 ([Mœ11a, Proposition 5.1]). *Let $\psi\in\Psi(G_n)$ with decomposition (2.6). Then, for any $\pi_{gp}\in\Pi_{\psi_{gp}}$, the parabolic induction $\tau_{\psi_{ngp}}\rtimes\pi_{gp}$ is irreducible. As a consequence,*

$$\Pi_\psi = \{\tau_{\psi_{ngp}}\rtimes\pi_{gp} \mid \pi_{gp}\in\Pi_{\psi_{gp}}\}. \tag{2.8}$$

2.6. **Extended multi-segments and operators.** In this subsection, we recall the notion of extended multi-segments and related operations, which provide important computation tools in the construction of the Arthur dual.

Definition 2.8 ([Ato22b, Definition 3.1]). *(Extended multi-segments)*

(1) *An* extended segment *is a triple $([A,B]_\rho,l,\eta)$, where*
- $[A,B]_\rho=\{\rho|\cdot|^A,\rho|\cdot|^{A-1},\ldots,\rho|\cdot|^B\}$ *is a segment for an irreducible unitary supercuspidal representation ρ of some $\mathrm{GL}_d(F)$;*
- $l\in\mathbb{Z}$ *with $0\le l\le\frac{b}{2}$, where $b=\#[A,B]_\rho=A-B+1$;*
- $\eta\in\{\pm1\}/E$, *where $E=\{\pm1\}$ if $b=2l$ and $E=\{+1\}$ if $b>2l$.*

In the statements and formulas in this section, we regard $\eta\in\{\pm1\}$ by fixing any of its preimage except in Definition 2.14, where the choice of the preimage is specified.

(2) *Consider a multi-set of extended segments of the form $\{([A_i,B_i]_\rho,l_i,\eta_i)\}_{i\in I_\rho}$. We say that a total order $>$ on I_ρ is* admissible *(or* satisfies (P)*) if*
$$A_i<A_j, B_i<B_j \Longrightarrow i<j.$$
We say that an admissible order $>$ satisfies (P') *if $B_i<B_j\Longrightarrow i<j$.*

(3) *An* extended multi-segment *for G_n is a union of multi-sets of extended segments indexed by a collection of total ordered sets $(I_\rho,>)$: $\mathcal{E}=\cup_\rho\{([A_i,B_i]_\rho,l_i,\eta_i)\}_{i\in(I_\rho,>)}$ such that*
   (a) *I_ρ is a totally ordered finite set with a fixed total order $>$ satisfying (P);*
   (b) *$A_i+B_i\ge0$ for all ρ and $i\in I_\rho$;*
   (c) *as a representation of $W_F\times\mathrm{SL}_2(\mathbb{C})\times\mathrm{SL}_2(\mathbb{C})$, $\psi_{\mathcal{E}}=\bigoplus_\rho\bigoplus_{i\in I_\rho}\rho\otimes S_{a_i}\otimes S_{b_i}$, where $(a_i,b_i)=(A_i+B_i+1,A_i-B_i+1)$, is a local Arthur parameter for G_n of good parity. We shall denote by $\psi_{\mathcal{E}}$ the local Arthur parameter associated with $\mathcal{E}$.*
   (d) *The sign condition*
   $$\prod_\rho\prod_{i\in I_\rho}(-1)^{[\frac{b_i}{2}]+l_i}\eta_i^{b_i}=1$$
   holds.

(4) *For each extended multi-segment $\mathcal{E}$, we denote by $\pi(\mathcal{E})$ the representation associated with $\mathcal{E}$ as in [Ato22b, §3.2], which is either irreducible or zero. We denote by $\underline{\mathrm{Rep}}$ the set of extended multi-segments that give nonzero representations, and by $\underline{\mathrm{Rep}}^{(P')}$*

the subset of $\underline{\mathrm{Rep}}$ *that consists of extended multi-segments whose total order on any* I_ρ *satisfies* (P'). *For a given* $\mathcal{E} = \cup_\rho \{([A_i, B_i]_\rho, l_i, \eta_i)\}_{i \in (I_\rho, >)} \in \underline{\mathrm{Rep}}$, *define that*

$$\mathcal{E}_\rho = \{([A_i, B_i]_\rho, l_i, \eta_i)\}_{i \in (I_\rho, >)} \quad \text{and} \quad \mathcal{E}^\rho = \cup_{\rho' \not\cong \rho} \{([A_i, B_i]_{\rho'}, l_i, \eta_i)\}_{i \in (I_{\rho'}, >)}.$$

Then $\mathcal{E} = \mathcal{E}^\rho \cup \mathcal{E}_\rho$.

(5) *Sometimes we write* $\mathcal{E}_\rho = \{([A_1, B_1]_\rho, l_1, \eta_1), \ldots, ([A_k, B_k]_\rho, l_k, \eta_k)\}$, *implying that the elements in* $\mathcal{E}_\rho$ *are listed increasingly with respect to the admissible order of* $\mathcal{E}_\rho$. *Assume that*

$$\mathcal{F} = \{([A_i, B_i]_\rho, l_i, \eta_i)\}_{i \in (I_\rho, >)} \quad \text{and} \quad \mathcal{F}' = \{([A_i, B_i]_\rho, l_i, \eta_i)\}_{i \in (I'_\rho, >)}.$$

Then we let $\mathcal{F} + \mathcal{F}' = \{([A_i, B_i]_\rho, l_i, \eta_i)\}_{i \in (I_\rho \sqcup I'_\rho, \gg)}$ *be the extended multi-segment, where the admissible order* $\gg$ *is defined by* $i \gg j$ *if and only if* $(i, j) \in I_\rho \times I_\rho \sqcup I'_\rho \times I'_\rho$ *and* $i > j$, *or,* $(i, j) \in I'_\rho \times I_\rho$.

(6) *Suppose that* $\mathcal{E} \in \underline{\mathrm{Rep}}^{(P')}$ *and denote*

$$\mathcal{F} = \mathcal{E}_\rho = \{([A_i, B_i]_\rho, l_i, \eta_i)\}_{i \in (I_\rho, >)}.$$

Let $x \in \mathbb{R}$. *We define*

$$\begin{aligned} \mathcal{F}_{>x} &:= \{([A_i, B_i]_\rho, l_i, \eta_i)\}_{i \in I_\rho, B_i > x}, \\ \mathcal{F}_{=x} &:= \{([A_i, B_i]_\rho, l_i, \eta_i)\}_{i \in I_\rho, B_i = x}, \\ \mathcal{F}_{<x} &:= \{([A_i, B_i]_\rho, l_i, \eta_i)\}_{i \in I_\rho, B_i < x}, \end{aligned}$$

with the admissible order inherited from $(I_\rho, >)$. *Note that* $\mathcal{F} = \mathcal{F}_{<x} + \mathcal{F}_{=x} + \mathcal{F}_{>x}$. *We also write* $\mathcal{F}_{\leq x} = \mathcal{F}_{<x} + \mathcal{F}_{=x}$ *and* $\mathcal{F}_{\geq x} = \mathcal{F}_{=x} + \mathcal{F}_{>x}$.

Atobe attached a symbol to each extended multi-segment $\mathcal{E}$ ([Ato22b, §3]). For example, when $\mathcal{E} = \{([A, B]_\rho, l, \eta)\}$ is a singleton, the symbol is as follows

$$\mathcal{E} = \left(\underbrace{\overset{B}{\lhd} \lhd \cdots \overset{B+l-1}{\lhd}}_{l} \; \overset{B+l}{\odot} \odot \cdots \odot \overset{A-l}{\odot} \; \underbrace{\overset{A-l+1}{\rhd} \cdots \rhd \overset{A}{\rhd}}_{l} \right)_\rho.$$

Here, $\odot \cdots \odot$ represents an alternating sequence of $\oplus$ and $\ominus$, starting with $\oplus$ if $\eta = 1$ (resp. $\ominus$ if $\eta = -1$). When $\mathcal{E}$ is not a singleton, we stack each row vertically. See the following for an example:

Example 2.9. *Let* ρ *be the trivial representation of* $\mathrm{GL}_1(F)$. *The symbol*

$$\mathcal{E} = \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{pmatrix} \lhd & \oplus & \ominus & \rhd \\ & & \lhd & \rhd \\ & & & \ominus \end{pmatrix}_\rho \end{array}$$

corresponds to the extended multi-segment $\mathcal{E} = \{([3, 0]_\rho, 1, 1), ([3, 2]_\rho, 1, 1), ([3, 3]_\rho, 0, -1)\}$ *of* Sp_{34}. *The associated local Arthur parameter is* $\psi_\mathcal{E} = \rho \otimes S_4 \otimes S_4 + \rho \otimes S_6 \otimes S_2 + \rho \otimes S_7 \otimes S_1$.

Given an extended multi-segment $\mathcal{E}$, one can use the algorithms in [HJLLZ25] to compute the representation $\pi(\mathcal{E})$. In particular, there are necessary conditions on $\mathcal{E}$ for $\pi(\mathcal{E})$ to be of Arthur type. This relies on the implementation of various operators on extended multi-segments as follows.

The first operator is called the *row exchange*. It is used to change the admissible order of an extended multi-segment, but does not affect the corresponding local Arthur parameters.

We say that $k < k+1$ are *adjacent* in a total order set $(I_\rho, >)$ if there does not exist $i \in I_\rho$ such that $k < i < k+1$.

Definition 2.10 ([Ato22b, Section 4.2], Row exchange). *Let $\mathcal{E} \in \underline{\mathrm{Rep}}^{(P')}$ with*

$$\mathcal{E}_\rho = \{([A_i, B_i]_\rho, l_i, \eta_i)\}_{i \in (I_\rho, >)}.$$

Assume that $k < k+1$ are adjacent in $(I_\rho, >)$. Let $\gg$ be the total order on I_ρ defined by $k \gg k+1$ and if $(i,j) \neq (k, k+1)$, then $i \gg j$ if and only if $i > j$.

If $\gg$ is not an admissible order on I_ρ, then we define $R_k(\mathcal{E}) = \mathcal{E}$. Otherwise, we define

$$R_k(\mathcal{E}_\rho) = \{([A_i, B_i]_\rho, l_i', \eta_i')\}_{i \in (I_\rho, \gg)},$$

where $(l_i', \eta_i') = (l_i, \eta_i)$ for $i \neq k, k+1$, and (l_k', η_k') and (l_{k+1}', η_{k+1}') are given as follows. Denote $\epsilon = (-1)^{A_k - B_k} \eta_k \eta_{k+1}$.

(1) Assume that $[A_k, B_k]_\rho \supset [A_{k+1}, B_{k+1}]_\rho$. We set $(l_{k+1}', \eta_{k+1}') = (l_{k+1}, (-1)^{A_k - B_k} \eta_{k+1})$ in this case.

(a) If $\epsilon = 1$ and $b_k - 2l_k < 2(b_{k+1} - 2l_{k+1})$, then

$$(l_k', \eta_k') = (b_k - (l_k + (b_{k+1} - 2l_{k+1})), (-1)^{A_{k+1} - B_{k+1}} \eta_k);$$

(b) If $\epsilon = 1$ and $b_k - 2l_k \geq 2(b_{k+1} - 2l_{k+1})$, then

$$(l_k', \eta_k') = (l_k + (b_{k+1} - 2l_{k+1}), (-1)^{A_{k+1} - B_{k+1} + 1} \eta_k);$$

(c) If $\epsilon = -1$, then $(l_k', \eta_k') = (l_k - (b_{k+1} - 2l_{k+1}), (-1)^{A_{k+1} - B_{k+1} + 1} \eta_k)$.

(2) Assume that $[A_k, B_k]_\rho \subseteq [A_{k+1}, B_{k+1}]_\rho$. We set $(l_k', \eta_k') = (l_k, (-1)^{A_{k+1} - B_{k+1}} \eta_k)$ in this case.

(a) If $\epsilon = 1$ and $b_{k+1} - 2l_{k+1} < 2(b_k - 2l_k)$, then

$$(l_{k+1}', \eta_{k+1}') = (b_{k+1} - (l_{k+1} + (b_k - 2l_k)), (-1)^{A_k - B_k} \eta_{k+1});$$

(b) If $\epsilon = 1$ and $b_{k+1} - 2l_{k+1} \geq 2(b_k - 2l_k)$, then

$$(l_{k+1}', \eta_{k+1}') = (l_{k+1} + (b_k - 2l_k), (-1)^{A_k - B_k + 1} \eta_{k+1});$$

(c) If $\epsilon = -1$, then $(l_{k+1}', \eta_{k+1}') = (l_{k+1} - (b_k - 2l_k), (-1)^{A_k - B_k + 1} \eta_{k+1})$.

Finally, we define that $R_k(\mathcal{E}) = \mathcal{E}^\rho \cup R_k(\mathcal{E}_\rho)$.

If $\gg$ is another admissible order on I_ρ, then we deform $\mathcal{E}_\rho$ into $\{([A_i, B_i]_\rho, (l_i)_\gg, (\eta_i)_\gg)\}_{i \in (I_\rho, \gg)}$ by applying a sequence of row exchanges on $\mathcal{E}_\rho$. We shall denote the resulting extended multi-segment by $\mathcal{E}_{\rho, \gg}$.

Next, we recall the definition of the operators sh_j^d, add_j^d on extended multi-segments. These operators can be useful in constructing new extended multi-segments.

Definition 2.11 (Shift, Add). *Let $\mathcal{E} = \cup_\rho \{([A_i, B_i]_\rho, l_i, \eta_i)\}_{i \in (I_\rho, >)}$ be an extended multi-segment. For $j \in I_{\rho'}$ and $d \in \mathbb{Z}$, we define the following operators.*

1. Define $sh_j^d(\mathcal{E}) = \cup_\rho \{([A_i', B_i']_\rho, l_i, \eta_i)\}_{i \in (I_\rho, >)}$ with

$$[A_i', B_i']_\rho = \begin{cases} [A_i + d, B_i + d]_\rho & \text{if } \rho = \rho' \text{ and } i = j, \\ [A_i, B_i]_\rho & \text{otherwise.} \end{cases}$$

Set $sh_{\rho'}^d = \sum_{j \in I_{\rho'}} sh_j^d$ and $sh^d := \sum_{\rho'} sh_{\rho'}^d$.

2. *Define* $add_j^d(\mathcal{E}) = \cup_\rho \{([A_i', B_i']_\rho, l_i', \eta_i)\}_{i \in (I_\rho, >)}$ *with*

$$([A_i', B_i']_\rho, l_i') = \begin{cases} ([A_i + d, B_i - d]_\rho, l_i + d) & \text{if } \rho = \rho' \text{ and } i = j, \\ ([A_i, B_i]_\rho, l_i) & \text{otherwise,} \end{cases}$$

Set $add_{\rho'}^d = \sum_{j \in I_{\rho'}} add_j^d$ *and* $add^d := \sum_{\rho'} add_{\rho'}^d$.

It is immediate that the operators commute with each other, so, we denote the composition by summation. We only use these notations in the case that the resulting object is still an extended multi-segment.

The next operator is called the *union-intersection*, which would change the corresponding local Arthur parameters if acting non-trivially.

Definition 2.12 ([Ato22b, Section 5.2], Union-intersection)**.** *Let* $\mathcal{E}$ *be an extended multi-segment with* $\mathcal{E}_\rho = \{([A_i, B_i]_\rho, l_i, \eta_i)\}_{i \in (I_\rho, >)}$. *For* $k < k+1$ *adjacent in* $(I_\rho, >)$, *we define an operator* ui_k, *called the* union-intersection, *on* $\mathcal{E}$ *as follows. Denote* $\epsilon = (-1)^{A_k - B_k} \eta_k \eta_{k+1}$. *If* $A_{k+1} > A_k$, $B_{k+1} > B_k$ *and any of the following cases holds:*

Case 1. $\epsilon = 1$ *and* $A_{k+1} - l_{k+1} = A_k - l_k$,
Case 2. $\epsilon = 1$ *and* $B_{k+1} + l_{k+1} = B_k + l_k$,
Case 3. $\epsilon = -1$ *and* $B_{k+1} + l_{k+1} = A_k - l_k + 1$,

we define that $ui_k(\mathcal{E}_\rho) = \{([A_i', B_i']_\rho, l_i', \eta_i')\}_{i \in (I_\rho, >)}$, *where* $([A_i', B_i']_\rho, l_i', \eta_i') = ([A_i, B_i]_\rho, l_i, \eta_i)$ *for* $i \neq k, k+1$, *and* $[A_k', B_k']_\rho = [A_{k+1}, B_k]_\rho$, $[A_{k+1}', B_{k+1}']_\rho = [A_k, B_{k+1}]_\rho$, *and* $(l_k', \eta_k', l_{k+1}', \eta_{k+1}')$ *are given case by case as follows:*

(1) *in Case 1,* $(l_k', \eta_k', l_{k+1}', \eta_{k+1}') = (l_k, \eta_k, l_{k+1} - (A_{k+1} - A_k), (-1)^{A_{k+1} - A_k} \eta_{k+1})$;
(2) *in Case 2, if* $b_k - 2l_k \geq A_{k+1} - A_k$, *then*

$$(l_k', \eta_k', l_{k+1}', \eta_{k+1}') = (l_k + (A_{k+1} - A_k), \eta_k, l_{k+1}, (-1)^{A_{k+1} - A_k} \eta_{k+1}),$$

and if $b_k - 2l_k < A_{k+1} - A_k$, *then* $(l_k', \eta_k', l_{k+1}', \eta_{k+1}') = (b_k - l_k, -\eta_k, l_{k+1}, (-1)^{A_{k+1} - A_k} \eta_{k+1})$;
(3) *in Case 3, if* $l_{k+1} \leq l_k$, *then* $(l_k', \eta_k', l_{k+1}', \eta_{k+1}') = (l_k, \eta_k, l_{k+1}, (-1)^{A_{k+1} - A_k} \eta_{k+1})$, *and if* $l_{k+1} > l_k$, *then* $(l_k', \eta_k', l_{k+1}', \eta_{k+1}') = (l_k, \eta_k, l_k, (-1)^{A_{k+1} - A_k + 1} \eta_{k+1})$;
(3′) *if we are in Case 3 and* $l_k = l_{k+1} = 0$, *then we delete* $([A_{k+1}', B_{k+1}']_\rho, l_{k+1}', \eta_{k+1}')$ *from* $ui_k(\mathcal{E}_\rho)$.

The union-intersection operator can be extended to non-adjacent extended segments as follows.

Definition 2.13. *Let* $\mathcal{E}$ *be an extended multi-segment with* $\mathcal{E}_\rho = \{([A_i, B_i]_\rho, l_i, \eta_i)\}_{i \in (I_\rho, >)}$. *Given* $i, j \in I_\rho$, *we define that* $ui_{i,j}(\mathcal{E}_\rho) = \mathcal{E}_\rho$ *unless the following conditions hold:*

1. $A_i < A_j$, $B_i < B_j$, *and* $j \gg i$ *are adjacent under some admissible order* $\gg$ *on* I_ρ.
2. *The operator* ui_i *is applicable on* $\mathcal{E}_{\rho, \gg}$.

In this case, we define that $ui_{i,j}(\mathcal{E}_\rho) := (ui_i(\mathcal{E}_{\rho,\gg}))_>$, *so that the admissible order of* $ui_{i,j}(\mathcal{E}_\rho)$ *and* $\mathcal{E}_\rho$ *are the same (if the* ui_i *is of type* $3'$, *then we delete the* j*-th row). Finally, we define* $ui_{i,j}(\mathcal{E}) := \mathcal{E}^\rho \cup ui_{i,j}(\mathcal{E}_\rho)$.

We say that $ui_{i,j}$ *is applicable on* $\mathcal{E}$ *if* $ui_{i,j}(\mathcal{E}) \neq \mathcal{E}$. *Furthermore, we say that* $ui_{i,j}$ *is of type 1, 2, 3, or* $3'$ *if the corresponding operator* ui_i *is of type 1, 2, 3, or* $3'$, *respectively, in Definition 2.12.*

Let π be an irreducible admissible representation of G_n. Aubert showed that there exists $\varepsilon \in \{\pm 1\}$ such that the virtual representation defined by

$$\widehat{\pi} := \varepsilon \sum_{P} (-1)^{\dim(A_P)} [\mathrm{Ind}_P^{G_n}(Jac_P(\pi))]$$

is an irreducible representation ([Aub95]). The above sum is over all standard parabolic subgroups P of G_n, where A_P denotes the maximal split torus in the center of the Levi subgroup of P, and Jac_P denotes the Jacquet module along P. We say that $\widehat{\pi}$ is the Aubert-Zelevinsky dual or Aubert-Zelevinsky involution of π.

The next operator is known as the dual, which sends a representation corresponding to an extended multi-segment to its Aubert-Zelevinsky dual.

Definition 2.14 ([Ato22b, Definition 6.1], Dual)**.** *Let $\mathcal{E} = \cup_\rho \{([A_i, B_i]_\rho, l_i, \eta_i)\}_{i \in (I_\rho, >)}$ be an extended multi-segment such that the admissible order $>$ on I_ρ satisfies (P') for all ρ. We define*

$$dual(\mathcal{E}) := \cup_\rho \{([A_i, -B_i]_\rho, l_i', \eta_i')\}_{i \in (I_\rho, >')},$$

as follows:

1. *The order $>'$ is defined by $i >' j$ if and only if $j > i$.*
2. *Set*
$$l_i' = \begin{cases} l_i + B_i & \text{if } B_i \in \mathbb{Z}, \\ l_i + B_i + \frac{1}{2}(-1)^{\alpha_i}\eta_i & \text{if } B_i \notin \mathbb{Z}, \end{cases} \quad \text{and} \quad \eta_i' = \begin{cases} (-1)^{\alpha_i+\beta_i}\eta_i & \text{if } B_i \in \mathbb{Z}, \\ (-1)^{\alpha_i+\beta_i+1}\eta_i & \text{if } B_i \notin \mathbb{Z}, \end{cases}$$
where $\alpha_i = \sum_{j \in I_\rho, j<i} a_j$, and $\beta_i = \sum_{j \in I_\rho, j>i} b_j$, $a_j = A_j + B_j + 1$, $b_j = A_j - B_j + 1$.
3. *When $B_i \notin \mathbb{Z}$ and $l_i = \frac{b_i}{2}$, set $\eta_i = (-1)^{\alpha_i+1}$.*

Finally, we define $dual(\mathcal{E}_\rho) := (dual(\mathcal{E}))_\rho$.

Theorem 2.15 ([Ato22b, Theorem 6.2])**.** *Let $\mathcal{E} \in \underline{\mathrm{Rep}}^{(P')}$. Then $\pi(dual(\mathcal{E})) = \widehat{\pi(\mathcal{E})}$ holds.*

The last operator we introduce is known as the partial dual.

Definition 2.16 ([HLL22, Definition 6.5], Partial dual)**.** *Let $\mathcal{E} \in \underline{\mathrm{Rep}}^{(P')}$, and let*

$$\mathcal{F} := \mathcal{E}_\rho = \{([A_i, B_i]_\rho, l_i, \eta_i)\}_{i \in (I_\rho, >)}.$$

For $i \in I_\rho$, denote

$$\alpha_i = \sum_{j<i} (A_j + B_j + 1) \quad \text{and} \quad \beta_i = \sum_{j>i} (A_j - B_j + 1).$$

Assume that there exists $k \in I_\rho$ such that

1. *$B_k = \frac{1}{2}, l_k = 0$;*
2. *$(-1)^{\alpha_k}\eta_k = -1$;*
3. *for any $i < k$, $B_i < \frac{1}{2}$.*

Then we define $dual_k^+(\mathcal{F})$ as follows. We write the decomposition

$$\mathcal{F} = \mathcal{F}_1 + \{([A_k, 1/2]_\rho, 0, \eta_k)\} + \mathcal{F}_2,$$

where $\mathcal{F}_1 = \mathcal{F}_{<1/2}$ and $\mathcal{F}_2 = \mathcal{F}_{>1/2}$, and then write

$$dual(\mathcal{F}) = \widetilde{\mathcal{F}_2} + \{([A_k, -1/2]_\rho, 0, (-1)^{\beta_k})\} + \widetilde{\mathcal{F}_1},$$

where $\widetilde{\mathcal{F}_2} = (dual(\mathcal{F}))_{<-1/2}$ *and* $\widetilde{\mathcal{F}_1} = (dual(\mathcal{F}))_{>-1/2}$*. Next, write*

$$dual(\mathcal{F}') := dual(\widetilde{\mathcal{F}_2} + \{([A_k, 1/2]_\rho, 0, (-1)^{\beta_k+1})\} + \widetilde{\mathcal{F}_1}) = \widetilde{\widetilde{\mathcal{F}_1}} + \{([A_k, -1/2]_\rho, 0, -\eta_k)\} + \widetilde{\widetilde{\mathcal{F}_2}},$$

where $\widetilde{\widetilde{\mathcal{F}_1}} = (dual(\mathcal{F}'))_{<-1/2}$ *and* $\widetilde{\widetilde{\mathcal{F}_2}} = (dual(\mathcal{F}'))_{>-1/2}$*. Then we define*

$$dual_k^+(\mathcal{F}) := \widetilde{\widetilde{\mathcal{F}_1}} + \{([A_k, -1/2]_\rho, 0, -\eta_k)\} + \mathcal{F}_2,$$

and say that $dual_k^+$ *is applicable on* $\mathcal{F}$*.*

Assume that $dual(\mathcal{F})$ *satisfies above conditions (1) – (3). Then we define*

$$dual_k^-(\mathcal{F}) := dual \circ dual_k^+ \circ dual(\mathcal{F}),$$

and say that $dual_k^-$ *is applicable on* $\mathcal{F}$*. We call this operators* $dual_k^+$*,* $dual_k^-$ *the* partial dual*, and denote by* $dual_k$ *if there is no ambiguity.*

Finally, we define $dual_k(\mathcal{E}) := \mathcal{E}^\rho \cup dual_k(\mathcal{E}_\rho)$*.*

Next, we introduce certain collections of the operators defined above. Note that one can easily construct the inverses of the add, shift, union-intersection, and dual operators, when they are applicable on an extended multi-segment.

Definition 2.17.

(1) *We say that an operator* T *defined above is a* raising operator *if it is of the form* $ui_{i,j}^{-1}$*,* $dual \circ ui_{j,i} \circ dual$*, or* $dual_k^-$*.*

(2) *We say that an extended multi-segment* $\mathcal{E} \in \underline{\mathrm{Rep}}$ *is* absolutely maximal *if there is no raising operator applicable on* $\mathcal{E}$*.*

The definition of raising operators is used to exhaust Arthur packets in [HLL22]. In later sections, they will be used to restrict the possible forms of extended multi-segments corresponding to representations of Arthur type, which allow us to exhaust the Arthur dual.

2.7. **Preservation of irreducibility and unitarizability.** In this subsection, we recall some important technical results regarding when and how irreducibility and unitarizability of a representation are preserved. We begin with an irreducibility criterion of Tadić, which is useful when considering parabolic inductions with a supercuspidal representation. For any $\pi \in \Pi(G_n)$, let $\mathrm{supp}(\pi)$ denote the supercuspidal support of π. Let $\mathcal{D}$ denote all irreducible essentially square integrable representations of $\mathrm{GL}_n(F)$, $n \geq 1$, and $e(\delta)$ denote the exponent of any representation $\delta \in \mathcal{D}$. Let $M(\mathcal{D})$ denote all finite multi-sets of representations in $\mathcal{D}$.

Theorem 2.18. [Tad23, (2.23)] *Let* $\sigma \in \mathcal{C}^{cl}, \pi \in \Pi(G_n)$*. Suppose now that* $\mathrm{supp}(\pi)$ *does not contain* $\rho|\cdot|^\alpha$ *or* $\rho|\cdot|^{-\alpha}$*. Assume that all members of* $\mathrm{supp}(\pi)$ *are contained in* $\{\rho|\cdot|^{k+x} : k \in \mathbb{Z}\}$*, for some fixed* $x \in \frac{1}{2}\mathbb{Z}$*. Write* $\pi = L(d)$*, for some* $d \in M(\mathcal{D})$*. Denote by* $d_{>0}$ *(resp.* $d_{<0}$*) the multiset consisting of all* δ *in* d *such that* $e(\delta) > 0$ *(resp* $e(\delta) < 0$*), counted with multiplicities. Then, if* π *is a ladder representation or if* $\alpha \leq 1$ *and all members of* $\mathrm{supp}(\pi)$ *are contained in* $\{\nu^{k+\alpha}\rho; k \in \mathbb{Z}\}$*, then holds*

$$L(d) \rtimes \sigma \text{ is reducible} \iff L(d_{>0}) \times L(d_{<0})^\vee \text{ is reducible.} \tag{2.9}$$

Next, we recall some basic notation on the support of representations, leading up to the result of Jantzen decomposition.

Definition 2.19. *Let* X *be a subset of* $\mathcal{C}$*.*

(1) X *is self-contragredient if for any* $\rho \in X$*, the contragredient of* ρ *is also in* X*.*

(2) For an irreducible admissible representation β of $\mathrm{GL}_d(F)$, we say that β is supported on X if the supercuspidal support of β is contained in X.

(3) Let π be an irreducible admissible representation of G_n that appears as an irreducible subquotient of $\theta_1 \times \cdots \times \theta_f \rtimes \sigma$, where $\theta_i \in \mathcal{C}$ and $\sigma \in \mathcal{C}_{cl}$. We say that π is supported on $X \cup \{\sigma\}$ for some self-contragredient $X \subseteq \mathcal{C}$ if $\theta_i \in X$ for $i = 1, \ldots, r$.

(4) Fix a supercuspidal representation σ of G_m and a self-contragredient subset $X \subseteq \mathcal{C}$. We denote by $\mathrm{Irr}(X;\sigma)$ the set of irreducible admissible representations π of G_n with $n \geq m$ such that π is supported on $X \cup \{\sigma\}$.

(5) Suppose that X is self-contragredient. Let $X = X_1 \sqcup X_2$ be a partition of X. Such a partition is called regular if X_1 is self-contragredient and

$$\theta \in X_1 \Longrightarrow \theta|\cdot|^1 \notin X_2.$$

That is, there is no reducibility among X_1 and X_2.

The following results are on the Jantzen decomposition, which give us another criterion to determine irreducibility.

Theorem 2.20 ([Jan97, Theorem 9.3])**.** *Let X be a subset of $\mathcal{C}$ that is self-contragredient and $X = X_1 \sqcup X_2$ is a regular partition. Let $\sigma \in \mathcal{C}_{cl}$.*

(i) For any $\pi \in \mathrm{Irr}(X;\sigma)$, there exist irreducible admissible representations β_1, β_2 of general linear groups and $X_1(\pi), X_2(\pi)$ of lower rank classical groups such that
- *β_i is supported on X_i and $X_i(\pi)$ is supported on $X_i \cup \{\sigma\}$; and*
- *there are injections*

$$\pi \hookrightarrow \beta_1 \rtimes X_2(\pi) \quad \text{and} \quad \pi \hookrightarrow \beta_2 \rtimes X_1(\pi).$$

The representations $X_i(\pi)$ are uniquely determined by π.

(ii) The map

$$\mathrm{Irr}(X;\sigma) \longrightarrow \mathrm{Irr}(X_1;\sigma) \times \mathrm{Irr}(X_2;\sigma), \quad \text{with } \pi \mapsto (X_1(\pi), X_2(\pi))$$

is a bijection. We denote the inverse map by Ψ_{X_1,X_2}.

(iii) For $i = 1, 2$, let β_i be an irreducible admissible representation of a general linear group supported on X_i and $\gamma_i \in \mathrm{Irr}(X_i;\sigma)$. Here we allow β_i to be the trivial representation of $\mathrm{GL}_0(F)$. Then

$$(\beta_1 \times \beta_2) \rtimes \Psi_{X_1,X_2}(\gamma_1, \gamma_2) \text{ is irreducible} \Longleftrightarrow \text{ both } \beta_i \rtimes \gamma_i \text{ are irreducible.}$$

(iv) For any $\pi \in \mathrm{Irr}(X;\sigma)$, π is tempered (resp. square-integrable) if and only if $X_1(\pi), X_2(\pi)$ are both tempered (resp. square-integrable).

To end this subsection, we recall when and how unitarizability of representations of G_n are preserved. These techniques provide the basic tools for our classification of the corank 4 unitary dual and motivate the definition of $\Pi_{A}^{\lim}(G_n)$. They can be used to prove or disprove unitarizability.

Proposition 2.21. *Fix a standard parabolic subgroup $P = MN$ of G with $M \cong \mathrm{GL}_{m_1}(F) \times \cdots \times \mathrm{GL}_{m_k}(F) \times G_0$. Let $\tau_i \in \mathrm{Irr}(\mathrm{GL}_{m_i}(F))$, $\pi_0 \in \mathrm{Irr}(G_0)$ and set $\pi_M = \tau_1 \boxtimes \cdots \boxtimes \tau_k \boxtimes \pi_0 \in \mathrm{Irr}(M)$.*

(1) (Unitary induction (UI)) If π_M is unitary, then $\mathrm{Ind}_P^G(\pi_M)$ is a direct sum of irreducible unitary representations of G.

(2) *(Unitary reduction (UR)) If π_M is hermitian and if $\mathrm{Ind}_P^G(\pi_M)$ is irreducible and unitary, then π_M is also unitary.*

(3) *(Complimentary series (CS)) Let $x_1(t), \ldots, x_r(t)\colon [0,1] \to \mathbb{R}$ be continuous functions, and set*
$$\Pi_t = \tau_1|\cdot|^{x_1(t)} \times \cdots \times \tau_r|\cdot|^{x_r(t)} \rtimes \pi_0.$$
If Π_t is irreducible and hermitian for $0 \le t < 1$, and if $\Pi_0 = \mathrm{Ind}_P^G(\pi_M)$ is unitary, then all irreducible subquotients of Π_t are unitary for $0 \le t \le 1$.

(4) *(Beyond the first reducibility point (RP1)) Suppose that*
- $k = 1$;
- $\tau_1 = \times_{i=1}^r \mathrm{Sp}(\rho_i, c_i, d_i)$ *is a product of unitary Speh representations with $\rho_i \cong \rho_i^\vee$ for $1 \le i \le r$;*
- π_0 *is of Arthur type of good parity.*

Let $\psi_M \in \Psi(M)$ be an A-parameter with $\pi_M \in \Pi_{\psi_M}$, and let $R_P(w, \pi_M, \psi_M)$ be the normalized intertwining operator defined by Arthur [Art13, Section 2.4] *with $w \in W(M,G)$. Assume further that $R_P(w, \pi_M, \psi_M)$ is not a scalar (so in particular $\mathrm{Ind}_P^G(\pi_M)$ is reducible). Then $\tau|\cdot|^s \rtimes \pi_0$ is not unitary for sufficiently small $s > 0$.*

2.8. **Definition of $\Pi_{\overline{A}}(G_n)$ and $\Pi_{\overline{A}}^{\lim}(G_n)$.** In this subsection, we recall the definitions of $\Pi_{\overline{A}}(G_n)$ and the unitary dual candidate set $\Pi_{\overline{A}}^{\lim}(G_n)$.

Definition 2.22. *(a) For $k \in \mathbb{Z}_{\ge 0}$, we define subsets $\Pi_{\overline{A}}^{(k)}(G_n)$ of $\Pi_u(G_n)$ inductively as follows. Set $\Pi_{\overline{A}}^{(0)}(G_n) := \Pi_A(G_n)$. For $k \ge 1$, and any $\pi \in \Pi(G_n)$, we say that $\pi \in \Pi_{\overline{A}}^{(k)}(G_n)$ if there exists a triple*

$$(\Pi_{\underline{x}} = \Pi_{x_1,\ldots x_s} = u_{\rho_1}(a_1,b_1)|\cdot|^{x_1} \times \cdots \times u_{\rho_s}(a_s,b_s)|\cdot|^{x_s} \rtimes \pi_A, \underline{y} \in \mathbb{R}^s, \underline{z} \in \mathbb{R}^s) \tag{2.10}$$

satisfying the following conditions

(1) $\pi_A \in \pi_{A,gp}(G_m)$ for some $m < n$, and the ρ_i's are irreducible unitary supercuspidal representations of the general linear groups (not necessarily self-dual).

(2) The point $\underline{y}$ lies in the set $U := \{\underline{x} \in \mathbb{R}^s | \Pi_{\underline{x}}$ is irreducible and Hermitian$\}$ and $\Pi_{\underline{y}} = \tau \rtimes \pi^{(k-1)}$ for some unitary representation τ of $GL_d(F)$ and $\pi^{(k-1)} \in \Pi_{\overline{A}}^{(k-1)}(G_{n-d})$.

(3) The points $\underline{z}$ lies in the unique connected component of U containing $\underline{y}$ and $\pi \cong \Pi_{\underline{z}}$.

Finally, we let $\Pi_{\overline{A}}(G_n) = \bigcup_{k \in \mathbb{Z}_{\ge 0}} \Pi_{\overline{A}}^{(k)}(G_n)$.

(b) We say that $\pi \in \Pi_{\overline{A}}^{\lim}(G_n)$ if there exists a triple as in (2.10) satisfying condition (1) *in Part* (a) *and*

(2') The point $\underline{y}$ lies in the open set $U := \{\underline{x} \in \mathbb{R}^s | \Pi_{\underline{x}}$ is irreducible$\}$ and $\Pi_{\underline{y}} \in \Pi_{\overline{A}}(G_n)$

*(3') The point $\underline{z}$ lies in the **closure** of the unique connected component of U containing $\underline{y}$ and π is a subquotient of $\Pi_{\underline{z}}$.*

3. Classification of tempered representations of corank 3 of good parity

As motivated by Definition 2.22, the first step to construct the unitary dual of corank 4 would be to construct the Arthur dual of corank 4. To do so, the first natural step is to classify all representations of Arthur type that are of good parity. In this section, we classify all tempered representations of corank 3, which will be used in the next section on the classification of non-tempered representations of corank 4. Let us first recall the notation in [HJLLZ25, Section 7] regarding tempered representations of good parity, that are in particular not supercuspidal. From now on, we let $m_\phi(\rho\otimes S_x)$ be the multiplicity of $\rho\otimes S_x$ inside the L-parameter ϕ, and let the set $\Psi(\pi)$ be the set of A-parameters ψ such that $\pi\in\Pi_\psi$.

Theorem 3.1. [HJLLZ25, Theorem 7.1] *Let $\pi=\pi(\phi,\varepsilon)$ be a tempered representation of good parity. The representation π is not supercuspidal if and only if at least one of the following holds for some ρ and $x\in\frac{1}{2}\mathbb{Z}_{\geq 0}$. Denote $m:=m_\phi(\rho\otimes S_{2x+1})$ for short.*

(I) Suppose that $x>0$, $m_\phi(\rho\otimes S_{2x+1})>0$ and $\varepsilon(\rho\otimes S_{2x+1})\varepsilon(\rho\otimes S_{2x-1})\neq -1$. In this case, there exists a unique tempered representation π_{temp} of smaller rank such that

$$\pi\hookrightarrow \underbrace{\rho|\cdot|^x\times\cdots\times\rho|\cdot|^x}_{m \text{ copies}}\rtimes\pi_{temp},$$

and we write $\pi=T^x_{I,m}(\pi_{temp})$. More precisely, we have $\pi_{temp}=\pi(\phi',\varepsilon')$ where

$$\phi'=\phi-(\rho\otimes S_{2x+1})^{\oplus m}+(\rho\otimes S_{2x-1})^{\oplus m},$$

and

$$\varepsilon'(\rho'\otimes S_a)=\begin{cases}\varepsilon(\rho\otimes S_{2x+1}) & \text{if } \rho'\otimes S_a\cong\rho\otimes S_{2x-1},\\ 0 & \text{if } \rho'\otimes S_a\cong\rho\otimes S_{2x+1},\\ \varepsilon(\rho'\otimes S_a) & \text{otherwise.}\end{cases}$$

*(II) Suppose that $x>0$, $m_\phi(\rho\otimes S_{2x+1})>1$ is **odd** and $\varepsilon(\rho\otimes S_{2x+1})\varepsilon(\rho\otimes S_{2x-1})=-1$. In this case, there exists a unique tempered representation π_{temp} of smaller rank such that*

$$\pi\hookrightarrow \underbrace{\rho|\cdot|^x\times\cdots\times\rho|\cdot|^x}_{m-1 \text{ copies}}\rtimes\pi_{temp},$$

and we write $\pi=T^x_{II,m}(\pi_{temp})$. More precisely, we have $\pi_{temp}=\pi(\phi',\varepsilon')$ where

$$\phi'=\phi-(\rho\otimes S_{2x+1})^{\oplus m-1}+(\rho\otimes S_{2x-1})^{\oplus m-1},$$

and $\varepsilon'(\rho'\otimes S_a)=\varepsilon(\rho'\otimes S_a)$.

*(III) Suppose that $x>0$, $m_\phi(\rho\otimes S_{2x+1})>1$ is **even** and $\varepsilon(\rho\otimes S_{2x+1})\varepsilon(\rho\otimes S_{2x-1})=-1$. In this case, there exists a unique tempered representation π_{temp} of smaller rank such that*

$$\pi\hookrightarrow \underbrace{\rho|\cdot|^x\times\cdots\times\rho|\cdot|^x}_{m-1 \text{ copies}}\times\rho|\cdot|^{x-1}\times\rho|\cdot|^{x-2}\times\cdots\times\rho|\cdot|^{-x}\rtimes\pi_{temp}$$

and we write $\pi=T^x_{III,m}(\pi_{temp})$. More precisely, we have that $\pi_{temp}=\pi(\phi',\varepsilon')$ where

$$\phi'=\phi-(\rho\otimes S_{2x+1})^{\oplus m}+(\rho\otimes S_{2x-1})^{\oplus m-2},$$

and $\varepsilon'(\rho'\otimes S_a)=\varepsilon(\rho'\otimes S_a)$.

(IV) Suppose that $x=0$, $m_\phi(\rho\otimes S_1)>1$ is odd. In this case, there exists a unique tempered representation π_{temp} of smaller rank such that

$$\pi\hookrightarrow \underbrace{\rho|\cdot|^x\times\cdots\times\rho|\cdot|^x}_{(m-1)/2 \text{ copies}}\rtimes\pi_{temp},$$

and we write $\pi = T_{IV,m}(\pi_{temp})$. *More precisely, we have that* $\pi_{temp} = \pi(\phi', \varepsilon')$ *where*

$$\phi' = \phi - (\rho \otimes S_{2x+1})^{\oplus m-1},$$

and $\varepsilon'(\rho' \otimes S_a) = \varepsilon(\rho' \otimes S_a)$.

(V) *Suppose that* $x = 0$, $m_\phi(\rho \otimes S_1) > 1$ *is even. In this case, there exists a unique tempered representation* π_{temp} *of smaller rank such that*

$$\pi \hookrightarrow \underbrace{\rho|\cdot|^x \times \cdots \times \rho|\cdot|^x}_{m/2 \text{ copies}} \rtimes \pi_{temp},$$

and we write $\pi = T_{V,m}^{\varepsilon(\rho\otimes S_1)}(\pi_{temp})$. *More precisely, we have that* $\pi_{temp} = \pi(\phi', \varepsilon')$ *where*

$$\phi' = \phi - (\rho \otimes S_{2x+1})^{\oplus m},$$

and

$$\varepsilon'(\rho' \otimes S_a) = \begin{cases} 0 & \text{if } \rho' \otimes S_a \cong \rho \otimes S_1, \\ \varepsilon(\rho' \otimes S_a) & \text{otherwise.} \end{cases}$$

The five cases above will give us a way of describing all tempered representations. By definition, they are only well-defined up to certain restrictions. We include these conditions below.

Remark 3.2. [HJLLZ25, Remark 7.3] *Let* $\pi_{temp} = \pi(\phi', \varepsilon')$.

- *The representation* $T_{I,m}^x(\pi_{temp})$ *is well-defined if and only if* $m_{\phi'}(\rho \otimes S_{2x+1}) = 0$ *and* $m_{\phi'}(\rho \otimes S_{2x-1}) \geq m$.
- *The representation* $T_{II,m}^x(\pi_{temp})$ *is well-defined if and only if* $m_{\phi'}(\rho \otimes S_{2x+1}) = 1$, $m_{\phi'}(\rho \otimes S_{2x-1}) \geq m$, *and* $\varepsilon'(\rho \otimes S_{2x+1})\varepsilon'(\rho \otimes S_{2x-1}) = -1$.
- *The representation* $T_{III,m}^x(\pi_{temp})$ *is well-defined if and only if* $m_{\phi'}(\rho \otimes S_{2x+1}) = 0$ *and* $m_{\phi'}(\rho \otimes S_{2x-1}) \geq m - 1$.
- *The representation* $T_{IV,m}(\pi_{temp})$ *is well-defined if and only if* $m_{\phi'}(\rho \otimes S_1) = 1$.
- *The representation* $T_{V,m}^{\epsilon}(\pi_{temp})$ *is well-defined if and only if* $m_{\phi'}(\rho \otimes S_1) = 0$.

With these tools we can classify tempered representations of good parity of arbitrary corank. Fix some $\rho \in \mathcal{C}$. In this section we classify all tempered representations of corank 3 of good parity, and in section 7, we do the same for tempered representations of corank 4. Suppose $\pi \in \Pi_{A,gp}(G_n)$ is tempered of corank 3. Then we split into three cases:

(A) $\pi \hookrightarrow \rho|\cdot|^{x_1} \rtimes \pi_{temp}$, where π_{temp} is tempered of corank 2. Then π is of the form $T_{I,1}^x(\pi_{temp}), T_{IV,3}(\pi_{temp})$ or $T_{V,2}^{\pm}(\pi_{temp})$.

(B) $\pi \hookrightarrow \rho|\cdot|^{x_1} \times \rho|\cdot|^{x_2} \rtimes \pi_{temp}$, where π_{temp} is tempered of corank 1. Then π is of the form $T_{I,2}^x(\pi_{temp}), T_{II,3}^x(\pi_{temp}), T_{III,2}^{\frac{1}{2}}(\pi_{temp}), T_{IV,5}(\pi_{temp})$ or $T_{V,4}^{\pm}(\pi_{temp})$.

(C) $\pi \hookrightarrow \rho|\cdot|^{x_1} \times \rho|\cdot|^{x_2} \times \rho|\cdot|^{x_3} \rtimes \pi_{sc}$, where π_{sc} is supercuspidal. Then π is of the form $T_{I,3}^x(\pi_{sc}), T_{III,2}^1(\pi_{sc}), T_{IV,7}(\pi_{sc}), T_{V,6}^{\pm}(\pi_{sc})$.

3.1. **Case** A : $\pi \hookrightarrow \rho|\cdot|^{x_1} \rtimes \pi_{temp}$**, where** π_{temp} **is tempered of corank** 2**.** We begin with case (A). For the rest of this section, let π_{sc} denote some self-dual supercuspidal representation and let $\alpha = \alpha_{\rho,\pi_{sc}}$ be its reducibility point. By [HJLLZ25, §10], here are the possible representations π_{temp} of corank 2:

$$T_{I,1}^{\alpha+1}(T_{I,1}^{\alpha}(\pi_{sc})), T_{I,1}^{\alpha-1}(T_{I,1}^{\alpha}(\pi_{sc})), T_{IV,3}(T_{I,1}^{\alpha}(\pi_{sc})),$$

$$T^{\pm}_{V,2}(T^{1}_{I,1}(\pi_{sc})), T^{1}_{I,1}(T^{\pm}_{V,2}(\pi_{sc})), T^{\frac{1}{2}}_{I,2}(\pi_{sc}), T^{\frac{1}{2}}_{II,3}(\pi_{sc}),$$
$$T^{\frac{1}{2}}_{III,2}(\pi_{sc}), T_{IV,5}(\pi_{sc}), T^{\pm}_{V,4}(\pi_{sc}).$$

There are 10 possibilities, so there are 30 cases in total in case (A). First we consider the case Let $\pi_{temp} = T^{\alpha+1}_{I,1}(T^{\alpha}_{I,1}(\pi_{sc}))$. From the classification in [HJLLZ25, Proposition 10.1], this is well-defined only when $\alpha > 0$.

Proposition 3.3. *Let $\pi_{temp} = T^{\alpha+1}_{I,1}(T^{\alpha}_{I,1}(\pi_{sc}))$ for $\alpha > 0$.*

(i) The representation $T^{x}_{I,1}(\pi_{temp})$ is well-defined if and only if $x = \alpha + 2$ for $\alpha > 0$, or $x = \alpha - 1$ for $\alpha > 1$.
(ii) The representation $T^{x}_{I,1}(\pi_{temp})$ is of critical type when $x = \alpha + 2$ for $\alpha > 0$, or $x = \alpha - 1$ for $\alpha > 1$.
(iii) Let $x \in \{\alpha + 2, \alpha - 1\}$, and define

$$\mathcal{E}_{\alpha+2} := \{([\alpha - 2, \epsilon_\rho]_\rho, 0, \eta), ([\alpha + 2, \alpha + 2], 0, (-1)^{\alpha-1-\epsilon_\rho}\eta)\},$$
$$\mathcal{E}_{\alpha-1} := \{([\alpha - 3, \epsilon_\rho]_\rho, 0, \eta), ([\alpha - 1, \alpha - 1], 0, *), ([\alpha + 1, \alpha + 1], 0, (-1)^{\alpha-1-\epsilon_\rho}\eta)\}.$$

Then we have $\pi(\mathcal{E}_x) = T^{x}_{I,1}(\pi_{temp})$. Here are the associated symbols:

$$\mathcal{E}_{\alpha+2} = \begin{array}{c} \begin{matrix} \epsilon_\rho & \cdots & \alpha-2 & \alpha-1 & \alpha & \alpha+1 & \alpha+2 \end{matrix} \\ \begin{pmatrix} \odot & \cdots & \odot & & & & \\ & & & & & & \odot \end{pmatrix} \end{array},$$

$$\mathcal{E}_{\alpha-1} = \begin{array}{c} \begin{matrix} \epsilon_\rho & \cdots & \alpha-3 & \alpha-2 & \alpha-1 & \alpha & \alpha+1 \end{matrix} \\ \begin{pmatrix} \odot & \cdots & \odot & & & & \\ & & & & \odot & & \\ & & & & & & \odot \end{pmatrix} \end{array}.$$

Proof. Part (i) follows directly from Remark 3.2. The condition $\alpha > 1$ for $x = \alpha - 1$ follow from the definition of $T^{x}_{I,1}$. Parts (ii) and (iii) follow from definition. □

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Proposition 3.4. *Let $\pi_{temp} = T^{\alpha+1}_{I,1}(T^{\alpha}_{I,1}(\pi_{sc}))$ for $\alpha > 0$.*

(i) The representation $T_{IV,3}(\pi_{temp})$ is well-defined if and only if $\alpha \in \mathbb{Z}_{>1}$.
(ii) The representation $T_{IV,3}(\pi_{temp})$ is not of critical type.
(iii) Define $\mathcal{E} = \{([0,0]_\rho, 0, \eta), ([0,0]_\rho, 0, \eta), ([\alpha-2, 0]_\rho, 0, \eta), ([\alpha+1, \alpha+1]_\rho, 0, (-1)^{\alpha-1}\eta)\}$. Then $\pi(\mathcal{E}) = T_{IV,3}(\pi_{temp})$. Here is the associated symbol:

$$\mathcal{E} = \begin{array}{c} \begin{matrix} 0 & \cdots & \alpha-2 & \alpha-1 & \alpha & \alpha+1 \end{matrix} \\ \begin{pmatrix} \odot & & & & & \\ \odot & & & & & \\ \odot & \cdots & \odot & & & \\ & & & & & \odot \end{pmatrix} \end{array}.$$

Proof. Let ϕ be the L-parameter associated with π_{temp}. By Remark 3.2, $T_{IV,3}(\pi_{temp})$ is well-defined if and only if $m_\phi(\rho \otimes S_1) = 1$, which happens only when $\alpha \neq 0, 1$. This proves part (i). Parts (ii) and (iii) follow from the definition. □

Proposition 3.5. *Let $\pi_{temp} = T^{\alpha+1}_{I,1}(T^{\alpha}_{I,1}(\pi_{sc}))$ for $\alpha > 0$.*

(i) The representation $T^{\pm}_{V,2}(\pi_{temp})$ is well-defined if and only if $\alpha = 1$.
(ii) When $\alpha = 1$, the representation $T^{\pm}_{V,2}(\pi_{temp})$ is of critical type.

(iii) Define

$$\mathcal{E}_\pm = \{([0,0],0,\pm 1))^2, ([2,2],0,\eta)\}.$$

Then $\pi(\mathcal{E}_\pm) = T^{\pm}_{V,2}(\pi_{temp})$. *Here are the associated symbols.*

$$\begin{array}{c} \\ \mathcal{E}_+ = \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{pmatrix} \oplus & & \\ \oplus & & \\ & & \odot \end{pmatrix} \end{array}, \quad \begin{array}{c} \\ \mathcal{E}_- = \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{pmatrix} \ominus & & \\ \ominus & & \\ & & \odot \end{pmatrix} \end{array}.$$

Proof. The proof is similar to Proposition 3.4, which we omit. □

Now let us consider the case $\pi_{temp} = T^{\alpha-1}_{I,1}(T^{\alpha}_{I,1}(\pi_{sc}))$. By the classification in [HJLLZ25, Proposition 10.1], this is well-defined only when $\alpha > 1$.

Proposition 3.6. *Let* $\pi_{temp} = T^{\alpha-1}_{I,1}(T^{\alpha}_{I,1}(\pi_{sc}))$ *and* $\alpha > 1$.

(i) The representation $T^{x}_{I,1}(\pi_{temp})$ *is well defined if and only if* $x = \alpha - 2$ *and* $\alpha > 2$, *or* $x = \alpha + 1$.

(ii) The representation $T^{x}_{I,1}(\pi_{temp})$ *is of critical type, when* $x = \alpha - 2, \alpha > 2$, *or* $x = \alpha + 1$. *When* $x = \alpha + 1$, *the representation* $T^{x}_{I,1}(\pi_{temp})$ *is the same as the representation* $T^{\alpha-1}_{I,1}(T^{\alpha+1}_{I,1}(T^{\alpha}_{I,1}(\pi_{sc}))$ *described in Proposition 3.3.*

(iii) Define

$$\mathcal{E} := \{([\alpha-4,\epsilon_\rho]_\rho,0,\eta), ([\alpha,\alpha-2]_\rho,0,-\eta\}.$$

Then $\pi(\mathcal{E}) = T^{\alpha-2}_{I,1}(\pi_{temp})$. *Here is the associated symbol.*

$$\begin{array}{c} \\ \mathcal{E} = \end{array} \begin{array}{c} \begin{array}{ccccccc} \epsilon_\rho & \cdots & \alpha-4 & \alpha-3 & \alpha-2 & \alpha-1 & \alpha \end{array} \\ \begin{pmatrix} \odot & \cdots & \odot & & & & \\ & & & & \odot & \odot & \odot \end{pmatrix} \end{array}.$$

Proof. The proof is similar to Proposition 3.3, which we omit. □

Proposition 3.7. *Let* $\pi_{temp} = T^{\alpha-1}_{I,1}(T^{\alpha}_{I,1}(\pi_{sc}))$ *and* $\alpha > 1$.

(i) The representation $T_{IV,3}(\pi_{temp})$ *is well-defined if and only if* $\alpha \in \mathbb{Z}_{>2}$.

(ii) The representation $T_{IV,3}(\pi_{temp})$ *is not of critical type.*

(iii) Define

$$\mathcal{E} = \{([0,0]_\rho,0,\eta), ([0,0]_\rho,0,\eta), ([\alpha-3,0]_\rho,0,\eta), ([\alpha,\alpha-1]_\rho,0,(-1)^{\alpha-2}\eta)\}.$$

Then $\pi(\mathcal{E}) = T_{IV,3}(\pi_{temp})$. *Here are the associated symbols.*

$$\begin{array}{c} \\ \mathcal{E} = \end{array} \begin{array}{c} \begin{array}{cccccc} 0 & \cdots & \alpha-3 & \alpha-2 & \alpha-1 & \alpha \end{array} \\ \begin{pmatrix} \odot & & & & & \\ \odot & & & & & \\ \odot & \cdots & \odot & & & \\ & & & & \odot & \odot \end{pmatrix} \end{array}.$$

Proof. The proof is similar to Proposition 3.4, which we omit. □

Proposition 3.8. *Let* $\pi_{temp} = T^{\alpha-1}_{I,1}(T^{\alpha}_{I,1}(\pi_{sc}))$ *and* $\alpha > 1$.

(i) The representation $T^{\pm}_{V,2}(\pi_{temp})$ *is well-defined if and only if* $\alpha = 2$.

(ii) When $\alpha = 2$, *the representation* $T^{\pm}_{V,2}(\pi_{temp})$ *is of critical type.*

(iii) *Define*
$$\mathcal{E}_{\pm} = \{([0,0],0,\pm 1)^2, ([2,1],0,\eta)\}.$$
Then $\pi(\mathcal{E}_{\pm}) = T_{V,2}(\pi_{temp})$. *Here are the associated symbols.*
$$\mathcal{E}_+ = \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{pmatrix} \oplus & & \\ \oplus & & \\ & \odot & \odot \end{pmatrix} \end{array}, \quad \mathcal{E}_- = \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{pmatrix} \ominus & & \\ \ominus & & \\ & \odot & \odot \end{pmatrix} \end{array}.$$

Proof. The proof is similar to Proposition 3.5, which we omit. $\square$

Now we consider the case $\pi_{temp} = T_{IV,3}(T_{I,1}^{\alpha}(\pi_{sc}))$. By the classification in [HJLLZ25, Proposition 10.2], this is well-defined only for $\alpha \in \mathbb{Z}_{>1}$.

Proposition 3.9. *Let* $\pi_{temp} = T_{IV,3}(T_{I,1}^{\alpha}(\pi_{sc}))$ *for* $\alpha \in \mathbb{Z}_{>1}$.

(i) *The representation* $T_{I,1}^{x}(\pi_{temp})$ *is well-defined if and only if* $x = \alpha + 1$ *or* $\alpha - 1$.

(ii) *When* $x = \alpha + 1$, *the representation* $T_{I,1}^{x}(\pi_{temp})$ *is not of critical type and is the same as the representation* $T_{IV,3}(T_{I,1}^{\alpha+1}(T_{I,1}^{\alpha}(\pi_{sc})))$ *described in Proposition 3.4. If* $x = \alpha - 1$, *and* $\alpha \neq 2$, *then the representation* $T_{I,1}^{x}(\pi_{temp})$ *is not of critical type and* $T_{I,1}^{x}(\pi_{temp}) = T_{IV,3}(T_{I,1}^{\alpha-1}(T_{I,1}^{\alpha}(\pi_{sc})))$ *described in Proposition 3.7.*

(iii) *When* $\alpha = 2$, *the representation* $\pi(\mathcal{E}) = T_{I,1}^{\alpha-1}(\pi_{temp})$ *is of critical type, where .*
$$\mathcal{E} = \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{pmatrix} \odot & & \\ \odot & & \\ & \odot & \odot \end{pmatrix} \end{array}.$$

(iv) *The representations* $T_{IV,3}(\pi_{temp})$ *and* $T_{V,2}^{\pm}(\pi_{temp})$ *are not well-defined.*

Proof. The proof of parts (i) through (iii) is similar to Proposition 3.3, which we omit. Part (iv) follows from the fact that $m_{\phi}(\rho \otimes S_1) = 2$, where ϕ is the L-parameter associated with π_{temp}. $\square$

The next case is $\pi_{temp} = T_{V,2}^{\pm}(T_{I,1}^{1}(\pi_{sc}))$. By [HJLLZ25, Proposition 10.3], this is well-defined if and only if $\alpha = 1$.

Proposition 3.10. *Let* $\pi_{temp}^{\pm} = T_{V,2}^{\pm}(T_{I,1}^{1}(\pi_{sc}))$ *and* $\alpha = 1$.

(i) *The representation* $T_{I,1}^{x}(\pi_{temp}^{\pm})$ *is well-defined if and only if* $x = 2$.

(ii) *The representation* $T_{I,1}^{2}(\pi_{temp}^{\pm})$ *is of critical type.*

(iii) *Define*
$$\mathcal{E}_{\pm} = \{([0,0]_{\rho},0,\pm 1), ([0,0]_{\rho},0,\pm 1), ([2,2]_{\rho},0,\eta)\}.$$
Then $\pi(\mathcal{E}_{\pm}) = T_{I,1}^{2}(\pi_{temp}^{\pm})$. *Here are the associated symbols.*
$$\mathcal{E}_+ = \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{pmatrix} \oplus & & \\ \oplus & & \\ & & \odot \end{pmatrix} \end{array},$$
$$\mathcal{E}_- = \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{pmatrix} \ominus & & \\ \ominus & & \\ & & \odot \end{pmatrix} \end{array}.$$

(iv) *The representations* $T_{IV,3}(\pi_{temp})$ *and* $T_{V,2}^{\pm}(\pi_{temp})$ *are not well-defined.*

Proof. The proof is similar to 3.9, which we omit. □

The next case to consider is $\pi_{temp} = T^1_{I,1}(T^{\pm}_{V,2}(\pi_{sc}))$. By the classification in [HJLLZ25, Proposition 10.5], this is well-defined only when $\alpha = 0$.

Proposition 3.11. *Let $\pi^{\pm}_{temp} = T^1_{I,1}(T^{\pm}_{V,2})(\pi_{sc})$ and $\alpha = 0$.*

(i) The representation $T^x_{I,1}(\pi^{\pm}_{temp})$ is well-defined if and only if $x = 2$.

(ii) When $x = 2$, the representation $T^x_{I,1}(\pi^{\pm}_{temp})$ is of critical type.

(iii) Define

$$\mathcal{E}_{\pm} = \{([0,0]_\rho, 0, \pm 1), ([2,2]_\rho, 0, \pm 1)\},$$

then $\pi(\mathcal{E}_{\pm}) = T^2_{I,1}(\pi^{\pm}_{temp})$. Here are the associated symbols.

$$\mathcal{E}_+ = \begin{array}{c} \begin{matrix} 0 & 1 & 2 \end{matrix} \\ \begin{pmatrix} \oplus & & \\ & & \oplus \end{pmatrix} \end{array}, \mathcal{E}_- = \begin{array}{c} \begin{matrix} 0 & 1 & 2 \end{matrix} \\ \begin{pmatrix} \ominus & & \\ & & \ominus \end{pmatrix} \end{array}.$$

Proof. The proof is similar to 3.10, which we omit. □

Proposition 3.12. *Let $\pi^{\pm}_{temp} = T^1_{I,1}(T^{\pm}_{V,2}(\pi_{sc}))$ and $\alpha = 0$.*

(i) The representation $T_{IV,3}(\pi_{temp})$ is well-defined.

(ii) The representation $T_{IV,3}(\pi_{temp})$ is of critical type.

(iii) Define

$$\mathcal{E}_{\pm} := ([0,0]_\rho, 0, \pm 1)^3, ([1,1]_\rho, 0, \pm 1)\}.$$

Then $\pi(\mathcal{E}_{\pm}) = T_{IV,3}(\pi_{temp})$. Here are the associated symbols.

$$\mathcal{E}_+ = \begin{array}{c} \begin{matrix} 0 & 1 \end{matrix} \\ \begin{pmatrix} \oplus & \\ \oplus & \\ \oplus & \\ & \oplus \end{pmatrix} \end{array}, \mathcal{E}_- = \begin{array}{c} \begin{matrix} 0 & 1 \end{matrix} \\ \begin{pmatrix} \ominus & \\ \ominus & \\ \ominus & \\ & \ominus \end{pmatrix} \end{array}.$$

(iv) The representation $T^{\pm}_{V,2}(\pi_{temp})$ is not well-defined.

Proof. The proof is similar to 3.9, which we omit. □

.

The next case to consider is $\pi_{temp} = T^{\frac{1}{2}}_{I,2}(\pi_{sc})$. By [HJLLZ25, Proposition 10.6], this is well-defined only when $\alpha = \frac{1}{2}$.

Proposition 3.13. *Let $\pi_{temp} = T^{\frac{1}{2}}_{I,2}(\pi_{sc})$ and $\alpha = \frac{1}{2}$.*

(i) The representation $T^x_{I,1}(\pi_{temp})$ is well-defined only when $x = \frac{3}{2}$.

(ii) The representation $T^{\frac{3}{2}}_{I,1}(\pi_{temp})$ is of critical type.

(iii) The set $\Psi(T^{\frac{3}{2}}_{I,1}(\pi_{temp}))$ is a singleton, and we have $T^{\frac{3}{2}}_{I,1}(\pi_{temp}) = \pi(\mathcal{E})$, where

$$\mathcal{E} = \begin{array}{c} \begin{matrix} \frac{1}{2} & \frac{3}{2} \end{matrix} \\ \begin{pmatrix} \oplus & \\ & \oplus \end{pmatrix} \end{array}.$$

(iv) The representations $T_{IV,3}(\pi_{temp})$ and $T^{\pm}_{V,2}(\pi_{temp})$ are not well-defined.

Proof. The proof for parts (i) and (ii) is similar to Proposition 3.3, which we omit. Part (iii) is easy to verify. For part (iv), since $\alpha = \frac{1}{2}$, the good parity condition implies that $x \in \frac{1}{2} + \mathbb{Z}$, so by definition the two representations are not well-defined. □

Now we consider the case $\pi_{temp} = T_{II,3}^{\frac{1}{2}}(\pi_{sc})$. By [HJLLZ25, Proposition 10.7], this is well-defined if and only if $\alpha \in \frac{1}{2} + \mathbb{Z}_{>0}$.

Proposition 3.14. *Let $\pi_{temp} = T_{II,3}^{\frac{1}{2}}(\pi_{sc})$ and let $\alpha \in \frac{1}{2} + \mathbb{Z}_{>0}$*

(i) The representation $T_{I,1}^{x}(\pi_{temp})$ is well-defined if and only if $x = \alpha$.

(ii) The representation $T_{I,1}^{\alpha}(\pi_{temp})$ is of critical type if and only if $\alpha = \frac{3}{2}$.

(iii) Define

$$\mathcal{E} = \{([1/2, 1/2]_\rho, 0, -1), ([1/2, 1/2]_\rho, 0, -1), ([\alpha-2, 1/2]_\rho, 0, -1), ([\alpha, \alpha], 0, \eta)\}.$$

Then $\pi(\mathcal{E}) = T_{I,1}^{\alpha}(\pi_{temp})$. Here is the associated symbol for $\alpha > \frac{3}{2}$.

$$\mathcal{E} = \begin{array}{c} \begin{matrix} \frac{1}{2} & \cdots & \alpha-2 & \alpha-1 & \alpha \end{matrix} \\ \begin{pmatrix} \ominus & & & & \\ \ominus & & & & \\ \ominus & \cdots & \odot & & \\ & & & & \odot \end{pmatrix} \end{array}.$$

For $\alpha = \frac{3}{2}$ the associated symbol is

$$\mathcal{E} = \begin{array}{c} \begin{matrix} \frac{1}{2} & \frac{2}{3} \end{matrix} \\ \begin{pmatrix} \ominus & \\ \ominus & \\ & \ominus \end{pmatrix} \end{array}.$$

(iv) The representations $T_{IV,3}(\pi_{temp})$ and $T_{V,2}(\pi_{temp})$ are not well-defined.

Proof. The proof is similar to Proposition 3.9, which we omit. □

The next case to consider is $\pi_{temp} = T_{III,2}^{\frac{1}{2}}(\pi_{sc})$. By [HJLLZ25, Proposition 10.8], this is well-defined only when $\alpha = \frac{1}{2}$.

Proposition 3.15. *Let $\pi_{temp} = T_{III,2}^{\frac{1}{2}}(\pi_{sc})$ and $\alpha = \frac{1}{2}$.*

(i) The representation $T_{I,1}^{x}(\pi_{temp})$ is well-defined if and only if $x = \frac{3}{2}$.

(ii) The representation $T_{I,1}^{\frac{3}{2}}(\pi_{temp})$ is of critical type.

(iii) The following is a complete list of extended multi-segments $\mathcal{E}$ (up to row exchanges) such that $\pi(\mathcal{E}) = T_{I,1}^{\frac{3}{2}}(\pi_{temp})$,

$$\left\{ \begin{array}{c} \begin{matrix} \frac{1}{2} & \frac{3}{2} \end{matrix} \\ \begin{pmatrix} \ominus & \\ & \ominus \end{pmatrix} \end{array}, \begin{array}{c} \begin{matrix} -\frac{1}{2} & \frac{1}{2} & \frac{3}{2} \end{matrix} \\ \begin{pmatrix} \oplus & \ominus & \\ & & \ominus \end{pmatrix} \end{array} \right\}.$$

(iv) The representations $T_{IV,3}(\pi_{temp})$ and $T_{V,2}^{\pm}(\pi_{temp})$ are not well-defined.

Proof. The proof is similar to Proposition 3.9, which we omit. □

Now we consider the case $\pi_{temp} = T_{IV,5}(\pi_{sc})$. From [HJLLZ25, Proposition 10.9], this is well-defined if and only if $\alpha \in \mathbb{Z}_{>0}$.

Proposition 3.16. *Let $\pi_{temp} = T_{IV,5}(\pi_{sc})$ and $\alpha \in \mathbb{Z}_{>0}$.*

(i) The representation $T_{I,1}^{x}(\pi_{temp})$ is well-defined if and only if $x = \alpha$.

(ii) The representation $T_{I,1}^{\alpha}(\pi_{temp})$ is of critical type if and only if $\alpha = 1$.

(iii) *Define*

$$\mathcal{E} = \{([0,0]_\rho, 0, \eta)^4, \} + \{([\alpha-2,0]_\rho, 0, \eta), ([\alpha,\alpha], 0, -\eta)\}.$$

Then $\pi(\mathcal{E}) = T_{I,1}^{\alpha}(\pi_{temp})$. *Here is the associated symbol.*

$$\mathcal{E} = \begin{array}{c} \begin{array}{cccccc} 0 & \dots & \alpha-2 & \alpha-1 & \alpha \end{array} \\ \begin{pmatrix} \odot & & & & \\ \odot & & & & \\ \odot & & & & \\ \odot & & & & \\ \odot & \dots & \odot & & \\ & & & & \odot \end{pmatrix} \end{array}.$$

(iv) *The representations* $T_{IV,3}(\pi_{temp})$ *and* $T_{V,2}^{\pm}(\pi_{temp})$ *are not well-defined.*

Proof. The proof is similar to Proposition 3.9, which we omit. □

The final case in case (A) is $\pi_{temp} = T_{V,4}^{\pm}(\pi_{sc})$, which is well-defined only when $\alpha = 0$ by [HJLLZ25, Proposition 10.10].

Proposition 3.17. *Let* $\pi_{temp}^{\pm} = T_{V,4}^{\pm}(\pi_{sc})$, *and* $\alpha = 0$.

(i) *The representation* $T_{I,1}^{x}(\pi_{temp}^{\pm})$ *is well-defined if and only if* $x = 1$.
(ii) *The representation* $T_{I,1}^{1}(\pi_{temp}^{\pm})$ *is of critical type.*
(iii) *The set* $\Psi(T_{V,4}^{\pm}(\pi_{sc}))$ *is a singleton, and* $\pi(\mathcal{E}_{\pm}) = T_{I,1}^{1}(\pi_{temp}^{\pm})$, *where*

$$\mathcal{E}_+ = \begin{array}{c} \begin{array}{cc} 0 & 1 \end{array} \\ \begin{pmatrix} \oplus & \\ \oplus & \\ \oplus & \\ & \oplus \end{pmatrix} \end{array}, \mathcal{E}_- = \begin{array}{c} \begin{array}{cc} 0 & 1 \end{array} \\ \begin{pmatrix} \ominus & \\ \ominus & \\ \ominus & \\ & \ominus \end{pmatrix} \end{array}.$$

(iv) *The representations* $T_{IV,3}(\pi_{temp}^{\pm})$ *and* $T_{V,2}(\pi_{temp})^{\pm}$ *are not well-defined.*

Proof. The proof is similar to Proposition 3.9, which we omit. □

3.2. **Case** $B : \pi \hookrightarrow \rho|\cdot|^{x_1} \times \rho|\cdot|^{x_2} \rtimes \pi_{temp}$**.** This concludes our discussion of case (A). Now we move on to case (B). The three possible tempered representations of corank 1 are $T_{I,1}^{\alpha}(\pi_{sc}), T_{IV,3}(\pi_{sc})$ and $T_{V,2}^{\pm}(\pi_{sc})$. First we consider the case $T_{I,1}^{\alpha}(\pi_{sc})$. By [HJLLZ25, Proposition 8.1], this is well-defined if and only if $\alpha > 0$.

Proposition 3.18. *Let* $\pi_{temp} = T_{I,1}^{\alpha}(\pi_{sc})$, $\alpha > 0$.

(i) *The representations* $T_{I,2}^{x}(\pi_{temp})$ *and* $T_{II,3}^{x}(\pi_{temp})$ *are not well-defined.*
(ii) *The representation* $T_{III,2}^{\frac{1}{2}}(\pi_{temp})$ *is not well-defined.*

Proof. Part (i) follows from the fact that the parameter ϕ corresponding to π_{temp} is multiplicity free. To see part (ii), suppose $T_{III,2}^{\frac{1}{2}}$ is well-defined, then we must have either $x = \alpha - 1 = \frac{1}{2}$, or $x = \alpha + 1 = \frac{1}{2}$. For the first case, we have $\alpha = \frac{3}{2}$, which means that the multiplicity of $\rho \otimes S_2$ in the parameter ϕ is 0, so it is not well-defined. For the second case we have $\alpha < 0$, which is impossible. This completes the proof. □

We check the remaining two possibilities for this tempered representation.

Proposition 3.19. *Let* $\pi_{temp} = T_{I,1}^{\alpha}(\pi_{sc})$, $\alpha > 0$.

(i) *The representation $T_{IV,5}(\pi_{temp})$ is-well-defined if and only if $\alpha \in \mathbb{Z}_{>0}$. It is the same as the representation $T_{I,1}^{\alpha}(T_{IV,5}(\pi_{sc}))$ described in Proposition 3.16.*

(ii) *The representation $T_{IV,5}(\pi_{temp})$ is of critical type if and only if $\alpha = 1$.*

Proof. Let ϕ be the L-parameter corresponding to π_{sc}. To ensure $m_\phi(\rho \otimes S_1) = 1$, we need $\alpha \in \mathbb{Z}$. The rest follows from definition.

Proposition 3.20. *Let $\pi_{temp} = T_{I,1}^{\alpha}(\pi_{sc})$, $\alpha > 0$.*

(i) *The representations $T_{V,4}^{\pm}(\pi_{temp})$ are well-defined if and only if $\alpha = 1$.*

(ii) *When $\alpha = 1$, the representations $T_{V,4}^{\pm}(\pi_{temp})$ are of critical type.*

(iii) *When $\alpha = 1$, define*

$$\mathcal{E}_{\pm} := \{([0,0],0,\pm 1)^4, ([1,1],0,\pm 1)\}.$$

Then $\pi(\mathcal{E}_{\pm}) = T_{V,4}^{\pm}(\pi_{temp})$. Here are the associated symbols.

$$\mathcal{E} = \begin{array}{c} \begin{matrix} 0 & 1 \end{matrix} \\ \begin{pmatrix} \oplus & \\ \oplus & \\ \oplus & \\ \oplus & \\ & \oplus \end{pmatrix} \end{array}, \mathcal{E}_{-} = \begin{array}{c} \begin{matrix} 0 & 1 \end{matrix} \\ \begin{pmatrix} \ominus & \\ \ominus & \\ \ominus & \\ \ominus & \\ & \ominus \end{pmatrix} \end{array}.$$

Proof. To ensure $S(\rho \otimes S_1) = 1$, we must have $\alpha - 1 = 0$ or $0 > \alpha$, this proves part (i). The rest follows from definition. □

The next case to consider in case (B) is $\pi_{temp} = T_{IV,3}(\pi_{sc})$, which is well-defined if and only if $\alpha \in \mathbb{Z}_{>0}$ by [HJLLZ25, Proposition 8.2].

Proposition 3.21. *Let $\pi_{temp} = T_{IV,3}(\pi_{sc})$, and $\alpha \in \mathbb{Z}_{>0}$*

(i) *The representations $T_{I,2}^{x}(\pi_{temp})$ is well-defined if and only if $\alpha = x = 1$.*

(ii) *When $\alpha = x = 1$, the representation $T_{I,2}^{1}(\pi_{temp})$ is of critical type.*

(iii) *When $\alpha = x = 1$, define*

$$\mathcal{E} := \{([0,0]_\rho, 0, \eta), ([1,1]_\rho, 0, \eta)^2\}.$$

Then $\pi(\mathcal{E}) = T_{I,2}^{1}(\pi_{temp})$. Here is the associated symbol.

$$\mathcal{E} = \begin{array}{c} \begin{matrix} 0 & 1 \end{matrix} \\ \begin{pmatrix} \odot & \\ & \odot \\ & \odot \end{pmatrix} \end{array}.$$

Proof. Let ϕ be the L-parameter associated to π_{temp}. To ensure $m_\phi(\rho \otimes S_{2(x-1)+1}) \geq m = 2$, we must have $x = 1$. In this case if $\alpha \neq 1$, then $m_\phi(\rho \otimes S_{2x+1}) \neq 0$. This proves part (i). The rest follows from definition. □

Proposition 3.22. *Let $\pi_{temp} = T_{IV,3}(\pi_{sc})$ and $\alpha \in \mathbb{Z}_{>0}$.*

(i) *The representation $T_{II,3}^{x}(\pi_{temp})$ is well-defined if and only if $x = 1$.*

(ii) *The representation $T_{II,3}^{1}(\pi_{temp})$ is of critical type.*

(iii) *Define*

$$\mathcal{E} := \{([\alpha - 1, 0]_\rho, 0, \eta), ([1,1]_\rho, 0, -\eta), ([1,1], 0, -\eta)\}.$$

Then $\pi(\mathcal{E}) = T^1_{II,3}(\pi_{temp})$. *Here is the associated symbol.*

$$\begin{array}{cc} & \begin{array}{cccc} 0 & 1 & \cdots & \alpha-1 \end{array} \\ \mathcal{E} = & \begin{pmatrix} \odot & \odot & \cdots & \odot \\ & \odot & & \\ & \odot & & \end{pmatrix}. \end{array}$$

(iv) *The representations* $T^{\frac{1}{2}}_{III,2}(\pi_{temp})$, $T_{IV,5}(\pi_{temp})$ *and* $T^{\pm}_{V,4}(\pi_{temp})$ *are not well-defined.*

Proof. Let ϕ be the L-parameter associated with π_{temp}. To ensure $m_\phi(\rho\otimes S_{2(x-1)+1}) \geq m = 3$, the only possibility is $x = 1$, in which case we can check it's well-defined. This proves part (i). Parts (ii) and (iii) follow from definition. For part (iv), the good parity condition tells us that $T^{\frac{1}{2}}_{III,2}$ is not well-defined, and since $m_\phi(\rho \otimes S_1) = 3$, $T_{IV,5}(\pi_{temp})$ and $T^{\pm}_{V,4}(\pi_{temp})$ are also not well-defined. □

The last case to consider in case (B) is $\pi_{temp} = T^{\pm}_{V,4}(\pi_{sc})$. This is well-defined if and only if $\alpha = 0$ by [HJLLZ25, Proposition 8.3].

Proposition 3.23. *Let* $\pi^{\pm}_{temp} = T^{\pm}_{V,2}(\pi_{sc})$ *and* $\alpha = 0$.

(i) *The representations* $T^x_{I,2}(\pi^{\pm}_{temp})$ *are well-defined if and only if* $x = 1$.
(ii) *The representations* $T^1_{I,2}(\pi^{\pm}_{temp})$ *are of critical type.*
(iii) *Define*

$$\mathcal{E}_{\pm} := \{([1, 1]_\rho, 0, \pm 1)^2\}.$$

Then $\pi(\mathcal{E}_\pm) = T^1_{I,1}(\pi^{\pm}_{temp})$. *Here are the associated symbols.*

$$\begin{array}{c} 1 \\ \mathcal{E}_+ = \begin{pmatrix} \oplus \\ \oplus \end{pmatrix}, \end{array} \quad \begin{array}{c} 1 \\ \mathcal{E}_- = \begin{pmatrix} \ominus \\ \ominus \end{pmatrix}. \end{array}$$

(iv) *The representations* $T^x_{II,3}(\pi_{temp})$, $T^{\frac{1}{2}}_{III,2}(\pi_{temp})$, $T_{IV,5}(\pi_{temp})$ *and* $T^{\pm}_{V,4}(\pi_{temp})$ *are all not well-defined.*

Proof. Part (i) follows from Remark 3.2. Parts (ii) and (iii) follows from definition. Let ϕ be the L-parameter associated with π_{temp}. Then no summand of ϕ has multiplicity ≥ 3, so $T^x_{II,3}(\pi_{temp})$ is not well-defined. By the good parity condition $T^{\frac{1}{2}}_{III,2}$ is not well-defined. Finally, since $m_\phi(\rho \otimes S_1) = 2$, $T_{IV,5}(\pi_{temp})$ and $T_{V,4}(\pi_{temp})$ are not well-defined. This proves part (iv). □

This concludes our discussion of case (B).

3.3. **Case** $C : \pi \hookrightarrow \rho|\cdot|^{x_1} \times \rho|\cdot|^{x_2} \times \rho|\cdot|^{x_3} \rtimes \pi_{sc}$**, where** π_{sc} **is supercuspidal.** Now let's move onto case (C). As listed above there are 6 total situations to consider. In the cases below we let $\pi_{sc} = \pi(\phi, \epsilon)$. Then $\phi = \oplus_\rho \oplus_{i=\epsilon_\rho}^{\alpha-1} \rho \otimes S_{2i+1}$

Proposition 3.24. (i) *The representation* $T^x_{I,3}(\pi_{sc})$ *is well-defined if and only if* $(x, \alpha) = (\frac{1}{2}, \frac{1}{2})$.

(ii) *When* $\alpha = \frac{1}{2}$, *the representation* $T^{\frac{1}{2}}_{I,3}(\pi_{sc})$ *is of critical type.*

(iii) When $\alpha = \frac{1}{2}$, we have that $\pi(\mathcal{E}) = T_{I,3}^{\frac{1}{2}}(\pi_{sc})$, where

$$\mathcal{E} = \begin{array}{c} \frac{1}{2} \\ \begin{pmatrix} \oplus \\ \oplus \\ \oplus \end{pmatrix} \end{array}.$$

Proof. Since π_{sc} is supercuspidal, ϕ must be multiplicity free. For $T_{I,3}^{x}(\pi_{sc})$ to be well-defined we must have $m_\phi(\rho \otimes S_{2x-1}) \geq 3$, which can only happen when $x = \frac{1}{2}$. We also require $m_\phi(\rho \otimes S_{2x+1}) = 0$, which additionally force $\alpha = \frac{1}{2}$ by the good parity condition. This proves part (i). Parts (ii) and (iii) follow from definition. □

Proposition 3.25. *(i) The representation $T_{III,2}^{1}(\pi_{sc})$ is well-defined if and only if $\alpha = 1$.*
(ii) When $\alpha = 1$, the representation $T_{III,2}^{1}(\pi_{sc})$ is of critical type.
(iii) When $\alpha = 1$, define:

$$\mathcal{E} := \{([0,0]_\rho, 0, \eta), ([1,1]_\rho, 0, -\eta), ([1,1]_\rho, 0, -\eta),$$

then $\pi(\mathcal{E}) = T_{III,2}^{1}(\pi_{sc})$. Here is the associated symbol.

$$\mathcal{E} = \begin{array}{cc} 0 & 1 \\ \multicolumn{2}{c}{\begin{pmatrix} \odot & \\ & \odot \\ & \odot \end{pmatrix}} \end{array}.$$

Note that this is different than the symbol in Proposition 3.21 since by definition, the signs in the first and second column of the symbol of $T_{III,2}^{1}(\pi_{sc})$ must be different whereas in Proposition 3.21 they are the same.

Proof. Part (i) follows from Remark 3.2. The rest follows from definition. □

Proposition 3.26. *(i) The representation $T_{IV,7}(\pi_{sc})$ is well-defined if and only if $\alpha \in \mathbb{Z}_{>0}$.*
(ii) The representation $T_{IV,7}(\pi_{sc})$ is not of critical type.
(iii) When $\alpha \in \mathbb{Z}_{>0}$, define

$$\mathcal{E} = \{([0,0]_\rho, 0, \eta)^6, \} + \{[\alpha - 1, 0]_\rho, 0, \eta)\}.$$

Then $\pi(\mathcal{E}) = T_{IV,7}(\pi_{sc})$. Here is the associated symbol.

$$\mathcal{E} = \begin{array}{ccc} 0 & \cdots & \alpha-1 \\ \multicolumn{3}{c}{\begin{pmatrix} \odot & & \\ \odot & & \\ \odot & & \\ \odot & & \\ \odot & & \\ \odot & & \\ \odot & \cdots & \odot \end{pmatrix}} \end{array}.$$

Proof. The proof is similar to Proposition 3.23, which we omit. □

Proposition 3.27. *(i) The representation $T_{V,6}^{\pm}(\pi_{sc})$ is well-defined if and only if $\alpha = 0$.*
(ii) When $\alpha = 0$, the representation $T_{V,6}^{\pm}(\pi_{sc})$ is of critical type.

(iii) When $\alpha = 0$*, the set* $\Psi(T^{\pm}_{V,6})$ *is a singleton and* $\pi(\mathcal{E}_{\pm}) = \pi^{\pm}_{V,6}(\pi_{sc})$*, where*

$$\mathcal{E}_+ = \begin{array}{c} 0 \\ \begin{pmatrix} \oplus \\ \oplus \\ \oplus \\ \oplus \\ \oplus \\ \oplus \end{pmatrix} \end{array}, \quad \mathcal{E}_- = \begin{array}{c} 0 \\ \begin{pmatrix} \ominus \\ \ominus \\ \ominus \\ \ominus \\ \ominus \\ \ominus \end{pmatrix} \end{array}.$$

Proof. The proof is similar to Proposition 3.23, which we omit. ☐

This concludes our classification of all tempered representations of corank 3.

4. Classification of non-tempered representations of corank 4 ($f(\pi) = 1$)

Now that we have the list of all tempered representations of corank 3, it is possible to give all $\pi \in \Pi_{gp}(G_n)$, which are non-tempered of corank 4. In particular, we will identify whether or not such representations are of Arthur type.

To exhaust all such representations π, we will classify them using the number of segments in their L-data, denoted by $f(\pi)$. By definition, the maximum value for $f(\pi)$ is 4 and the case $f(\pi) = 0$ corresponds to the case when π is tempered, so it suffices to consider the case where $f(\pi) = 1, 2, 3, 4$. In this section, we will focus on the case $f(\pi) = 1$.

To determine exactly when a non-tempered representation π is of Arthur type, we will reduce the number of segments in their L-data to obtain a representation $\pi^{\rho,-}$ of smaller corank, and use the known algorithms in [HJLLZ25, Section 6]. We state the relevant definitions below.

Definition 4.1. [HJLLZ25, Definition 6.2] *Suppose* $\pi \in \Pi(G_n)$ *is non-tempered. Write* $\pi = L(\Delta_{\rho_1}[x_1, -y_1], \ldots, \Delta_{\rho_f}[x_f, -y_f]; \pi(\phi, \epsilon))$.

1. *We define* $\pi^{\rho,-}$ *to be the representation whose* L*-data is obtained by removing all copies of* $\Delta_\rho[x, -y]$ *from* π *such that*
$$x = \min\{x_i | x_i - y_i = \min\{x_j - y_j | \rho_j \cong \rho\}\}$$
$$y = x - \min\{x_j - y_j | \rho_j \cong \rho\}$$
We write $\pi = \pi^{\rho,-} + \{(\Delta_\rho[x, -y])^r\}$*, where* r *denotes the multiplicity.*

Definition 4.2. [HJLLZ25, Definition 6.8] *Suppose* π *is a rep of* G_n *of good parity, and* $\pi = \pi^{\rho,-} + \{(\Delta_\rho[x, -y]^r)\}$.

1. *We denote by* $\Psi(\pi^{\rho,-}; \Delta_\rho[x, -y], r)$ *the set of local Arthur parameters such that*
   - $\pi^{\rho,-} \in \Pi_\psi$
   - *If* $y - x - 1 > 0$*, then* ψ *contains* r *copies of* $\rho \otimes S_{x+y+1} \otimes S_{y-x-1}$*;*
   - *Any summand of* ψ *of the form* $\rho \otimes S_a \otimes S_b$ *satisfies* $b \leq y - x + 1$*, and* $a > x + y + 1$ *if* $b = y - x + 1$

   For any $\psi \in \Psi(\pi^{\rho,-}, \Delta_\rho[x, -y], r)$*, we define*
$$\psi^+ := \psi - (\rho \otimes S_{x+y+1} \otimes S_{y-x-1})^{\oplus r} + (\rho \otimes S_{x+y+1} \otimes S_{y-x+1})^{\oplus r}$$
2. *We denote by* $\mathscr{E}(\pi^{\rho,-}; \Delta_\rho[x, -y], r)$ *the set of extended multi-segments* $\mathcal{E} \in \underline{Rep}^{(P')}$ *such that* $\pi(\mathcal{E}) = \pi^{\rho,-}$ *and* $\psi_{\mathcal{E}} \in \Psi(\pi^{\rho,-}; \Delta_\rho[x, -y], r)$*. For each* $\mathcal{E} \in \mathscr{E}(\pi^{\rho,-}; \Delta_\rho[x, -y], r)$ *we define* $\mathcal{E}^{\rho,+}$ *as follows.*

(a) *If $y-x=1$. then we define $\mathcal{E}^{\rho,+}$ by inserting r copies of $([x,x-1]_\rho,1,1)$ in $\mathcal{E}$ with admissible order $\gg$ on $I_\rho \sqcup \{j_1,\dots,j_r\}$, where j_k corresponds to the k-th copy we inserted, as follows:* $\begin{cases} j_r \gg j_{r-1} \gg \dots \gg j_1, & \\ \alpha \gg \beta \Longleftrightarrow \alpha > \beta & \text{for } \alpha,\beta \in I_\rho \\ \alpha \gg j_k \Longleftrightarrow B_\alpha > x-1 & \text{for } \alpha \in I_\rho \end{cases}$

(b) *If $y-x>1$, then change the admissible order if necessary so that there exists $j_1 \dots j_r \in I_\rho$ such that*
- *$j_1 < \dots < j_r$ are adjacent under the admissible order on I_ρ,*
- *$[A_{j_1},B_{j_1}]_\rho = \dots = [A_{j_r},B_{j_r}]_\rho = [y-1,x+1]_\rho$*
- *$j_1 = \min\{i \in I_\rho | B_j = B_{j_1}\}$,*
- *$A_{j_1} - B_{j_1} + 3 \geq A_i - B_i + 1$ for all $i \in I_\rho$, and the equality does not hold for $i < j_1$.*

Then we define $\mathcal{E}^{\rho,+} := \sum_{k=1}^r add^1_{j_k}(\mathcal{E}) \in \underline{Rep}^{(P')}$.

We will mostly be concerned with the reduction from π to $\pi^{\rho,-}$, so the following result is crucial.

Theorem 4.3. [HJLLZ25, Theorem 6.10] *The rep π is of Arthur type if and only if $\Psi(\pi^{\rho,-};\Delta_\rho[x,-y],r)$ is nonempty. Moreover, for any $\mathcal{E} \in \mathscr{E}(\pi^{\rho,-};\Delta_\rho[x,-y],r)$, we have $\pi = \pi(\mathcal{E}^{\rho,+})$. In other words, for any $\psi \in \Psi(\pi^{\rho,-};\Delta_\rho[x,-y],r)$, we have $\psi^+ \in \Psi(\pi)$.*

Explicitly, this gives us a way to construct the extended multi-segment corresponding to π if we know the representation $\pi^{\rho,-}$ is of Arthur type.

The following theorem gives certain restrictions on the L-data of $\pi^{\rho,-}$ for the original representation π to be of Arthur type. Here, we make no distinction between a representation and its L-data.

Lemma 4.4. [HJLLZ25, Lemma 7.8] *Suppose $\pi \in \Pi_{gp}(G_n)$ and $\pi = \pi^{\rho,-} + \{\Delta_\rho[x,-y]^s\}$. Then π is of Arthur type only if $y-x=1$ or there exists at least s copies of $\{\rho|\cdot|^z\}_{z=|x+1|}^{y-1}$ in $|\Omega|(\pi^{\rho,-})_\rho$.*

To show that certain representations are not of Arthur type, we state and prove the following lemma using the definition of raising operators. This is a generalization of [HJLLZ25, Lemma 4.23].

Lemma 4.5. *Suppose $\mathcal{E} \in \underline{Rep}$ with $\mathcal{E}_\rho = \{([A_i,B_i]_\rho,l_i,\eta_i)\}_{i\in(I_\rho,>)} + \{([X_k,X_k],0,*)^{s_k}\}_{k=1}^n$ satisfying the following conditions:*

(1) $X_k > \frac{1}{2}$ for $1 \leq k \leq n$
(2) For all $1 \leq k \leq n$ and $i \in I_\rho$, $X_k > A_i + 1$.

Then any extended multi-segment $\mathcal{E}'$ such that $\pi(\mathcal{E}') \cong \pi(\mathcal{E})$ must also contain s_k copies of $([X_k,X_k]_\rho,0,*)$ and does not contain any extended segment of the form $([X_k,-X_k]_\rho,*,*)$.

Proof. It suffices to check that if T or T^{-1} is a raising operator, then $T(\mathcal{E})$ or $T^{-1}(\mathcal{E})$ satisfies the given conditions. We use induction on n. When $n=1$, this is exactly the same statement of [HJLLZ25, Lemma 4.23].

Now assume the statement holds for $n-1$ segments of the form $([X_k,X_k],0,*)$, then applying the raising operators $ui^{-1}_{i,j}$, $dual \circ ui_{j,i} \circ dual$, or $dual^-_k$ or their inverses to the segment $([X_n,X_n],0,*)$ will give back another segment of the form $([X_n,X_n]_\rho,0,*)$, and there will be no segments of the form $([X_n,-X_n]_\rho,*,*)$, since the two conditions guarantee that the segment $[X_n,X_n]$ is separated from the other segments. This completes the proof □

When $f(\pi) = 1$, we have the following four cases:

(A) $\pi = L(\Delta_\rho[-x,-x];\pi_{temp})$, where $x > 0$ and π_{temp} is tempered of corank 3.
(B) $\pi = L(\Delta_\rho[-x,-x-1];\pi_{temp})$, where $x > -\frac{1}{2}$ and π_{temp} is tempered of corank 2.
(C) $\pi = L(\Delta_\rho[-x,-x-2];\pi_{temp})$, where $x > -1$ and π_{temp} is tempered of corank 1.
(D) $\pi = L(\Delta_\rho[-x,-x-3];\pi_{sc})$, where $x > -\frac{3}{2}$ and π_{sc} is supercuspidal.

4.1. **Case** (A)**:** $\pi = L(\Delta_\rho[-x,-x];\pi_{temp})$**.** Let us begin with case (A). According to our classification of tempered representations of corank 3, there are many situations to consider.

Proposition 4.6. *Consider the representation*

$$\pi_x = L(\Delta_\rho[-x,-x];\pi_{temp}),$$

where $\pi_{temp} = T_{I,1}^{\alpha+2}(T_{I,1}^{\alpha+1}(T_{I,1}^{\alpha}(\pi_{sc})))$ *for* $x > 0$ *and* $\alpha > 0$ *as described in Proposition 3.3.*

(i) The representation π_x *is of Arthur type if and only if* $\frac{1}{2} \le x \le \alpha - 1$ *when* $\alpha \ge \frac{3}{2}$, *or* $x = \alpha = \frac{1}{2}$.
(ii) The representation π_x *is of critical type when* $x \in \{\alpha-1, \alpha, \alpha+1, \alpha+2, \alpha+3\}$.
(iii) Define $\mathcal{E}_x$ *in various cases as follows. Then* $\pi_x = \pi(\mathcal{E}_x)$. *When* $\alpha \ge 2$, *set*

$$\begin{aligned}\mathcal{E}_x := &\{([x,-x]_\rho, \lfloor x \rfloor, (-1)^{2x}\eta), ([x-2,\epsilon_\rho]_\rho, 0, -\eta)\}\\ &+ \{([z,z]_\rho, 0, (-1)^{z-\epsilon_\rho}\eta)\}_{z=x}^{\alpha-2} + \{([\alpha+2,\alpha+2]_\rho, 0, (-1)^{\alpha-1-\epsilon_\rho}\eta)\}.\end{aligned}$$

Here is the associated symbol.

$$\mathcal{E}_x = \begin{array}{c} \begin{array}{ccccccccccccc} -x & -(x-1) & \cdots & \epsilon_\rho & \cdots & x-2 & x-1 & x & \cdots & \alpha-2 & \alpha-1 & \alpha & \alpha+1 & \alpha+2 \end{array} \\ \left(\begin{array}{cccccccccccccc} \lhd & \lhd & \cdots & \cdots & \cdots & \rhd & \rhd & \rhd & & & & & & \\ & & & \odot & \cdots & \odot & & & & & & & & \\ & & & & & & & \odot & & & & & & \\ & & & & & & & & \ddots & & & & & \\ & & & & & & & & & \odot & & & & \\ & & & & & & & & & & & & & \odot \end{array}\right) \end{array}.$$

When $\alpha = \frac{3}{2}$ *(resp.* $\frac{1}{2}$*),* $\mathcal{E}_{\frac{1}{2}}$ *are given as follows respectively,*

$$\mathcal{E}_{\frac{1}{2}} = \begin{array}{c}\begin{array}{ccccc} -\frac{1}{2} & \frac{1}{2} & \frac{3}{2} & \frac{5}{2} & \frac{7}{2} \end{array}\\ \left(\begin{array}{ccccc} \lhd & \rhd & & & \\ & & & & \oplus \end{array}\right)\end{array}, \left(resp. \mathcal{E}_{\frac{1}{2}} = \begin{array}{c}\begin{array}{cccc} -\frac{1}{2} & \frac{1}{2} & \frac{3}{2} & \frac{5}{2} \end{array}\\ \left(\begin{array}{cccc} \lhd & \rhd & & \\ & & & \oplus \end{array}\right)\end{array}\right).$$

Proof: We first show part (iii). When $\alpha < 2$, the assertion can be easily verified. For $\alpha \ge 2$, Proposition 3.3 tells us that $\pi(add_1^{-1}(\mathcal{E}_x)) = \pi_{temp}$ after applying a sequence of operators (e.g. take the dual of the first row and take the union-intersection of the first two rows) on $\pi(add_1^{-1}(\mathcal{E}_x))$. Therefore by Theorem 4.3, we obtain:

$$\pi(\mathcal{E}_x) = \pi(add_1^{-1}(\mathcal{E}_x) + \{\Delta_\rho[-x,-x]\}) = \pi_x.$$

This proves part (iii) and the sufficient direction of part (i).

Now let's show the necessary direction of part (i). Since $|\Omega|(\pi_{temp}) = \{\rho|\cdot|^y : \epsilon_\rho \le y \le \alpha-2\} + \{\rho|\cdot|^{\alpha+2}\}$. By Lemma 4.4, the representation π_x is of Arthur type only in one of the following situations:

(1) $x = \frac{1}{2}$ and $\alpha \in \mathbb{Z}_{>0}$,
(2) $\frac{1}{2} \le x \le \alpha - 1$ and $\alpha \ge \frac{3}{2}$,
(3) $x = \alpha + 3$ with $\alpha \in x + \mathbb{Z}$.

Therefore it suffices to show that π_x is not of Arthur type in situation (3). Let $x = \alpha + 3$ and $\alpha \in x + \mathbb{Z}$. A priori, we have an extended multi-segment $\mathcal{E}_{temp}$ listed in Proposition 3.3 (iii) such that $\pi(\mathcal{E}_{temp}) = \pi_{temp}$. Then by Lemma 4.5, there does not exist any multi-segment $\mathcal{E}$ that contains $([\alpha+2, -\alpha-2]_\rho, *, *)$ such that $\pi(\mathcal{E}) = \pi_{temp}$.

However, by Definition 4.2, any element in $\Psi(\pi_{temp}, \Delta_\rho[-x,-x], 1)$ must contain 1 copy of $\rho \otimes S_1 \otimes S_{2\alpha+5}$, which is equivalent to saying that the corresponding extended multi-segment must contain some element of the form $([\alpha+2, -\alpha-2]_\rho, *, *)$. This implies that the set $\Psi(\pi_{temp}; \Delta_\rho[-x,-x], 1)$ must be empty, so π_x is not of Arthur type by Theorem 4.3. This completes the proof. $\square$

Next we consider the case $\pi_{temp} = T_{I,1}^{\alpha-1}(T_{I,1}^{\alpha+1}(T_{I,1}^{\alpha}(\pi_{sc})))$.

Proposition 4.7. *Consider the representation*

$$\pi_x = L(\Delta_\rho[-x,-x]; \pi_{temp}),$$

where $\pi_{temp} = T_{I,1}^{\alpha-1}(T_{I,1}^{\alpha+1}(T_{I,1}^{\alpha}(\pi_{sc})))$ *for* $x > 0$ *and* $\alpha > 1$ *as described in Proposition 3.3.*

(i) The representation π_x *is of Arthur type if and only if* $\frac{1}{2} \leq x \leq \alpha - 2$ *when* $\alpha \geq \frac{5}{2}$, *or when* $x = \alpha = \frac{3}{2}$.

(ii) The representation π_x *is of critical type when* $x \in \{\alpha-2, \alpha-1, \alpha, \alpha+1, \alpha+2\}$.

(iii) When $\frac{1}{2} \leq x \leq \alpha - 2$ *for* $\alpha \geq \frac{5}{2}$, *or* $x = \alpha = \frac{3}{2}$, *define*

$$\begin{aligned}\mathcal{E}_x := &\{([x,-x]_\rho, \lfloor x \rfloor, (-1)^{2x}\eta), ([x-2, \epsilon_\rho]_\rho, 0, -\eta)\} + \{([z,z]_\rho, 0, (-1)^{z-\epsilon_\rho}\eta)\}_{z=x}^{\alpha-3} \\ &+ \{([\alpha-1,\alpha-1]_\rho, 0, (-1)^{\alpha-1-\epsilon_\rho}\eta), ([\alpha+1,\alpha+1]_\rho, 0, (-1)^{\alpha-2-\epsilon_\rho}\eta\}.\end{aligned}$$

Then $\pi_x = \pi(\mathcal{E}_x)$. *Here is the associated symbol.*

$$\mathcal{E} = \begin{array}{c} \begin{array}{ccccccccccccccc} -x & -(x-1) & \cdots & \epsilon_\rho & \cdots & x-2 & x-1 & x & \cdots & \alpha-3 & \alpha-2 & \alpha-1 & \alpha & \alpha+1 \end{array} \\ \begin{pmatrix} \lhd & \lhd & \cdots & \odot & \cdots & \rhd & \rhd & \rhd & & & & & & \\ & & & \odot & \cdots & \odot & & & & & & & & \\ & & & & & & & \odot & & & & & & \\ & & & & & & & & \ddots & & & & & \\ & & & & & & & & & \odot & & & & \\ & & & & & & & & & & & & & \odot \end{pmatrix} \end{array}.$$

Proof. The proof is similar to Proposition 4.6, which we omit. $\square$

The next case to consider is $\pi_{temp} = T_{IV,3}(T_{I,1}^{\alpha+1}(T_{I,1}^{\alpha}(\pi_{sc})))$ for $\alpha \in \mathbb{Z}_{>1}$.

Proposition 4.8. *Consider the representation*

$$\pi_x = L(\Delta_\rho[-x,-x]; \pi_{temp})$$

where $\pi_{temp} = T_{IV,3}(T_{I,1}^{\alpha+1}(T_{I,1}^{\alpha}(\pi_{sc})))$ *for* $x > 0$ *and* $\alpha \in \mathbb{Z}_{>1}$, *as described in Proposition 3.4.*

(i) The representation π_x *is of Arthur type if and only if* $1 \leq x \leq \alpha - 1$.

(ii) The representation π_x *is of critical type when* $(x, \alpha) = (1, 2)$.

(iii) When $1 \leq x \leq \alpha - 1$, *define*

$$\begin{aligned}\mathcal{E}_x := &\{([x,-x]_\rho, \lfloor x \rfloor, (-1)^{2x}\eta)\} + \{([0,0]_\rho, 0, \eta))^2\} + ([x-2, 0]_\rho, 0, \eta)\} \\ &+ \{([z,z]_\rho, 0, (-1)^{z-\epsilon_\rho}\eta)\}_{z=x}^{\alpha-2} + \{([\alpha+1,\alpha+1], 0, (-1)^{\alpha-2-\epsilon_\rho}\eta\}.\end{aligned}$$

Then $\pi_x = \pi(\mathcal{E}_x)$. Here is the associated symbol.

$$\mathcal{E} = \begin{array}{c} \begin{array}{cccccccccccccc} -x & -(x-1) & \cdots & 0 & \cdots & x-2 & x-1 & x & \cdots & \alpha-2 & \alpha-1 & \alpha & \alpha+1 \end{array} \\ \begin{pmatrix} \lhd & \lhd & \cdots & \odot & \cdots & \rhd & \rhd & \rhd & & & & & \\ & & & \odot & & & & & & & & & \\ & & & \odot & & & & & & & & & \\ & & & \odot & \cdots & \odot & & & & & & & \\ & & & & & & & \odot & & & & & \\ & & & & & & & & \ddots & & & & \\ & & & & & & & & & \odot & & & \\ & & & & & & & & & & & & \odot \end{pmatrix} \end{array}.$$

Proof. The proof for parts (i) and (iii) is similar to Proposition 4.6, which we omit. Part (ii) follows from Proposition 3.4 (ii). $\square$

Now we consider the case $\pi_{temp} = T^{\pm}_{V,2}(T^2_{I,1}(T^1_{I,1}(\pi_{sc})))$.

Proposition 4.9. *Consider the representation*

$$\pi^{\pm}_x = L(\Delta_\rho[-x,-x];\pi^{\pm}_{temp}),$$

where $\pi^{\pm}_{temp} = T^{\pm}_{V,2}(T^2_{I,1}(T^1_{I,1}(\pi_{sc})))$ and $\alpha = 1$, as described in Proposition 3.5.

(i) The representation $\pi^{\pm}_x$ is of Arthur type if and only if $x = 1$.
(ii) The representation $\pi^{\pm}_x$ is of critical type for $x \in \{1,2,3\}$.
(iii) Define

$$\mathcal{E}_{\pm} = \{([-1,-1],1,\eta)\}\{([0,0],0,\pm 1))^3,\} + \{([2,2],0,\eta)\},$$

Then $\pi^{\pm}_1 = \pi(\mathcal{E}_{\pm})$. Here are the associated symbols.

$$\mathcal{E}_+ = \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{pmatrix} \lhd & \odot & \rhd \\ \oplus & & \\ \oplus & & \\ \oplus & & \\ & & \odot \end{pmatrix} \end{array}, \quad \mathcal{E}_- = \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{pmatrix} \lhd & \odot & \rhd \\ \ominus & & \\ \ominus & & \\ \ominus & & \\ & & \odot \end{pmatrix} \end{array}.$$

Proof. The proof is similar to Proposition 4.6, which we omit. $\square$

The next case is $\pi_{temp} = T^{\alpha-2}_{I,1}(T^{\alpha-1}_{I,1}(T^{\alpha}_{I,1}(\pi_{sc})))$.

Proposition 4.10. *Consider the representation*

$$\pi_x = L(\Delta_\rho[-x,-x];\pi_{temp})$$

where $\pi_{temp} = T^{\alpha-2}_{I,1}(T^{\alpha-1}_{I,1}(T^{\alpha}_{I,1}(\pi_{sc})))$ for $x > 0$ and $\alpha > 2$ as described in Proposition 3.6.

(i) The representation π_x is of Arthur type if and only if $0 < x \le \alpha - 3$ for $\alpha \ge \frac{7}{2}$.
(ii) The representation π_x is of critical type when $x \in \{\alpha-3, \alpha-2, \alpha-1, \alpha, \alpha+1\}$.
(iii) Define

$$\mathcal{E}_x = \{([x,-x]_\rho, \lfloor x \rfloor, (-1)^{2x}\eta), ([x-2,\epsilon_\rho]_\rho, 0, \eta), ([\alpha-4, x]_\rho, 0, (-1)^{x-\epsilon_\rho}\eta), ([\alpha,\alpha-2]_\rho, 0, -\eta)\}.$$

Then $\pi_x = \pi(\mathcal{E}_x)$. Here is the associated symbol.

$$\mathcal{E}_x = \begin{array}{c} \begin{array}{cccccccccccccc} -x & -(x-1) & \cdots & \epsilon_\rho & \cdots & x-2 & x-1 & x & \cdots & \alpha-4 & \alpha-3 & \alpha-2 & \alpha-1 & \alpha \end{array} \\ \begin{pmatrix} \lhd & \lhd & \cdots & \odot & \cdots & \rhd & \rhd & \rhd & & & & & & \\ & & & \odot & \cdots & \odot & & & & & & & & \\ & & & & & & & & \odot & & & & & \\ & & & & & & & & & \ddots & & & & \\ & & & & & & & & & & \odot & & & \\ & & & & & & & & & & & \odot & \odot & \odot \end{pmatrix} \end{array}.$$

Proof. By Lemma 4.4, it suffices to prove that π_x is not of Arthur type when $x = \alpha-1, \alpha, \alpha+1$. If $x = \alpha - 1$, then by Proposition 3.6 and Lemma 4.5, there does not exist any extended multi-segment $\mathcal{E}$ that contains $([\alpha-2, -\alpha+2]_\rho, *, *)$ such that $\pi(\mathcal{E}) = \pi_{temp}$.

However, by Definition 4.2, any element in $\Psi(\pi_{temp}, \Delta_\rho[-x,-x], 1)$ must contain 1 copy of $\rho \otimes S_1 \otimes S_{2\alpha-1}$, which means that the corresponding extended multi-segment must contain a segment of the form $([\alpha-2, -\alpha+2]_\rho, *, *)$. Therefore the set $\Psi(\pi_{temp}, \Delta_\rho[-x,-x], 1)$ must be empty, so π_x is not of Arthur type by Theorem 4.3. This completes the proof for $x = \alpha - 1$. The cases are $x = \alpha, \alpha+1$ are exactly the same. □

Now we consider the case $\pi_{temp} = T_{IV,3}(T_{I,1}^{\alpha-1}(T_{I,1}^{\alpha}(\pi_{sc})))$.

Proposition 4.11. *Consider the representation*

$$\pi_x = L(\Delta_\rho[-x,-x]; \pi_{temp}),$$

where $\pi_{temp} = T_{IV,3}(T_{I,1}^{\alpha-1}(T_{I,1}^{\alpha}(\pi_{sc})))$ for $x > 0$ and $\alpha \in \mathbb{Z}_{>2}$, as described in Proposition 3.7.

(i) The representation π_x is of Arthur type if and only if $0 < x \leq \alpha - 2$.

(ii) The representation π_x is of critical type when $(x, \alpha) = (1, 3)$.

(iii) When $0 \leq x \leq \alpha - 2$, define

$$\begin{aligned} \mathcal{E}_x = \{&([x,-x]_\rho, \lfloor x \rfloor, (-1)^{2x}\eta), ([0,0]_\rho, 0, \eta)^2, ([x-2, 0]_\rho, 0, \eta), \\ &([\alpha-3, x]_\rho, 0, (-1)^x\eta), ([\alpha-1, \alpha-1]_\rho, 0, -\eta), ([\alpha,\alpha]_\rho, 0, \eta)\} \end{aligned}$$

Then $\pi(\mathcal{E}_x) = \pi_x$. Here is the associated symbol.

$$\mathcal{E}_x = \begin{array}{c} \begin{array}{ccccccccccccc} -x & -(x-1) & \cdots & 0 & \dots & x-2 & x-1 & x & \cdots & \alpha-3 & \alpha-2 & \alpha-1 & \alpha \end{array} \\ \begin{pmatrix} \lhd & \lhd & \cdots & \odot & \cdots & \rhd & \rhd & \rhd & & & & & \\ & & & \odot & & & & & & & & & \\ & & & \odot & & & & & & & & & \\ & & & \odot & \cdots & \odot & & & & & & & \\ & & & & & & & \odot & & & & & \\ & & & & & & & & \ddots & & & & \\ & & & & & & & & & \odot & & & \\ & & & & & & & & & & & \odot & \\ & & & & & & & & & & & & \odot \end{pmatrix} \end{array}.$$

Proof. The proof for parts (ii) and (iii) are similar to Proposition 4.8, which we omit. Part (ii) follows from Proposition 3.7. □

The next case is $\pi_{temp} = T_{V,2}^{\pm}(T_{I,1}^{1}(T_{I,1}^{2}(\pi_{sc})))$.

Proposition 4.12. *Consider the representation*

$$\pi_x^{\pm} = L(\Delta_\rho[-x,-x]; \pi_{temp}^{\pm}),$$

where $\pi_{temp}^{\pm} = T_{V,2}^{\pm}(T_{I,1}^{1}(T_{I,1}^{2}(\pi_{sc})))$ for $x > 0$, $\alpha = 2$, as described in Proposition 3.8.

(1) The representation $\pi_x^{\pm}$ is of Arthur type if and only if $x = 1$.

(2) The representation $\pi_x^{\pm}$ is of critical type for $x \in \{1,2,3\}$.
(3) Define

$$\mathcal{E}_{\pm} := \{([1,-1]_\rho, 1, \pm 1), ([0,0]_\rho, 0, \pm 1)^2, ([2,1]_\rho, 0, \eta)\}.$$

Then $\pi(\mathcal{E}_{\pm}) = \pi_1^{\pm}$. Here are the associated symbols.

$$\mathcal{E}_+ = \begin{array}{c} \begin{matrix} -1 & 0 & 1 & 2 \end{matrix} \\ \begin{pmatrix} \lhd & \oplus & \rhd & \\ & \oplus & & \\ & & \oplus & \ominus \end{pmatrix} \end{array}, \quad \mathcal{E}_- = \begin{array}{c} \begin{matrix} -1 & 0 & 1 & 2 \end{matrix} \\ \begin{pmatrix} \lhd & \ominus & \rhd & \\ & \ominus & & \\ & & \ominus & \oplus \end{pmatrix} \end{array}.$$

Proof. The proof is similar to the proof of Proposition 4.9, which we omit. □

Now we move onto the case $\pi_{temp}^{\pm} = T_{I,1}^{1}(T_{V,2}^{\pm}(T_{I,1}^{1}(\pi_{sc})))$.

Proposition 4.13. *Consider the representation*

$$\pi_x^{\pm} = L(\Delta_\rho[-x,-x]; \pi_{temp}^{\pm}),$$

where $\pi_{temp}^{\pm} = T_{I,1}^{2}(T_{V,2}^{\pm}(T_{I,1}^{1}(\pi_{sc})))$ for $x > 0$ and $\alpha = 1$, as described in Proposition 3.10.

(i) The representation π_x is of Arthur type if and only if $x = 1$.
(ii) The representation π_x is of critical type when $x \in \{1,2,3\}$.
(iii) Define

$$\mathcal{E}_{\pm} := \{([1,-1]_\rho, 1, \pm 1), ([0,0]_\rho, 0, \pm 1)^2, ([2,1]_\rho, 0, \eta)\}.$$

Then $\pi(\mathcal{E}_{\pm}) = \pi_1^{\pm}$. Here are the associated symbols.

$$\mathcal{E}_+ = \begin{array}{c} \begin{matrix} -1 & 0 & 1 & 2 \end{matrix} \\ \begin{pmatrix} \lhd & \oplus & \rhd & \\ & \oplus & & \\ & & & \odot \end{pmatrix} \end{array}, \quad \mathcal{E}_- = \begin{array}{c} \begin{matrix} -1 & 0 & 1 & 2 \end{matrix} \\ \begin{pmatrix} \lhd & \ominus & \rhd & \\ & \ominus & & \\ & & & \odot \end{pmatrix} \end{array}.$$

Proof. The proof is similar to the proof of Proposition 4.12, which we omit. □

Now we consider the case $\pi_{temp} = T_{I,1}^{2}(T_{I,1}^{1}(T_{V,2}^{\pm}(\pi_{sc})))$.

Proposition 4.14. *Consider the representation*

$$\pi_x^{\pm} = L(\Delta_\rho[-x,-x]; \pi_{temp}^{\pm}),$$

where $\pi_{temp}^{\pm} = T_{I,1}^{2}(T_{I,1}^{1}(T_{V,2}^{\pm}(\pi_{sc})))$ for $x > 0$ and $\alpha = 0$, as described in Proposition 3.11.

(i) The representation $\pi_x^{\pm}$ is of Arthur type if and only if $x = 1$.
(ii) The representation $\pi_x^{\pm}$ is of critical type when $x \in \{1,2,3\}$
(iii) Define

$$\mathcal{E}_{\pm} := \{([1,-1]_\rho, 1, \pm 1), ([2,2]_\rho, 0, \pm 1)\}.$$

Then $\pi(\mathcal{E}_{\pm}) = \pi_1^{\pm}$. Here are the associated symbols.

$$\mathcal{E}_+ = \begin{array}{c} \begin{matrix} -1 & 0 & 1 & 2 \end{matrix} \\ \begin{pmatrix} \lhd & \oplus & \rhd & \\ & & & \oplus \end{pmatrix} \end{array}, \quad \mathcal{E}_- = \begin{array}{c} \begin{matrix} -1 & 0 & 1 & 2 \end{matrix} \\ \begin{pmatrix} \lhd & \ominus & \rhd & \\ & & & \ominus \end{pmatrix} \end{array}.$$

Proof. The proof is similar to that of Proposition 4.13, which we omit. □

The next case is $\pi_{temp} = T_{IV,3}(T_{I,1}^{1}(T_{V,2}^{\pm}(\pi_{sc})))$.

Proposition 4.15. *Consider the representation*

$$\pi_x^{\pm} = L(\Delta_\rho[-x,-x]; \pi_{temp}^{\pm}),$$

where $\pi_{temp}^{\pm} = T_{IV,3}(T_{I,1}^{1}(T_{V,2}^{\pm}(\pi_{sc})))$ for $\alpha = 0$, as described in Proposition 3.12.

(i) The representation $\pi_x^{\pm}$ is of Arthur type if and only if $x = 1$.
(ii) The representation $\pi_x^{\pm}$ is of critical type when $x \in \{1, 2\}$.
(iii) Define
$$\mathcal{E}_{\pm} := \{([1,-1]_\rho, 1, \pm 1), ([0,0]_\rho, 0, \pm 1), ([1,1]_\rho, 0, \pm 1)\},$$
Then $\pi(\mathcal{E}_{\pm}) = \pi_1^{\pm}$. Here are the associated symbols.

$$\mathcal{E}_+ = \begin{array}{c} \begin{matrix} -1 & 0 & 1 \end{matrix} \\ \begin{pmatrix} \triangleleft & \odot & \triangleright \\ & \odot & \\ & \oplus & \\ & & \oplus \end{pmatrix} \end{array}, \quad \mathcal{E}_- = \begin{array}{c} \begin{matrix} -1 & 0 & 1 \end{matrix} \\ \begin{pmatrix} \triangleleft & \odot & \triangleright \\ & \odot & \\ & \ominus & \\ & & \ominus \end{pmatrix} \end{array}.$$

Proof. The proof is similar to that of Proposition 4.14, which we omit. ☐

Now let's look at $\pi_{temp} = T_{I,1}^{\frac{3}{2}}(T_{I,2}^{\frac{1}{2}}(\pi_{sc}))$.

Proposition 4.16. *Consider the representation*
$$\pi_x = L(\Delta_\rho[-x,-x]; \pi_{temp})$$
where $\pi_{temp} = T_{I,1}^{\frac{3}{2}}(T_{I,2}^{\frac{1}{2}}(\pi_{sc}))$ for $x > 0$ and $\alpha = \frac{1}{2}$, as described in Proposition 3.13.
(i) The representation π_x is not of Arthur type for any x.
(ii) The representation π_x of critical type when $x \in \{\frac{1}{2}, \frac{3}{2}, \frac{5}{2}\}$.

Proof. By Lemma 4.4, π_x can only be of Arthur type if $x = \frac{5}{2}$. When $x = \frac{5}{2}$, we see from Proposition 3.13 that the support of the extended multi-segment corresponding to π_x does not include $\frac{1}{2}$, so by Definition 4.2 and Theorem 4.3, it cannot be of Arthur type. ☐

The next case we have is $\pi_{temp} = T_{I,1}^{\alpha}(T_{II,3}^{\frac{1}{2}}(\pi_{sc}))$.

Proposition 4.17. *Consider the representation*
$$\pi_x = L(\Delta_\rho[-x,-x]; \pi_{temp}),$$
where $\pi_{temp} = T_{I,1}^{\alpha}(T_{II,3}^{\frac{1}{2}}(\pi_{sc}))$ for $x > 0$ and $\alpha \in \frac{1}{2} + \mathbb{Z}_{>0}$, as described in Proposition 3.14.
(i) The representation π_x is of Arthur type if and only if $\frac{1}{2} \leq x \leq \alpha - 1$.
(ii) The representation π_x is of critical type when $(x, \alpha) = (\frac{1}{2}, \frac{3}{2}), (\frac{3}{2}, \frac{3}{2}), (\frac{5}{2}, \frac{3}{2})$ or $(\frac{3}{2}, \frac{5}{2})$.
(iii) Define
$$\begin{aligned} \mathcal{E}_x := &\{([x,-x]_\rho, \lfloor x \rfloor, (-1)^{x+\frac{1}{2}}), ([\frac{1}{2}, \frac{1}{2}]_\rho, 0, -1)^2, \\ &([x-2, \frac{3}{2}]_\rho, 0, 1), ([\alpha-2, x]_\rho, 0, (-1)^{x+\frac{1}{2}}), ([\alpha,\alpha]_\rho, 0, (-1)^{\alpha-\frac{1}{2}})\}. \end{aligned}$$
Then $\pi(\mathcal{E}_x) = \pi_x$. Here is the associated symbol.

$$\mathcal{E}_x = \begin{array}{c} \begin{matrix} -x & -(x-1) & \cdots & \frac{1}{2} & \frac{3}{2} & \cdots & x-2 & x-1 & x & \cdots & \alpha-2 & \alpha-1 & \alpha \end{matrix} \\ \begin{pmatrix} \triangleleft & \triangleleft & \cdots & \ominus & \cdots & \triangleright & \triangleright & \triangleright & & & & & \\ & & & \ominus & & & & & & & & & \\ & & & \ominus & & & & & & & & & \\ & & & & \oplus & \cdots & \odot & & & & & & \\ & & & & & & & & \odot & & & & \\ & & & & & & & & & \ddots & & & \\ & & & & & & & & & & \odot & & \\ & & & & & & & & & & & & \odot \end{pmatrix} \end{array}.$$

Proof. The proof is similar to Proposition 4.14, which we omit. ☐

Now we move onto the case $\pi_{temp} = T_{I,1}^{\frac{3}{2}}(T_{III,2}^{\frac{1}{2}}(\pi_{sc}))$.

Proposition 4.18. *Consider the representation*

$$\pi_x = L(\Delta_\rho[-x,-x];\pi_{temp}),$$

where $\pi_{temp} = T_{I,1}^{\frac{3}{2}}(T_{III,2}^{\frac{1}{2}}(\pi_{sc}))$, *for* $x > 0$ *and* $\alpha = \frac{1}{2}$, *as described in Proposition 3.15.*

(i) The representation π_x *is of Arthur type if and only if* $x = \frac{1}{2}$ *or* $\frac{3}{2}$.
(ii) The representation π_x *is of critical type when* $x \in \{\frac{1}{2},\frac{3}{2},\frac{5}{2}\}$.
(iii) For $x \in \{\frac{1}{2},\frac{3}{2}\}$, *we have* $\pi(\mathcal{E}_x) = \pi_x$, *where* $\mathcal{E}_x$ *are given as follows.*

$$\mathcal{E}_{\frac{1}{2}} = \begin{array}{c} \begin{array}{ccc} -\frac{1}{2} & \frac{1}{2} & \frac{3}{2} \end{array} \\ \left(\begin{array}{ccc} \lhd & \ominus & \rhd \\ & \ominus & \\ & & \ominus \end{array}\right) \end{array}, \quad \mathcal{E}_{\frac{3}{2}} = \begin{array}{c} \begin{array}{cccc} -\frac{3}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{3}{2} \end{array} \\ \left(\begin{array}{cccc} \lhd & \oplus & \ominus & \rhd \\ & & & \ominus \end{array}\right) \end{array}.$$

Proof. By Lemma 4.4 and Proposition 3.15, the only possible x for π_x to be of Arthur type are $x = \frac{1}{2},\frac{3}{2},\frac{5}{2}$. By Definition 4.1, the set $\Psi(\pi_{temp},[-x,-x],1)$ is nonempty if and only if it contains some parameter ψ containing $\rho\otimes S_1\otimes S_{2x-1}$, or that the corresponding extended multisegment contains the segment $([x-1,-(x-1)]_\rho,*,*)$.

Looking at the possible extended multisegments in Proposition 3.15, this only happens when $x = \frac{1}{2},\frac{3}{2}$. By Theorem 4.3, this is exactly when π_x is of Arthur type. This proves part (i). Part (ii) follows from definition, and part (iii) follows from Proposition 3.15 (iii). ☐

The next case is $\pi_{temp} = T_{I,1}^{\alpha}(T_{IV,5}(\pi_{sc}))$.

Proposition 4.19. *Consider the representation*

$$\pi_x = L(\Delta_\rho[-x,-x];\pi_{temp}),$$

where $\pi_{temp} = T_{I,1}^{\alpha}(T_{IV,5}(\pi_{sc}))$ *for* $x > 0$ *and* $\alpha \in \mathbb{Z}_{>0}$, *as described in Proposition 3.16.*

(i) The representation π_x *if of Arthur type if and only if* $1 \le x \le \alpha - 1$ *for* $\alpha \ge 2$, *or when* $x = \alpha = 1$.
(ii) The representation π_x *is of critical type when* $(x,\alpha) = (1,1)$ *or* $(1,2)$.
(iii) Define

$$\begin{aligned}\mathcal{E}_x :=&\{([x,-x]_\rho,\lfloor x\rfloor,(-1)^x\eta),([0,0]_\rho,0,\eta)^5,\\ &([x-2,1]_\rho,0,-\eta),([\alpha-2,x]_\rho,0,(-1)^x\eta),([\alpha,\alpha]_\rho,0,(-1)^{\alpha-1}\eta)\}.\end{aligned}$$

Then $\pi(\mathcal{E}_x) = \pi_x$. *Here is the associated symbol.*

$$\mathcal{E}_x = \begin{array}{c} \begin{array}{ccccccccccccc} -x & -(x-1) & \cdots & 0 & 1 & \cdots & x-2 & x-1 & x & \cdots & \alpha-2 & \alpha-1 & \alpha \end{array} \\ \left(\begin{array}{ccccccccccccc} \lhd & \lhd & \cdots & \ominus & \oplus & \cdots & \rhd & \rhd & \rhd & & & & \\ & & & \odot & & & & & & & & & \\ & & & \odot & & & & & & & & & \\ & & & & \odot & \cdots & \odot & & & & & & \\ & & & & & & & & \odot & & & & \\ & & & & & & & & & \ddots & & & \\ & & & & & & & & & & \odot & & \\ & & & & & & & & & & & & \odot \end{array}\right) \end{array}.$$

Proof. When $x = \alpha = 1$, it's easy to verify that π_x is of Arthur type. The rest of the proof is similar to Proposition 4.17, which we omit. ☐

Now we move onto the case $\pi_{temp} = T^1_{I,1}(T^{\pm}_{V,4}(\pi_{sc}))$.

Proposition 4.20. *Consider the representation*

$$\pi_x^{\pm} = L(\Delta_\rho[-x,-x]; \pi_{temp}^{\pm}),$$

where $\pi_{temp}^{\pm} = T^1_{I,1}(T^{\pm}_{V,4}(\pi_{sc}))$ *for* $x > 0$ *and* $\alpha = 0$*, as described in Proposition 3.17.*

(i) The representation $\pi_x^{\pm}$ *is of Arthur type if and only if* $x = 1$*.*
(ii) The representation $\pi_x^{\pm}$ *is of critical type when* $x \in \{1,2\}$*.*
(iii) We have that $\pi(\mathcal{E}_{\pm}) = \pi_x^{\pm}$*, where*

$$\mathcal{E}_+ = \begin{array}{ccc} -1 & 0 & 1 \end{array} \begin{pmatrix} \lhd & \oplus & \rhd \\ & \oplus & \\ & \oplus & \\ & & \oplus \end{pmatrix}, \quad \mathcal{E}_- = \begin{array}{ccc} -1 & 0 & 1 \end{array} \begin{pmatrix} \lhd & \ominus & \rhd \\ & \ominus & \\ & \ominus & \\ & & \ominus \end{pmatrix}.$$

Proof. The proof is similar to Proposition 4.18, which we omit. $\square$

The next relevant case is $\pi_{temp} = T^{\pm}_{V,4}(T^1_{I,1}(\pi_{sc}))$.

Proposition 4.21. *Consider the representation*

$$\pi_x^{\pm} = L(\Delta_\rho[-x,-x]; \pi_{temp}^{\pm}),$$

where $\pi_{temp}^{\pm} = T^{\pm}_{V,4}(T^1_{I,1}(\pi_{sc}))$ *for* $x > 0$ *and* $\alpha = 1$*, as described in Proposition 3.20.*

(i) The representation $\pi_x^{\pm}$ *is of Arthur type if and only if* $x = 1$*.*
(ii) The representation $\pi_x^{\pm}$ *is of critical type for* $x \in \{1,2\}$*.*
(iii) We have $\pi(\mathcal{E}_{\pm}) = \pi_x^{\pm}$*, where*

$$\mathcal{E}_+ = \begin{array}{ccc} -1 & 0 & 1 \end{array} \begin{pmatrix} \lhd & \oplus & \rhd \\ & \oplus & \\ & \oplus & \\ & \oplus & \\ & & \oplus \end{pmatrix}, \quad \mathcal{E}_- = \begin{array}{ccc} -1 & 0 & 1 \end{array} \begin{pmatrix} \lhd & \ominus & \rhd \\ & \ominus & \\ & \ominus & \\ & \ominus & \\ & & \ominus \end{pmatrix}.$$

Proof. The proof is similar to that of Proposition 4.20, which we omit. $\square$

Now we move onto the case $\pi_{temp} = T^1_{I,2}(T_{IV,3}(\pi_{sc}))$.

Proposition 4.22. *Consider the representation*

$$\pi_x = L(\Delta_\rho[-x,-x]; \pi_{temp}),$$

where $\pi_{temp} = T^1_{I,2}(T_{IV,3}(\pi_{sc}))$ *for* $x > 0$ *and* $\alpha = 1$*, as described in Proposition 3.21.*

(i) The representation π_x *is of Arthur type if and only if* $x = 1$*.*
(ii) The representation π_x *is of critical type when* $x \in \{1,2\}$*.*
(iii) We have $\pi(\mathcal{E}) = \pi_1$*, where*

$$\mathcal{E} = \begin{array}{ccc} -1 & 0 & 1 \end{array} \begin{pmatrix} \lhd & \odot & \rhd \\ & & \odot \\ & & \odot \end{pmatrix}.$$

Proof. The proof is similar to Proposition 4.22, which we omit. $\square$

The next case to consider is $\pi_{temp} = T^1_{II,3}(T_{IV,3}(\pi_{sc}))$.

Proposition 4.23. *Consider the representation*

$$\pi_x = L(\Delta_\rho[-x,-x]; \pi_{temp}),$$

where $\pi_{temp} = T^1_{II,3}(T_{IV,3}(\pi_{sc}))$*, for* $x > 0$ *and* $\alpha \in \mathbb{Z}_{>0}$*, as described in Proposition 3.22.*

(i) The representation π_x is of Arthur type if and only if $1 \leq x \leq \alpha$.
(ii) The representation π_x is of critical type when $(x,\alpha) = (1,1), (2,1), (2,2)$.
(iii) Define

$$\begin{aligned}\mathcal{E}_x :=& \{([x,-x]_\rho, \lfloor x \rfloor, (-1)^x\eta), ([x-2,0]_\rho, 0, -\eta), \\ & ([1,1]_\rho, 0, -\eta)^2, ([\alpha-1,x]_\rho, 0, (-1)^x\eta)\}.\end{aligned}$$

Then $\pi(\mathcal{E}_x) = \pi_x$. Here is the associated symbol.

$$\mathcal{E}_x = \begin{array}{c} \begin{array}{ccccccccccc} -x & -(x-1) & \cdots & 0 & 1 & \cdots & x-2 & x-1 & x & \cdots & \alpha-1 \end{array} \\ \left(\begin{array}{ccccccccccc} \lhd & \lhd & \cdots & \odot & \odot & \cdots & \rhd & \rhd & \rhd & & \\ & & & \odot & \odot & \cdots & \odot & & & & \\ & & & & \odot & & & & & & \\ & & & & \odot & & & & & & \\ & & & & & & & & \odot & & \\ & & & & & & & & & \ddots & \\ & & & & & & & & & & \odot \end{array}\right) \end{array}.$$

Proof. The proof is similar to Proposition 4.19, which we omit □

. There are four more cases to consider in Case (A). Next we have $\pi_{temp} = T^1_{I,2}(T^{\pm}_{V,2}(\pi_{sc}))$.

Proposition 4.24. *Consider the representation*

$$\pi^{\pm}_x = L(\Delta_\rho[-x,-x]; \pi^{\pm}_{temp}),$$

where $\pi^{\pm}_{temp} = T^1_{I,2}(T^{\pm}_{V,2}(\pi_{sc}))$ *for* $x > 0$ *and* $\alpha = 0$*, as described in Proposition 3.23.*

(i) The representation $\pi^{\pm}_x$ *is not of Arthur type for any* x*.*
(ii) The representation $\pi^{\pm}_x$ *is of critical type when* $x \in \{1,2\}$*.*

Proof. This follows from the fact that the set $\Psi(\pi^{\pm}_{temp})$ is a singleton and it doesn't contain the interval $[1,-1]$. □

The next case is $\pi_{temp} = T^1_{III,2}(\pi_{sc})$.

Proposition 4.25. *Consider the representation*

$$\pi_x = L(\Delta_\rho[-x,-x]; \pi_{temp}),$$

where $\pi_{temp} = T^1_{III,2}(\pi_{sc})$ *for* $\alpha = 1$*, as described in Proposition 3.24.*

(i) The representation π_x *is of Arthur type if and only if* $x \in \{1,2\}$*.*
(ii) The representation π_x *is of critical type when* $x \in \{1,2\}$*.*
(iii) We have that $\pi(\mathcal{E}_x) = \pi_x$*, where*

$$\mathcal{E}_1 = \begin{array}{c} \begin{array}{ccc} -1 & 0 & 1 \end{array} \\ \left(\begin{array}{ccc} \lhd & \odot & \rhd \\ & & \odot \\ & & \odot \end{array}\right) \end{array}, \quad \mathcal{E}_2 = \begin{array}{c} \begin{array}{ccccc} -2 & -1 & 0 & 1 & 2 \end{array} \\ \left(\begin{array}{ccccc} \lhd & \lhd & \odot & \rhd & \rhd \\ & & \odot & & \\ & & & \odot & \end{array}\right) \end{array}.$$

Proof. By Lemma 4.4, the only possible x for π_x to be of Arthur type are $x = 0$ or 1. It is easy to verify that both cases hold. The rest follows from definition. □

We continue to the case $\pi_{temp} = T_{I,3}^{\frac{1}{2}}(\pi_{sc})$, which is well-defined if and only if $\alpha = \frac{1}{2}$ by Proposition 3.24.

Proposition 4.26. *Consider the representation*

$$\pi_x = L(\Delta_\rho[-x,-x];\pi_{temp}),$$

where $\pi_{temp} = T_{I,3}^{\frac{1}{2}}(\pi_{sc})$ *and* $\alpha = \frac{1}{2}$, *as described in Proposition 3.24.*

(i) The representation π_x *is of Arthur type if and only if* $x = \frac{1}{2}$.
(ii) The representation π_x *is of critical type if and only if* $x = \frac{1}{2}$ *or* $x = \frac{3}{2}$.
(iii) W have $\pi(\mathcal{E}_{\frac{1}{2}}) = \pi_{\frac{1}{2}}$, *where*

$$\mathcal{E}_{\frac{1}{2}} = \begin{array}{c} \begin{array}{cc} -\frac{1}{2} & \frac{1}{2} \end{array} \\ \left(\begin{array}{cc} \lhd & \rhd \\ & \ominus \\ & \ominus \\ & \ominus \end{array}\right) \end{array}.$$

Proof. Part (i) follows directly from Theorem 4.3. Parts (ii) and (iii) follow from definition. $\square$

Two more cases remain in Case (A). The next one is $\pi_{temp} = T_{IV,7}(\pi_{sc})$.

Proposition 4.27. *Consider the representation*

$$\pi_x = L(\Delta_\rho[-x,-x];\pi_{temp}),$$

where $\pi_{temp} = T_{IV,7}(\pi_{sc})$ *for* $x > 0$ *and* $\alpha \in \mathbb{Z}_{>0}$, *as described in Proposition 3.26.*

(i) The representation π_x *is of Arthur type if and only if* $1 \leq x \leq \alpha$.
(ii) The representation is of critical type when $(x,\alpha) = (1,1)$.
(iii) Define

$$\begin{aligned}\mathcal{E}_x :=& \{([x,-x]_\rho, \lfloor x \rfloor, (-1)^x\eta), ([0,0]_\rho, 0, \eta)^6, ([x-2,0]_\rho, 0, -\eta),\\ & ([\alpha-1,x]_\rho, 0, (-1)^x\eta)\}.\end{aligned}$$

Then $\pi(\mathcal{E}_x) = \pi_x$. *Here is the associated symbol.*

$$\mathcal{E}_x = \begin{array}{c} \begin{array}{cccccccccc} -x & -(x-1) & \cdots & 0 & \cdots & x-2 & x-1 & x & \cdots & \alpha-1 \end{array} \\ \left(\begin{array}{cccccccccc} \lhd & \lhd & \cdots & \odot & \odot & \cdots & \rhd & \rhd & \rhd & \\ & & & \odot & & & & & & \\ & & & \odot & & & & & & \\ & & & \odot & & & & & & \\ & & & \odot & & & & & & \\ & & & \odot & & & & & & \\ & & & \odot & & & & & & \\ & & & \odot & \ldots & \odot & & & & \\ & & & & & & & \odot & \cdots & \odot \end{array}\right) \end{array}.$$

Proof. The proof is similar to Proposition 4.23, which we omit. $\square$

The final case remaining in case (A) is $\pi_{temp} = T_{V,6}^{\pm}(\pi_{sc})$.

Proposition 4.28. *Consider the representation*

$$\pi_x^{\pm} = L(\Delta_\rho[-x,-x];\pi_{temp}^{\pm}),$$

where $\pi_{temp}^{\pm} = T_{V,6}^{\pm}(\pi_{sc})$, *for* $x > 0$ *and* $\alpha = 0$, *as described in Proposition 3.27.*

(i) The representation $\pi_x^{\pm}$ is of Arthur type if and only if $x = 1$.
(ii) The representation $\pi_x^{\pm}$ is of critical type when $x = 1$.
(iii) We have $\pi(\mathcal{E}_{\pm}) = \pi_x^{\pm}$, where

$$\mathcal{E}_+ = \begin{array}{c} \begin{array}{ccc} -1 & 0 & 1 \end{array} \\ \left(\begin{array}{ccc} \lhd & \oplus & \rhd \\ & \oplus & \\ & \oplus & \\ & \oplus & \\ & \oplus & \\ & \oplus & \end{array}\right) \end{array}, \quad \mathcal{E}_- = \begin{array}{c} \begin{array}{ccc} -1 & 0 & 1 \end{array} \\ \left(\begin{array}{ccc} \lhd & \ominus & \rhd \\ & \ominus & \\ & \ominus & \\ & \ominus & \\ & \ominus & \\ & \ominus & \end{array}\right) \end{array}.$$

Proof. The proof is similar to Proposition 4.21, which we omit. □

This concludes our discussion of Case (A). Now let 's move onto Case (B).

4.2. **Case** $(B) : \pi = L(\Delta_\rho[-x,-x-1]; \pi_{temp})$**.** In this case, we are interested in the representation $\pi_{[-x,-x-1]} = L(\Delta_\rho[-x,-x-1]; \pi_{temp})$ where π_{temp} is tempered of corank 2. By Langlands classification we have the restriction $x > -\frac{1}{2}$.

Similar to what we've done during the classification of tempered representations, there are 10 total cases to consider. We start with the case $\pi_{temp} = T_{I,1}^{\alpha+1}(T_{I,1}^{\alpha}(\pi_{sc}))$.

Proposition 4.29. *Consider the representation*

$$\pi_{[-x,-x-1]} = L(\Delta_\rho[-x,-x-1]; \pi_{temp}),$$

for $x > -\frac{1}{2}$, where $\pi_{temp} = T_{I,1}^{\alpha+1}(T_{I,1}^{\alpha}(\pi_{sc}))$ for $\alpha > 0$.

(i) The representation $\pi_{[-x,-x-1]}$ is of Arthur type if and only if $\epsilon_\rho \leq x \leq \alpha - 2$ when $\alpha \geq 2$, or when $(x,\alpha) = (0,1)$.
(ii) The representation $\pi_{[-x,-x-1]}$ is of critical type if and only if $x \in \{\alpha-2, \alpha-1, \alpha, \alpha+1, \alpha+2\}$.
(iii) Define $\mathcal{E}_{[-x,-x-1]}$ in various cases as follows. When $\alpha \geq 2$, let

$$\begin{aligned} \mathcal{E}_{[-x,-x-1]} :=& \{([x+1,-x]_\rho, \lfloor x \rfloor, (-1)^x \eta), ([x-2,\epsilon_\rho]_\rho, 0, \eta), \\ & ([\alpha-2, x+1]_\rho, 0, (-1)^{x+1-\epsilon_\rho}\eta), ([\alpha+1,\alpha+1]_\rho, 0, (-1)^{\alpha-\epsilon_\rho}\eta)\}. \end{aligned}$$

Here is the associated symbol.

$$\mathcal{E}_{[-x,-x-1]} = \begin{array}{c} \begin{array}{cccccccccccccc} -x & -(x-1) & \cdots & \epsilon_\rho & \cdots & x-2 & x-1 & x & x+1 & \cdots & \alpha-2 & \alpha-1 & \alpha & \alpha+1 \end{array} \\ \left(\begin{array}{cccccccccccccc} \lhd & \lhd & \cdots & \odot & \cdots & \rhd & \rhd & \rhd & \rhd & & & & & \\ & & & \odot & \cdots & \odot & & & & & & & & \\ & & & & & & & & \odot & \cdots & \odot & & & \\ & & & & & & & & & & & & & \odot \end{array}\right) \end{array}.$$

When $(x,\alpha) = (0,1)$, let

$$\mathcal{E}_{[0,-1]} = \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \left(\begin{array}{ccc} \lhd & \odot & \rhd \\ & & \odot \end{array}\right) \end{array}.$$

Then we have $\pi(\mathcal{E}_{[-x,-x-1]}) = \pi_{[-x,-x-1]}(\pi_{temp})$.

Proof. The sufficient direction for part (i) can be proven in a similar way as Proposition 4.6, which we omit. For the necessary direction, we see that [HJLLZ25, Lemma 7.8] gives us the restriction that $\pi_{[-x,-x-1]}$ is of Arthur type only if

(1) $x = 0$

(2) $x = \frac{1}{2}$ and $|\Omega|(\pi_{temp})_\rho$ contains $\{\rho|\cdot|^{\frac{1}{2}}\}$.
(3) $x > \frac{1}{2}$ and $|\Omega|(\pi_{temp})_\rho$ contains $\{\rho|\cdot|^{x-1}, \rho|\cdot|^{x}\}$.

Since $|\Omega|(\pi_{temp})_\rho = \{\rho|\cdot|^y : \epsilon_\rho \leq y \leq \alpha - 2\} \cup \{\rho|\cdot|^{\alpha+1}\}$, we can easily check that the three cases combined gives us the criterion in (i). This proves part (i). Part (ii) and (iii) follow from definition. This proves the proposition. $\square$

Now we move onto the case $\pi_{temp} = T_{I,1}^{\alpha-1}(T_{I,1}^{\alpha}(\pi_{sc}))$.

Proposition 4.30. *Consider the representation*

$$\pi_{[-x,-x-1]} = L(\Delta_\rho[-x,-x-1]; \pi_{temp}),$$

for $x > -\frac{1}{2}$, where $\pi_{temp} = T_{I,1}^{\alpha-1}(T_{I,1}^{\alpha}(\pi_{sc}))$ and $\alpha > 1$.

(i) The representation $\pi_{[-x,-x-1]}$ is of Arthur type if and only if $\epsilon_\rho \leq x \leq \alpha - 3$ for $\alpha \geq 3$, or when $(x, \alpha) = (0, 2)$.

(ii) The representation $\pi_{[-x,-x-1]}$ is of critical type if and only if $x \in \{\alpha-2, \alpha-1, \alpha, \alpha+1\}$.

(iii) Define $\mathcal{E}_{[-x,-x-1]}$ in various cases as follows. When $\alpha \geq 3$, let

$$\begin{aligned}\mathcal{E}_{[-x,-x-1]} :=\{&([x+1,-x]_\rho, \lfloor x \rfloor, (-1)^x\eta), ([x-2,\epsilon_\rho]_\rho, 0, \eta),\\ &([\alpha-3, x+1]_\rho, 0, (-1)^{x+1-\epsilon_\rho}\eta), ([\alpha-1,\alpha-1]_\rho, 0, (-1)^{\alpha-\epsilon_\rho}\eta),\\ &([\alpha,\alpha]_\rho, 0, (-1)^{\alpha-1-\epsilon_\rho}\eta)\}.\end{aligned}$$

Here is the associated symbol.

$$\mathcal{E}_{[-x,-x-1]} = \begin{array}{c}\begin{array}{cccccccccccccc} -x & -(x-1) & \cdots & \epsilon_\rho & \cdots & x-2 & x-1 & x & x+1 & \cdots & \alpha-3 & \alpha-2 & \alpha-1 & \alpha \end{array}\\ \left(\begin{array}{cccccccccccccc} \lhd & \lhd & \cdots & \odot & \ldots & \rhd & \rhd & \rhd & \rhd & & & & & \\ & & & \odot & \cdots & \odot & & & & & & & & \\ & & & & & & & & \odot & \cdots & \odot & & & \\ & & & & & & & & & & & & \odot & \\ & & & & & & & & & & & & & \odot \end{array}\right)\end{array}.$$

When $(x, \alpha) = (0, 2)$, let

$$\mathcal{E}_{[0,-1]} = \begin{array}{c}\begin{array}{ccc} 0 & 1 & 2 \end{array}\\ \left(\begin{array}{ccc} \lhd & \odot & \rhd \\ & \odot & \\ & & \odot \end{array}\right)\end{array}.$$

Then we have $\pi(\mathcal{E}_{[-x,-x-1]}) = \pi_{[-x,-x-1]}(\pi_{temp})$.

Proof. The proof is similar to Proposition 4.29, except that we also have to eliminate the case where $x = \alpha$ for $\alpha \geq 3$. This can be done in the same way as the proof of Proposition 4.6, which we omit. $\square$

The next case is $\pi_{temp} = T_{IV,3}(T_{I,1}^{\alpha}(\pi_{sc}))$.

Proposition 4.31. *Consider the representation*

$$\pi_{[-x,-x-1]} = L(\Delta_\rho[-x,-x-1]; \pi_{temp}),$$

for $x > -\frac{1}{2}$, where $\pi_{temp} = T_{IV,3}(T_{I,1}^{\alpha}(\pi_{sc}))$ and $\alpha \in \mathbb{Z}_{>1}$.

(i) The representation $\pi_{[-x,-x-1]}$ is of Arthur type if and only if $0 \leq x \leq \alpha - 2$.

(ii) The representation $\pi_{[-x,-x-1]}$ is of critical type if and only if $(x, \alpha) = (0, 2)$ or $(1, 3)$.

(iii) Define

$$\mathcal{E}_{[-x,-x-1]} := \{([x+1,-x]_\rho, \lfloor x \rfloor, (-1)^x\eta), ([0,0]_\rho, 0, \eta)^2, ([x-2,0]_\rho, 0, \eta), \\ ([\alpha-2, x+1], 0, (-1)^{x+1-\epsilon_\rho}\eta), ([\alpha,\alpha]_\rho, 0, (-1)^{\alpha-1-\epsilon_\rho}\eta)\}.$$

Then $\pi(\mathcal{E}_{[-x,-x-1]}) = \pi_{[-x,-x-1]}(\pi_{temp})$. *Here is the associated symbol.*

$$\mathcal{E}_{[-x,-x-1]} = \begin{array}{c} \begin{array}{ccccccccccccc} -x & -(x-1) & \cdots & 0 & \cdots & x-2 & x-1 & x & x+1 & \cdots & \alpha-2 & \alpha-1 & \alpha \end{array} \\ \begin{pmatrix} \lhd & \lhd & \cdots & \odot & \cdots & \rhd & \rhd & \rhd & \rhd & & & & \\ & & & \odot & & & & & & & & & \\ & & & \odot & & & & & & & & & \\ & & & \odot & \cdots & \odot & & & & & & & \\ & & & & & & & & \odot & \cdots & \odot & & \\ & & & & & & & & & & & & \odot \end{pmatrix} \end{array}.$$

Proof. The proof is similar to Proposition 4.30, which we omit. □

Let's look at the next case, which is $\pi_{temp} = T^{\pm}_{V,2}(T^1_{I,1}(\pi_{sc}))$.

Proposition 4.32. *Consider the representation*

$$\pi^{\pm}_{[-x,-x-1]} = L(\Delta_\rho[-x,-x-1]; \pi^{\pm}_{temp}),$$

for $x > -\frac{1}{2}$, *where* $\pi^{\pm}_{temp} = T^{\pm}_{V,2}(T^1_{I,1}(\pi_{sc}))$ *and* $\alpha = 1$.

(i) The representation $\pi^{\pm}_{[-x,-x-1]}$ *is of Arthur type if and only if* $x = 0$, *or* $x = 1$ *and* $\epsilon_{sc}(\rho \otimes S_3) = \mp 1$.

(ii) The representation $\pi^{\pm}_{[-x,-x-1]}$ *of critical type if and only if* $x \in \{0, 1, 2\}$.

(iii) When $x = 0$, *we have* $\pi(\mathcal{E}^{\pm}_{[0,-1]}) = \pi^{\pm}_{[0,-1]}$, *where*

$$\mathcal{E}^{+}_{[0,-1]} = \begin{array}{c} \begin{array}{cc} 0 & 1 \end{array} \\ \begin{pmatrix} \oplus & \\ \oplus & \ominus \\ \ominus & \oplus \end{pmatrix} \end{array}, \quad \mathcal{E}^{-}_{[0,-1]} = \begin{array}{c} \begin{array}{cc} 0 & 1 \end{array} \\ \begin{pmatrix} \ominus & \\ \ominus & \oplus \\ \oplus & \ominus \end{pmatrix} \end{array}.$$

When $x = 1$ *and* $\epsilon_{sc}(\rho \otimes S_3) = \mp 1$, *we have* $\pi(\mathcal{E}^{\pm}_{[-1,-2]}) = \pi^{\pm}_{[-1,-2]}$, *where*

$$\mathcal{E}^{+}_{[-1,-2]} = \begin{array}{c} \begin{array}{cccc} -1 & 0 & 1 & 2 \end{array} \\ \begin{pmatrix} \lhd & \lhd & \rhd & \rhd \\ & \ominus & & \end{pmatrix} \end{array}, \quad \mathcal{E}^{-}_{[-1,-2]} = \begin{array}{c} \begin{array}{cccc} -1 & 0 & 1 & 2 \end{array} \\ \begin{pmatrix} \lhd & \lhd & \rhd & \rhd \\ & \oplus & & \end{pmatrix} \end{array}.$$

Proof. By Lemma 4.4, the representation $\pi_{[-x,-x-1]}$ can only be of Arthur type when $x = 0$ or 1. When $x = 0$, it's easy to verify that the representation is of Arthur type.
When $x = 1$, Definition 4.2 and Theorem 4.3 tells us that $\pi_{[-1,-2]}$ is of Arthur type if and only if the segment $[1,0]_\rho$ is contained in the extended multi-segment corresponding to π_{temp}. By [HJLLZ25, Proposition 10.3], this can only happen when you can apply the *ui* operator on the second and third row, and by [HLL22, Definition 3.23], we have the condition stated. □

The next case is $\pi_{temp} = T^1_{I,1}(T_{V,2}(\pi_{sc}))$.

Proposition 4.33. *Consider the representation*

$$\pi^{\pm}_{[-x,-x-1]} = L(\Delta_\rho[-x,-x-1]; \pi^{\pm}_{temp}),$$

for $x > -\frac{1}{2}$, *where* $\pi^{\pm}_{temp} = T^1_{I,1}(T^{\pm}_{V,2}(\pi_{sc}))$ *and* $\alpha = 0$.

(i) The representation $\pi^{\pm}_{[-x,-x-1]}$ *is of Arthur type if and only if* $x = 0$.

(ii) The representation $\pi^{\pm}_{[-x,-x-1]}$ *is of critical type if and only if* $x \in \{0, 1, 2\}$.

(iii) We have $\pi(\mathcal{E}^{\pm}_{[0,-1]}) = \pi^{\pm}_{[0,-1]}$*, where*

$$\mathcal{E}^{+}_{[0,-1]} = \begin{array}{c} \begin{array}{cc} 0 & 1 \end{array} \\ \begin{pmatrix} \oplus & \ominus \\ \ominus & \oplus \end{pmatrix} \end{array}, \quad \mathcal{E}^{-}_{[0,-1]} = \begin{array}{c} \begin{array}{cc} 0 & 1 \end{array} \\ \begin{pmatrix} \ominus & \oplus \\ \oplus & \ominus \end{pmatrix} \end{array}.$$

Proof. By [HJLLZ25, Proposition 10.5], the set $\Psi(\pi^{\pm}_{temp})$ is a singleton, and it does not contain the segment $([1,0]_\rho,*,*)$, so when $x=1$, the representation $\pi_{[-x,-x-1]}$ is not of Arthur type. The rest of the proof is similar to Proposition 4.9, which we omit. □

Now we move onto the case $\pi_{temp} = T^{\frac{1}{2}}_{I,2}(\pi_{sc})$.

Proposition 4.34. *Consider the representation*

$$\pi_{[-x,-x-1]} = L(\Delta_\rho[-x,-x-1];\pi_{temp}),$$

for $x > -\frac{1}{2}$*, where* $\pi_{temp} = T^{\frac{1}{2}}_{I,2}(\pi_{sc})$ *and* $\alpha = \frac{1}{2}$*.*

(i) The representation $\pi_{[-x,-x-1]}$ *is of Arthur type if and only if* $x = \frac{1}{2}$*.*
(ii) The representation $\pi_{[-x,-x-1]}$ *is of critical type if and only if* $x \in \{\frac{1}{2}, \frac{3}{2}\}$*.*
(iii) We have that $\pi(\mathcal{E}_{[-\frac{1}{2},-\frac{3}{2}]}) = \pi_{[-\frac{1}{2},-\frac{3}{2}]}$*, where*

$$\mathcal{E}_{[-\frac{1}{2},-\frac{3}{2}]} = \begin{array}{c} \begin{array}{ccc} -\frac{1}{2} & \frac{1}{2} & \frac{3}{2} \end{array} \\ \begin{pmatrix} \lhd & \oplus & \rhd \\ & \oplus & \end{pmatrix} \end{array}.$$

Proof. The proof is similar to Proposition 4.33, which we omit. □

The next case to consider is $\pi_{temp} = T^{\frac{1}{2}}_{II,3}(\pi_{sc})$.

Proposition 4.35. *Consider the representation*

$$\pi_{[-x,-x-1]} = L(\Delta_\rho[-x,-x-1];\pi_{temp}),$$

for $x > -\frac{1}{2}$*, where* $\pi_{temp} = T^{\frac{1}{2}}_{II,3}(\pi_{sc})$ *and* $\alpha \in \frac{1}{2} + \mathbb{Z}_{>0}$*.*

(i) The representation $\pi_{[-x,-x-1]}$ *is of Arthur type if and only if* $\frac{1}{2} \le x \le \alpha - 1$*.*
(ii) The representation $\pi_{[-x,-x-1]}$ *is of critical type if and only if* $(x,\alpha) = (\frac{1}{2},\frac{3}{2}), (\frac{3}{2},\frac{3}{2}), (\frac{3}{2},\frac{5}{2})$*.*
(iii) Define

$$\begin{aligned}\mathcal{E}_{[-x,-x-1]} :=\{&([x+1,-x]_\rho, \lfloor x \rfloor, (-1)^{x+\frac{1}{2}}), ([\frac{1}{2},\frac{1}{2}]_\rho, 0, -1)^2, ([x-2,\frac{1}{2}]_\rho, 0, -1),\\ &([\alpha-1, x+1]_\rho, 0, (-1)^{x-\frac{1}{2}})\}.\end{aligned}$$

Then $\pi(\mathcal{E}_{[-x,-x-1]}) = \pi_{[-x,-x-1]}$*. Here is the associated symbol.*

$$\mathcal{E}_{[-x,-x-1]} = \begin{array}{c} \begin{array}{ccccccccccc} -x & -(x-1) & \dots & \frac{1}{2} & \cdots & x-2 & x-1 & x & x+1 & \cdots & \alpha-1 \end{array} \\ \begin{pmatrix} \lhd & \lhd & \cdots & \odot & \cdots & \rhd & \rhd & \rhd & \rhd & & \\ & & & \odot & & & & & & & \\ & & & \odot & & & & & & & \\ & & & \odot & \cdots & \odot & & & & & \\ & & & & & & & & \odot & \cdots & \odot \end{pmatrix} \end{array}.$$

Proof. The proof is similar to Proposition 4.31, which we omit. □

Three cases remain in Case (B). The next one is $\pi_{temp} = T^{\frac{1}{2}}_{III,2}(\pi_{sc})$.

Proposition 4.36. *Consider the representation*

$$\pi_{[-x,-x-1]} = L(\Delta_\rho[-x,-x-1];\pi_{temp}),$$

for $x > -\frac{1}{2}$, *where* $\pi_{temp} = T^{\frac{1}{2}}_{III,2}(\pi_{sc})$ *and* $\alpha = \frac{1}{2}$.

(i) *The representation* $\pi_{[-x,-x-1]}$ *is of Arthur type if and only if* $x = \frac{1}{2}$.
(ii) *The representation* $\pi_{[-x,-x-1]}$ *is of critical type if and only if* $x \in \{\frac{1}{2}, \frac{3}{2}\}$.
(iii) *We have* $\pi(\mathcal{E}_{[-\frac{1}{2},-\frac{3}{2}]}) = \pi_{[-\frac{1}{2},-\frac{3}{2}]}$, *where*

$$\mathcal{E}_{[-\frac{1}{2},-\frac{3}{2}]} = \begin{array}{c} \begin{array}{ccc} -\frac{1}{2} & \frac{1}{2} & \frac{3}{2} \end{array} \\ \begin{pmatrix} \lhd & \ominus & \rhd \\ & \ominus & \end{pmatrix} \end{array}.$$

Proof. The proof is similar to Proposition 4.30, which we omit. □

The next remaining case is $\pi_{temp} = T_{IV,5}(\pi_{sc})$.

Proposition 4.37. *Consider the representation*

$$\pi_{[-x,-x-1]} = L(\Delta_\rho[-x,-x-1];\pi_{temp}),$$

for $x > -\frac{1}{2}$, *where* $\pi_{temp} = T_{IV,5}(\pi_{sc})$ *and* $\alpha \in \mathbb{Z}_{>0}$.

(i) *The representation* $\pi_{[-x,-x-1]}$ *is of Arthur type if and only if* $0 \leq x \leq \alpha - 1$.
(ii) *The representation* $\pi_{[-x,-x-1]}$ *is of critical type if and only if* $(x,\alpha) = (0,1), (1,1)$, *or* $(1,2)$.
(iii) *Define*

$$\begin{aligned}\mathcal{E}_{[-x,-x-1]} :=\{&([x+1,-x]_\rho, \lfloor x \rfloor, (-1)^x\eta), ([0,0]_\rho, 0, \eta)^4, ([x-2,0]_\rho, 0, \eta),\\ &([\alpha-1, x+1]_\rho, 0, (-1)^{x+1}\eta)\}.\end{aligned}$$

Then $\pi(\mathcal{E}_{[-x,-x-1]}) = \pi_{[-x,-x-1]}$. *Here is the associated symbol.*

$$\mathcal{E}_{[-x,-x-1]} = \begin{array}{c} \begin{array}{cccccccccccc} -x & -(x-1) & \cdots & 0 & \cdots & x-2 & x-1 & x & x+1 & \cdots & \alpha-1 \end{array} \\ \begin{pmatrix} \lhd & \lhd & \cdots & \odot & \cdots & \rhd & \rhd & \rhd & \rhd & & \\ & & & \odot & & & & & & & \\ & & & \odot & & & & & & & \\ & & & \odot & & & & & & & \\ & & & \odot & & & & & & & \\ & & & \odot & \cdots & \odot & & & & & \\ & & & & & & & & \odot & \cdots & \odot \end{pmatrix} \end{array}.$$

Proof. The proof is similar to Proposition 4.29, which we omit. □

The final case to consider in Case (B) is $\pi_{temp} = T^{\pm}_{V,4}(\pi_{sc})$.

Proposition 4.38. *Consider the representation*

$$\pi^{\pm}_{[-x,-x-1]} = L(\Delta_\rho[-x,-x-1];\pi^{\pm}_{temp}),$$

for $x > -\frac{1}{2}$, *where* $\pi^{\pm}_{temp} = T^{\pm}_{V,4}(\pi_{sc})$ *and* $\alpha = 0$.

(i) *The representation* $\pi^{\pm}_{[-x,-x-1]}$ *is of Arthur type if and only if* $x = 0$.
(ii) *The representation* $\pi^{\pm}_{[-x,-x-1]}$ *is of critical type when* $x \in \{0, 1\}$.

(iii) We have $\pi(\mathcal{E}^{\pm}_{[0,-1]}) = \pi^{\pm}_{[0,-1]}$*, where*

$$\mathcal{E}^{+}_{[0,-1]} = \begin{array}{c} \begin{matrix} 0 & 1 \end{matrix} \\ \begin{pmatrix} \lhd & \rhd \\ \oplus & \\ \oplus & \\ \oplus & \\ \oplus & \end{pmatrix} \end{array}, \quad \mathcal{E}^{-}_{[0,-1]} = \begin{array}{c} \begin{matrix} 0 & 1 \end{matrix} \\ \begin{pmatrix} \lhd & \rhd \\ \ominus & \\ \ominus & \\ \ominus & \\ \ominus & \end{pmatrix} \end{array}.$$

Proof. The proof is similar to 4.33, which we omit. $\square$

This concludes our discussion of Case (B). Now we move onto Case (C).

4.3. **Case** (C)**:** $\pi = L(\Delta_\rho[-x,-x-2];\pi_{temp})$**.** In this section we will consider representations of the form $\pi_{[-x,-x-2]} = L(\Delta_\rho[-x,-x-2](\pi_{temp}))$, where π_{temp} is tempered of corank 1. By the classification in [HJLLZ25, Section 8], there are three total cases to consider. By Langlands classification we have the natural restriction $\pi > -1$.

The first case is $\pi_{temp} = T^{\alpha}_{I,1}(\pi_{sc})$.

Proposition 4.39. *Consider the representation*

$$\pi_{[-x,-x-2]} = L(\Delta_\rho[-x,-x-2];\pi_{temp}),$$

for $x > -1$*, where* $\pi_{temp} = T^{\alpha}_{I,1}(\pi_{sc})$ *and* $\alpha > 0$*.*

(i) The representation $\pi_{[-x,-x-2]}$ *is of Arthur type if and only if one of the following cases hold:*
  (a) $x = -\frac{1}{2}$*,*
  (b) $x = 0$ *and* $\alpha \in \mathbb{Z}_{>0} \setminus \{2\}$*,*
  (c) $x = \frac{1}{2}$ *and* $\alpha \in \frac{1}{2} + \mathbb{Z}_{>2}$*,*
  (d) $\epsilon_\rho + 1 \leq x \leq \alpha - 3$ *for* $\alpha \geq 1$*.*

(ii) The representation $\pi_{[-x,-x-2]}$ *is of critical type if and only if* $x \in \{\alpha-3, \alpha-2, \alpha-1, \alpha, \alpha+1\}$*.*

(iii) Define $\mathcal{E}_{[} - x, -x-2]$ *in various cases as follows. For* $x = -\frac{1}{2}$*, let*

$$\mathcal{E}_{[\frac{1}{2},-\frac{3}{2}]} = \begin{array}{c} \begin{matrix} \frac{1}{2} & \frac{3}{2} & \dots & \alpha-2 & \alpha-1 & \alpha \end{matrix} \\ \begin{pmatrix} \odot & & & & & \\ \lhd & \rhd & & & & \\ & \odot & \dots & \odot & & \\ & & & & & \odot \end{pmatrix} \end{array}.$$

For $x = 0$*, let*

$$\mathcal{E}_{[0,-2]} = \begin{array}{c} \begin{matrix} 0 & 1 & 2 & \dots & \alpha-2 & \alpha-1 & \alpha \end{matrix} \\ \begin{pmatrix} \odot & & & & & & \\ \lhd & \odot & \rhd & & & & \\ & & \odot & \dots & \odot & & \\ & & & & & & \odot \end{pmatrix} \end{array}.$$

For $x = \frac{1}{2}$*, let*

$$\mathcal{E}_{[-\frac{1}{2},-\frac{5}{2}]} = \begin{array}{c} \begin{matrix} -\frac{1}{2} & \frac{1}{2} & \frac{3}{2} & \frac{5}{2} & \cdots & \alpha-2 & \alpha-1 & \alpha \end{matrix} \\ \begin{pmatrix} \lhd & \odot & \odot & \rhd & & & & \\ & & & \odot & \cdots & \odot & & \\ & & & & & & & \odot \end{pmatrix} \end{array}.$$

For $\epsilon_\rho + 1 \leq x = \alpha - 3$ *and* $\alpha \geq 1$*, define*

$$\mathcal{E}_{[-x,-x-2]} = \{([x+2,-x]_\rho, 0, (-1)^{x-\epsilon_\rho}\eta), ([\epsilon_\rho,\epsilon_\rho]_\rho, 0, \eta),$$

$$([\alpha-2,x+2]_\rho,0,(-1)^{x-\epsilon_\rho}\eta),([\alpha,\alpha]_\rho,0,(-1)^{\alpha-\epsilon_\rho}\eta)\}.$$

Then we have $\pi(\mathcal{E}_{[-x,-x-2]}):=\pi_{[-x,-x-2]}$. *Here is the associated symbol.*

$$\mathcal{E}_{[-x,-x-2]}=\begin{array}{c} \begin{array}{ccccccccc} -x & \cdots & \epsilon_\rho & \cdots & x+2 & \ldots & \alpha-2 & \alpha-1 & \alpha \end{array} \\ \begin{pmatrix} \lhd & \cdots & \odot & \cdots & \rhd & & & & \\ & & \odot & & & & & & \\ & & & & \odot & \ldots & \odot & & \\ & & & & & & & & \odot \end{pmatrix} \end{array}.$$

Proof. Part (iii) and the sufficient direction of part (i) can be proven in a similar way as Proposition 4.29, which we omit. For the necessary condition, we have the following restriction from Lemma 4.4: π is of Arthur type only if $x=-\frac{1}{2}$ or the set $\{\rho|\cdot|^z\}_{z=|-x+1|}^{x+1}$ lies in $|\Omega|(\pi_{temp})$. For $\pi_{temp}=T_{I,1}^{\alpha}(\pi_{sc})$ we have that $|\Omega|(\pi_{temp})=\{\rho|\cdot|^y\}_{y=\epsilon_\rho}^{\alpha-2}\cup\{\rho|\cdot|^\alpha\}$. Therefore we reduce to the four cases listed in the proposition. Part (ii) follows from definition. □

The second case in Case (C) is $\pi_{temp}=T_{IV,3}(\pi_{sc})$.

Proposition 4.40. *Consider the representation*

$$\pi_{[-x,-x-2]}=L(\Delta_\rho[-x,-x-2];\pi_{temp}),$$

for $x>-1$, *where* $\pi_{temp}=T_{IV,3}(\pi_{sc})$ *and* $\alpha\in\mathbb{Z}_{>0}$.

(i) The representation $\pi_{[-x,-x-2]}$ *is of Arthur type if and only if one of the following cases hold:*
  (a) $x=0$ *and* $\alpha\in\mathbb{Z}_{>1}$,
  (b) $1\le x\le\alpha-2$ *for* $\alpha\ge 3$.

(ii) The representation $\pi_{[-x,-x-2]}$ *is of critical type if and only if*

$$(x,\alpha)=(0,1),(0,2),(1,1),(1,2),(1,3).$$

(iii) Define $\mathcal{E}_{[-x,-x-2]}$ *in various cases as follows. For* $x=0$ *and* $\alpha\in\mathbb{Z}_{>1}$, *let*

$$\mathcal{E}_{[0,-2]}=\begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & \cdots & \alpha-1 \end{array} \\ \begin{pmatrix} \odot & & & & \\ \odot & & & & \\ \odot & & & & \\ \lhd & \odot & \rhd & & \\ & & \odot & \cdots & \odot \end{pmatrix} \end{array}.$$

For $1\le x\le\alpha-2$ *and* $\alpha\ge 3$, *define*

$$\begin{aligned}\mathcal{E}_{[-x,-x-2]}:=&\{([x+2,-x]_\rho,0,(-1)^x\eta),([0,0]_\rho,0,\eta)^2,\\ &([\alpha-1,x+2]_\rho,0,(-1)^x\eta)\}.\end{aligned}$$

Then we have $\pi(\mathcal{E}_{[-x,-x-2]})=\pi_{[-x,-x-2]}$. *Here is the associated symbol.*

$$\mathcal{E}_{[-x,-x-2]}=\begin{array}{c} \begin{array}{ccccccc} -x & \cdots & 0 & \ldots & x+2 & \cdots & \alpha-1 \end{array} \\ \begin{pmatrix} \lhd & \cdots & \odot & \cdots & \rhd & & \\ & & \odot & & & & \\ & & \odot & & & & \\ & & & & \odot & \cdots & \odot \end{pmatrix} \end{array}.$$

Proof. The proof is similar to Proposition 4.39, which we omit. □

The final case in Case (C) is $\pi_{temp}=T_{V,2}^{\pm}(\pi_{sc})$.

Proposition 4.41. *Consider the representation*

$$\pi^{\pm}_{[-x,-x-2]} = L(\Delta_\rho[-x,-x-2];\pi^{\pm}_{temp}),$$

where $\pi^{\pm}_{temp} = T^{\pm}_{V,2}(\pi_{sc})$ *for* $x > -1$ *and* $\alpha = 0$.

(i) The representation $\pi^{\pm}_{[-x,-x-2]}$ *is not of Arthur type for any* x.
(ii) The representation $\pi^{\pm}_{[-x,-x-2]}$ *is of critical type if and only if* $x \in \{0,1\}$.

Proof. This follows from Lemma 4.4 and the fact that $|\Omega|(\pi^{\pm}_{temp})$ is the singleton $\{\rho\}$. $\square$

This concludes our discussion of Case (C). Let us now move onto the final case when $f(\pi) = 1$.

4.4. **Case** $(D) : \pi = L(\Delta_\rho[-x,-x-3],\pi_{sc})$**.** In this section, we are interested in representations of the form $\pi_{[-x,-x-3]} = L(\Delta_\rho[-x,-x-3],\pi_{sc})$. The natural restriction from Langlands classification in this case gives us $x > -\frac{3}{2}$.

Since we are working with supercuspidal representations, we will introduce the following result to simplify our computations.

Proposition 4.42. [HJLLZ25, Proposition 8.4] *The representation (of good parity)*

$$\pi = L(\Delta_\rho[x,-y];\pi_{sc})$$

is of Arthur type if and only if $x - y = -1$ *or* $y \le \alpha$. *Note that* x *is not arbitrary since we require* $x - y < 0$ *for the Langlands classification.*

From this, we can easily derive the following result.

Proposition 4.43. *Consider the representation*

$$\pi_{[-x,-x-3]} = L(\Delta_\rho[-x,-x-3];\pi_{sc}),$$

where $x > -\frac{3}{2}$.

(1) The representation $\pi_{[-x,-x-3]}$ *is of Arthur type if and only if* $x = -1$ *or* $-\frac{1}{2} \le x \le \alpha - 3$.
(2) The representation $\pi_{[-x,-x-3]}$ *is of critical type if and only if* $x \in \{\alpha-3,\alpha-2,\alpha-1,\alpha\}$.
(3) Define $\mathcal{E}_{sc}$ *and* $\mathcal{E}_{[-x,-x-3]}$ *in various cases as follows. Then* $\pi(\mathcal{E}_{sc}) = \pi_{sc}$ *and* $\pi(\mathcal{E}_{[-x,-x-3]}) = \pi_{[-x,-x-3]}$. *When* $x = -1$, *let*

$$\mathcal{E}_{sc} = \begin{array}{c} \begin{matrix} 0 & 1 & 2 & \dots & \alpha-1 \end{matrix} \\ \begin{pmatrix} \odot & \odot & & & \\ & & \odot & \dots & \odot \end{pmatrix} \end{array},$$

$$\mathcal{E}_{[-x,-x-3]} = \begin{array}{c} \begin{matrix} -1 & 0 & 1 & 2 & \cdots & \alpha-1 \end{matrix} \\ \begin{pmatrix} \lhd & \odot & \odot & \rhd & & \\ & & & \odot & & \\ & & & & \ddots & \\ & & & & & \odot \end{pmatrix} \end{array}.$$

When $x = -\frac{1}{2}$, *let*

$$\mathcal{E}_{sc} = \begin{array}{c} \begin{matrix} \frac{1}{2} & \dots & \alpha-1 \end{matrix} \\ \begin{pmatrix} \odot & \dots & \odot \end{pmatrix} \end{array},$$

$$\mathcal{E}_{[-x,-x-3]} = \begin{array}{c} \begin{array}{cccc} \frac{1}{2} & \frac{3}{2} & \cdots & \alpha-1 \end{array} \\ \begin{pmatrix} \odot & & & \\ \lhd & \rhd & & \\ & \odot & & \\ & & \ddots & \\ & & & \odot \end{pmatrix} \end{array}.$$

When $0 \leq x \leq \alpha - 3$, *define:*

$$\begin{aligned} \mathcal{E}_{sc} :=& \{([x+2,-x+1]_\rho, \lfloor x-1 \rfloor, (-1)^{x-\epsilon_\rho}\eta), ([x-4,\epsilon_\rho]_\rho, 0, \eta), \\ & ([\alpha-1, x+1]_\rho, 0, (-1)^{x+1-\epsilon_\rho}\eta)\}, \\ \mathcal{E}_{[-x,-x-3]} :=& \{([x+3,-x]_\rho, \lfloor x \rfloor, (-1)^{x-\epsilon_\rho}\eta), ([x-4,\epsilon_\rho]_\rho, 0, \eta), \\ & ([\alpha-1, x+1]_\rho, 0, (-1)^{x+1-\epsilon_\rho}\eta)\}. \end{aligned}$$

Here are the associated symbols.

$$\mathcal{E}_{sc} = \begin{array}{c} \begin{array}{ccccccccccccc} -x+1 & \cdots & \epsilon_\rho & \cdots & x-4 & x-3 & x-2 & x-1 & x & x+1 & x+2 & \cdots & \alpha-1 \end{array} \\ \begin{pmatrix} \lhd & \cdots & \odot & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \rhd & & \\ & & \odot & \cdots & \odot & & & & & & & & \\ & & & & & & & & & \odot & \odot & \cdots & \odot \end{pmatrix} \end{array},$$

$$\mathcal{E}_{[-x,-x-3]} = \begin{array}{c} \begin{array}{cccccccccccccc} -x & \cdots & \epsilon_\rho & \cdots & x-4 & x-3 & x-2 & x-1 & x & x+1 & x+2 & x+3 & \cdots & \alpha-1 \end{array} \\ \begin{pmatrix} \lhd & \cdots & \odot & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \rhd & & \\ & & \odot & \cdots & \odot & & & & & & & & & \\ & & & & & & & & & \odot & \odot & \odot & \cdots & \odot \end{pmatrix} \end{array}.$$

Proof. From Proposition 4.42, we see directly that $\pi_{[-x,-x-3]}$ is of Arthur type if and only if $x = -1$ or $x + 3 \leq \alpha$. This proves part (i). Part (ii) follows from definition and part (iii) can be proven using the similar kinds of calculation as Proposition 4.6 for supercuspidal representations. We omit the details. □

This concludes our discussion for the case $f(\pi) = 1$.

5. Classification of non-tempered representations of corank 4 ($f(\pi) = 2$)

In this section we will look at non-tempered representations with two segments in their L-data, which are of corank 4. Under the restriction imposed by Langlands classification, here are the different cases we need to consider:

(A) $\pi = L(\Delta_\rho[-x_1,-x_1], \Delta_\rho[-x_2,-x_2]; \pi_{temp})$, where $x_1 > x_2 \geq \frac{1}{2}$ and π_{temp} is tempered of corank 2.

(B) $\pi = L(\Delta_\rho[-x,-x], \Delta_\rho[-x,-x]; \pi_{temp})$, where $x \geq \frac{1}{2}$ and π_{temp} is tempered of corank 2.

(C) $\pi = L(\Delta_\rho[-x_1,-x_1-1], \Delta_\rho[-x_2,-x_2]; \pi_{temp})$, where $x_1 > -\frac{1}{2}$, $x_2 > 0$,$x_2 - x_1 \leq \frac{1}{2}$ and π_{temp} is tempered of corank 1.

(D) $\pi = L(\Delta_\rho[-x_1,-x_1], \Delta_\rho[-x_2,-x_2-1]; \pi_{temp})$, where $x_1 \geq \frac{1}{2}, x_2 > -\frac{1}{2}$, $x_1 - x_2 \geq \frac{1}{2}$ and π_{temp} is tempered of corank 1.

(E) $\pi = L(\Delta_\rho[-x_1,-x_1-1], \Delta_\rho[-x_2,-x_2-1]; \pi_{sc})$, where $x_1, x_2 > -\frac{1}{2}$ and $x_1 \geq x_2$.

(F) $\pi = L(\Delta_\rho[-x_1,-x_1-2], \Delta_\rho[-x_2,-x_2]; \pi_{sc})$, where $x_1 > -1$, $x_2 \geq \frac{1}{2}$, and $x_2 - x_1 \leq 1$.

(G) $\pi = L(\Delta_\rho[-x_1,-x_1], \Delta_\rho[-x_2,-x_2-2]; \pi_{sc})$, where $x_1 \geq \frac{1}{2}$, $x_2 > -1$ and $x_1 - x_2 \geq 1$.

5.1. **Case**(A) : $\pi = L(\Delta_\rho[-x_1,-x_1],\Delta_\rho[-x_2,-x_2];\pi_{temp})$**.** We begin with $Case(A)$, where $\pi = L(\Delta_\rho[-x_1,-x_1],\Delta_\rho[-x_2,-x_2];\pi_{temp})$ for π_{temp} tempered of corank 2. We can assume $x_1 > x_2$ in this case, and the natural restriction imposed by Langlands classification tells us $x_1, x_2 \geq \frac{1}{2}$. The case of $x_1 < x_2$ will be equivalent, since it is well-known that the representations $\Delta_\rho[x_1,y_1]$ and $\Delta_\rho[x_2,y_2]$ commute when the segments $[x_1,y_1]$ and $[x_2,y_2]$ are not linked ([HLL22, Lemma 2.1]).

Therefore, similar to before, there are a total of 10 subcases to consider. Let us start with $\pi_{temp} = T_{I,1}^{\alpha+1}(T_{I,1}^{\alpha}(\pi_{sc}))$.

Proposition 5.1. *Consider the representation*

$$\pi_{x_1,x_2} = L(\Delta_\rho[-x_1,-x_1],\Delta_\rho[-x_2,-x_2];\pi_{temp}),$$

where $x_1 > x_2 \geq \frac{1}{2}$ *and* $\pi_{temp} = T_{I,1}^{\alpha+1}(T_{I,1}^{\alpha}(\pi_{sc}))$ *for* $\alpha > 0$.

(i) *The representation* π_{x_1,x_2} *is of Arthur type if and only if* $\frac{1}{2} \leq x_2 < x_1 \leq \alpha - 1$ *or* $(x_1,x_2) = (\alpha,\alpha-1)$ *when* $\alpha \geq \frac{3}{2}$, *or when* $(x_1,x_2,\alpha) = (\frac{3}{2},\frac{1}{2},\frac{1}{2})$.

(ii) *The representation* π_{x_1,x_2} *if of critical type in the following cases:*
- $(x_1,x_2) = (\alpha-1,\alpha-2)$ *for* $\alpha \geq \frac{3}{2}$,
- $(x_1,x_2) = (\alpha,\alpha-1)$ *when* $\alpha \geq \frac{3}{2}$,
- $(x_1,x_2) = (\alpha+1,\alpha)$, $(\alpha+2,\alpha+1)$, $(\alpha+3,\alpha+2)$, $(\alpha+1,\alpha-1)$, $(\alpha+2,\alpha)$, $(\alpha+2,\alpha-1)$ *for* $\alpha \geq \frac{1}{2}$.

(iii) *Define* $\mathcal{E}_{x_1,x_2}$ *in various cases as follows. Then* $\pi(\mathcal{E}_{x_1,x_2}) = \pi_{x_1,x_2}$. *When* $\alpha \geq \frac{3}{2}$, *and* $\frac{1}{2} \leq x_2 < x_1 \leq \alpha - 1$, *define*

$$\begin{aligned}\mathcal{E}_{x_1,x_2} :=& \{([x_1,-x_1]_\rho,\lfloor x_1 \rfloor,(-1)^{x_1+1-\epsilon_\rho}\eta),([x_2-2,\epsilon_\rho]_\rho,0,-\eta),\\ &([\alpha-2,x_2]_\rho,0,(-1)^{x_2+1-\epsilon_\rho}\eta),([\alpha+1,\alpha+1]_\rho,0,(-1)^{\alpha-\epsilon_\rho}\eta)\}.\end{aligned}$$

Here is the associated symbol.

$$\mathcal{E}_{x_1,x_2} = \begin{array}{c} \begin{array}{cccccccccccccc} -x_1 & \cdots & \epsilon_\rho & \cdots & x_2-2 & x_2-1 & x_2 & \cdots & x_1 & \cdots & \alpha-2 & \alpha-1 & \alpha & \alpha+1 \end{array} \\ \begin{pmatrix} \lhd & \cdots & \odot & \cdots & \cdots & \cdots & \cdots & \cdots & \rhd & & & & & \\ & & \odot & \cdots & \odot & & & & & & & & & \\ & & & & & & \odot & \cdots & \odot & \cdots & \odot & & & \\ & & & & & & & & & & & & & \odot \end{pmatrix} \end{array}.$$

When $\alpha \geq \frac{3}{2}$ *and* $(x_1,x_2) = (\alpha,\alpha-1)$, *define*

$$\begin{aligned}\mathcal{E}_{x_1,x_2} :=& \{([x_1,-x_1]_\rho,\lfloor x_1 \rfloor,(-1)^{x_1+1-\epsilon_\rho}\eta),([\alpha-3,\epsilon_\rho]_\rho,0,-\eta),\\ &([\alpha+1,\alpha+1]_\rho,0,(-1)^{\alpha-\epsilon_\rho}\eta)\}.\end{aligned}$$

Here is the associated symbol.

$$\mathcal{E}_{x_1,x_2} = \begin{array}{c} \begin{array}{ccccccccc} -\alpha & \cdots & \epsilon_\rho & \cdots & \alpha-3 & \alpha-2 & \alpha-1 & \alpha & \alpha+1 \end{array} \\ \begin{pmatrix} \lhd & \cdots & \odot & \cdots & \cdots & \cdots & \cdots & \rhd & \\ & & \odot & \cdots & \odot & & & & \\ & & & & & & & & \odot \end{pmatrix} \end{array}.$$

Finally, when $(x_1,x_2,\alpha) = (\frac{3}{2},\frac{1}{2},\frac{1}{2})$, *let*

$$\mathcal{E}_{\frac{3}{2},\frac{1}{2}} = \begin{array}{c} \begin{array}{cccc} -\frac{3}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{3}{2} \end{array} \\ \begin{pmatrix} \lhd & \lhd & \rhd & \rhd \\ & & & \odot \end{pmatrix} \end{array}.$$

Proof. Let us first show the sufficient direction for (i). The case $(x_1, x_2, \alpha) = (\frac{3}{2}, \frac{1}{2}, \frac{1}{2})$ is easy to verify. Now consider the case when $\alpha \geq \frac{3}{2}$ and $x_2 + 1 = x_1$, $x_2 \leq \alpha - 1$. Here we can construct an extended multi-segment $\mathcal{E}_{x_2}$ of the form:

$$\mathcal{E}_{x_2} = \begin{array}{c} \begin{array}{cccccccccccc} -x_2 & \cdots & \epsilon_\rho & \cdots & x_2-2 & x_2-1 & x_2 & \ldots & \alpha-2 & \alpha-1 & \alpha & \alpha+1 \end{array} \\ \left(\begin{array}{cccccccccccc} \lhd & \cdots & \odot & \cdots & \ldots & \cdots & \rhd & & & & & \\ & & \odot & \cdots & \odot & & & & & & & \\ & & & & & & \odot & \cdots & \odot & & & \\ & & & & & & & & & & & \odot \end{array} \right) \end{array}$$

such that $\mathcal{E}_{x_2}$ lies in $\mathscr{E}(\pi^{\rho,-}_{x_1,x_2}; \Delta_\rho[-x_1, -x_1], 1)$, and thus by Theorem 4.3, π_{x_1,x_2} is of Arthur type with $\pi_{x_1,x_2} = \pi(\mathcal{E}^{\rho,+}_{x_2})$. Now when $x_2 + 1 < x_1 \leq \alpha - 1$, we see from the discussion in [HJLLZ25, Section 9], that there exists a multi-segment $\mathcal{E}'$ containing $([\alpha-1, \alpha-1]_\rho, 0, *)$ in its bottom row such that

$$\pi(\mathcal{E}') = L(\Delta_\rho[-x_1, -x_1], \Delta_\rho[-x_2, -x_2]; \pi_{sc})$$

Define a new multi-segment $\mathcal{E}$ which is identical to $\mathcal{E}'$ except that the bottom row is now replaced with $([\alpha+1, \alpha+1]_\rho, 0, *)$. It follows that $\pi(\mathcal{E}) = \pi_{x_1,x_2}$, and thus π_{x_1,x_2} is of Arthur type. This concludes the proof of the sufficient direction.

Now we show the necessary direction. By definition we have

$$\pi^{\rho,-}_{x_1,x_2} = L(\Delta_\rho[-x_2, -x_2]; \pi_{temp})$$

Then Theorem 4.3 suggests that π_{x_1,x_2} is of Arthur type only if $\pi^{\rho,-}_{x_1,x_2}$ is also of Arthur type. By [HJLLZ25, Proposition 11.1], this happens when $\frac{1}{2} \leq x_2 \leq \alpha - 1$ when $\alpha \geq \frac{3}{2}$, or when $x_2 = \frac{1}{2} = \alpha$.

Furthermore, we have

$$|\Omega|(\pi^{\rho,-}_{x_1,x_2}) \subseteq \{\rho|\cdot|^y : \epsilon_\rho \leq y \leq \alpha - 2\} \cup \{\rho|\cdot|^{\alpha+1}\} \cup \{\rho|\cdot|^{x_2}, \rho|\cdot|^{-x_2}\}$$

Lemma 4.4 gives the constriction that π_{x_1,x_2} is of Arthur type only if $x_1 = \frac{1}{2}$ or $\rho|\cdot|^{x_1-1} \in |\Omega|(\pi^{\rho,-})$. Since $x_1 > \frac{1}{2}$ by definition, we may conclude that $\pi^{\rho,-}$ is of Arthur type only if one of the following holds:

- $x_2 + 1 \leq x_1 \leq \alpha - 1$, or
- $x_1 = \alpha + 2$, or
- $x_2 = \alpha - 1$, $x_1 = \alpha$

It suffices to show that $\pi^{\rho,-}$ is not of Arthur type when $x_1 = \alpha + 2$. We resolve this case in the same way as we did before. When $x_1 = \alpha + 2$, by Lemma 4.5, there does not exist any extended multi-segment $\mathcal{E}$ containing the segment $([\alpha+1, -(\alpha+1)]_\rho, *, *)$ such that $\pi(\mathcal{E}) = \pi^{\rho-}_{x_1,x_2}$. By Definition 4.2, this means that the set $\Psi(\pi^{\rho,-}_{x_1,x_2}; \Delta_\rho[x_1, -x_1], 1)$ is empty, so π_{x_1,x_2} is not of Arthur type by Proposition 4.3. The rest follows from definition. $\square$

Now we move onto the case $\pi_{temp} = T^{\alpha-1}_{I,1}(T^{\alpha}_{I,1}(\pi_{sc}))$.

Proposition 5.2. *Consider the representation*

$$\pi_{x_1,x_2} = L(\Delta_\rho[-x_1, -x_1], \Delta_\rho[-x_2, -x_2]; \pi_{temp}),$$

where $x_1 > x_2 \geq \frac{1}{2}$ and $\pi_{temp} = T^{\alpha-1}_{I,1}(T^{\alpha}_{I,1}(\pi_{sc}))$ for $\alpha > 1$.

(i) The representation π_{x_1,x_2} is of Arthur type if and only if $\frac{1}{2} < x_2 < x_1 \leq \alpha - 2$ or $(x_1, x_2) = (\alpha - 1, \alpha - 2)$ when $\alpha \geq \frac{5}{2}$, or when $(x_1, x_2, \alpha) = (\frac{3}{2}, \frac{1}{2}, \frac{3}{2})$.

(ii) The representation is of critical type in the following cases:

- $(x_1, x_2) = (\alpha - 2, \alpha - 3)$ *for* $\alpha \geq \frac{7}{2}$,
- $(x_1, x_2) = (\alpha - 1, \alpha - 2)$ *for* $\alpha \geq \frac{5}{2}$,
- $(x_1, x_2) = (\alpha, \alpha-1), (\alpha+1, \alpha), (\alpha+2, \alpha+1), (\alpha+1, \alpha-1), (\alpha, \alpha-2), (\alpha+1, \alpha-2)$.

(iii) *Define* $\mathcal{E}_{x_1,x_2}$ *in various cases as follows. Then* $\pi(\mathcal{E}_{x_1,x_2}) = \pi_{x_1,x_2}$. *When* $\alpha \geq \frac{3}{2}$, *and* $\frac{1}{2} \leq x_2 < x_1 \leq \alpha - 1$, *define*

$$\begin{aligned}\mathcal{E}_{x_1,x_2} :=& \{([-x_1, x_1]_\rho, \lfloor x_1 \rfloor, (-1)^{x_1+1-\epsilon_\rho}\eta), ([x_2 - 2, \epsilon_\rho]_\rho, 0, -\eta),\\ & ([\alpha - 3, x_2]_\rho, 0, (-1)^{x_2+1-\epsilon_\rho}\eta), ([\alpha, \alpha]_\rho, 0, (-1)^{\alpha+1-\epsilon_\rho}\eta)\}.\end{aligned}$$

Here is the associated symbol:

$$\mathcal{E}_{x_1,x_2} = \begin{array}{c} \begin{array}{cccccccccccccc} -x_1 & \cdots & \epsilon_\rho & \cdots & x_2-2 & x_2-1 & x_2 & \cdots & x_1 & \cdots & \alpha-3 & \alpha-2 & \alpha-1 & \alpha \end{array} \\ \left(\begin{array}{cccccccccccccc} \lhd & \cdots & \odot & \cdots & \cdots & \ldots & \cdots & \cdots & \rhd & & & & & \\ & & \odot & \cdots & \odot & & & & & & & & & \\ & & & & & & \odot & \cdots & \odot & \cdots & \odot & & & \\ & & & & & & & & & & & & & \odot \end{array}\right) \end{array}.$$

When $\alpha \geq \frac{3}{2}$ *and* $(x_1, x_2) = (\alpha - 1, \alpha - 2)$, *define*

$$\begin{aligned}\mathcal{E}_{x_1,x_2} :=& \{([-x_1, x_1]_\rho, \lfloor x_1 \rfloor, (-1)^{x_1+1-\epsilon_\rho}\eta), ([\alpha - 4, \epsilon_\rho]_\rho, 0, -\eta),\\ & ([\alpha, \alpha]_\rho, 0, (-1)^{\alpha+1-\epsilon_\rho}\eta)\}.\end{aligned}$$

Here is the associated symbol:

$$\mathcal{E}_{x_1,x_2} = \begin{array}{c} \begin{array}{ccccccccc} -(\alpha-1) & \cdots & \epsilon_\rho & \cdots & \alpha-4 & \alpha-3 & \alpha-2 & \alpha-1 & \alpha \end{array} \\ \left(\begin{array}{ccccccccc} \lhd & \cdots & \odot & \cdots & \cdots & \ldots & \cdots & \rhd & \\ & & \odot & \cdots & \odot & & & & \\ & & & & & & & & \odot \end{array}\right) \end{array}.$$

When $(x_1, x_2, \alpha) = (\frac{3}{2}, \frac{1}{2}, \frac{3}{2})$, *then define* $\mathcal{E}_{x_1,x_2}$ *to be the same extended multi-segment defined in Proposition 5.1, when* $(x_1, x_2, \alpha) = (\frac{3}{2}, \frac{1}{2}, \frac{1}{2})$

Proof. To prove the necessary direction, we note that $\pi_{x_1,x_2}^{\rho,-}$ is of Arthur type if and only if $\frac{1}{2} \leq x_2 \leq \alpha - 2$ for $\alpha \geq \frac{3}{2}$, or $(x, \alpha) = (\frac{1}{2}, \frac{3}{2})$ by [HJLLZ25, Proposition 11.3]. The rest of the proof is similar to Proposition 5.1, which we omit. □

The next case is $\pi_{temp} = T_{IV,3}(T_{I,1}^{\alpha}(\pi_{sc}))$.

Proposition 5.3. *Consider the representation*

$$\pi_{x_1,x_2} = L(\Delta_\rho[-x_1, -x_1], \Delta_\rho[-x_2, -x_2]; \pi_{temp}),$$

where $x_1 > x_2 \geq \frac{1}{2}$ *and* $\pi_{temp} = T_{IV,3}(T_{I,1}^{\alpha}(\pi_{sc}))$ *for* $\alpha \in \mathbb{Z}_{>1}$.

(i) *The representation* $\pi_{[x_1,x_2]}$ *is of Arthur type if and only if* $1 \leq x_2 < x_1 \leq \alpha - 1$, *or* $(x_1, x_2) = (\alpha, \alpha - 1)$.

(ii) *The representation* $\pi_{[x_1,x_2]}$ *is of critical type when* $(x_1, x_2, \alpha) = (2, 1, 2)$, $(2, 1, 3)$, $(3, 1, 2)$.

(iii) *Define* $\mathcal{E}_{x_1,x_2}$ *in various cases as follows. Then* $\pi(\mathcal{E}_{x_1,x_2}) = \pi_{x_1,x_2}$. *When* $1 \leq x_2 < x_1 \leq \alpha - 1$, *define*

$$\begin{aligned}\mathcal{E}_{x_1,x_2} :=& \{([x_1, -x_1]_\rho, \lfloor x_1 \rfloor, (-1)^{x_1+1-\epsilon_\rho}\eta), ([0, 0]_\rho, 0, -\eta)^2, ([x_2 - 2, \epsilon_\rho]_\rho, 0, -\eta),\\ & ([\alpha - 2, x_2]_\rho, 0, (-1)^{x_2+1-\epsilon_\rho}\eta), ([\alpha, \alpha]_\rho, 0, (-1)^{\alpha+1-\epsilon_\rho}\eta)\}.\end{aligned}$$

Here is the associated symbol.

$$\mathcal{E}_{x_1,x_2} = \begin{array}{c} \begin{array}{ccccccccccccc} -x_1 & \cdots & 0 & \cdots & x_2-2 & x_2-1 & x_2 & \cdots & x_1 & \cdots & \alpha-2 & \alpha-1 & \alpha \end{array} \\ \begin{pmatrix} \lhd & \cdots & \odot & \cdots & \cdots & \cdots & \cdots & \cdots & \rhd & & & & \\ & & \odot & & & & & & & & & & \\ & & \odot & & & & & & & & & & \\ & & \odot & \cdots & \odot & & & & & & & & \\ & & & & & & \odot & \cdots & \odot & \cdots & \odot & & \\ & & & & & & & & & & & & \odot \end{pmatrix} \end{array}.$$

When $(x_1, x_2) = (\alpha, \alpha-1)$, define

$$\begin{aligned}\mathcal{E}_{x_1,x_2} :=& \{([-x_1, x_1]_\rho, \lfloor x_1 \rfloor, (-1)^{x_1+1-\epsilon_\rho}\eta), ([0,0]_\rho, 0, -\eta)^2, \\ & ([\alpha-3, \epsilon_\rho]_\rho, 0, -\eta), ([\alpha,\alpha]_\rho, 0, (-1)^{\alpha+1-\epsilon_\rho}\eta)\}.\end{aligned}$$

Here is the associated symbol.

$$\mathcal{E}_{x_1,x_2} = \begin{array}{c} \begin{array}{cccccccc} -\alpha & \cdots & \epsilon_\rho & \cdots & \alpha-3 & \alpha-2 & \alpha-1 & \alpha \end{array} \\ \begin{pmatrix} \lhd & \cdots & \odot & \cdots & \cdots & \cdots & \cdots & \rhd \\ & & \odot & \cdots & \odot & & & \\ & & & & & & & \odot \end{pmatrix} \end{array}.$$

Proof. To prove the necessary direction, we note that $\pi^{\rho,-}_{x_1,x_2}$ is of Arthur type if and only if $1 \leq x_2 \leq \alpha - 1$ by [HJLLZ25, Proposition 11.3]. The rest of the proof is similar to Proposition 5.1, which we omit. □

Now we move onto the case $\pi_{temp} = T^{\pm}_{V,2}(T^1_{I,1}(\pi_{sc}))$.

Proposition 5.4. *Consider the representation*

$$\pi^{\pm}_{x_1,x_2} = L(\Delta_\rho[-x_1, -x_1], \Delta_\rho[-x_2, -x_2]; \pi^{\pm}_{temp}),$$

where $x_1 > x_2 \geq \frac{1}{2}$ and $\pi^{\pm}_{temp} = T^{\pm}_{V,2}(T^1_{I,1}(\pi_{sc}))$ for $\alpha = 0$.

(i) The representation $\pi^{\pm}_{[x_1,x_2]}$ is of Arthur type if and only if $(x_1, x_2) = (2, 1)$.

(ii) The representation $\pi^{\pm}_{[x_1,x_2]}$ is of critical type if and only if $(x_1, x_2) = (2, 1)$ or $(3, 2)$.

(iii) We have $\pi(\mathcal{E}^{\pm}) = \pi^{\pm}_{2,1}$, , where

$$\mathcal{E}^+ = \begin{array}{c} \begin{array}{ccccc} -2 & -1 & 0 & 1 & 2 \end{array} \\ \begin{pmatrix} \lhd & \lhd & \oplus & \rhd & \rhd \\ & & \oplus & \ominus & \end{pmatrix} \end{array}, \quad \mathcal{E}^- = \begin{array}{c} \begin{array}{ccccc} -2 & -1 & 0 & 1 & 2 \end{array} \\ \begin{pmatrix} \lhd & \lhd & \ominus & \rhd & \rhd \\ & & \ominus & \oplus & \end{pmatrix} \end{array}.$$

Proof. By [HJLLZ25, Proposition 11.4], the representation $(\pi^{\pm}_{x_1,x_2})^{\rho,-} = L(\Delta_\rho[-x_2, -x_2]; \pi_{temp})$ is of Arthur type if and only if $x_2 = 1$. The additional requirement from Lemma 4.4 tells us that we must have $x_1 = 2$. By Theorem 4.3, this is the only case where $\pi^{\pm}_{x_1,x_2}$ is of Arthur type. The rest follows from definition. □

The next relevant case is $\pi_{temp} = T^1_{I,1}(T^{\pm}_{V,2}(\pi_{sc}))$.

Proposition 5.5. *Consider the representation*

$$\pi^{\pm}_{x_1,x_2} = L(\Delta_\rho[-x_1, -x_1], \Delta_\rho[-x_2, -x_2]; \pi^{\pm}_{temp}),$$

where $x_1 > x_2 \geq \frac{1}{2}$ and $\pi^{\pm}_{temp} = T^1_{I,1}(T^{\pm}_{V,2}(\pi_{sc}))$ for $\alpha = 0$.

(i) The representation $\pi^{\pm}_{x_1,x_2}$ is of Arthur type if and only if $(x_1, x_2) = (2, 1)$.

(ii) The representation $\pi^{\pm}_{[x_1,x_2]}$ is of critical type if and only if $(x_1, x_2) = (2, 1)$ or $(3, 2)$.

(iii) *We have* $\pi(\mathcal{E}^{\pm}) = \pi^{\pm}_{2,1}$, *, where*

$$\mathcal{E}^{+} = \begin{array}{c} \begin{array}{ccccc} -2 & -1 & 0 & 1 & 2 \end{array} \\ \begin{pmatrix} \lhd & \lhd & \oplus & \rhd & \rhd \\ & & & \oplus & \end{pmatrix} \end{array}, \quad \mathcal{E}^{-} = \begin{array}{c} \begin{array}{ccccc} -2 & -1 & 0 & 1 & 2 \end{array} \\ \begin{pmatrix} \lhd & \lhd & \ominus & \rhd & \rhd \\ & & & \ominus & \end{pmatrix} \end{array}.$$

Proof. By [HJLLZ25, Proposition 11.5], the representation $(\pi^{\pm}_{x_1,x_2})^{\rho,-} = L(\Delta_\rho[-x_2,-x_2];\pi_{temp})$ is of Arthur type if and only if $x_2 = 1$. The rest follows from the same argument as Proposition 5.4, which we omit. $\square$

We move onto the next case, which is $\pi_{temp} = T^{\frac{1}{2}}_{I,2}(\pi_{sc})$.

Proposition 5.6. *Consider the representation*

$$\pi_{x_1,x_2} = L(\Delta_\rho[-x_1,-x_1],\Delta_\rho[-x_2,-x_2];\pi_{temp}),$$

where $x_1 > x_2 \geq \frac{1}{2}$ *and* $\pi_{temp} = T^{\frac{1}{2}}_{I,2}(\pi_{sc})$ *for* $\alpha = \frac{1}{2}$.

(i) *The representation* π_{x_1,x_2} *is of Arthur type if and only if* $(x_1,x_2) = (\frac{3}{2},\frac{1}{2})$.

(ii) *The representation* π_{x_1,x_2} *is of critical type if and only if* $(x_1,x_2) = (\frac{3}{2},\frac{1}{2})$ *or* $(\frac{5}{2},\frac{3}{2})$.

(iii) *We have* $\pi(\mathcal{E}) = \pi_{\frac{3}{2},\frac{1}{2}}$, *where*

$$\mathcal{E} = \begin{array}{c} \begin{array}{cc} -\frac{1}{2} & \frac{1}{2} \end{array} \\ \begin{pmatrix} \lhd & \rhd \\ & \oplus \\ & \oplus \end{pmatrix} \end{array}.$$

Proof. By [HJLLZ25, Proposition 11.6], the representation $(\pi_{x_1,x_2})^{\rho,-} = L(\Delta_\rho[-x_2,-x_2];\pi_{temp})$ is of Arthur type if and only if $x_2 = \frac{1}{2}$. The rest follows from the same argument as Proposition 5.4, which we omit. $\square$

Now let us consider $\pi_{temp} = T^{\frac{1}{2}}_{II,3}(\pi_{sc})$.

Proposition 5.7. *Consider the representation*

$$\pi_{x_1,x_2} = L(\Delta_\rho[-x_1,-x_1],\Delta_\rho[-x_2,-x_2];\pi_{temp}),$$

where $x_1 > x_2 \geq \frac{1}{2}$ *and* $\pi_{temp} = T^{\frac{1}{2}}_{II,3}(\pi_{sc})$ *and* $\alpha \in \frac{1}{2} + \mathbb{Z}_{>0}$.

(i) *The representation* π_{x_1,x_2} *is of Arthur type if and only if* $\frac{1}{2} \leq x_2 < x_1 \leq \alpha$ *or* $(x_1,x_2) = (\alpha+1,\alpha)$.

(ii) *The representation* π_{x_1,x_2} *is of critical type if and only if* $(x_1,x_2,\alpha) = (\frac{3}{2},\frac{1}{2},\frac{3}{2}),(\frac{5}{2},\frac{3}{2},\frac{3}{2})$ *or* $(\frac{5}{2},\frac{3}{2},\frac{5}{2})$.

(iii) *Define* $\mathcal{E}_{x_1,x_2}$ *in various cases as follows. Then* $\pi(\mathcal{E}_{x_1,x_2}) = \pi_{x_1,x_2}$. *When* $\frac{1}{2} \leq x_2 < x_1 \leq \alpha$, *define*

$$\begin{aligned}\mathcal{E}_{x_1,x_2} :=\{&([x_1,-x_1]_\rho,\lfloor x_1 \rfloor,1),([\frac{1}{2},\frac{1}{2}]_\rho,0,1)^2,([x_2-2,\frac{1}{2}]_\rho,0,1),\\ &([\alpha-1,x_2]_\rho,0,(-1)^{x_2+\frac{3}{2}}).\end{aligned}$$

Here is the associated symbol.

$$\mathcal{E}_{x_1,x_2} = \begin{array}{c} \begin{array}{ccccccccccc} -x_1 & \cdots & \frac{1}{2} & \cdots & x_2-2 & x_2-1 & x_2 & \cdots & x_1 & \cdots & \alpha-1 \end{array} \\ \begin{pmatrix} \lhd & \cdots & \oplus & \cdots & \cdots & \cdots & \cdots & \cdots & \rhd & & \\ & & \oplus & & & & & & & & \\ & & \oplus & & & & & & & & \\ & & \oplus & \cdots & \odot & & & & & & \\ & & & & & & \odot & \cdots & \cdots & \cdots & \odot \end{pmatrix} \end{array}.$$

When $(x_1, x_2) = (\alpha+1, \alpha)$, *define*

$$\mathcal{E}_{x_1,x_2} := \{([-x_1, x_1]_\rho, \lfloor x_1 \rfloor, 1), ([\frac{1}{2}, \frac{1}{2}]_\rho, 0, 1)^2, ([\alpha-2, \frac{1}{2}]_\rho, 0, 1)\}.$$

Here is the associated symbol.

$$\mathcal{E}_{x_1,x_2} = \begin{array}{c} \begin{array}{ccccccc} -(\alpha+1) & \cdots & \frac{1}{2} & \cdots & \alpha-2 & \alpha-1 & \alpha+1 \end{array} \\ \left(\begin{array}{ccccccc} \lhd & \cdots & \oplus & \cdots & \cdots & \cdots & \rhd \\ & & \oplus & & & & \\ & & \oplus & & & & \\ & & \oplus & \cdots & \odot & & \end{array} \right) \end{array}.$$

Proof. By [HJLLZ25, Proposition 11.7], the representation $\pi^{\rho,-}_{x_1,x_2} = L(\Delta_\rho[-x_2, -x_2]; \pi_{temp})$ is of Arthur type if and only if $\frac{1}{2} \leq x_2 \leq \alpha$. Since

$$|\Omega|(\pi^{\rho,-}_{x_1,x_2})_\rho \subseteq \{\rho|\cdot|^y : \frac{1}{2} \leq y \leq \alpha - 1\} \cup \{\rho|\cdot|^{x_2}, \rho|\cdot|^{-x_2}\}$$

the necessary direction follows from Lemma 4.4 and Proposition 4.3. The rest of the proof is similar to Proposition 5.1, which we omit. □

There are three remaining cases we need to consider. The next one is $\pi_{temp} = T^{\frac{1}{2}}_{III,2}(\pi_{sc})$.

Proposition 5.8. *Consider the representation*

$$\pi_{x_1,x_2} = L(\Delta_\rho[-x_1, -x_1], \Delta_\rho[-x_2, -x_2]; \pi_{temp}),$$

where $x_1 > x_2 \geq \frac{1}{2}$ *and* $\pi_{temp} = T^{\frac{1}{2}}_{III,2}(\pi_{sc})$ *for* $\alpha = \frac{1}{2}$.

(i) The representation π_{x_1,x_2} *is of Arthur type if and only if* $(x_1, x_2) = (\frac{3}{2}, \frac{1}{2})$ *or* $(\frac{5}{2}, \frac{3}{2})$.
(ii) The representation π_{x_1,x_2} *is of critical type if and only if* $(x_1, x_2) = (\frac{3}{2}, \frac{1}{2})$ *or* $(\frac{5}{2}, \frac{3}{2})$.
(iii) We have $\pi(\mathcal{E}) = \pi_{x_1,x_2}$, *where*

$$\mathcal{E}_{\frac{3}{2},\frac{1}{2}} = \begin{array}{c} \begin{array}{cccc} -\frac{3}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{3}{2} \end{array} \\ \left(\begin{array}{cccc} \lhd & \lhd & \rhd & \rhd \\ & & \ominus & \\ & & \ominus & \end{array} \right) \end{array},$$

$$\mathcal{E}_{\frac{5}{2},\frac{3}{2}} = \begin{array}{c} \begin{array}{cccccc} -\frac{5}{2} & -\frac{3}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{3}{2} & \frac{5}{2} \end{array} \\ \left(\begin{array}{cccccc} \lhd & \lhd & \oplus & \ominus & \rhd & \rhd \\ & & & \ominus & & \end{array} \right) \end{array}.$$

Proof. By [HJLLZ25, Proposition 10.8], the representation $\pi^{\rho,-}_{x_1,x_2} = L(\Delta_\rho[-x_2, -x_2]; \pi_{temp})$ is of Arthur type if and only if $x_2 = \frac{1}{2}$ or $\frac{3}{2}$. The rest of the proof is similar to Proposition 5.4, which we omit. □

The second last case is $\pi_{temp} = T_{IV,5}(\pi_{sc})$.

Proposition 5.9. *Consider the representation*

$$\pi_{x_1,x_2} = L(\Delta_\rho[-x_1, -x_1], \Delta_\rho[-x_2, -x_2]; \pi_{temp}),$$

where $x_1 > x_2 \geq \frac{1}{2}$ *and* $\pi_{temp} = T_{IV,5}(\pi_{sc})$ *for* $\alpha \in \mathbb{Z}_{>0}$.

(i) The representation π_{x_1,x_2} *is of Arthur type if and only if* $1 \leq x_2 < x_1 \leq \alpha$, *or when* $(x_1, x_2) = (\alpha+1, \alpha)$.
(ii) The π_{x_1,x_2} *is of critical type if and only if* $(x_1, x_2, \alpha) = (2, 1, 1)$ *or* $(2, 1, 2)$.

(iii) Define $\mathcal{E}_{x_1,x_2}$ in various cases as follows. Then $\pi(\mathcal{E}_{x_1,x_2})=\pi_{x_1,x_2}$. When $1\le x_2<x_1\le\alpha$, define

$$\begin{aligned}\mathcal{E}_{x_1,x_2}:=&\{([x_1,-x_1]_\rho,\lfloor x_1\rfloor,(-1)^{x_1+1}\eta),([0,0]_\rho,0,-\eta)^4,([x_2-2,0]_\rho,0,(-1)^{x_2-1}\eta),\\&([\alpha-1,x_2]_\rho,0,(-1)^{x_2+1}\eta)\}.\end{aligned}$$

Here is the associated symbol.

$$\mathcal{E}_{x_1,x_2}=\begin{array}{c}\begin{array}{ccccccccccc} -x_1 & \cdots & 0 & \cdots & x_2-2 & x_2-1 & x_2 & \cdots & x_1 & \cdots & \alpha-1\end{array}\\ \left(\begin{array}{ccccccccccc} \lhd & \cdots & \odot & \cdots & \cdots & \cdots & \cdots & \cdots & \rhd & & \\ & & \odot & & & & & & & & \\ & & \odot & & & & & & & & \\ & & \odot & & & & & & & & \\ & & \odot & & & & & & & & \\ & & \odot & \cdots & \odot & & & & & & \\ & & & & & & \odot & \cdots & \cdots & \cdots & \odot \end{array}\right)\end{array}.$$

When $(x_1,x_2)=(\alpha+1,\alpha)$, define

$$\mathcal{E}_{x_1,x_2}:=\{([-x_1,x_1]_\rho,\lfloor x_1\rfloor,(-1)^{\alpha}\eta),([0,0]_\rho,0,-\eta)^4,([\alpha-2,0]_\rho,0,(-1)^{\alpha-1}\eta)\}.$$

Here is the associated symbol.

$$\mathcal{E}_{x_1,x_2}=\begin{array}{c}\begin{array}{ccccccc} -(\alpha+1) & \cdots & 0 & \cdots & \alpha-2 & \alpha-1 & \alpha+1\end{array}\\ \left(\begin{array}{ccccccc} \lhd & \cdots & \odot & \cdots & \cdots & \cdots & \rhd \\ & & \odot & & & & \\ & & \odot & & & & \\ & & \odot & & & & \\ & & \odot & & & & \\ & & \odot & \cdots & \odot & & \end{array}\right)\end{array}.$$

Proof. By [HJLLZ25, Proposition 11.9], the representation $\pi_{x_1,x_2}^{\rho,-}=L(\Delta_\rho[-x_2,-x_2];\pi_{temp})$ is of Arthur type if and only if $1\le x_2\le\alpha$. The rest of the proof is similar to Proposition 5.7, which we omit. □

The final case we need to consider in Case (A) is $\pi_{temp}=T_{V,4}^{\pm}(\pi_{sc})$.

Proposition 5.10. *Consider the representation*

$$\pi_{x_1,x_2}^{\pm}=L(\Delta_\rho[-x_1,-x_1],\Delta_\rho[-x_2,-x_2];\pi_{temp}^{\pm}),$$

where $x_1>x_2\ge\frac{1}{2}$ and $\pi_{temp}^{\pm}=T_{V,4}^{\pm}(\pi_{sc})$ for $\alpha=0$.

(i) The representation $\pi_{x_1,x_2}^{\pm}$ is of Arthur type if and only if $(x_1,x_2)=(2,1)$.
(ii) The representation $\pi_{x_1,x_2}^{\pm}$ is of critical type if and only if $(x_1,x_2)=(2,1)$.
(iii) We have $\pi(\mathcal{E}^{\pm})=\pi_{2,1}^{\pm}$, where

$$\mathcal{E}^{+}=\begin{array}{c}\begin{array}{ccccc} -2 & -1 & 0 & 1 & 2\end{array}\\ \left(\begin{array}{ccccc} \lhd & \lhd & \oplus & \rhd & \rhd \\ & & \oplus & & \\ & & \oplus & & \\ & & \oplus & & \end{array}\right)\end{array},\quad \mathcal{E}^{-}=\begin{array}{c}\begin{array}{ccccc} -2 & -1 & 0 & 1 & 2\end{array}\\ \left(\begin{array}{ccccc} \lhd & \lhd & \ominus & \rhd & \rhd \\ & & \ominus & & \\ & & \ominus & & \\ & & \ominus & & \end{array}\right)\end{array}.$$

Proof. By [HJLLZ25, Proposition 11.10], the representation $(\pi_{x_1,x_2}^{\pm})^{\rho,-}=L(\Delta_\rho[-x_2,-x_2];\pi_{temp}^{\pm})$ is of Arthur type if and only if $x_2=1$. The rest of the proof is similar to Proposition 5.5, which we omit. □

This concludes our discussion of Case (A).

5.2. **Case** (B) : $\pi = L(\Delta_\rho[-x,-x],\Delta_\rho[-x,-x];\pi_{temp})$**.** In this section we will consider representations of the form $\pi_x = L(\Delta_\rho[-x,-x],\Delta_\rho[-x,-x];\pi_{temp})$, where $x \geq \frac{1}{2}$ and π_{temp} is tempered of corank 2. Therefore we consider the exact same representations as Case (A). The first case is $\pi_{temp} = T_{I,1}^{\alpha+1}(T_{I,1}^{\alpha}(\pi_{sc}))$.

Proposition 5.11. *Consider the representation*

$$\pi_x = L(\Delta_\rho[-x,-x],\Delta_\rho[-x,-x];\pi_{temp}),$$

for $x \geq \frac{1}{2}$*, where* $\pi_{temp} = T_{I,1}^{\alpha+1}(T_{I,1}^{\alpha}(\pi_{sc}))$ *and* $\alpha > 0$.

(i) The representation π_x *is of Arthur type if and only if* $x = \frac{1}{2}$.
(ii) The representation π_x *is of critical type if and only if* $x \in \{\alpha-1,\alpha,\alpha+1,\alpha+2\}$.
(iii) We have $\pi(\mathcal{E}) = \pi_{\frac{1}{2}}$ *where*

$$\mathcal{E} = \begin{array}{c} \begin{array}{cccccccc} -\frac{1}{2} & \frac{1}{2} & \cdots & \alpha-2 & \alpha-1 & \alpha & \alpha+1 \end{array} \\ \begin{pmatrix} \lhd & \rhd & & & & & \\ \lhd & \rhd & & & & & \\ & \odot & \cdots & \odot & & & \\ & & & & & & \odot \end{pmatrix} \end{array}.$$

Proof. The sufficient direction is easy to check. For the necessary direction, by Lemma 4.4, the representation π_x is of Arthur type only if $x = \frac{1}{2}$ or $|\Omega|(\pi_{temp})$ contains 2 copies of $\rho|\cdot|^{x-1}$, but $|\Omega|(\pi_{temp})$ is multiplicity free, so the conclusion follows. The rest of the proposition follows from definition. $\square$

The next case is $\pi_{temp} = T_{I,1}^{\alpha-1}(T_{I,1}^{\alpha}(\pi_{sc}))$, which is similar.

Proposition 5.12. *Consider the representation*

$$\pi_x = L(\Delta_\rho[-x,-x],\Delta_\rho[-x,-x];\pi_{temp}),$$

for $x \geq \frac{1}{2}$*, where* $\pi_{temp} = T_{I,1}^{\alpha-1}(T_{I,1}^{\alpha}(\pi_{sc}))$ *and* $\alpha > 1$.

(i) The representation π_x *is of Arthur type if and only if* $x = \frac{1}{2}$.
(ii) The representation π_x *is of critical type if and only if* $x \in \{\alpha-2,\alpha-1,\alpha,\alpha+1\}$.
(iii) We have $\pi(\mathcal{E}) = \pi_{\frac{1}{2}}$*, where*

$$\mathcal{E} = \begin{array}{c} \begin{array}{cccccccc} -\frac{1}{2} & \frac{1}{2} & \cdots & \alpha-3 & \alpha-2 & \alpha-1 & \alpha \end{array} \\ \begin{pmatrix} \lhd & \rhd & & & & & \\ \lhd & \rhd & & & & & \\ & \odot & \cdots & \odot & & & \\ & & & & & \odot & \\ & & & & & & \odot \end{pmatrix} \end{array}.$$

Proof. The proof is similar to Proposition 5.11, which we omit. $\square$

The next case we have is $\pi_{temp} = T_{IV,3}(T_{I,1}^{\alpha}(\pi_{sc}))$.

Proposition 5.13. *Consider the representation*

$$\pi_x = L(\Delta_\rho[-x,-x],\Delta_\rho[-x,-x];\pi_{temp}),$$

for $x \geq \frac{1}{2}$*, where* $\pi_{temp} = T_{IV,3}(T_{I,1}^{\alpha}(\pi_{sc}))$ *and* $\alpha \in \mathbb{Z}_{>1}$.

(i) The representation π_x *is of Arthur type if and only if* $x = 1$.
(ii) The representation π_x *is of critical type if and only if* $(x,\alpha) = (1,2)$.

(iii) We have $\pi(\mathcal{E}) = \pi_1$ *where*

$$\mathcal{E} = \begin{array}{c} \begin{array}{ccccccc} -1 & 0 & 1 & \cdots & \alpha-2 & \alpha-1 & \alpha \end{array} \\ \begin{pmatrix} \lhd & \odot & \rhd & & & & \\ \lhd & \odot & \rhd & & & & \\ & \odot & \odot & \cdots & \odot & & \\ & & & & & & \odot \end{pmatrix} \end{array}.$$

Proof. By Lemma 4.4, π_x is of Arthur type only if $|\Omega|(\pi_{temp})$ contains 2 copies of $\rho|\cdot|^{x-1}$. This is only possible if $x = 1$. The rest is easy to verify. □

Now we move onto the case $\pi_{temp} = T_{V,2}^{\pm}(T_{I,1}^{1}(\pi_{sc}))$.

Proposition 5.14. *Consider the representation*

$$\pi_x^{\pm} = L(\Delta_\rho[-x,-x], \Delta_\rho[-x,-x]; \pi_{temp}^{\pm}),$$

for $x \geq \frac{1}{2}$, *where* $\pi_{temp}^{\pm} = T_{V,2}^{\pm}(T_{I,1}^{1}(\pi_{sc}))$ *and* $\alpha = 1$.

(i) The representation π_x *is of Arthur type if and only if* $x = 1$.
(ii) The representation π_x *is of critical type if and only if* $x \in \{1, 2\}$.
(iii) We have $\pi(\mathcal{E}^{\pm}) = \pi_1^{\pm}$, *where*

$$\mathcal{E}^{+} = \begin{array}{c} \begin{array}{ccc} -1 & 0 & 1 \end{array} \\ \begin{pmatrix} \lhd & \oplus & \rhd \\ \lhd & \oplus & \rhd \\ & & \odot \end{pmatrix} \end{array}, \quad \mathcal{E}^{-} = \begin{array}{c} \begin{array}{ccc} -1 & 0 & 1 \end{array} \\ \begin{pmatrix} \lhd & \ominus & \rhd \\ \lhd & \ominus & \rhd \\ & & \odot \end{pmatrix} \end{array}.$$

Proof. The proof is similar to Proposition 5.13, which we omit. □

The next case to consider is $\pi_{temp} = T_{I,1}^{1}(T_{V,2}^{\pm}(\pi_{sc}))$.

Proposition 5.15. *Consider the representation*

$$\pi_x^{\pm} = L(\Delta_\rho[-x,-x], \Delta_\rho[-x,-x]; \pi_{temp}^{\pm}),$$

for $x \geq \frac{1}{2}$, *where* $\pi_{temp}^{\pm} = T_{I,1}(T_{V,2}^{\pm}(\pi_{sc}))$ *and* $\alpha = 0$.

(i) The representation π_x *is not of Arthur type for any* x.
(ii) The representation π_x *is of critical type if and only if* $x \in \{1, 2\}$.

Proof. Since $|\Omega|(\pi_{temp}) = \{\rho, \rho|\cdot|^1\}$, the conclusion follows from Lemma 4.4. □

We move onto the case $\pi_{temp} = T_{I,2}^{\frac{1}{2}}(\pi_{sc})$.

Proposition 5.16. *Consider the representation*

$$\pi_x = L(\Delta_\rho[-x,-x], \Delta_\rho[-x,-x]; \pi_{temp}),$$

for $x \geq \frac{1}{2}$, *where* $\pi_{temp} = T_{I,2}^{\frac{1}{2}}(\pi_{sc})$ *and* $\alpha = \frac{1}{2}$.

(i) The representation π_x *is of Arthur type if and only if* $x = \frac{1}{2}$.
(ii) The representation π_x *is of critical type if and only if* $x \in \{\frac{1}{2}, \frac{3}{2}\}$.
(iii) We have that $\pi(\mathcal{E}) = \pi_{\frac{1}{2}}$, *where*

$$\mathcal{E} = \begin{array}{c} \begin{array}{cc} -\frac{1}{2} & \frac{1}{2} \end{array} \\ \begin{pmatrix} \lhd & \rhd \\ \lhd & \rhd \\ & \oplus \\ & \oplus \end{pmatrix} \end{array}.$$

Proof. The proof is similar to Proposition 5.11, which we omit. □

The next case is $\pi_{temp} = T_{II,3}^{\frac{1}{2}}(\pi_{sc})$.

Proposition 5.17. *Consider the representation*

$$\pi_x = L(\Delta_\rho[-x,-x], \Delta_\rho[-x,-x]; \pi_{temp}),$$

for $x \geq \frac{1}{2}$, where $\pi_{temp} = T_{II,3}^{\frac{1}{2}}(\pi_{sc})$ and $\alpha \in \frac{1}{2} + \mathbb{Z}_{>0}$.

(i) The representation π_x is of Arthur type if and only if $x = \frac{1}{2}$.
(ii) The representation π_x is of critical type if and only if $(x, \alpha) = (\frac{3}{2}, \frac{3}{2})$.
(iii) We have that $\pi(\mathcal{E}) = \pi_{\frac{1}{2}}$, where

$$\mathcal{E} = \begin{array}{c} \begin{array}{cccc} -\frac{1}{2} & \frac{1}{2} & \cdots & \alpha-1 \end{array} \\ \begin{pmatrix} \lhd & \rhd & & \\ \lhd & \rhd & & \\ & \ominus & & \\ & \ominus & & \\ & \ominus & \cdots & \odot \end{pmatrix} \end{array}.$$

Proof. The sufficient direction is easy to verify. By Lemma 4.4, the representation π_x can only be of Arthur type if $x = \frac{1}{2}$ or $x = \frac{3}{2}$. Therefore, to prove the necessary direction, it suffices to show that the case $x = \frac{3}{2}$ is not of Arthur type.

Suppose it is, then by Definition 4.2, the extended multi-segment corresponding to π_{temp} must contain at least two segments of the form $([\frac{1}{2}, -\frac{1}{2}]_\rho, *, *)$. If this is true, then by looking at $\Omega(\pi_{temp})$, we see that there can be at most 1 more segment of the form $([A_i, B_i]_\rho, *, *)$ with $B_i = \frac{1}{2}$. This contradicts [HLL22, Proposition 3.5] since π_{temp} has an order 2 nonzero derivative at $x = \frac{1}{2}$. This proves part (i). The rest follows from definition. □

Three more cases remain. The next one is $\pi_{temp} = T_{III,2}^{\frac{1}{2}}(\pi_{sc})$.

Proposition 5.18. *Consider the representation*

$$\pi_x = L(\Delta_\rho[-x,-x], \Delta_\rho[-x,-x]; \pi_{temp}),$$

for $x \geq \frac{1}{2}$, where $\pi_{temp} = T_{III,2}^{\frac{1}{2}}(\pi_{sc})$ and $\alpha = \frac{1}{2}$.

(1) The representation π_x is of Arthur type if and only if $x = \frac{1}{2}$.
(2) The representation π_x is of critical type if and only if $x \in \{\frac{1}{2}, \frac{3}{2}\}$.
(3) We have $\pi(\mathcal{E}) = \pi_{\frac{1}{2}}$, where

$$\mathcal{E} = \begin{array}{c} \begin{array}{cc} -\frac{1}{2} & \frac{1}{2} \end{array} \\ \begin{pmatrix} \lhd & \rhd \\ \lhd & \rhd \\ & \ominus \\ & \ominus \end{pmatrix} \end{array}.$$

Proof. The proof is similar to Proposition 5.17, which we omit. □

The next tempered representation of corank 2 is $\pi_{temp} = T_{IV,5}(\pi_{sc})$.

Proposition 5.19. *Consider the representation*

$$\pi_x = L(\Delta_\rho[-x,-x], \Delta_\rho[-x,-x]; \pi_{temp}),$$

for $x \geq \frac{1}{2}$, where $\pi_{temp} = T_{IV,5}(\pi_{sc})$ and $\alpha \in \mathbb{Z}_{>0}$.

(i) The representation π_x is of Arthur type if and only if $x = 1$.
(ii) The representation π_x is of critical type if and only if $(x, \alpha) = (1, 1)$.
(iii) We have that $\pi(\mathcal{E}) = \pi_1$, where

$$\mathcal{E} = \begin{array}{c} \begin{matrix} -1 & 0 & 1 & \cdots & \alpha-1 \end{matrix} \\ \begin{pmatrix} \lhd & \odot & \rhd & & \\ \lhd & \odot & \rhd & & \\ & \odot & & & \\ & \odot & & & \\ & \odot & \odot & \cdots & \odot \end{pmatrix} \end{array}.$$

Proof. The proof is similar to Proposition 5.13, which we omit. □

The final case in this section is $\pi_{temp} = T^{\pm}_{V,4}(\pi_{sc})$.

Proposition 5.20. *Consider the representation*

$$\pi^{\pm}_x = L(\Delta_\rho[-x, -x], \Delta_\rho[-x, -x]; \pi^{\pm}_{temp}),$$

for $x \geq \frac{1}{2}$, where $\pi^{\pm}_{temp} = T^{\pm}_{V,4}(\pi_{sc})$ and $\alpha = 0$.
(1) The representation π_x is of Arthur type if and only if $x = 1$.
(2) The representation π_x is of critical type if and only if $x = 1$.
(3) We have $\pi(\mathcal{E}^{\pm}) = \pi^{\pm}_1$, where

$$\mathcal{E}^{+} = \begin{array}{c} \begin{matrix} -1 & 0 & 1 \end{matrix} \\ \begin{pmatrix} \lhd & \oplus & \rhd \\ \lhd & \oplus & \rhd \\ & \oplus & \\ & \oplus & \end{pmatrix} \end{array}, \quad \mathcal{E}^{-} = \begin{array}{c} \begin{matrix} -1 & 0 & 1 \end{matrix} \\ \begin{pmatrix} \lhd & \ominus & \rhd \\ \lhd & \ominus & \rhd \\ & \ominus & \\ & \ominus & \end{pmatrix} \end{array}.$$

Proof. The proof is similar to Proposition 5.14, which we omit. □

This concludes our work in Case (B). Now we move onto Case (C).

5.3. **Case** (C)**:** $\pi = L(\Delta_\rho[-x_1, -x_1 - 1], \Delta_\rho[-x_2, -x_2]; \pi_{temp})$**.** In this subsection we are concerned with representations of the form $\pi_{x_1,x_2} = L(\Delta_\rho[-x_1, -x_1-1], \Delta_\rho[-x_2, -x_2]; \pi_{temp})$, where π_{temp} is tempered of corank 1. Here we require $x_1 \geq 0, x_2 \geq \frac{1}{2}$ and $x_2 - x_1 \leq \frac{1}{2}$ by Langlands classification of classical groups. We assume our representations are of good parity, so the last condition can be rewritten as $x_2 \leq x_1$

Since we're dealing with tempered representations of corank 1, there are three cases to consider. Let us begin with $\pi_{temp} = T^{\alpha}_{I,1}(\pi_{sc})$.

Proposition 5.21. *Consider the representation*

$$\pi_{x_1,x_2} = L(\Delta_\rho[-x_1, -x_1 - 1], \Delta_\rho[-x_2, -x_2]; \pi_{temp}),$$

where $x_1 \geq 0, x_2 \geq \frac{1}{2}$, $x_2 \leq x_1$, and $\pi_{temp} = T^{\alpha}_{I,1}(\pi_{sc})$ for $\alpha > 0$.
(i) The representation π_{x_1,x_2} is of Arthur type if and only if $\frac{1}{2} \leq x_2 \leq \alpha - 3$, $\epsilon_\rho + 1 \leq x_1 \leq \alpha - 2$ and $x_1 > x_2$ for $\alpha \geq \frac{7}{2}$.
(ii) π_{x_1,x_2} is of critical type when $(x_1, x_2) = (\alpha-2, \alpha-3), (\alpha-2, \alpha-2), (\alpha-1, \alpha-1), (\alpha-1, \alpha-2), (\alpha, \alpha), (\alpha, \alpha-1), (\alpha+1, \alpha+1), (\alpha+1, \alpha), (\alpha+1, \alpha-1)$ or $(\alpha+2, \alpha+1)$.
(iii) Define $\mathcal{E}_{x_1,x_2}$ in various cases as follows, then we have $\pi(\mathcal{E}_{x_1,x_2}) = \pi_{x_1,x_2}$. When $\frac{1}{2} \leq x_2 \leq \alpha - 3$, $\epsilon_\rho + 1 \leq x_1 \leq \alpha - 2$ and $x_1 = x_2 + 1$, define

$$\mathcal{E}_{x_1,x_2} := \{([x_1 + 1, -x_1]_\rho, \lfloor x_1 \rfloor, (-1)^{x_1+1}\eta), ([x_1 - 3, \epsilon_\rho]_\rho, 0, \eta), ([x_1 - 1, x_1 - 1]_\rho, 0, (-1)^{x_1-\epsilon_\rho}\eta),$$

$([\alpha-2,x_1+1]_\rho,0,(-1)^{x_1+1-\epsilon_\rho}\eta),([\alpha,\alpha]_\rho,0,(-1)^{x_2+1}\eta)\}.$

Here is the associated symbol.

$$\mathcal{E}_{x_1,x_2}=\begin{array}{c}\begin{array}{ccccccccccccc} -x_1 & \cdots & \epsilon_\rho & \cdots & x_1-3 & x_1-2 & x_1-1 & x_1 & x_1+1 & \cdots & \alpha-2 & \alpha-1 & \alpha \end{array}\\ \begin{pmatrix} \lhd & \cdots & \odot & \cdots & \cdots & \cdots & \cdots & \cdots & \rhd & & & & \\ & & \odot & \cdots & \odot & & & & & & & & \\ & & & & & & \odot & & & & & & \\ & & & & & & & & \odot & \cdots & \odot & & \\ & & & & & & & & & & & & \odot \end{pmatrix}\end{array}.$$

When $\frac{1}{2}\leq x_2\leq \alpha-3$, $\epsilon_\rho+1\leq x_1\leq\alpha-2$ and $x_1-x_1>1$, define

$$\begin{aligned}\mathcal{E}_{x_1,x_2}:=\{&([x_1+1,-x_1]_\rho,\lfloor x_1\rfloor,(-1)^{x_1+1}\eta),([x_2,-x_2]_\rho,\lfloor x_2\rfloor,(-1)^{x_2-\epsilon_\rho}\eta),\\ &([x_2-2,0]_\rho,0,-\eta),([x_1-2,x_2]_\rho,0,(-1)^{x_2}\eta),\\ &([\alpha-2,x_1+1]_\rho,0,(-1)^{x_1+1}\eta),([\alpha,\alpha]_\rho,0,(-1)^{x_2+1}\eta)\}.\end{aligned}$$

Here is the associated symbol.

$$\mathcal{E}_{x_1,x_2}=\begin{array}{c}\begin{array}{ccccccccccccccccccc} -x_1 & \cdots & -x_2 & \cdots & \epsilon_\rho & \cdots & x_2-2 & x_2-1 & x_2 & \cdots & x_1-2 & x_1-1 & x_1 & x_1+1 & \cdots & \alpha-2 & \alpha-1 & \alpha \end{array}\\ \begin{pmatrix} \lhd & \cdots & \lhd & \cdots & \odot & \cdots & \cdots & \cdots & \rhd & \cdots & \rhd & \rhd & \rhd & \rhd & & & & \\ & & \lhd & \cdots & \odot & \cdots & \cdots & \cdots & \rhd & & & & & & & & & \\ & & & & \odot & \cdots & \odot & & & & & & & & & & & \\ & & & & & & \odot & & & & & & & & & & & \\ & & & & & & & & \odot & \cdots & \odot & & & & & & & \\ & & & & & & & & & & & & & \odot & \cdots & \odot & & \\ & & & & & & & & & & & & & & & & & \odot \end{pmatrix}\end{array}.$$

Proof. The sufficient condition can be proven in a similar way as Proposition 5.11, which we omit. Let us now show the necessary direction. First by definition we must have:

$$\pi^{\rho,-}_{x_1,x_2}=L(\Delta_\rho[-x_2,-x_2];\pi_{temp}),$$

which is of Arthur type if and only if $\frac{1}{2}\leq x_2\leq\alpha-1$ for $\alpha\geq\frac{3}{2}$, or $x_2=\alpha=\frac{1}{2}$, which is not possible in this case. Furthermore, since

$$|\Omega|(\pi^{\rho,-}_{x_1,x_2})\subseteq\{\rho|\cdot|^y:\epsilon_\rho\leq y\leq\alpha-2\}\cup\{\rho|\cdot|^\alpha\}\cup\{\rho|\cdot|^{x_2},\rho|\cdot|^{-x_2}\}$$

we must have either

- $\epsilon_\rho+1\leq x_1\leq\alpha-2$, or
- $x_1=x_2$ and $\epsilon_\rho\leq x_2-1\leq\alpha-2$

It suffices to show that the case $x_1=x_2$ will not give a representation of Arthur type. Suppose the contrary, then there exists some segment $([x_1,-x_1+1]_\rho,*,*)$ inside the extended multi-segment $\mathcal{E}_{x_1,x_2}$ corresponding to π_{x_1,x_2} by Definition 4.2. This is impossible since the first segment inside $\mathcal{E}_{x_1,x_2}$ is of the form $([x_1,-x_1],\lfloor x_1\rfloor,*)$, and applying the raising operators will not produce the desired segment. The other segments will contain a gap at x_1-1, so producing a segment of the form $([x_1,-x_1+1]_\rho,*,*)$ is impossible. This proves the sufficient direction. Considering our restrictions on x_1,x_2 gives the desired conditions. This proves part (i). Part (ii) follows from definition. ∎

The next case is $\pi_{temp}=T_{IV,3}(\pi_{sc})$.

Proposition 5.22. *Consider the representation*

$$\pi_{x_1,x_2}=L(\Delta_\rho[-x_1,-x_1-1],\Delta_\rho[-x_2,-x_2];\pi_{temp}),$$

where $x_1\geq 0, x_2\geq\frac{1}{2}$, $x_2\leq x_1$, and $\pi_{temp}=T_{IV,3}(\pi_{sc})$ for $\alpha\in\mathbb{Z}_{>0}$.

(i) *The representation π_{x_1,x_2} is of Arthur type if and only if $1 < x_1 \leq \alpha - 1$ and $1 \leq x_2 \leq \alpha - 2$ with $x_2 < x_1$, or when $x_1 = x_2 = 1$.*

(ii) *The representation π_{x_1,x_2} is of critical type if and only if $(x_1, x_2, \alpha) = (2,1,3), (2,1,2), (2,1,1), ($.*
.

(iii) *Define π_{x_1,x_2} in various cases as follows, then we have $\pi(\mathcal{E}_{x_1,x_2}) = \pi_{x_1,x_2}$. When $1 < x_1 \leq \alpha - 1$, $1 \leq x_2 \leq \alpha - 2$ and $x_1 = x_2 + 1$, define*

$$\begin{aligned}\mathcal{E}_{x_1,x_2} :=& \{([x_1+1, -x_1]_\rho, \lfloor x_1 \rfloor, (-1)^{x_1+1}\eta), ([0,0]_\rho, 0, \eta)^2, ([x_1-3, 0]_\rho, 0, \eta),\\ & ([x_1-1, x_1-1]_\rho, 0, (-1)^{x_1}\eta), ([\alpha-1, x_1+1]_\rho, 0, (-1)^{x_1+1}\eta).\end{aligned}$$

Here is the associated symbol.

$$\mathcal{E}_{x_1,x_2} = \begin{array}{c} \begin{array}{cccccccccccc} -x_1 & \cdots & 0 & \cdots & x_1-3 & x_1-2 & x_1-1 & x_1 & x_1+1 & \cdots & \alpha-2 & \alpha-1 \end{array} \\ \begin{pmatrix} \lhd & \cdots & \odot & \cdots & \cdots & \rhd & \rhd & \rhd & \rhd & & & \\ & & \odot & & & & & & & & & \\ & & \odot & & & & & & & & & \\ & & \odot & \cdots & \odot & & & & & & & \\ & & & & & & \odot & & & & & \\ & & & & & & & & \odot & \cdots & \odot & \odot \end{pmatrix} \end{array}.$$

When $\frac{1}{2} \leq x_2 \leq \alpha - 3$, $\epsilon_\rho + 1 \leq x_1 \leq \alpha - 2$ and $x_1 - x_1 > 1$, define

$$\begin{aligned}\mathcal{E}_{x_1,x_2} :=& \{([x_1+1, -x_1]_\rho, \lfloor x_1 \rfloor, (-1)^{x_1+1}\eta), ([x_2, -x_2]_\rho, \lfloor x_2 \rfloor, (-1)^{x_2}\eta),\\ & ([0,0]_\rho, 0, -\eta)^2, ([x_2-2, 0]_\rho, 0, \eta), ([x_1-2, x_2]_\rho, 0, (-1)^{x_2}\eta),\\ & ([\alpha-1, x_1+1]_\rho, 0, (-1)^{x_1+1}\eta)\}.\end{aligned}$$

Here is the associated symbol.

$$\mathcal{E}_{x_1,x_2} = \begin{array}{c} \begin{array}{ccccccccccccccccc} -x_1 & \cdots & -x_2 & \cdots & 0 & \cdots & x_2-2 & x_2-1 & x_2 & \cdots & x_1-2 & x_1-1 & x_1 & x_1+1 & \cdots & \alpha-2 & \alpha-1 \end{array} \\ \begin{pmatrix} \lhd & \cdots & \lhd & \cdots & \odot & \cdots & \cdots & \cdots & \rhd & \cdots & \rhd & \rhd & \rhd & \rhd & & & \\ & & \lhd & \cdots & \odot & \cdots & \cdots & \cdots & \rhd & & & & & & & & \\ & & & & \odot & \cdots & \odot & & & & & & & & & & \\ & & & & & & \odot & & & & & & & & & & \\ & & & & & & & & \odot & \cdots & \odot & & & & & & \\ & & & & & & & & & & & & & \odot & \cdots & \odot & \odot \end{pmatrix} \end{array}.$$

When $x_1 = x_2 = 1$, we have that $\pi_{1,1} = \pi(\mathcal{E}_{1,1})$, where

$$\mathcal{E}_{1,1} = \begin{array}{c} \begin{array}{ccccccc} -1 & 0 & 1 & 2 & \dots & \alpha-1 & \alpha \end{array} \\ \begin{pmatrix} \lhd & \odot & \rhd & & & & \\ \lhd & \lhd & \odot & \odot & \dots & \rhd & \rhd \\ & \odot & & & & & \\ & & & \odot & \dots & \odot & \odot \end{pmatrix} \end{array}.$$

Proof. From [HJLLZ25, Proposition 10.13], we see that the representation

$$\pi^{\rho,-}_{x_1,x_2} = L(\Delta_\rho[-x_2, -x_2]; \pi_{temp})$$

is of Arthur type if and only if $1 \leq x_2 \leq \alpha$. The rest of the proof is similar to Proposition 5.21, which we omit. □

The last tempered representation of corank 1 is $\pi_{temp} = T^{\pm}_{V,2}(\pi_{sc})$.

Proposition 5.23. *Consider the representation*

$$\pi^{\pm}_{x_1,x_2} = L(\Delta_\rho[-x_1, -x_1-1], \Delta_\rho[-x_2, -x_2]; \pi^{\pm}_{temp}),$$

where $x_1 \geq 0, x_2 \geq \frac{1}{2}$, $x_2 \leq x_1$, and $\pi^{\pm}_{temp} = T^{\pm}_{V,2}(\pi_{sc})$ for $\alpha = 0$.

(i) *The representation* $\pi^{\pm}_{x_1,x_2}(\pi_{sc})$ *is not of Arthur type.*
(ii) *The representation* $\pi^{\pm}_{x_1,x_2}$ *is of critical type if and only if* $(x_1, x_2) = (2, 1)$ *or* $(1, 1)$.

Proof. By [HJLLZ25, Proposition 10.14], the representation

$$(\pi^{\pm}_{x_1,x_2})^{\rho,-} = L(\Delta_\rho[-x_2, -x_2]; \pi^{\pm}_{temp})$$

is of Arthur type if and only if $x_2 = 1$. In this case, $\Psi((\pi^{\pm}_{x_1,x_2})^{\rho,-})$ are singletons with

$$|\Omega|(L(\Delta_\rho[-x_2, -x_2]; \pi^{\pm}_{temp})) \subseteq \{\rho|\cdot|^y, -1 \leq y \leq 1\} \cup \{\rho|\cdot|^{x_2}, \rho|\cdot|^{-x_2}\}.$$

From Lemma 4.4, the only possible x_1 for $\pi^{\pm}_{x_1,x_2}$ to be of Arthur type is $x_1 = x_2 = 1$, but this case can also be eliminated by Definition 4.2 and Theorem 4.3. This proves part (i). Part (ii) follows from definition. ☐

.

This concludes our discussion of Case (C). Now let's move onto Case (D).

5.4. **Case** (D) : $\pi = L(\Delta_\rho[-x_1, -x_1], \Delta_\rho[-x_2, -x_2 - 1]; \pi_{temp})$**.** In this subsection we consider representations of the form $\pi_{x_1,x_2} = L(\Delta_\rho[-x_1, -x_1], \Delta_\rho[-x_2, -x_2 - 1]; \pi_{temp})$, where π_{temp} is tempered of corank 1. Here the situation is similar to Case (C) except the orders of the segments are exchanged. Because the two segments can be linked, they are not necessarily commutative. Just like in Case (C), we have $x_1 > 0, x_2 > -\frac{1}{2}$ and $x_1 - x_2 \geq \frac{1}{2}$ by the Langlands classification of classical groups. By the good parity condition, the last statement can be translated to $x_1 > x_2$.
We'll work with the same three cases as before, starting with $\pi_{temp} = T^{\alpha}_{I,1}(\pi_{sc})$.

Proposition 5.24. *Consider the representation*

$$\pi_{x_1,x_2} = L(\Delta_\rho[-x_1, -x_1], \Delta_\rho[-x_2, -x_2 - 1]; \pi_{temp})$$

where $x_1 > x_2 \geq 0$ *and* $\pi_{temp} = T^{\alpha}_{I,1}(\pi_{sc})$ *for* $\alpha > 0$.

(i) *The representation* π_{x_1,x_2} *is of Arthur type if and only if* $\epsilon_\rho + 1 \leq x_1 \leq \alpha$, $0 \leq x_2 \leq \alpha - 2$ *and* $x_1 - x_2 \geq 2$ *for* $\alpha \geq 3$, *or* $(x_1, x_2) = (1, 0)$ *for* $\alpha \in \mathbb{Z}_{>1}$, *or* $(x_1, x_2, \alpha) = (\frac{5}{2}, \frac{1}{2}, \frac{1}{2})$ *and* $\epsilon_{sc}(\rho \otimes S_2) = -1$.
(ii) *The representation* π_{x_1,x_2} *is of critical type if and only if* $(x_1, x_2) = (\alpha-1, \alpha-3), (\alpha-1, \alpha-2), (\alpha, \alpha-2), (\alpha, \alpha-1), (\alpha+1, \alpha-1), (\alpha+1, \alpha), (\alpha+2, \alpha), (\alpha+2, \alpha+1), (\alpha+3, \alpha+1)$.
(iii) *Define* $\mathcal{E}_{x_1,x_2}$ *in various cases as follows, then* $\pi(\mathcal{E}_{x_1,x_2}) = \pi_{x_1,x_2}$. *When* $\epsilon_\rho + 1 \leq x_1 \leq \alpha - 1$, $0 \leq x_2 \leq \alpha - 2$ *and* $x_1 - x_2 \geq 2$ *for* $\alpha \geq 3$, *define*

$$\begin{aligned}\mathcal{E}_{x_1,x_2} :=& \{([x_1, -x_1]_\rho, \lfloor x_1 \rfloor, \eta), ([x_2 + 1, -x_2]_\rho, \lfloor x_2 \rfloor, -\eta), ([x_2 - 2, \epsilon_\rho]_\rho, 0, \eta), \\ & ([\alpha - 2, x_2 + 1]_\rho, 0, (-1)^{x_2+1-\epsilon_\rho}\eta), ([\alpha, \alpha]_\rho, 0, (-1)^{\alpha-\epsilon_\rho}\eta)\}.\end{aligned}$$

Here is the associated symbol.

$$\mathcal{E}_{x_1,x_2} = \begin{array}{c}\begin{array}{cccccccccccccccc} -x_1 & \cdots & -x_2 & \cdots & \epsilon_\rho & \cdots & x_2-2 & x_2-1 & x_2 & x_2+1 & \cdots & x_1 & \cdots & \alpha-2 & \alpha-1 & \alpha \end{array} \\ \left(\begin{array}{cccccccccccccccc} \lhd & \cdots & \lhd & \cdots & \odot & \cdots & \rhd & \rhd & \rhd & \rhd & \cdots & \rhd & & & & \\ & & \lhd & \cdots & \odot & \cdots & \rhd & \rhd & \rhd & \rhd & & & & & & \\ & & & & \odot & \cdots & \odot & & & & & & & & & \\ & & & & & & & & & \odot & \cdots & \cdots & \cdots & \odot & & \\ & & & & & & & & & & & & & & & \odot \end{array}\right)\end{array}.$$

When $(x_1, x_2) = (1, 0)$ *and* $\alpha \in \mathbb{Z}_{>1}$, *define*

$$\mathcal{E}_{x_1,x_2} := \{([1, -1]_\rho, 1, \eta), ([1, 0]_\rho, 1, -\eta), ([\alpha - 2, 1]_\rho, 0, \eta),$$

$$([\alpha,\alpha]_\rho, 0, (-1)^{\alpha-1}\eta)\}$$

Here is the associated symbol.

$$\mathcal{E}_{x_1,x_2} = \begin{array}{ccccccc} -1 & 0 & 1 & \cdots & \alpha-2 & \alpha-1 & \alpha \end{array} \\ \begin{pmatrix} \lhd & \odot & \rhd & & & & \\ & \lhd & \rhd & & & & \\ & & \odot & \cdots & \odot & & \\ & & & & & & \odot \end{pmatrix}.$$

When $(x_1,x_2,\alpha)=(\frac{5}{2},\frac{1}{2},\frac{1}{2})$ *and* $\epsilon_{sc}(\rho\otimes S_2)=-1$*, let*

$$\mathcal{E}_{x_1,x_2} = \begin{array}{cccccc} -\frac{5}{2} & -\frac{3}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{3}{2} & \frac{5}{2} \end{array} \\ \begin{pmatrix} \lhd & \lhd & \oplus & \ominus & \rhd & \rhd \\ & & \lhd & \rhd & & \end{pmatrix}.$$

Proof. The sufficient direction can be proven in the same way as Proposition 5.1, which we omit. Now we show the necessary direction. By Proposition [HJLLZ25, Proposition 11.11], the representation

$$\pi^{\rho,-}_{x_1,x_2} = L(\Delta_\rho[-x_2,-x_2-1];\pi_{temp})$$

is of Arthur type if and only if $0\le x_2\le \alpha-2$ for $\alpha\ge 2$, or when $(x_2,\alpha)=(0,1)$ or $(x_2,\alpha)=(\frac{1}{2},\frac{1}{2})$. Matching this with our restrictions gives the condition on x_2. For the restrictions on x_1, we see from Lemma 4.4 that either $\epsilon_\rho+1\le x_1\le\alpha-1$, or $x_1=\alpha$. Therefore, it remains to show that the case $x_2=x_1-1$ is not of Arthur type, when $x_1>1, x_2\ge 0$ satisfies the restrictions above.

Suppose π_{x_1,x_2} is of Arthur type in this case. By Definition 4.2, there exists an $\mathcal{E}$ with $\pi(\mathcal{E})=L(\Delta_\rho[-x_2,-x_2-1];\pi_{temp})$ that contains a segment of the form $([x_1-1,-(x_1-1)]_\rho,*,*)$. However, any such $\mathcal{E}$ must have the first segment (by the admissible order) be of the form $([x_1,-(x_1-1)]_\rho,*,*)$. This strictly contains our desired segment, and we cannot reduce the segment any further by definition, which gives a contradiction. This proves the necessary direction and part (i). Parts (ii) and (iii) follow from definition. ∎

The next case is $\pi_{temp}=T_{IV,3}(\pi_{sc})$.

Proposition 5.25. *Consider the representation*

$$\pi_{x_1,x_2} = L(\Delta_\rho[-x_1,-x_1],\Delta_\rho[-x_2,-x_2-1];\pi_{temp}),$$

where $x_1>x_2\ge 0$ *and* $\pi_{temp}=T_{IV,3}(\pi_{sc})$ *for* $\alpha\in\mathbb{Z}_{>0}$*.*

(i) The representation π_{x_1,x_2} *is of Arthur type if and only if* $2\le x_1\le\alpha$, $0\le x_2\le\alpha-2$, *and* $x_1-x_2\ge 2$, *or when* $(x_1,x_2)=(1,0)$ *or* $(\alpha+1,\alpha-1)$.

(ii) The representation π_{x_1,x_2} *is of critical type if and only if*

$$(x_1,x_2,\alpha)=(1,0,1),(2,1,1),(3,1,1),(2,1,2),(3,1,2),(3,1,3).$$

(iii) Define $\mathcal{E}_{x_1,x_2}$ *in various cases as follows, then* $\pi(\mathcal{E}_{x_1,x_2})=\pi_{x_1,x_2}$*. When* $2\le x_1\le\alpha, 0\le x_2\le\alpha-2$ *and* $x_1-x_2\ge 2$*, define*

$$\begin{aligned}\mathcal{E}_{x_1,x_2} :=&\{([x_1,-x_1]_\rho,\lfloor x_1\rfloor,\eta),([x_2,-x_2]_\rho,\lfloor x_2\rfloor,-\eta),([0,0]_\rho,0,\eta)^2\\ &([x_2-2,0]_\rho,0,\eta),([\alpha-1,x_2+1]_\rho,0,(-1)^{x_2+1}\eta)\}.\end{aligned}$$

Here is the associated symbol.

$$\mathcal{E}_{x_1,x_2} = \begin{array}{c} \begin{array}{cccccccccccccc} -x_1 & \cdots & -x_2 & \cdots & 0 & \cdots & x_2-2 & x_2-1 & x_2 & x_2+1 & \cdots & x_1 & \cdots & \alpha-1 \end{array} \\ \begin{pmatrix} \lhd & \cdots & \lhd & \cdots & \odot & \cdots & \rhd & \rhd & \rhd & \rhd & \cdots & \rhd & & \\ & & \lhd & \cdots & \odot & \cdots & \rhd & \rhd & \rhd & & & & & \\ & & & & \odot & & & & & & & & & \\ & & & & \odot & & & & & & & & & \\ & & & & \odot & \cdots & \odot & & & & & & & \\ & & & & & & & & & \odot & \cdots & \cdots & \cdots & \odot \end{pmatrix} \end{array}.$$

When $(x_1, x_2) = (\alpha+1, \alpha-1)$, *define*

$$\begin{aligned} \mathcal{E}_{x_1,x_2} :=& \{([x_1, -x_1]_\rho, \lfloor x_1 \rfloor, \eta), ([x_2, -x_2]_\rho, \lfloor x_2 \rfloor, -\eta), ([0, 0]_\rho, 0, \eta)^2 \\ & ([x_2-2, 0]_\rho, 0, \eta)\}. \end{aligned}$$

Here is the associated symbol.

$$\mathcal{E}_{x_1,x_2} = \begin{array}{c} \begin{array}{ccccccccccc} -x_1 & -\alpha & -x_2 & \cdots & 0 & \cdots & x_2-2 & x_2-1 & x_2 & \alpha & x_1 \end{array} \\ \begin{pmatrix} \lhd & \lhd & \lhd & \cdots & \odot & \cdots & \rhd & \rhd & \rhd & \rhd & \rhd \\ & & \lhd & \cdots & \odot & \cdots & \rhd & \rhd & \rhd & & \\ & & & & \odot & & & & & & \\ & & & & \odot & & & & & & \\ & & & & \odot & \cdots & \odot & & & & \end{pmatrix} \end{array}.$$

When $(x_1, x_2) = (1, 0)$, *define*

$$\mathcal{E}_{x_1,x_2} := \{([1, -1]_\rho, 1, \eta), ([0, 0]_\rho, 0, \eta)^2, ([\alpha-1, 0]_\rho, 1, -\eta), ([1, 1]_\rho, 0, \eta)\}.$$

Here is the associated symbol.

$$\mathcal{E}_{x_1,x_2} = \begin{array}{c} \begin{array}{cccccc} -1 & 0 & 1 & \cdots & \alpha-1 & \alpha \end{array} \\ \begin{pmatrix} \lhd & \odot & \rhd & & & \\ & \odot & & & & \\ & \odot & & & & \\ & \lhd & \odot & \cdots & \rhd & \\ & & \odot & & & \end{pmatrix} \end{array}.$$

Proof. From [HJLLZ25, Proposition 11.12], we see that the representation

$$\pi^{\rho,-}_{x_1,x_2} = L(\Delta_\rho[-x_2, -x_2-1]; \pi_{temp})$$

is of Arthur type if and only if $0 \leq x_2 \leq \alpha - 1$. From here, the rest of the proof follows the exact same way as Proposition 5.24, which we omit. □

The last tempered representation of corank 1 is $\pi_{temp} = T^{\pm}_{V,2}(\pi_{sc})$.

Proposition 5.26. *Consider the representation*

$$\pi^{\pm}_{x_1,x_2} = L(\Delta_\rho[-x_1, -x_1], \Delta_\rho[-x_2, -x_2-1]; \pi^{\pm}_{temp}),$$

where $x_1 > x_2 \geq 0$ *and* $\pi^{\pm}_{temp} = T^{\pm}_{V,2}(\pi_{sc})$ *for* $\alpha = 0$.

(i) The representation $\pi^{\pm}_{x_1,x_2}$ *is of Arthur type if and only if* $(x_1, x_2) = (1, 0)$.
(ii) The representation $\pi^{\pm}_{x_1,x_2}$ *is of critical type if and only if* $(x_1, x_2) = (1, 0), (2, 0)$.
(iii) We have $\pi(\mathcal{E}^{\pm}) = \pi^{\pm}_{0,1}$, *where*

$$\mathcal{E}^{+} = \begin{array}{c} \begin{array}{ccc} -1 & 0 & 1 \end{array} \\ \begin{pmatrix} \lhd & \oplus & \rhd \\ & \oplus & \\ & \oplus & \ominus \end{pmatrix} \end{array}, \quad \mathcal{E}^{-} = \begin{array}{c} \begin{array}{ccc} -1 & 0 & 1 \end{array} \\ \begin{pmatrix} \lhd & \ominus & \rhd \\ & \ominus & \\ & \ominus & \oplus \end{pmatrix} \end{array}.$$

Proof. This follows immediately from [HJLLZ25, Proposition 11.13], which states that the representation
$$(\pi^{\pm}_{x_1,x_2})^{\rho,-} = L(\Delta_\rho[-x_2,-x_2-1];\pi^{\pm}_{temp})$$
is of Arthur type if and only if $x_2 = 0$. □

This concludes our discussion in Case (D). In the next subsection, we will look at Cases $(E),(F),(G)$ together.

5.5. **Cases $(E),(F),(G)$ involving supercuspidal representations.** In this subsection, we will look at the remaining three cases together, since they all involve supercuspidal representations. The first case we'll examine is Case (E), where $\pi_{x_1,x_2} = L(\Delta_\rho[-x_1,-x_1-1],\Delta_\rho[-x_2,-x_2-1];\pi_{sc})$.
From Langlands classification we obtain the bounds $x_1,x_2 > -\frac{1}{2}$ and $x_1 \geq x_2$.

Proposition 5.27. *Consider the representation*
$$\pi_{x_1,x_2} = L(\Delta_\rho[-x_1,-x_1-1],\Delta_\rho[-x_2,-x_2-1];\pi_{sc}),$$
for $x_1,x_2 > -\frac{1}{2}$ and $x_1 \geq x_2$.

(1) The representation π_{x_1,x_2} is of Arthur type if and only if $1 \leq x_1 \leq \alpha-1$ and $x_1 > x_2$, or when $(x_1,x_2) = (0,0)$ or $(\alpha,\alpha-1)$.
(2) The representation π_{x_1,x_2} is of critical type if and only if $(x_1,x_2) = (\alpha-1,\alpha-3),(\alpha-1,\alpha-2),(\alpha-1,\alpha-1),(\alpha,\alpha-2),(\alpha,\alpha-1),(\alpha,\alpha),(\alpha+1,\alpha-1),(\alpha+1,\alpha),(\alpha+2,\alpha)$.
(3) When $1 \leq x_1 \leq \alpha-1$ and $x_1 > x_2$, define
$$\mathcal{E}_{x_1,x_2} := \{([x_1,-x_1]_\rho,\lfloor x_1\rfloor,\eta),([x_2-2,\epsilon_\rho]_\rho,0,\eta),([\alpha-1,x_2+1]_\rho,0,(-1)^{x_2+1}\eta).$$
Then $\pi(\mathcal{E}_{x_1,x_2}) = \pi_{x_1,x_2}$. Here is the associated symbol.
$$\begin{array}{cc} & \begin{array}{ccccccccccccc} -x_1 & \cdots & -x_2 & \cdots & \epsilon_\rho & \cdots & x_2-2 & x_2-1 & x_2 & x_2+1 & \cdots & x_1 & \cdots & \alpha-1 \end{array} \\ \mathcal{E}_{x_1,x_2} = & \left(\begin{array}{cccccccccccccc} \lhd & \cdots & \lhd & \cdots & \odot & \cdots & \rhd & \rhd & \rhd & \rhd & \cdots & \rhd & & \\ & & & & \odot & \cdots & \odot & & & & & & & \\ & & & & & & & & & \odot & \cdots & \cdots & \cdots & \odot \end{array}\right). \end{array}$$

Proof. The sufficient direction can be proved in the same way as Proposition 5.1, which we omit. For the necessary direction, first note that from Proposition 4.42, we have that the representation
$$\pi^{\rho,-}_{x_1,x_2} = L(\Delta_\rho[-x_2,-x_2-1];\pi_{sc})$$
is of Arthur type if and only if $x_2 \leq \alpha-1$. Furthermore, Lemma 4.4 gives us the constraint that either $(x_1,x_2) = (\alpha,\alpha-1),(0,0)$ or $x_1 \leq \alpha-1$ since we have $|\Omega|(\pi_{sc}) = \{\rho|\cdot|^y : \epsilon_\rho \leq y \leq \alpha-1\}$. Therefore, it suffices to show that the case $x_1 = x_2$ does not give a representation of Arthur type.

Suppose on the contrary that π_{x_1,x_2} is of Arthur type when $x_1 = x_2$. Then by Theorem 4.3, there exists an extended multi-segment $\mathcal{E}$ containing 2 copies of the segment $([x_1,-x_1+1],*,*)$ such that $\pi(\mathcal{E}) = \pi_{sc}$. However, we know that $|\Omega|(\pi_{sc})$ is multiplicity-free, which gives a contradiction. This proves part (i). The rest follows from definition. □

Now we move onto Case (F), where $\pi_{x_1,x_2} = L(\Delta_\rho[-x_1,-x_1-2],\Delta_\rho[-x_2,-x_2];\pi_{sc})$, where the Langlands classification gives us the natural constraint $x_1 > -1, x_2 \geq \frac{1}{2}$ and $x_2 - x_1 \leq 1$.

Proposition 5.28. *Consider the representation*

$$\pi_{x_1,x_2} = L(\Delta_\rho[-x_1, -x_1-2], \Delta_\rho[-x_2, -x_2]; \pi_{sc}),$$

where $x_1 > -1, x_2 \geq \frac{1}{2}$ *and* $x_2 - x_1 \leq 1$.

(i) The representation π_{x_1,x_2} *of Arthur type if and only if* $\epsilon_\rho + 1 \leq x_1 \leq \alpha - 2$ *and* $x_2 < x_1$.

(ii) The representation π_{x_1,x_2} *is of critical type if and only if* $(x_1, x_2) = (\alpha-2, \alpha-3), (\alpha-2, \alpha-2), (\alpha-2, \alpha-1), (\alpha-1, \alpha-2), (\alpha-1, \alpha-1), (\alpha-1, \alpha), (\alpha, \alpha-1), (\alpha, \alpha), (\alpha, \alpha+1), (\alpha+1, \alpha)$.

(iii) Define $\mathcal{E}_{x_1,x_2}$ *in various cases as follows, then* $\pi(\mathcal{E}_{x_1,x_2}) = \pi_{x_1,x_2}$. *When* $\epsilon_\rho + 1 \leq x_1 \leq \alpha - 2$ *and* $x_2 = x_1 - 1$, *define*

$$\begin{aligned}\mathcal{E}_{x_1,x_2} :=& \{([x_1+2, -x_1]_\rho, \lfloor x_1 \rfloor, \eta), ([x_2-2, \epsilon_\rho])_\rho, 0, \eta), ([x_1-1, x_2]_\rho, 0, (-1)^{x_2+1-\epsilon_\rho}\eta),\\ &([\alpha-1, x_1+2]_\rho, 0, (-1)^{x_1+1-\epsilon_\rho}\eta).\end{aligned}$$

Here is the associated symbol.

$$\mathcal{E}_{x_1,x_2} = \begin{array}{c} \begin{array}{cccccccccccccccc} -x_1 & \cdots & -x_2 & \cdots & \epsilon_\rho & \cdots & x_2-2 & x_2-1 & x_2 & \cdots & x_1-1 & x_1 & x_1+1 & x_1+2 & \cdots & \alpha-1 \end{array} \\ \left(\begin{array}{cccccccccccccccc} \lhd & \cdots & \lhd & \cdots & \odot & \cdots & \rhd & \rhd & \rhd & \cdots & \rhd & \rhd & \rhd & \rhd & & \\ & & & & \odot & \cdots & \odot & & & & & & & & & \\ & & & & & & & & \odot & \cdots & \odot & & & & & \\ & & & & & & & & & & & & & \odot & \cdots & \odot \end{array}\right) \end{array}.$$

When $\epsilon_\rho + 1 \leq x_1 \leq \alpha - 2$ *and* $x_1 - x_2 > 1$, *define*

$$\begin{aligned}\mathcal{E}_{x_1,x_2} :=& \{([x_1+2, -x_1]_\rho, \lfloor x_1 \rfloor, (-1)^{x_1-\epsilon_\rho}\eta), ([x_2, -x_2]_\rho, \lfloor x_2 \rfloor, (-1)^{x_2-\epsilon_\rho}\eta),\\ &([x_2-2, \epsilon_\rho]_\rho, 0, \eta), ([x_1-2, x_2]_\rho, 0, (-1)^{x_2-\epsilon_\rho}\eta),\\ &([\alpha-1, x_1+2]_\rho, 0, (-1)^{x_1+1-\epsilon_\rho}\eta)\},\end{aligned}$$

Here is the associated symbol.

$$\mathcal{E}_{x_1,x_2} = \begin{array}{c} \begin{array}{cccccccccccccccc} -x_1 & \cdots & -x_2 & \cdots & \epsilon_\rho & \cdots & x_2-2 & x_2-1 & x_2 & \cdots & x_1-2 & x_1-1 & x_1 & x_1+1 & x_1+2\cdots & \alpha-1 \end{array} \\ \left(\begin{array}{cccccccccccccccc} \lhd & \cdots & \lhd & \cdots & \odot & \cdots & \cdots & \cdots & \rhd & \cdots & \rhd & \rhd & \rhd & \rhd & \rhd & \\ & & \lhd & \cdots & \odot & \cdots & \cdots & \cdots & \rhd & & & & & & & \\ & & & & \odot & \cdots & \odot & & & & & & & & & \\ & & & & & & & & \odot & \cdots & \odot & & & & & \\ & & & & & & & & & & & & & \odot & \cdots & \odot \end{array}\right) \end{array}.$$

Proof. The sufficient direction can be proven in a similar way as Proposition 5.27, which we omit. Now we show the necessary direction. Assume $x_2 = x_1 + 1$, and π_{x_1,x_2} is of Arthur type, then by Definition 4.2 and Theorem 4.3, there exists an extended multi-segment $\mathcal{E}$ that contains two segments of the form $([x_1, -x_1]_\rho, *, *)$ and $([x_1+1, -(x_1-1)]_\rho, *, *)$ such that $\pi(\mathcal{E}) = \pi_{sc}$. This is impossible, since the L-parameter of π_{sc} must be multiplicity free. Similarly, we see that the representation π_{x_1,x_2} where $x_1 = x_2$ cannot be of Arthur type. Therefore, it follows that $x_2 < x_1$. Furthermore, by Lemma 4.4 and Theorem 4.42, we have that π_{x_1,x_2} is of Arthur type only if $\epsilon_\rho + 1 \leq x_1 \leq \alpha - 2$. This proves the necessary direction. The rest follows from definition. □

To wrap up the case where $f(\pi) = 2$, we look at the final subcase, Case (G), where $\pi_{x_1,x_2} = L(\Delta_\rho[-x_1, -x_1], \Delta_\rho[-x_2, -x_2-2]; \pi_{sc})$. The natural restriction from Langlands classification gives us $x_1 \geq \frac{1}{2}, x_2 > -1$ and $x_1 - x_2 \geq 1$.

Proposition 5.29. *Consider the representation*

$$\pi_{x_1,x_2} = L(\Delta_\rho[-x_1,-x_1], \Delta_\rho[-x_2,-x_2-2]; \pi_{sc}),$$

where $x_1 \geq \frac{1}{2}, x_2 > -1$ *and* $x_1 - x_2 \geq 1$.

(1) *The representation* π_{x_1,x_2} *is of Arthur type if and only if* $\epsilon_\rho + 1 \leq x_1 \leq \alpha$ *and* $x_2 \leq x_1 - 3$, *or when* $(x_1,x_2) = (\alpha+1, \alpha-2)$ *or* $(1,0)$.

(2) *The representation* π_{x_1,x_2} *is of critical type if and only if* $(x_1,x_2) = (\alpha-1,\alpha-2), (\alpha,\alpha-2), (\alpha+1,\alpha-2), (\alpha,\alpha-1), (\alpha+1,\alpha-1), (\alpha+2,\alpha-1), (\alpha+1,\alpha), (\alpha+2,\alpha), (\alpha+3,\alpha), (\alpha,\alpha-3)$.

(3) *Define* $\mathcal{E}_{x_1,x_2}$ *in various cases as follows, then* $\pi(\mathcal{E}_{x_1,x_2}) = \pi_{x_1,x_2}$. *When* $\epsilon_\rho + 1 \leq x_1 \leq \alpha$ *and* $x_2 = x_1 - 3$, *define*

$$\begin{aligned}\mathcal{E}_{x_1,x_2} :=& \{([x_1,-x_1]_\rho, \lfloor x_1 \rfloor, \eta), ([x_1-2,-(x_1-3)])_\rho, 0, -\eta), ([x_1-5,\epsilon_\rho]_\rho, 0, -\eta),\\ &([\alpha-1, x_1-1]_\rho, 0, (-1)^{x_2+1-\epsilon_\rho}\eta).\end{aligned}$$

Here is the associated symbol.

$$\mathcal{E}_{x_1,x_2} = \begin{array}{c}\begin{array}{cccccccccccccc} -x_1 & -(x_1-1) & -(x_1-2) & -(x_1-3) & \cdots & \epsilon_\rho & \cdots & x_1-5 & \cdots & x_1-2 & x_1-1 & x_1 & \cdots & \alpha-1 \end{array}\\ \begin{pmatrix} \lhd & \lhd & \lhd & \lhd & \cdots & \odot & \cdots & \rhd & \cdots & \rhd & \rhd & \rhd & & \\ & & & \lhd & \cdots & \odot & \cdots & \rhd & \cdots & \rhd & & & & \\ & & & & & \odot & \cdots & \odot & & & & & & \\ & & & & & & & \odot & \cdots & \odot & & & & \\ & & & & & & & & & & \odot & \odot & \cdots & \odot \end{pmatrix}\end{array}.$$

When $\epsilon_\rho + 1 \leq x_1 \leq \alpha$ *and* $x_1 - x_2 > 3$, *define*

$$\begin{aligned}\mathcal{E}_{x_1,x_2} :=& \{([x_1,-x_1]_\rho, \lfloor x_1 \rfloor, (-1)^{x_1-\epsilon_\rho}\eta), ([x_2+2,-x_2]_\rho, \lfloor x_2 \rfloor, (-1)^{x_2-\epsilon_\rho}\eta),\\ &([x_2-2,\epsilon_\rho]_\rho, 0, \eta), ([x_1-2, x_2+2]_\rho, 0, (-1)^{x_2-\epsilon_\rho}\eta),\\ &([\alpha-1, x_1]_\rho, 0, (-1)^{x_1+1-\epsilon_\rho}\eta).\}\end{aligned}$$

Here is the associated symbol.

$$\mathcal{E}_{x_1,x_2} = \begin{array}{c}\begin{array}{ccccccccccccccc} -x_1 & \cdots & -x_2 & \cdots & \epsilon_\rho & \cdots & x_2-2 & \cdots & x_2+2 & \cdots & x_1-2 & x_1-1 & x_1 & \cdots & \alpha-1 \end{array}\\ \begin{pmatrix} \lhd & \cdots & \lhd & \cdots & \odot & \cdots & \odot & \cdots & \rhd & \cdots & \rhd & \rhd & \rhd & & \\ & & \lhd & \cdots & \odot & \cdots & \cdots & \cdots & \rhd & & & & & & \\ & & & & \odot & \cdots & \odot & & & & & & & & \\ & & & & & & & & \odot & \cdots & \odot & & & & \\ & & & & & & & & & & & & \odot & \cdots & \odot \end{pmatrix}\end{array}.$$

When $(x_1,x_2) = (\alpha+1, \alpha-2)$, *define*

$$\mathcal{E}_{x_1,x_2} := \{([x_1,-x_1]_\rho, \lfloor x_1 \rfloor, \eta), ([x_1-2,-(x_1-3)])_\rho, 0, -\eta), ([x_1-4,\epsilon_\rho]_\rho, 0, -\eta)\}.$$

Here is the associated symbol. Here is the associated symbol.

$$\mathcal{E}_{x_1,x_2} = \begin{array}{c}\begin{array}{ccccccccccccc} -x_1 & -(x_1-1) & -(x_1-2) & -(x_1-3) & \cdots & \epsilon_\rho & \cdots & x_1-5 & x_1-4 & x_1-3 & x_1-2 & x_1-1 & x_1 \end{array}\\ \begin{pmatrix} \lhd & \lhd & \lhd & \lhd & \cdots & \odot & \cdots & \rhd & \rhd & \rhd & \rhd & \rhd & \rhd \\ & & & \lhd & \cdots & \odot & \cdots & \rhd & \rhd & \rhd & \rhd & & \\ & & & & & \odot & \cdots & \odot & \odot & & & & \end{pmatrix}\end{array}.$$

When $(x_1,x_2) = (1,0)$, *define*

$$\mathcal{E} = \{([1,-1]_\rho, 1, \eta), ([\alpha-1, 0]_\rho, 1, -\eta), ([2,2]_\rho, 0, \eta)\}.$$

Then $\pi_{1,0} = \pi(\mathcal{E})$. Here is the associated symbol.

$$\mathcal{E}_{x_1,x_2} = \begin{array}{c} \begin{matrix} -1 & 0 & 1 & 2 & \ldots & \alpha-1 \end{matrix} \\ \begin{pmatrix} \lhd & \odot & \rhd & & & \\ & \lhd & \odot & \odot & \ldots & \rhd \\ & & & \odot & & \end{pmatrix} \end{array}.$$

Proof. The proof is similar to Proposition 5.27, which we omit. □

With this, we are done with the classification of non-tempered representations π with $f(\pi) = 2$. We will move onto the cases $f(\pi) = 3$ and $f(\pi) = 4$ now.

6. Classification of non-tempered representations of corank 4 ($f(\pi) = 3, 4$)

In this section, we consider non-tempered representations π of corank 4, where there are exactly 3 or 4 segments in the L-data of π, i.e. $f(\pi) = 3, 4$.

6.1. The case $f(\pi) = 3$.

We begin with the case where $f(\pi) = 3$. With these restrictions, we can group them into the following subcases:

(A) $\pi = L(\Delta_\rho[-x_1, -x_1], \Delta_\rho[-x_2, -x_2], \Delta_\rho[-x_3, -x_3]; \pi_{temp})$, where $x_1 > x_2 > x_3 \geq \frac{1}{2}$ and π_{temp} is tempered of corank 1.
(B) $\pi = L(\Delta_\rho[-x_1, -x_1], \Delta_\rho[-x_1, -x_1], \Delta_\rho[-x_2, -x_2]; \pi_{temp})$, where $x_1 > x_2 \geq \frac{1}{2}$, and π_{temp} is tempered of corank 1.
(C) $\pi = L(\Delta_\rho[-x_1, -x_1], \Delta_\rho[-x_2, -x_2], \Delta_\rho[-x_2, -x_2]; \pi_{temp})$, where $x_1 > x_2 \geq \frac{1}{2}$, and π_{temp} is tempered of corank 1.
(D) $\pi = L(\Delta_\rho[-x, -x], \Delta_\rho[-x, -x], \Delta_\rho[-x, -x]; \pi_{temp})$, where $x \geq \frac{1}{2}$ and π_{temp} is tempered of corank 1.
(E) $\pi = L(\Delta_\rho[-x_1, -x_1 - 1], \Delta_\rho[-x_2, -x_2], \Delta_\rho[-x_3, -x_3]; \pi_{sc})$, where $x_1 \geq x_2 \geq x_3$.
(F) $\pi = L(\Delta_\rho[-x_1, -x_1], \Delta_\rho[-x_2, -x_2 - 1], \Delta_\rho[-x_3, -x_3]; \pi_{sc})$, where $x_1 > x_2 \geq x_3$.
(G) $\pi = L(\Delta_\rho[-x_1, -x_1], \Delta_\rho[-x_2, -x_2], \Delta_\rho[-x_3, -x_3 - 1]; \pi_{sc})$, where $x_1 \geq x_2 > x_3$.

6.2. Case (A) : $\pi = L(\Delta_\rho[-x_2, -x_2], \Delta_\rho[-x_1, -x_1], \Delta_\rho[-x_3, -x_3]; \pi_{temp})$.

We begin with Case (A), where we denote $\pi_{x_1,x_2,x_3} = L(\Delta_\rho[-x_1, -x_1], \Delta_\rho[-x_2, -x_2], \Delta_\rho[-x_3, -x_3]; \pi_{temp})$ for $x_1 > x_2 > x_3 \geq \frac{1}{2}$ and π_{temp} tempered of corank 1. There are 3 cases to consider. The first one is $\pi_{temp} = T_{I,1}^{\alpha}(\pi_{sc})$

Proposition 6.1. *Consider the representation*

$$\pi_{x_1,x_2,x_3} = L(\Delta_\rho[-x_1, -x_1], \Delta_\rho[-x_2, -x_2], \Delta_\rho[-x_3, -x_3]; \pi_{temp}),$$

for $x_1 > x_2 > x_3 \geq \frac{1}{2}$, where $\pi_{temp} = T_{I,1}^{\alpha}(\pi_{sc})$ and $\alpha > 0$.

(i) The representation π_{x_1,x_2,x_3} is of Arthur type if and only if one of the following holds:
- $\epsilon_\rho + 1 \leq x_1 \leq \alpha - 1$ *and* $\frac{1}{2} \leq x_3 < x_2 \leq \alpha - 2$.
- $(x_1, x_2) = (\alpha, \alpha - 1)$ *and* $\frac{1}{2} < x_3 \leq \alpha - 2$.
- $(x_1, x_2, x_3) = (\alpha + 1, \alpha, \alpha - 1)$.
- $(x_1, x_2, x_3, \alpha) = (\frac{5}{2}, \frac{3}{2}, \frac{1}{2}, \frac{1}{2})$.

(ii) The representation π_{x_1,x_2,x_3} is of critical type in the following cases:
- $(x_1, x_2, x_3) = (\alpha - 1, \alpha - 2, \alpha - 3)$.
- $(x_1, x_2, x_3) = (\alpha, \alpha - 1, \alpha - 2)$.
- $(x_1, x_2, x_3) = (\alpha + 1, \alpha, \alpha - 1)$.
- $(x_1, x_2, x_3) = (\alpha + 2, \alpha + 1, \alpha)$ *or* $(\alpha + 3, \alpha + 2, \alpha + 1)$.

(iii) Define $\mathcal{E}_{x_1,x_2,x_3}$ in various cases as follows. Then $\pi(\mathcal{E}_{x_1,x_2,x_3}) = \pi_{x_1,x_2,x_3}$. When $\epsilon_\rho + 1 \leq x_1 \leq \alpha + 1$ and $\frac{1}{2} \leq x_3 < x_2 \leq \alpha$, with $x_1 = x_2 + 1 = x_3 + 2$, define

$$\begin{aligned}\mathcal{E}_{x_1,x_2,x_3} :=&\{[x_1,-x_1]_\rho, \lfloor x_1 \rfloor, \eta), ([x_3-2,\epsilon_\rho]_\rho, 0, -\eta),\\ &([\alpha-2, x_2]_\rho, 0, (-1)^{x_2-\epsilon_\rho}\eta), ([\alpha,\alpha]_\rho, 0, (-1)^{\alpha-\epsilon_\rho}\eta)\}.\end{aligned}$$

Here is the associated symbol.

$$\mathcal{E}_{x_1,x_2,x_3} = \begin{array}{c} \begin{array}{cccccccccccc} -x_1 & \cdots & \epsilon_\rho & \cdots & x_3-2 & \cdots & x_2 & x_1 & \cdots & \alpha-2 & \alpha-1 & \alpha \end{array} \\ \begin{pmatrix} \triangleleft & \cdots & \odot & \cdots & \triangleright & \cdots & \triangleright & \triangleright & & & & \\ & & \odot & \cdots & \odot & & & & & & & \\ & & & & & & \odot & \odot & \cdots & \odot & & \\ & & & & & & & & & & & \odot \end{pmatrix} \end{array}.$$

When $\epsilon_\rho + 1 \leq x_1 \leq \alpha + 1$ and $\frac{1}{2} \leq x_3 < x_2 \leq \alpha$, with $x_1 = x_2 + 1 > x_3 + 2$, define

$$\begin{aligned}\mathcal{E}_{x_1,x_2,x_3} :=&\{[x_1,-x_1]_\rho, \lfloor x_1 \rfloor, \eta), ([x_3,-x_3]_\rho, \lfloor x_3 \rfloor, -\eta), ([x_3-2,\epsilon_\rho]_\rho, 0, -\eta),\\ &([x_2-2, x_3]_\rho, 0, (-1)^{x_3+1-\epsilon_\rho}\eta), ([\alpha-2, x_2]_\rho, 0, (-1)^{x_2-\epsilon_\rho}\eta), ([\alpha,\alpha]_\rho, 0, (-1)^{\alpha-\epsilon_\rho}\eta)\}.\end{aligned}$$

Here is the associated symbol.

$$\mathcal{E}_{x_1,x_2,x_3} = \begin{array}{c} \begin{array}{cccccccccccccccccc} -x_1 & \cdots & -x_3 & \cdots & \epsilon_\rho & \cdots & x_3-2 & x_3-1 & x_3 & \cdots & x_2-2 & x_2-1 & x_2 & x_1 & \cdots & \alpha-2 & \alpha-1 & \alpha \end{array} \\ \begin{pmatrix} \triangleleft & \cdots & \triangleleft & \cdots & \odot & \cdots & \triangleright & \triangleright & \triangleright & \cdots & \triangleright & \triangleright & \triangleright & \triangleright & \triangleright & & & \\ & & \triangleleft & \cdots & \odot & \cdots & \triangleright & \triangleright & \triangleright & & & & & & & & & \\ & & & & \odot & \cdots & \odot & & & & & & & & & & & \\ & & & & & & & & \odot & \cdots & \odot & & & & & & & \\ & & & & & & & & & & & & \odot & \odot & \cdots & \odot & & \\ & & & & & & & & & & & & & & & & & \odot \end{pmatrix} \end{array}.$$

When $\epsilon_\rho + 1 \leq x_1 \leq \alpha - 1$ and $\frac{1}{2} \leq x_3 < x_2 \leq \alpha - 2$, with $x_1 > x_2 + 1 = x_3 + 2$, define

$$\begin{aligned}\mathcal{E}_{x_1,x_2,x_3} :=&\{[x_1,-x_1]_\rho, \lfloor x_1 \rfloor, \eta), ([x_2,-x_2]_\rho, \lfloor x_2 \rfloor, -\eta), ([x_3-2,\epsilon_\rho]_\rho, 0, -\eta), ([x_1-2, x_3]_\rho, 0, (-1)^{x_3+1-\epsilon_\rho}\eta),\\ &([\alpha-2, x_1]_\rho, 0, (-1)^{x_1-\epsilon_\rho}\eta), ([\alpha,\alpha]_\rho, 0, (-1)^{\alpha-\epsilon_\rho}\eta)\}.\end{aligned}$$

Here is the associated symbol.

$$\mathcal{E}_{x_1,x_2,x_3} = \begin{array}{c} \begin{array}{cccccccccccccccccc} -x_1 & \cdots & -x_2 & \cdots & \epsilon_\rho & \cdots & x_3-2 & x_3-1 & x_3 & x_2 & \cdots & x_1-2 & x_1-1 & x_1 & \cdots & \alpha-2 & \alpha-1 & \alpha \end{array} \\ \begin{pmatrix} \triangleleft & \cdots & \triangleleft & \cdots & \odot & \cdots & \triangleright & \triangleright & \triangleright & \triangleright & \cdots & \triangleright & \triangleright & \triangleright & & & & \\ & & \triangleleft & \cdots & \odot & \cdots & \triangleright & \triangleright & \triangleright & \triangleright & & & & & & & & \\ & & & & \odot & \cdots & \odot & & & & & & & & & & & \\ & & & & & & & & \odot & \odot & \cdots & \odot & & & & & & \\ & & & & & & & & & & & & & \odot & \cdots & \odot & & \\ & & & & & & & & & & & & & & & & & \odot \end{pmatrix} \end{array}.$$

When $\epsilon_\rho + 1 \leq x_1 \leq \alpha - 1$ and $\frac{1}{2} \leq x_3 < x_2 \leq \alpha - 2$, with $x_1 > x_2 + 1 > x_3 + 2$, define

$$\begin{aligned}\mathcal{E}_{x_1,x_2,x_3} :=&\{[x_1,-x_1]_\rho, \lfloor x_1 \rfloor, \eta), ([x_2,-x_2]_\rho, \lfloor x_2 \rfloor, -\eta), ([x_3,-x_3]_\rho, \lfloor x_3 \rfloor, \eta), ([x_3-2,\epsilon_\rho]_\rho, 0, -\eta)\\ &([x_1-2, x_3]_\rho, 0, (-1)^{x_3+1-\epsilon_\rho}\eta), ([x_2-2, x_3]_\rho, 0, (-1)^{x_3-\epsilon_\rho}), ([x_1-2, x_2]_\rho, 0, (-1)^{x_2-\epsilon_\rho}\eta),\\ &([\alpha-2, x_1]_\rho, 0, (-1)^{x_1-\epsilon_\rho}\eta), ([\alpha,\alpha]_\rho, 0, (-1)^{\alpha-\epsilon_\rho}\eta)\}.\end{aligned}$$

Here is the associated symbol.

$$\mathcal{E}_{x_1,x_2,x_3} = \begin{array}{c} \begin{array}{ccccccccccccccccccccccc} -x_1 & \cdots & -x_2 & \cdots & -x_3 & \cdots & \epsilon_\rho & \cdots & x_3-2 & x_3-1 & x_3 & \cdots & x_2-2 & x_2-1 & x_2 & \cdots & x_1-2 & x_1-1 & x_1 & \cdots & \alpha-2 & \alpha-1 & \alpha \end{array} \\ \left(\begin{array}{ccccccccccccccccccccccc} \lhd & \cdots & \lhd & \cdots & \lhd & \cdots & \odot & \cdots & \rhd & \rhd & \rhd & \cdots & \rhd & \rhd & \rhd & \cdots & \rhd & \rhd & \rhd & & & & \\ & & \lhd & \cdots & \lhd & \cdots & \odot & \cdots & \rhd & \rhd & \rhd & \cdots & \rhd & \rhd & \rhd & & & & & & & & \\ & & & & \lhd & \cdots & \odot & \cdots & \rhd & \rhd & \rhd & & & & & & & & & & & & \\ & & & & & & \odot & \cdots & \odot & & & & & & & & & & & & & & \\ & & & & & & & & & & \odot & \cdots & \odot & & & & & & & & & & \\ & & & & & & & & & & & & & & \odot & \cdots & \odot & & & & & & \\ & & & & & & & & & & & & & & & & & & \odot & \cdots & \odot & & \\ & & & & & & & & & & & & & & & & & & & & & & \odot \end{array} \right) \end{array}.$$

When $(x_1, x_2, x_3, \alpha) = (\frac{5}{2}, \frac{3}{2}, \frac{1}{2}, \frac{1}{2})$, *define*

$$\mathcal{E}_{\frac{5}{2},\frac{3}{2},\frac{1}{2}} = \begin{array}{c} \begin{array}{cccccc} -\frac{5}{2} & -\frac{3}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{3}{2} & \frac{5}{2} \end{array} \\ \left(\begin{array}{cccccc} \lhd & \lhd & \lhd & \rhd & \rhd & \rhd \\ & & & \oplus & & \end{array} \right) \end{array}$$

Proof. The sufficient direction can be proven in a similar way as Proposition 5.1, which we omit. Now let's show the necessary direction.

First, by [HJLLZ25, Proposition 11.15], we have that:

$$\pi^{\rho,-}_{x_1,x_2,x_3} = L(\Delta_\rho[-x_2, -x_2], \Delta_\rho[-x_3, -x_3]; \pi_{temp})$$

is of Arthur type if and only if $\frac{1}{2} \leq x_3 < x_2 \leq \alpha - 1$, $(x_2, x_3) = (\alpha, \alpha - 1)$ or $(x_2, x_3, \alpha) = (\frac{3}{2}, \frac{1}{2}, \frac{1}{2})$. Furthermore, if π_{x_1,x_2,x_3} is of Arthur type, then by Lemma 4.4, we require that

$$x_1 - 1 \in |\Omega|(\pi_{x_1,x_2,x_3}(\pi_{sc})^{\rho,-}) \subseteq \{\rho|\cdot|^y : \epsilon_\rho \leq y \leq \alpha - 2\} \cup \{\rho|\cdot|^\alpha\} \cup \{\rho|\cdot|^{x_2}.\rho|\cdot|^{-x_2}, \rho|\cdot|^{x_3}, \rho|\cdot|^{-x_3}\}$$

We conclude that π_{x_1,x_2,x_3} is of Arthur type only if one of the following holds:

- $\epsilon_\rho + 1 \leq x_1 \leq \alpha - 1$ and $\frac{1}{2} \leq x_3 < x_2 \leq \alpha - 2$.
- $(x_1, x_2) = (\alpha, \alpha - 1)$ and $\frac{1}{2} \leq x_3 \leq \alpha - 2$
- $(x_1, x_2, x_3) = (\alpha + 1, \alpha, \alpha - 1)$
- $x_1 = \alpha + 1$ and $\frac{1}{2} \leq x_3 < x_2 \leq \alpha - 1$.

Therefore it suffices to eliminate the last case. This follows from the exact same argument used in Proposition 5.1, which we omit. This proves part (*i*) of the proposition. Part (*ii*) follows from definition. □

The next case to consider is $\pi_{temp} = T_{IV,3}(\pi_{sc})$.

Proposition 6.2. *Consider the representation*

$$\pi_{x_1,x_2,x_3} = L(\Delta_\rho[-x_1, -x_1], \Delta_\rho[-x_2, -x_2], \Delta_\rho[-x_3, -x_3]; \pi_{temp}),$$

for $x_1 > x_2 > x_3 \geq \frac{1}{2}$, *where* $\pi_{temp} = T_{IV,3}(\pi_{sc})$ *and* $\alpha \in \mathbb{Z}_{>0}$.

(*i*) *The representation* $\pi_{x_1,x_2,x_3}(\pi_{sc})$ *is of Arthur type if and only if one of the following holds:*
- $1 \leq x_1 \leq \alpha$ *and* $1 \leq x_3 < x_2 \leq \alpha - 1$.
- $(x_1, x_2) = (\alpha + 1, \alpha)$ *and* $1 \leq x_3 \leq \alpha - 1$.
- $(x_1, x_2, x_3) = (\alpha + 2, \alpha + 1, \alpha)$.

(*ii*) *The representation* $\pi_{x_1,x_2,x_3}(\pi_{sc})$ *is of critical type in the following cases:*
- $(x_1, x_2, x_3) = (\alpha, \alpha - 1, \alpha - 2)$ *for* $\alpha = 3$.
- $(x_1, x_2, x_3) = (\alpha + 1, \alpha, \alpha - 1)$ *for* $\alpha = 2$.
- $(x_1, x_2, x_3) = (\alpha + 2, \alpha + 1, \alpha)$ *for* $\alpha = 1$.

(iii) Define $\mathcal{E}_{x_1,x_2,x_3}$ in various cases as follows. Then $\pi(\mathcal{E}_{x_1,x_2,x_3}) = \pi_{x_1,x_2,x_3}$. When $1 \leq x_1 \leq \alpha+2$ and $1 \leq x_3 < x_2 \leq \alpha+1$, with $x_1 = x_2+1 = x_3+2$, define

$$\begin{aligned}\mathcal{E}_{x_1,x_2,x_3} :=& \{[x_1,-x_1]_\rho, \lfloor x_1 \rfloor, \eta), ([0,0]_\rho, 0, \eta)^2, ([x_3-2,0]_\rho, 0, -\eta),\\ &([\alpha-1,x_2]_\rho, 0, (-1)^{x_2}\eta).\end{aligned}$$

Here is the associated symbol.

$$\mathcal{E}_{x_1,x_2,x_3} = \begin{array}{c}\begin{array}{cccccccccc} -x_1 & \cdots & 0 & \cdots & x_3-2 & \cdots & x_2 & x_1 & \cdots & \alpha-1 \end{array}\\ \left(\begin{array}{cccccccccc} \lhd & \cdots & \odot & \cdots & \rhd & \cdots & \rhd & \rhd & & \\ & & \odot & & & & & & & \\ & & \odot & & & & & & & \\ & & \odot & \cdots & \odot & & & & & \\ & & & & & & \odot & \odot & \cdots & \odot \end{array}\right)\end{array}$$

When $1 \leq x_1 \leq \alpha+1$ and $1 \leq x_3 < x_2 \leq \alpha$, with $x_1 = x_2+1 > x_3+2$, define

$$\begin{aligned}\mathcal{E}_{x_1,x_2,x_3} :=& \{[x_1,-x_1]_\rho, \lfloor x_1 \rfloor, \eta), ([x_3,-x_3]_\rho, \lfloor x_3 \rfloor, -\eta), ([0,0]_\rho, 0, -\eta)^2, ([x_3-2,0]_\rho, 0, -\eta),\\ &([x_2-2,x_3]_\rho, 0, (-1)^{x_3+1}\eta), ([\alpha-1,x_2]_\rho, 0, (-1)^{x_2}\eta).\end{aligned}$$

Here is the associated symbol.

$$\mathcal{E}_{x_1,x_2,x_3} = \begin{array}{c}\begin{array}{cccccccccccccccc} -x_1 & \cdots & -x_3 & \cdots & 0 & \cdots & x_3-2 & x_3-1 & x_3 & \cdots & x_2-2 & x_2-1 & x_2 & x_1 & \cdots & \alpha-1 \end{array}\\ \left(\begin{array}{cccccccccccccccc} \lhd & \cdots & \lhd & \cdots & \odot & \cdots & \rhd & \rhd & \rhd & \cdots & \rhd & \rhd & \rhd & \rhd & & \\ & & \lhd & \cdots & \odot & \cdots & \rhd & \rhd & \rhd & & & & & & & \\ & & & & \odot & & & & & & & & & & & \\ & & & & \odot & & & & & & & & & & & \\ & & & & \odot & \cdots & \odot & & & & & & & & & \\ & & & & & & & & \odot & \cdots & \odot & & & & & \\ & & & & & & & & & & & & \odot & \odot & \cdots & \odot \end{array}\right)\end{array}$$

When $1 \leq x_1 \leq \alpha-1$ and $1 \leq x_3 < x_2 \leq \alpha-2$, with $x_1 > x_2+1 = x_3+2$, define

$$\begin{aligned}\mathcal{E}_{x_1,x_2,x_3} :=& \{[x_1,-x_1]_\rho, \lfloor x_1 \rfloor, \eta), ([x_2,-x_2]_\rho, \lfloor x_2 \rfloor, -\eta), ([0,0]_\rho, 0, -\eta)^2, ([x_3-2,0]_\rho, 0, -\eta),\\ &([x_1-2,x_3]_\rho, 0, (-1)^{x_3+1}\eta), ([\alpha-1,x_1]_\rho, 0, (-1)^{x_1}\eta)\}.\end{aligned}$$

Here is the associated symbol.

$$\mathcal{E}_{x_1,x_2,x_3} = \begin{array}{c}\begin{array}{cccccccccccccccc} -x_1 & \cdots & -x_2 & \cdots & 0 & \cdots & x_3-2 & x_3-1 & x_3 & x_2 & \cdots & x_1-2 & x_1-1 & x_1 & \cdots & \alpha-1 \end{array}\\ \left(\begin{array}{cccccccccccccccc} \lhd & \cdots & \lhd & \cdots & \odot & \cdots & \rhd & \rhd & \rhd & \rhd & \cdots & \rhd & \rhd & \rhd & & \\ & & \lhd & \cdots & \odot & \cdots & \rhd & \rhd & \rhd & \rhd & & & & & & \\ & & & & \odot & & & & & & & & & & & \\ & & & & \odot & & & & & & & & & & & \\ & & & & \odot & \cdots & \odot & & & & & & & & & \\ & & & & & & & & \odot & \odot & \cdots & \odot & & & & \\ & & & & & & & & & & & & & \odot & \cdots & \odot \end{array}\right)\end{array}.$$

When $1 \leq x_1 \leq \alpha-1$ and $1 \leq x_3 < x_2 \leq \alpha-2$, with $x_1 > x_2+1 > x_3+2$, define

$$\begin{aligned}\mathcal{E}_{x_1,x_2,x_3} :=& \{[x_1,-x_1]_\rho, \lfloor x_1 \rfloor, \eta), ([x_2,-x_2]_\rho, \lfloor x_2 \rfloor, -\eta), ([x_3,-x_3]_\rho, \lfloor x_3 \rfloor, \eta), ([x_3-2,0]_\rho, 0, -\eta)\\ &([x_1-2,x_3]_\rho, 0, (-1)^{x_3+1}\eta), ([x_2-2,x_3]_\rho, 0, (-1)^{x_3-\epsilon_\rho}), ([x_1-2,x_2]_\rho, 0, (-1)^{x_2}\eta),\\ &([\alpha-1,x_1]_\rho, 0, (-1)^{x_1}\eta).\end{aligned}$$

Here is the associated symbol.

$$\mathcal{E}_{x_1,x_2,x_3} = \begin{array}{c} \begin{array}{ccccccccccccccccccccc} -x_1 & \cdots & -x_2 & \cdots & -x_3 & \cdots & \epsilon_\rho & \cdots & x_3-2 & x_3-1 & x_3 & \cdots & x_2-2 & x_2-1 & x_2 & \cdots & x_1-2 & x_1-1 & x_1 & \cdots & \alpha-1 \end{array} \\ \left(\begin{array}{ccccccccccccccccccccc} \lhd & \cdots & \lhd & \cdots & \lhd & \cdots & \odot & \cdots & \rhd & \rhd & \rhd & \cdots & \rhd & \rhd & \rhd & \cdots & \rhd & \rhd & \rhd & & \\ & & \lhd & \cdots & \lhd & \cdots & \odot & \cdots & \rhd & \rhd & \rhd & \cdots & \rhd & \rhd & \rhd & & & & & & \\ & & & & \lhd & \cdots & \odot & \cdots & \rhd & \rhd & \rhd & & & & & & & & & & \\ & & & & & & \odot & \cdots & \odot & & & & & & & & & & & & \\ & & & & & & & & & & \odot & \cdots & \odot & & & & & & & & \\ & & & & & & & & & & & & & & \odot & \cdots & \odot & & & & \\ & & & & & & & & & & & & & & & & & & \odot & \cdots & \odot \end{array} \right) \end{array}.$$

Proof. By [HJLLZ25, Proposition 11.16], we know that the representation

$$\pi^{\rho,-}_{x_1,x_2,x_3} = L(\Delta_\rho[-x_2,-x_2], \Delta_\rho[-x_3,-x_3]; \pi_{temp})$$

is of Arthur type if and only if $1 \leq x_3 < x_2 \leq \alpha$ or $(x_2, x_3) = (\alpha+1, \alpha)$. The rest of the proof follows in a similar way as Proposition 6.1, which we omit. □

The last case to consider in Case (A) is $\pi_{temp} = T^{\pm}_{V,2}(\pi_{sc})$.

Proposition 6.3. *Consider the representation*

$$\pi^{\pm}_{x_1,x_2,x_3} = L(\Delta_\rho[-x_1,-x_1], \Delta_\rho[-x_2,-x_2], \Delta_\rho[-x_3,-x_3]; \pi^{\pm}_{temp}),$$

for $x_1 > x_2 > x_3 \geq \frac{1}{2}$, where $\pi^{\pm}_{temp} = T_{V,2}(\pi_{sc})^{\pm}$ and $\alpha = 0$.

(1) The representation $\pi^{\pm}_{x_1,x_2,x_3}$ is of Arthur type if and only if $(x_1,x_2,x_3) = (3,2,1)$.
(2) The representation $\pi^{\pm}_{3,2,1}$ is of critical type if and only if $(x_1,x_2,x_3) = (3,2,1)$.
(3) We have $\pi(\mathcal{E}^{\pm}) = \pi^{\pm}_{x_1,x_2,x_3}$, where

$$\mathcal{E}^{+} = \begin{array}{c} \begin{array}{ccccccc} -3 & -2 & -1 & 0 & 1 & 2 & 3 \end{array} \\ \left(\begin{array}{ccccccc} \lhd & \lhd & \lhd & \oplus & \rhd & \rhd & \rhd \\ & & & \oplus & & & \end{array} \right) \end{array}, \quad \mathcal{E}^{-} = \begin{array}{c} \begin{array}{ccccccc} -3 & -2 & -1 & 0 & 1 & 2 & 3 \end{array} \\ \left(\begin{array}{ccccccc} \lhd & \lhd & \lhd & \oplus & \rhd & \rhd & \rhd \\ & & & \ominus & & & \end{array} \right) \end{array}.$$

Proof. This follows immediately from Lemma 4.4 and [HJLLZ25, Proposition 11.17], which state that the representation

$$(\pi^{\pm}_{x_1,x_2,x_3})^{\rho,-} = L(\Delta_\rho[-x_2,-x_2], \Delta_\rho[-x_3,-x_3]; \pi^{\pm}_{temp})$$

is of Arthur type if and only if $(x_2, x_3) = (2, 1)$. □

This concludes all the cases in Case (A). Now we move onto Case (B).

6.3. **Case** $(B): \pi = L(\Delta_\rho[-x_1,-x_1], \Delta_\rho[-x_1,-x_1], \Delta_\rho[-x_2,-x_2]; \pi_{temp})$**.** In this subsection we will investigate representations of the form

$$\pi_{x_1,x_2} = L(\Delta_\rho[-x_1,-x_1], \Delta_\rho[-x_1,-x_1], \Delta_\rho[-x_2,-x_2]; \pi_{temp})$$

for $x_1 > x_2 \geq \frac{1}{2}$ where π_{temp} is tempered of corank 1. Again, there are three cases to consider. The first one is $\pi_{temp} = T^{\alpha}_{I,1}(\pi_{sc})$.

Proposition 6.4. *Consider the representation*

$$\pi_{x_1,x_2} = L(\Delta_\rho[-x_1,-x_1], \Delta_\rho[-x_1,-x_1], \Delta_\rho[-x_2,-x_2]; \pi_{temp}),$$

for $x_1 > x_2 \geq \frac{1}{2}$, where $\pi_{temp} = T^{\alpha}_{I,1}(\pi_{sc})$ and $\alpha > 0$.

(i) The representation π_{x_1,x_2} is of Arthur type if and only if $(x_1,x_2,\alpha) = (\frac{3}{2}, \frac{1}{2}, \frac{1}{2})$.
(ii) The representation π_{x_1,x_2} is of critical type if and only if $(x_1,x_2) = (\alpha+2,\alpha+1), (\alpha+1,\alpha), (\alpha+1,\alpha-1), (\alpha,\alpha-1)$ or $(\alpha-1,\alpha-2)$.

(iii) *When* $\alpha = \frac{1}{2}$, *we have* $\pi(\mathcal{E}) = \pi_{\frac{3}{2},\frac{1}{2}}$, *where*

$$\mathcal{E} = \begin{array}{c} \begin{array}{cccc} -\frac{3}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{3}{2} \end{array} \\ \left(\begin{array}{cccc} \lhd & \odot & \odot & \rhd \\ \lhd & \odot & \odot & \rhd \end{array} \right) \end{array}.$$

Proof. We only need to show the necessary direction. By [HJLLZ25, Proposition 10.12], the representation

$$\pi^{\rho,-}_{x_1,x_2} = L(\Delta_\rho[-x_2,-x_2];\pi_{temp})$$

is of Arthur type if and only if $\frac{1}{2} \leq x_2 \leq \alpha - 1$ when $\alpha \geq \frac{3}{2}$, or $x_2 = \frac{1}{2} = \alpha$. On the other hand, Lemma 4.4 tells us that π_{x_1,x_2} is of Arthur type only if $|\Omega|(\pi^{\rho,-}_{x_1,x_2})$ contains two copies of $\rho|\cdot|^{x_1-1}$. We know that:

$$|\Omega|(\pi^{\rho,-}_{x_1,x_2}) \subset \{\rho|\cdot|^y : 0 \leq y \leq \alpha - 2\} \cup \{\rho|\cdot|^\alpha\} \cup \{\rho|\cdot|^{x_2}, \rho|\cdot|^{-x_2}\}$$

By our assumption we cannot have $x_1 = 1$. Therefore, from this we may conclude that π_{x_1,x_2} can only be of Arthur type in the following cases:

- $1 \leq x_2 \leq \alpha - 2$ and $x_1 = x_2 + 1$ for $\alpha \geq \frac{3}{2}$
- $(x_1, x_2, \alpha) = (\frac{3}{2}, \frac{1}{2}, \frac{1}{2})$

We will show that the first case fails. Suppose $1 \leq x_2 \leq \alpha - 2$ and $x_1 = x_2 + 1$ for $\alpha \geq \frac{3}{2}$ and π_{x_1,x_2} is of Arthur type. Then by Theorem 4.3 and Definition 4.2, there exists an extended multi-segment $\mathcal{E}$ that contains two segments of the form $([x_2, -x_2]_\rho, *, *)$ such that $\pi(\mathcal{E}) = L(\Delta_\rho[-x_2,-x_2];\pi_{temp})$. However, by [HJLLZ25, Proposition 10.12], the multiplicity of $x_2 - 1$ inside the extended multi-segment $\mathcal{E}$ can only be 1, so we reach a contradiction. This proves part (i). Part (ii) follows from definition. □

The second case we need to consider is $\pi_{temp} = T_{IV,3}(\pi_{sc})$.

Proposition 6.5. *Consider the representation*

$$\pi_{x_1,x_2} = L(\Delta_\rho[-x_1,-x_1], \Delta_\rho[-x_1,-x_1], \Delta_\rho[-x_2,-x_2];\pi_{temp}),$$

for $x_1 > x_2 \geq 1$, *where* $\pi_{temp} = T_{IV,3}(\pi_{sc})$ *and* $\alpha \in \mathbb{Z}_{>0}$.

(i) *The representation* π_{x_1,x_2} *is not of Arthur type for any* x_1, x_2.
(ii) *The representation* π_{x_1,x_2} *is of critical type if and only if one of the following holds:*
  - $(x_1, x_2) = (\alpha + 1, \alpha)$ *and* $\alpha = 1$.
  - $(x_1, x_2) = (\alpha, \alpha - 1)$ *and* $\alpha = 2$.

Proof. From [HJLLZ25, Proposition 10.13], the representation

$$\pi^{\rho,-}_{x_1,x_2} = L(\Delta_\rho[-x_2,-x_2];\pi_{temp})$$

can only be of Arthur type when $1 \leq x_2 \leq \alpha$. On the other hand, Lemma 4.4 tells us that π_{x_1,x_2} is of Arthur type only if $|\Omega|(\pi^{\rho,-}_{x_1,x_2})$ contains two copies of $\rho|\cdot|^{x_1-1}$. We know that:

$$|\Omega|(\pi^{\rho,-}_{x_1,x_2}) \subset \{\rho|\cdot|^y : 0 \leq y \leq \alpha - 1\} \cup \{\rho, \rho\} \cup \{\rho|\cdot|^{x_2}, \rho|\cdot|^{-x_2}.\}$$

By assumption we cannot have $x_1 = 1$, which means that π_{x_1,x_2} can only be of Arthur type if $1 \leq x_2 \leq \alpha - 1$ and $x_1 = x_2 + 1$. However, by the same argument as in the proof of Proposition 6.4, we can disregard this case as well. Therefore π_{x_1,x_2} is not of Arthur type for any x_1, x_2. □

The last case to consider is $\pi_{temp} = T^{\pm}_{V,2}(\pi_{sc})$.

Proposition 6.6. *Consider the representation*

$$\pi^{\pm}_{x_1,x_2} = L(\Delta_\rho[-x_1,-x_1], \Delta_\rho[-x_1,-x_1], \Delta_\rho[-x_2,-x_2]; \pi^{\pm}_{temp}),$$

for $x_1 > x_2 \geq 1$, where $\pi^{\pm}_{temp} = T^{\pm}_{V,2}(\pi_{sc})$ and $\alpha = 0$.

(1) The representation $\pi^{\pm}_{x_1,x_2}$ is not of Arthur type for any x_1, x_2.
(2) The representation π_{x_1,x_2} is of critical type if and only if $(x_1, x_2) = (2, 1)$.

Proof. By Lemma 4.4 and [HJLLZ25, Proposition 10.14], the representation $\pi^{\pm}_{x_1,x_2}$ can only be of Arthur type if $x_1 = 2, x_2 = 1$. It's easy to verify that this case does not work. This proves the proposition. $\square$

This concludes our discussion in Case (B). We move onto Case (C).

6.4. **Case** (C) : $\pi = L(\Delta_\rho[-x_1,-x_1], \Delta_\rho[-x_2,-x_2], \Delta_\rho[-x_2,-x_2]; \pi_{temp})$**.** In this subsection, we will consider representations of the form

$$\pi_{x_1,x_2} = L(\Delta_\rho[-x_1,-x_1], \Delta_\rho[-x_2,-x_2], \Delta_\rho[-x_2,-x_2]; \pi_{temp}),$$

where $x_1 > x_2 \geq \frac{1}{2}$ and π_{temp} is tempered of corank 1. This may look similar to the previous case, but since the Steinberg segments do not commute in general, we need to deal with them separately. The first case to consider is $\pi_{temp} = T^{\alpha}_{I,1}(\pi_{sc})$.

Proposition 6.7. *Consider the representation*

$$\pi_{x_1,x_2} = L(\Delta_\rho[-x_1,-x_1], \Delta_\rho[-x_2,-x_2], \Delta_\rho[-x_2,-x_2]; \pi_{temp}),$$

where $x_1 > x_2 \geq \frac{1}{2}$ and $\pi_{temp} = T^{\alpha}_{I,1}(\pi_{sc})$ for $\alpha > 0$.

(i) The representation π_{x_1,x_2} is of Arthur type if and only if $(x_1, x_2) = (\frac{3}{2}, \frac{1}{2})$, or if $x_2 = \frac{1}{2}, \frac{3}{2} \leq x_1 \leq \alpha - 1$.
(ii) The representation π_{x_1,x_2} is of critical type if and only if

$$(x_1, x_2) = (\alpha+2, \alpha+1), (\alpha+1, \alpha), (\alpha+1, \alpha-1), (\alpha, \alpha-1), (\alpha-1, \alpha-2).$$

(iii) We have $\pi(\mathcal{E}_{\frac{3}{2},\frac{1}{2}}) = \pi_{\frac{3}{2},\frac{1}{2}}$, where

$$\mathcal{E}_{\frac{3}{2},\frac{1}{2}} = \begin{array}{c} \begin{array}{cccccc} -\frac{3}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{3}{2} & \cdots & \alpha-1 \end{array} \\ \begin{pmatrix} \lhd & \lhd & \rhd & \rhd & & \\ & \lhd & \rhd & & & \\ & & \odot & \odot & \cdots & \odot \end{pmatrix} \end{array}.$$

Proof. The sufficient direction is easy to verify. Now we show the necessary direction. By [HJLLZ25, Proposition 11.18], the representation

$$\pi^{\rho,-}_{x_1,x_2} = L(\Delta_\rho[-x_2,-x_2], \Delta_\rho[-x_2,-x_2]; \pi_{temp})$$

is of Arthur type if and only if $x_2 = \frac{1}{2}$. By 4.4, the representation π_{x_1,x_2} is only of Arthur type when the set $|\Omega|(\pi^{\rho,-}_{x_1,x_2})$ contains $\rho|\cdot|^{x_1}$. This happens only when $x_1 = \frac{3}{2}$, $x_1 \leq \alpha - 1$, or $x_1 = \alpha + 1$. The last case may be eliminated via a similar argument as before. $\square$

The next case to consider is $\pi_{temp} = T_{IV,3}(\pi_{sc})$.

Proposition 6.8. *Consider the representation*

$$\pi_{x_1,x_2} = L(\Delta_\rho[-x_1,-x_1], \Delta_\rho[-x_2,-x_2], \Delta_\rho[-x_2,-x_2]; \pi_{temp}),$$

where $x_1 > x_2 \geq \frac{1}{2}$ and $\pi_{temp} = T_{IV,3}(\pi_{sc})$ for $\alpha \in \mathbb{Z}_{>0}$.

(i) The representation π_{x_1,x_2} is of Arthur type if and only if $1 < x_1 \leq \alpha - 1$ and $x_2 = 1$, or when $(x_1, x_2) = (2, 1)$.

(ii) The representation π_{x_1,x_2} is of critical type when $(x_1, x_2, \alpha) = (2, 1, 1)$ or $(2, 1, 2)$.

(iii) Define $\mathcal{E}_{x_1,x_2}$ in various cases as follows. Then $\pi(\mathcal{E}_{x_1,x_2}) = \pi_{x_1,x_2}$. When $(x_1, x_2) = (2, 1)$, define

$$\mathcal{E}_{2,1} = \begin{array}{c} \begin{array}{cccccc} -2 & -1 & 0 & 1 & 2 & \cdots \ \alpha-1 \end{array} \\ \begin{pmatrix} \lhd & \lhd & \odot & \rhd & \rhd & & \\ & \lhd & \odot & \rhd & & & \\ & & \odot & \odot & \odot & \cdots & \odot \end{pmatrix} \end{array}.$$

When $x_2 = 1, x_1 > 2$, define

$$\mathcal{E}_{x_1,x_2} := \{([x_1, -x_1]_\rho, \lfloor x_1 \rfloor, \eta), ([1, -1]_\rho, 1, \eta)^2, ([x_1 - 2, 0]_\rho, 0, \eta), ([\alpha - 1, x_1]_\rho, 0, (-1)^{x_1}\eta).$$

Here is the associated symbol.

$$\mathcal{E}_{x_1,x_2} = \begin{array}{c} \begin{array}{ccccccccccc} -x_1 & \cdots & -1 & 0 & 1 & \cdots & x_1-2 & x_1-1 & x_1 & \cdots & \alpha-1 \end{array} \\ \begin{pmatrix} \lhd & \cdots & \lhd & \odot & \rhd & \cdots & \rhd & \rhd & \rhd & & \\ & & \lhd & \odot & \rhd & & & & & & \\ & & \lhd & \odot & \rhd & & & & & & \\ & & & \odot & \odot & \cdots & \odot & \odot & \odot & & \\ & & & & & & & & \odot & \cdots & \odot \end{pmatrix} \end{array}.$$

Proof. The sufficient direction is easy to verify. The necessary direction follows from Lemma 4.4 and [HJLLZ25, Proposition 11.19]. The rest follows from definition. □

The final case we need to consider in Case (C) is $\pi_{temp} = T^{\pm}_{V,2}(\pi_{sc})$

Proposition 6.9. *Consider the representation*

$$\pi^{\pm}_{x_1,x_2} = L(\Delta_\rho[-x_1, -x_1], \Delta_\rho[-x_2, -x_2], \Delta_\rho[-x_2, -x_2]; \pi^{\pm}_{temp}),$$

where $x_1 > x_2 \geq \frac{1}{2}$ and $\pi^{\pm}_{temp} = T^{\pm}_{V,2}(\pi_{sc})$ for $\alpha = 0$.

(i) The representation $\pi^{\pm}_{x_1,x_2}$ is of Arthur type if and only if $(x_1, x_2) = (2, 1)$.

(ii) The representation $\pi^{\pm}_{x_1,x_2}$ is of critical type if and only if $(x_1, x_2) = (2, 1)$.

(iii) We have $\pi(\mathcal{E}^{\pm}) = \pi^{\pm}_{2,1}$, where

$$\mathcal{E}_+ = \begin{array}{c} \begin{array}{ccccc} -2 & -1 & 0 & 1 & 2 \end{array} \\ \begin{pmatrix} \lhd & \lhd & \oplus & \rhd & \rhd \\ & \lhd & \oplus & \rhd & \end{pmatrix} \end{array}, \quad \mathcal{E}_- = \begin{array}{c} \begin{array}{ccccc} -2 & -1 & 0 & 1 & 2 \end{array} \\ \begin{pmatrix} \lhd & \lhd & \ominus & \rhd & \rhd \\ & \lhd & \ominus & \rhd & \end{pmatrix} \end{array}.$$

Proof. This follows directly from [HJLLZ25, Proposition 11.20] and Lemma 4.4. □

This concludes our discussion of Case (C). Now we proceed to Case (D).

6.5. **Case** $(D) : \pi = L(\Delta_\rho[-x, -x], \Delta_\rho[-x, -x], \Delta_\rho[-x, -x]; \pi_{temp})$**.** In this subsection we will consider the situation where all three segments inside the L-data of π are identical, i.e. when $\pi_x = L(\Delta_\rho[-x, -x], \Delta_\rho[-x, -x], \Delta_\rho[-x, -x]; \pi_{temp})$ for $x \geq \frac{1}{2}$ and π_{temp} is tempered of corank 1. As usual, there are three cases to consider. First is $\pi_{temp} = T^{\alpha}_{I,1}(\pi_{sc})$.

Proposition 6.10. *Consider the representation*

$$\pi_x = L(\Delta_\rho[-x, -x], \Delta_\rho[-x, -x], \Delta_\rho[-x, -x]; \pi_{temp}),$$

where $x \geq \frac{1}{2}$ and $\pi_{temp} = T^{\alpha}_{I,1}(\pi_{sc})$ for $\alpha > 0$.

(i) The representation π_x is not of Arthur type for any x.

(ii) The representation π_x is of critical type when $x \in \{\alpha - 1, \alpha, \alpha + 1\}$.

Proof. Part (i) follows directly from Lemma 4.4. The key point is that π_{temp} is multiplicity-free. Part (ii) follows from definition. $\square$

The next case is $\pi_{temp} = T_{IV,3}(\pi_{sc})$, which will give us the only nontrivial case in this section.

Proposition 6.11. *Consider the representation*

$$\pi_x = L(\Delta_\rho[-x,-x], \Delta_\rho[-x,-x], \Delta_\rho[-x,-x]; \pi_{temp}),$$

where $x \geq \frac{1}{2}$ *and* $\pi_{temp} = T_{IV,3}(\pi_{sc})$ *for* $\alpha \in \mathbb{Z}_{>0}$.

(i) The representation π_x *is of Arthur type if and only if* $x = 1$.
(ii) The representation π_x *is of critical type if and only if* $x = \alpha = 1$.
(iii) We have $\pi(\mathcal{E}) = \pi_1$, *where*

$$\mathcal{E} = \begin{array}{c} \begin{array}{ccccc} -1 & 0 & 1 & \cdots & \alpha-1 \end{array} \\ \left(\begin{array}{ccccc} \lhd & \odot & \rhd & & \\ \lhd & \odot & \rhd & & \\ \lhd & \odot & \rhd & & \\ & & \odot & \cdots & \odot \end{array}\right) \end{array}.$$

Proof. By Lemma 4.4, the representation π_x is of Arthur type only if there exists three copies of $\rho|\cdot|^{x-1}$ inside $|\Omega|(\pi_x^{\rho,-})$. This can only happen when $x = 1$. This proves the necessary direction. The sufficient direction is easy to verify and the rest follows from definition. $\square$

The last case we have is $\pi_{temp} = T_{V,2}^{\pm}(\pi_{sc})$.

Proposition 6.12. *Consider the representation*

$$\pi_x^{\pm} = L(\Delta_\rho[-x,-x], \Delta_\rho[-x,-x], \Delta_\rho[-x,-x]; \pi_{temp}^{\pm}]),$$

where $x \geq \frac{1}{2}$ *and* $\pi_{temp}^{\pm} = T_{V,2}^{\pm}(\pi_{sc})$ *for* $\alpha = 0$.

(i) The representation π_x *is not of Arthur type for any* x.
(ii) The representation π_x *is of critical type if and only if* $x = 1$.

Proof. This follows directly from Lemma 4.4. $\square$

This concludes our discussion of Case (D). Now we are ready to move on to Cases $(E), (F), (G)$, where the three segments inside the L-data of π exhaust the corank of π by 4.

6.6. **Cases** $(E), (F), (G)$ **involving supercuspidal representations.** The last three subcases in this section all involve supercuspidal representations. We start with Case (E), where $\pi = L(\Delta_\rho[-x_1, -x_1-1], \Delta_\rho[-x_2,-x_2], \Delta_\rho[-x_3,-x_3]; \pi_{sc})$, where $x_1 \geq x_2 \geq x_3$ by Langlands classification and the good parity condition.

Proposition 6.13. *Consider the representation*

$$\pi_{x_1,x_2,x_3} = L(\Delta_\rho[-x_1,-x_1-1], \Delta_\rho[-x_2,-x_2], \Delta_\rho[-x_3,-x_3]; \pi_{sc}),$$

where $x_1 \geq x_2 \geq x_3 \geq \frac{1}{2}$.

(i) The representation π_{x_1,x_2,x_3} *is of Arthur type if and only if* $\frac{1}{2} \leq x_3 < x_2 < x_1 \leq \alpha - 1$, *or* $(x_1,x_2,x_3) = (\frac{3}{2}, \frac{1}{2}, \frac{1}{2})$ *or* $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$.
(ii) The representation π_{x_1,x_2,x_3} *is of critical type if and only if* $(x_1,x_2,x_3) = (\alpha,\alpha,\alpha), (\alpha,\alpha,\alpha-1), (\alpha,\alpha-1,\alpha-1), (\alpha,\alpha-1,\alpha-2), (\alpha-1,\alpha-1,\alpha-1), (\alpha-1,\alpha-1,\alpha-2), (\alpha-1,\alpha-2,\alpha-2), (\alpha-1,\alpha-2,\alpha-3), (\alpha+1,\alpha,\alpha), (\alpha+1,\alpha,\alpha-1), (\alpha+1,\alpha+1,\alpha), (\alpha+2,\alpha+1,\alpha)$.

(iii) Define $\mathcal{E}_{x_1,x_2,x_3}$ in various cases as follows. Then $\pi(\mathcal{E}_{x_1,x_2,x_3}) = \pi_{x_1,x_2,x_3}$. When $\frac{1}{2} \leq x_3 < x_2 < x_1 \leq \alpha - 1$, and $x_1 = x_2 + 1 = x_3 + 2$, define

$$\begin{aligned}\mathcal{E}_{x_1,x_2,x_3} :=& \{([x_1+1,-x_1]_\rho, \lfloor x_1 \rfloor, \eta), ([x_3-2,\epsilon_\rho]_\rho, 0, \eta), ([x_1-1,x_3]_\rho, 0, (-1)^{x_3-\epsilon_\rho}\eta),\\ & ([\alpha-1,x_1+1]_\rho, 0, (-1)^{x_1+1-\epsilon_\rho}\eta)\}.\end{aligned}$$

Here is the associated symbol.

$$\mathcal{E}_{x_1,x_2,x_3} = \begin{array}{c} \begin{array}{ccccccccccccc} -x_1 & \cdots & \epsilon_\rho & \cdots & x_3-2 & x_3-1 & x_3 & \cdots & x_1-1 & x_1 & x_1+1 & \cdots & \alpha-1 \end{array} \\ \left(\begin{array}{ccccccccccccc} \lhd & \cdots & \odot & \cdots & \rhd & \rhd & \rhd & \cdots & \rhd & \rhd & \rhd & & \\ & & \odot & \cdots & \odot & & & & & & & & \\ & & & & & & \odot & \cdots & \odot & & & & \\ & & & & & & & & & & \odot & \cdots & \odot \end{array}\right) \end{array}.$$

When $\frac{1}{2} \leq x_3 < x_2 < x_1 \leq \alpha - 1$, and $x_1 > x_2 + 1 = x_3 + 2$, define

$$\begin{aligned}\mathcal{E}_{x_1,x_2,x_3} :=& \{([x_1+1,-x_1]_\rho, \lfloor x_1 \rfloor, \eta), ([x_2,-x_2]_\rho, \lfloor x_2 \rfloor, \eta), ([x_3-2,\epsilon_\rho]_\rho, 0, \eta),\\ & ([x_1-2,x_3]_\rho, 0, (-1)^{x_3-\epsilon_\rho}\eta), ([\alpha-1,x_1+1]_\rho, 0, (-1)^{x_1+1-\epsilon_\rho}\eta)\}.\end{aligned}$$

Here is the associated symbol.

$$\mathcal{E}_{x_1,x_2,x_3} = \begin{array}{c} \begin{array}{ccccccccccccccccc} -x_1 & \cdots & -x_2 & \cdots & \epsilon_\rho & \cdots & x_3-2 & x_3-1 & x_3 & x_2 & \cdots & x_1-2 & x_1-1 & x_1 & x_1+1 & \cdots & \alpha-1 \end{array} \\ \left(\begin{array}{ccccccccccccccccc} \lhd & \cdots & \lhd & \cdots & \odot & \cdots & \rhd & \rhd & \rhd & \rhd & \cdots & \rhd & \rhd & \rhd & \rhd & & \\ & & \lhd & \cdots & \odot & \cdots & \rhd & \rhd & \rhd & \rhd & & & & & & & \\ & & & & \odot & \cdots & \odot & & & & & & & & & & \\ & & & & & & & & \odot & \odot & \cdots & \odot & & & & & \\ & & & & & & & & & & & & & & \odot & \cdots & \odot \end{array}\right) \end{array}.$$

When $\frac{1}{2} \leq x_3 < x_2 < x_1 \leq \alpha - 1$, and $x_1 > x_2 + 1 > x_3 + 1$, define

$$\begin{aligned}\mathcal{E}_{x_1,x_2,x_3} :=& \{([x_1+1,-x_1]_\rho, \lfloor x_1 \rfloor, \eta), ([x_2,-x_2]_\rho, \lfloor x_2 \rfloor, \eta), ([x_3,-x_3]_\rho, \lfloor x_3 \rfloor, \eta), ([x_3-2,\epsilon_\rho]_\rho, 0, \eta),\\ & ([x_2-2,x_3]_\rho, 0, (-1)^{x_3-\epsilon_\rho}\eta), ([x_1-2,x_2]_\rho, 0, (-1)^{x_2-\epsilon_\rho}\eta), ([\alpha-1,x_1+1]_\rho, 0, (-1)^{x_1+1-\epsilon_\rho}\eta)\}.\end{aligned}$$

Here is the associated symbol.

$$\mathcal{E}_{x_1,x_2,x_3} = \begin{array}{c} \begin{array}{cccccccccccccccccccccc} -x_1 & \cdots & -x_2 & \cdots & -x_3 & \cdots & \epsilon_\rho & \cdots & x_3-2 & x_3-1 & x_3 & \cdots & x_2-2 & x_2-1 & x_2 & \cdots & x_1-2 & x_1-1 & x_1 & x_1+1 & \cdots & \alpha-1 \end{array} \\ \left(\begin{array}{cccccccccccccccccccccc} \lhd & \cdots & \lhd & \cdots & \lhd & \cdots & \odot & \cdots & \rhd & \rhd & \rhd & \cdots & \rhd & \rhd & \rhd & \cdots & \rhd & \rhd & \rhd & \rhd & & \\ & & \lhd & \cdots & \lhd & \cdots & \odot & \cdots & \rhd & \rhd & \rhd & \cdots & \rhd & \rhd & \rhd & & & & & & & \\ & & & & \lhd & \cdots & \odot & \cdots & \rhd & \rhd & \rhd & & & & & & & & & & & \\ & & & & & & \odot & \cdots & \odot & & & & & & & & & & & & & \\ & & & & & & & & & & \odot & \cdots & \odot & & & & & & & & & \\ & & & & & & & & & & & & & & \odot & \cdots & \odot & & & & & \\ & & & & & & & & & & & & & & & & & & & \odot & \cdots & \odot \end{array}\right) \end{array}.$$

When $(x_1, x_2, x_3) = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$, let

$$\mathcal{E}_{x_1,x_2,x_3} = \begin{array}{c} \begin{array}{ccccc} -\frac{1}{2} & \frac{1}{2} & \frac{3}{2} & \cdots & \alpha-1 \end{array} \\ \left(\begin{array}{ccccc} \lhd & \rhd & & & \\ \lhd & \rhd & & & \\ \lhd & \odot & \rhd & & \\ & & \odot & \cdots & \odot \end{array}\right) \end{array}.$$

When $(x_1, x_2, x_3) = (\frac{3}{2}, \frac{1}{2}, \frac{1}{2})$, *let*

$$\mathcal{E}_{x_1,x_2,x_3} = \begin{array}{c} \begin{array}{ccccccc} -\frac{3}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{3}{2} & \frac{5}{2} & \cdots & \alpha-1 \end{array} \\ \left(\begin{array}{ccccccc} \lhd & \lhd & \odot & \rhd & \rhd & & \\ & \lhd & \rhd & & & & \\ & & \odot & & & & \\ & & & & \odot & \cdots & \odot \end{array} \right) \end{array}.$$

Proof. The sufficient direction can be proved in a similar way as Proposition 6.1, which we omit. Now we show the necessary direction. First assume $x_1 = x_2$ and π_{x_1,x_2,x_3} is of Arthur type. Then by Definition 4.2 and Theorem 4.3, there exists an extended multi-segment $\mathcal{E}$ containing a segment of the form $([x, -(x-1)]_\rho, *, *)$ and another segment of the form $[x-1, -(x-1)]_\rho$ such that $\pi(\mathcal{E}) = \pi_{sc})$. This cannot happen since the L-parameter of π_{sc} is multiplicity-free. Thus we may conclude that $x_1 > x_2$.

Then by [HJLLZ25, Proposition 9.12], the representation

$$\pi^{\rho,-}_{x_1,x_2,x_3} = L(\Delta_\rho[-x_2, -x_2], \Delta_\rho[-x_3, -x_3]; \pi_{sc})$$

is of Arthur type if and only if one of the following holds:

- $\frac{1}{2} \leq x_3 < x_2 \leq \alpha$,
- $(x_2, x_3) = (\alpha+1, \alpha)$,
- $(x_2, x_3) = (\alpha, \alpha-1)$,
- $x_2 = x_3 = \frac{1}{2}$.

Combining these conditions with Lemma 4.4, we may conclude that π_{x_1,x_2,x_3} is of Arthur type only when $\frac{1}{2} \leq x_3 < x_2 < x_1 \leq \alpha - 1$ and when $x_2 = x_3 = \frac{1}{2}$. In the second case, we must have $x_1 = \frac{1}{2}$ or $\frac{3}{2}$, since otherwise there exists an extended multi-segment $\mathcal{E}$ containing a segment of the form $([x_1, -(x_1-1)]_\rho$ such that $\pi(\mathcal{E}) = L(\Delta_\rho[-\frac{1}{2}, -\frac{1}{2}], \Delta_\rho[-\frac{1}{2}, -\frac{1}{2}]; \pi_{sc})$ with $x_1 > \frac{3}{2}$, which cannot happen. This proves part (i). The rest follows from definition ☐

This concludes our discussion of Case (E). Now we move on to Case (F), where

$$\pi = L(\Delta_\rho[-x_1, -x_1], \Delta_\rho[-x_2, -x_2-1], \Delta_\rho[-x_3, -x_3]; \pi_{sc}),$$

for $x_1 > x_2 \geq x_3$.

Proposition 6.14. *Consider the representation*

$$\pi_{x_1,x_2,x_3} = L(\Delta_\rho[-x_1, -x_1], \Delta_\rho[-x_2, -x_2-1], \Delta_\rho[-x_3, -x_3]; \pi_{sc}),$$

where $x_1 > x_2 \geq x_3 \geq \frac{1}{2}$.

(i) The representation π_{x_1,x_2,x_3} *is of Arthur type if and only if one of the following holds:*
  - $\epsilon_\rho + 1 < x_1 \leq \alpha$, $\frac{1}{2} \leq x_3 < x_2 \leq \alpha - 2$ *and* $x_1 - x_2 > 1$.
  - $(x_1, x_2) = (\alpha+1, \alpha-1)$, *and* $\frac{1}{2} \leq x_3 < x_2$.
  - $(x_1, x_2, x_3) = (\frac{3}{2}, \frac{1}{2}, \frac{1}{2})$ *and* $\alpha \in \frac{1}{2} + \mathbb{Z}_{>0}$.

(ii) The representation π_{x_1,x_2,x_3} *is of critical type if and only if* $(x_1, x_2, x_3) = (\alpha+3, \alpha+1, \alpha), (\alpha+2, \alpha+1, \alpha), (\alpha+2, \alpha, \alpha), (\alpha+2, \alpha, \alpha-1), (\alpha+1, \alpha, \alpha), (\alpha+1, \alpha, \alpha-1), (\alpha+1, \alpha-1, \alpha-1), (\alpha+1, \alpha-1, \alpha-2), (\alpha, \alpha-1, \alpha-1), (\alpha, \alpha-1, \alpha-2), (\alpha, \alpha-2, \alpha-2), (\alpha, \alpha-2, \alpha-3)$.

(iii) Define $\mathcal{E}_{x_1,x_2,x_3}$ *in various cases as follows. Then* $\pi(\mathcal{E}_{x_1,x_2,x_3}) = \pi_{x_1,x_2,x_3}$. *When* $\epsilon_\rho + 1 < x_1 \leq \alpha, x_1 - x_2 = 2$, *and* $\frac{1}{2} \leq x_3 = x_2 - 1 \leq \alpha - 3$, *define*

$$\mathcal{E}_{x_1,x_2,x_3} := \{([x_1, -x_1]_\rho, \lfloor x_1 \rfloor, \eta), ([x_2, -x_2]_\rho, \lfloor x_2 \rfloor, -\eta), ([x_3 - 2, \epsilon_\rho]_\rho, 0, \eta),$$

$$([x_3, x_3]_\rho, 0, (-1)^{x_3-\epsilon_\rho}\eta), ([\alpha-1, x_2+1]_\rho, 0, (-1)^{x_2+1-\epsilon_\rho}\eta)\}.$$

Here is the associated symbol.

$$\mathcal{E}_{x_1,x_2,x_3} = \begin{array}{c} \begin{array}{ccccccccccccc} -x_1 & \cdots & -x_2 & \cdots & \epsilon_\rho & \cdots & x_3-2 & x_3-1 & x_3 & x_2 & x_1-1 & x_1 & \cdots & \alpha-1 \end{array} \\ \begin{pmatrix} \lhd & \cdots & \lhd & \cdots & \odot & \cdots & \rhd & \rhd & \rhd & \rhd & \rhd & \rhd & & \\ & & \lhd & \cdots & \odot & \cdots & \rhd & \rhd & \rhd & \rhd & & & & \\ & & & & \odot & \cdots & \odot & & & & & & & \\ & & & & & & & & \odot & & & & & \\ & & & & & & & & & & \odot & \odot & \cdots & \odot \end{pmatrix} \end{array}.$$

When $\epsilon_\rho + 1 < x_1 \leq \alpha, x_1 - x_2 > 2,$ *and* $\frac{1}{2} \leq x_3 = x_2 - 1 \leq \alpha - 3,$ *define*

$$\begin{aligned}\mathcal{E}_{x_1,x_2,x_3} :=&\{([x_1, -x_1]_\rho, \lfloor x_1 \rfloor, \eta), ([x_2+1, -x_2]_\rho, \lfloor x_2 \rfloor, -\eta), ([x_3-2, \epsilon_\rho]_\rho, 0, -\eta),\\ &([x_3, x_3]_\rho, 0, (-1)^{x_3-\epsilon_\rho}\eta), ([x_1-2, x_2+1]_\rho, 0, (-1)^{x_2-\epsilon_\rho}\eta), ([\alpha-1, x_1]_\rho, 0, (-1)^{x_1-\epsilon_\rho}\eta)\}.\end{aligned}$$

Here is the associated symbol.

$$\mathcal{E}_{x_1,x_2,x_3} = \begin{array}{c} \begin{array}{ccccccccccccccccc} -x_1 & \cdots & -x_2 & \cdots & \epsilon_\rho & \cdots & x_3-2 & x_3-1 & x_3 & x_2 & x_2+1 & \cdots & x_1-2 & x_1-1 & x_1 & \cdots & \alpha-1 \end{array} \\ \begin{pmatrix} \lhd & \cdots & \lhd & \cdots & \odot & \cdots & \rhd & \rhd & \rhd & \rhd & \rhd & \cdots & \rhd & \rhd & \rhd & & \\ & & \lhd & \cdots & \odot & \cdots & \rhd & \rhd & \rhd & \rhd & \rhd & & & & & & \\ & & & & \odot & \cdots & \odot & & & & & & & & & & \\ & & & & & & & & \odot & & & & & & & & \\ & & & & & & & & & & \odot & \cdots & \odot & & & & \\ & & & & & & & & & & & & & & \odot & \cdots & \odot \end{pmatrix} \end{array}.$$

When $\epsilon_\rho + 1 < x_1 \leq \alpha, x_1 - x_2 > 2,$ *and* $\frac{1}{2} \leq x_3 < x_2 - 1 \leq \alpha - 3,$ *define*

$$\begin{aligned}\mathcal{E}_{x_1,x_2,x_3} :=&\{([x_1, -x_1]_\rho, \lfloor x_1 \rfloor, \eta), ([x_2+1, -x_2]_\rho, \lfloor x_2 \rfloor, -\eta), ([x_2-2, -x_3]_\rho, 0, (-1)^{x_3-\epsilon_\rho}\eta),\\ &([x_3-2, \epsilon_\rho]_\rho, 0, -\eta), ([x_2-2, x_3]_\rho, 0, (-1)^{x_3-\epsilon_\rho}\eta), ([x_1-2, x_2+1]_\rho, 0, (-1)^{x_2-\epsilon_\rho}\eta),\\ &([\alpha-1, x_1]_\rho, 0, (-1)^{x_1-\epsilon_\rho}\eta)\}.\end{aligned}$$

Here is the associated symbol.

$$\mathcal{E}_{x_1,x_2,x_3} = \begin{array}{c} \begin{array}{ccccccccccccccccccc} -x_1 & \cdots & -x_2 & \cdots & \epsilon_\rho & \cdots & x_3-2 & x_3-1 & x_3 & \cdots & x_2-2 & x_2-1 & x_2 & x_2+1 & \cdots & x_1-2 & x_1-1 & x_1 & \cdots & \alpha-1 \end{array} \\ \begin{pmatrix} \lhd & \cdots & \lhd & \cdots & \odot & \cdots & \rhd & \rhd & \rhd & \cdots & \rhd & \rhd & \rhd & \rhd & \cdots & \rhd & \rhd & \rhd & & \\ & & \lhd & \cdots & \odot & \cdots & \rhd & \rhd & \rhd & \rhd & \rhd & & & & & & & & & \\ & & & & \odot & \cdots & \odot & & & & & & & & & & & & & \\ & & & & & & & & \odot & \cdots & \odot & & & & & & & & & \\ & & & & & & & & & & & & & \odot & \cdots & \odot & & & & \\ & & & & & & & & & & & & & & & & & \odot & \cdots & \odot \end{pmatrix} \end{array}.$$

When $(x_1, x_2, x_3) = (\frac{3}{2}, \frac{1}{2}, \frac{1}{2})$ *and* $\alpha \in \frac{1}{2} + \mathbb{Z}_{>0},$ *let*

$$\mathcal{E}_{x_1,x_2,x_3} = \begin{array}{c} \begin{array}{cccccc} -\frac{3}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{3}{2} & \cdots & \alpha-1 \end{array} \\ \begin{pmatrix} \lhd & \lhd & \rhd & \rhd & & \\ & \lhd & \odot & \rhd & & \\ & & & \odot & \cdots & \odot \end{pmatrix} \end{array}.$$

Proof. The sufficient direction can be proved in a similar way as Proposition 6.13, which we omit. Now we show the necessary direction. First, for $x_2 > \frac{1}{2}$, we assume that $x_1 = x_2 + 1$, and π_{x_1,x_2,x_3} is of Arthur type. Then by Definition 4.2 and Theorem 4.3, there exists an extended multi-segment $\mathcal{E}$ that contains a segment of the form $\{[x_1 - 1, -(x_1 - 1)]_\rho, *, *\}$ and another segment of the form $([x_2, -(x_2 - 1)]_\rho, *, *)$, such that $\pi(\mathcal{E}) = \pi_{sc}$. This gives a contradiction since the L-parameter of π_{sc} is multiplicity-free. Thus, we may conclude that $x_1 - x_2 > 1$.

Now by [HJLLZ25, Proposition 11.21 and 11.22], the representation

$$\pi^{\rho,-}_{x_1,x_2,x_3} = L(\Delta_\rho[-x_2, -x_2 - 1], \Delta_\rho[-x_3, -x_3]; \pi_{sc})$$

is of Arthur type if and only if one of the following holds:

- $\frac{1}{2} \leq x_3 < x_2 \leq \alpha - 1$,
- $x_2 = x_3 = \frac{1}{2}$ and $\alpha > \frac{1}{2}$.

In order for π_{x_1,x_2,x_3} to be of Arthur type, we must also have $\epsilon_\rho + 1 \leq x_1 \leq \alpha$ for the first case, or $(x_1, x_2) = (\alpha+1, \alpha-1)$ by Lemma 4.4. In the second case, if $x_1 \geq \frac{5}{2}$, then π_{x_1,x_2,x_3} is of Arthur type only if there exists an extended multi-segment $\mathcal{E}$ containing the segment $([x_1 - 1, -(x_1-1)]_\rho, *, *)$ such that $\pi(\mathcal{E}) = L(\Delta_\rho[-\frac{1}{2}, -\frac{3}{2}], \Delta_\rho[-\frac{1}{2}, -\frac{1}{2}])$. You can show that this cannot be the case by exhausting the possible Arthur packets the representation can appear in, using the algorithm described in [HLL22]. This proves the necessary direction. The rest follows from definition. □

This takes care of Case (F). Now we will tackle the final case, Case (G), where

$$\pi = L(\Delta_\rho[-x_1, -x_1], \Delta_\rho[-x_2, -x_2], \Delta_\rho[-x_3, -x_3 - 1]; \pi_{sc}).$$

Proposition 6.15. *Consider the representation*

$$\pi_{x_1,x_2,x_3} = L(\Delta_\rho[-x_1, -x_1], \Delta_\rho[-x_2, -x_2], \Delta_\rho[-x_3, -x_3 - 1]; \pi_{sc}),$$

where $x_1 \geq x_2 > x_3 \geq 0$.

(i) *The representation* π_{x_1,x_2,x_3} *is of Arthur type if and only if one of the following holds:*
- $1 \leq x_3 + 1 < x_2 < x_1 \leq \alpha$,
- $(x_1, x_2) = (\alpha+1, \alpha)$ *and* $0 \leq x_3 < \alpha - 1$,
- $(x_1, x_2, x_3) = (\alpha+2, \alpha+1, \alpha-1)$.
- $(x_1, x_2, x_3, \alpha) = (2, 1, 0, 1)$.

(ii) *The representation* π_{x_1,x_2,x_3} *is of critical type if and only if* $(x_1, x_2, x_3) = (\alpha+1, \alpha+1, \alpha), (\alpha+2, \alpha+1, \alpha), (\alpha+2, \alpha+2, \alpha), (\alpha+3, \alpha+2, \alpha), (\alpha, \alpha, \alpha-1), (\alpha+1, \alpha, \alpha-1), (\alpha+1, \alpha+1, \alpha-1), (\alpha+2, \alpha+1, \alpha-1), (\alpha, \alpha, \alpha-2), (\alpha, \alpha-1, \alpha-2), (\alpha, \alpha-1, \alpha-3), (\alpha+1, \alpha, \alpha-2)$.

(iii) *Define* $\mathcal{E}_{x_1,x_2,x_3}$ *in various cases as follows, then we have* $\pi(\mathcal{E}_{x_1,x_2,x_3}) = \pi_{x_1,x_2,x_3}$. *When* $1 \leq x_3 + 2 = x_2 = x_1 - 1 \leq \alpha + 1$, *define*

$$\begin{aligned}\mathcal{E}_{x_1,x_2,x_3} :=& \{([x_1, -x_1]_\rho, \lfloor x_1 \rfloor, \eta), ([x_3, -x_3]_\rho, \lfloor x_3 \rfloor, -\eta), ([x_3 - 2, \epsilon_\rho]_\rho, 0, \eta) \\ & ([\alpha - 1, x_2]_\rho, 0, (-1)^{x_2 - \epsilon_\rho}\eta)\end{aligned}$$

Here is the associated symbol.

$$\mathcal{E}_{x_1,x_2,x_3} = \begin{array}{c} \begin{array}{ccccccccccccc} -x_1 & -x_2 & -x_3 & \cdots & \epsilon_\rho & \cdots & x_3-2 & x_3-1 & x_3 & x_2 & x_1 & \cdots & \alpha-1 \end{array} \\ \left(\begin{array}{ccccccccccccc} \lhd & \lhd & \lhd & \cdots & \odot & \cdots & \rhd & \rhd & \rhd & \rhd & \rhd & & \\ & & \lhd & \cdots & \odot & \cdots & \rhd & \rhd & \rhd & & & & \\ & & & & \odot & \cdots & \odot & & & & & & \\ & & & & & & & & & \odot & \odot & \cdots & \odot \end{array}\right) \end{array}.$$

When $1 \leq x_3 + 2 < x_2 = x_1 - 1 \leq \alpha$, *define*

$$\begin{aligned}\mathcal{E}_{x_1,x_2,x_3} :=& \{([x_1, -x_1]_\rho, \lfloor x_1 \rfloor, \eta), ([x_3 + 1, -x_3]_\rho, \lfloor x_3 \rfloor, -\eta), ([x_3 - 2, \epsilon_\rho]_\rho, 0, \eta) \\ & ([x_2 - 2, x_3 + 1]_\rho, 0, (-1)^{x_3 + 1 - \epsilon_\rho}\eta), ([\alpha - 1, x_2]_\rho, 0, (-1)^{x_2 - \epsilon_\rho}\eta)\}.\end{aligned}$$

Here is the associated symbol.

$$\mathcal{E}_{x_1,x_2,x_3} = \begin{array}{c} \begin{array}{ccccccccccccccccc} -x_1 & \cdots & -x_3 & \cdots & \epsilon_\rho & \cdots & x_3-2 & x_3-1 & x_3 & x_3+1 & \cdots & x_2-2 & x_2-1 & x_2 & x_1 & \cdots & \alpha-1 \end{array} \\ \left(\begin{array}{ccccccccccccccccc} \lhd & \cdots & \lhd & \cdots & \odot & \cdots & \rhd & \rhd & \rhd & \rhd & \rhd & \cdots & \rhd & \rhd & \rhd & & \\ & & \lhd & \cdots & \odot & \cdots & \rhd & \rhd & \rhd & \rhd & & & & & & & \\ & & & & \odot & \cdots & \odot & & & & & & & & & & \\ & & & & & & & & & \odot & \cdots & \odot & & & & & \\ & & & & & & & & & & & & & \odot & \odot & \cdots & \odot \end{array} \right) \end{array}.$$

Finally, when $1 \leq x_3 + 2 < x_2 < x_1 - 1 \leq \alpha$, define

$$\begin{aligned} \mathcal{E}_{x_1,x_2,x_3} :=& \{([x_1, -x_1]_\rho, \lfloor x_1 \rfloor, \eta), ([x_2, -x_2]_\rho, \lfloor x_2 \rfloor, -\eta), ([x_3 + 1, -x_3]_\rho, \lfloor x_3 \rfloor, -\eta), ([x_3 - 2, \epsilon_\rho]_\rho, 0, \eta) \\ & ([x_2 - 2, x_3 + 1]_\rho, 0, (-1)^{x_3+1-\epsilon_\rho}\eta), ([x_1 - 2, x_2]_\rho, 0, (-1)^{x_2+1-\epsilon_\rho}\eta), ([\alpha - 1, x_1]_\rho, 0, (-1)^{x_1-\epsilon_\rho}\eta). \end{aligned}$$

Here is the associated symbol.

$$\mathcal{E}_{x_1,x_2,x_3} = \begin{array}{c} \begin{array}{cccccccccccccccccccccc} -x_1 & \cdots & -x_2 & \cdots & -x_3 & \cdots & \epsilon_\rho & \cdots & x_3-2 & x_3-1 & x_3 & x_3+1 & \cdots & x_2-2 & x_2-1 & x_2 & \cdots & x_1-2 & x_1-1 & x_1 & \cdots & \alpha-1 \end{array} \\ \left(\begin{array}{cccccccccccccccccccccc} \lhd & \cdots & \lhd & \cdots & \lhd & \cdots & \odot & \cdots & \rhd & \rhd & \rhd & \rhd & \cdots & \rhd & \rhd & \rhd & \cdots & \rhd & \rhd & \rhd & & \\ & & \lhd & \cdots & \lhd & \cdots & \odot & \cdots & \rhd & \rhd & \rhd & \rhd & \cdots & \rhd & \rhd & \rhd & & & & & & \\ & & & & \lhd & \cdots & \odot & \cdots & \rhd & \rhd & \rhd & & & & & & & & & & & \\ & & & & & & \odot & \cdots & \odot & & & & & & & & & & & & & \\ & & & & & & & & & & & \odot & \cdots & \odot & & & & & & & & \\ & & & & & & & & & & & & & & & \odot & \cdots & \odot & & & & \\ & & & & & & & & & & & & & & & & & & & \odot & \cdots & \odot \end{array} \right) \end{array}.$$

Proof. The sufficient direction can be proven in a similar way as Proposition 6.13, which we omit. Now we prove the necessary direction.

Suppose π_{x_1,x_2,x_3} is of Arthur type, then we must have $x_1 > x_2$. Otherwise, by Definition 4.2 and Theorem 4.3, there exists an extended multi-segment $\mathcal{E}$ containing two copies of $([x_1 - 1, -(x_1 - 1)]_\rho$ such that $\pi(\mathcal{E})$ such that $\pi(\mathcal{E}) = \pi_{sc}$. This is impossible since the L-parameter of π_{sc} is multiplicity-free.

Next, by [HJLLZ25, Proposition 11.23 and 11.24], the representation

$$\pi^{\rho,-}_{x_1,x_2,x_3} = L(\Delta_\rho[-x_2, -x_2], \Delta_\rho[-x_3, -x_3 - 1])$$

is of Arthur type if and only if $1 \leq x_3 + 1 < x_2 \leq \alpha$ or $(x_2, x_3) = (\alpha + 1, \alpha - 1)$. Combining this with Lemma 4.4, we see that π_{x_1,x_2,x_3} is of Arthur type only if one of the following holds:

- $1 < x_3 + 1 < x_2 < x_1 \leq \alpha$
- $(x_1, x_2) = (\alpha + 1, \alpha)$ and $0 \leq x_3 < \alpha - 1$
- $(x_1, x_2, x_3) = (\alpha + 2, \alpha + 1, \alpha - 1)$
- $(x_1, x_2, x_3) = (2, 1, 0)$ and $\alpha = 1$

This gives the desired condition and proves the necessary direction. The rest follows from definition. □

This concludes our discussion of Case (F) and of all the non-tempered representations π of corank 4 with $f(\pi) = 3$.

6.7. **The case** $f(\pi) = 4$**.** We move onto the final case in this section, where there are 4 segments in the L-data of π. This situation is a lot easier to deal with compared to the previous cases, since the number of segments match the corank of π, we must have

$$\pi_{x_1,x_2,x_3,x_4} = L(\Delta_\rho[-x_1, -x_1], \Delta_\rho[-x_2, -x_2], \Delta_\rho[-x_3, -x_3], \Delta_\rho[-x_4, -x_4]; \pi_{sc})$$

There is a way to directly characterize which one of such representations lie in an Arthur packet, introduced as Condition $(\mathcal{A})$ in [HJLLZ25]. We reference their result below. In general, we consider representations of the form

$$\pi = L(\Delta_\rho[-x_1, -x_1]^{m_{x_1}}, \dots, \Delta_\rho[-x_f, -x_f]^{m_{x_f}}; \pi_{sc}), \tag{6.1}$$

where $\frac{1}{2} \leq x_f < \cdots < x_1$. Here the multiplicities m_{x_i}s are the multiplicities of a given segment inside the L-data. For continuity, we set $m_x = 0$ for $x \in \frac{1}{2}\mathbb{Z}$ if $x \neq x_i$ for any $1 \leq i \leq f$.

Definition 6.16. [HJLLZ25, Definition 9.1] *Suppose the L-data of π is of the form (6.1). We say the L-data of π satisfies condition $(\mathcal{A})$ if the following holds.*

- *If $\alpha \geq 1$, then for any $1 \leq x \leq \alpha$, we have $\lfloor \frac{m_x}{2} \rfloor \leq \lfloor \frac{m_{x-1}}{2} \rfloor$.*
- *For any $x > \alpha$, we have $m_x \leq m_{x-1}$.*

The characterization is given as follows.

Theorem 6.17. [HJLLZ25, Theorem 9.2] *A representation π of the form (6.1) is of Arthur type (hence unitary) if and only if its L-data satisfies condition $(\mathcal{A})$.*

This theorem handles all representations π where $f(\pi) = \mathrm{corank}(\pi)$. In particular, we can apply it to the case where $\mathrm{corank}(\pi) = 4$.

Proposition 6.18. *Let*

$$\pi_{x_1,x_2,x_3,x_4} = L(\Delta_\rho[-x_1,-x_1], \Delta_\rho[-x_2,-x_2], \Delta_\rho[-x_3,-x_3], \Delta_\rho[-x_4,-x_4]; \pi_{sc})$$

with $x_1 \geq x_2 \geq x_3 \geq x_4 > 0$ be of good parity.

(1) *When $x_1 > x_2 > x_3 > x_4$, π_{x_1,x_2,x_3,x_4} is of Arthur type if and only if one of the following holds:*
   - $x_1 \leq \alpha$,
   - $(x_1, x_2) = (\alpha+1, \alpha)$,
   - $(x_1, x_2, x_3) = (\alpha+2, \alpha+1, \alpha)$,
   - $(x_1, x_2, x_3, x_4) = (\alpha+3, \alpha+2, \alpha+1, \alpha)$.

   It's of critical type when $(x_1, x_2, x_3, x_4) = (\alpha+3, \alpha+2, \alpha+1, \alpha), (\alpha+2, \alpha+1, \alpha, \alpha-1), (\alpha+1, \alpha, \alpha-1, \alpha-2), (\alpha, \alpha-1, \alpha-2, \alpha-3)$.

(2) *When $x_1 = x_2 > x_3 > x_4$, π_{x_1,x_2,x_3,x_4} is not of Arthur type. It is of critical type when $(x_1, x_2, x_3, x_4) = (\alpha+2, \alpha+2, \alpha+1, \alpha), (\alpha+1, \alpha+1, \alpha, \alpha-1), (\alpha, \alpha, \alpha-1, \alpha-2)$.*

(3) *When $x_1 > x_2 = x_3 > x_4$, π_{x_1,x_2,x_3,x_4} is not of Arthur type. It is of critical type if and only if $(x_1, x_2, x_3, x_4) = (\alpha+2, \alpha+1, \alpha+1, \alpha), (\alpha+1, \alpha, \alpha, \alpha-1), (\alpha, \alpha-1, \alpha-1, \alpha-2)$.*

(4) *When $x_1 > x_2 > x_3 = x_4$, π_{x_1,x_2,x_3,x_4} is of Arthur type in the following cases:*
   - $\alpha > 1, \frac{1}{2} = x_3 = x_4 < x_2 < x_1 \leq \alpha+1$,
   - $\alpha = \frac{1}{2}, (x_1, x_2, x_3, x_4) = (\frac{5}{2}, \frac{3}{2}, \frac{1}{2}, \frac{1}{2})$.

   It is of critical type if and only if $(x_1, x_2, x_3, x_4) = (\alpha+2, \alpha+1, \alpha, \alpha), (\alpha+1, \alpha, \alpha-1, \alpha-1), (\alpha, \alpha-1, \alpha-2, \alpha-2)$.

(5) *When $x_1 = x_2 = x_3 > x_4$, π_{x_1,x_2,x_3,x_4} is not of Arthur type. It is of critical type when $(x_1, x_2, x_3, x_4) = (\alpha, \alpha, \alpha, \alpha-1)$ or $(\alpha+1, \alpha+1, \alpha+1, \alpha)$.*

(6) *When $x_1 = x_2 > x_3 = x_4$, π_{x_1,x_2,x_3,x_4} is not of Arthur type. It is of critical type when $(x_1, x_2, x_3, x_4) = (\alpha, \alpha, \alpha-1, \alpha-1)$ or $(\alpha+1, \alpha+1, \alpha, \alpha)$.*

(7) *When $x_1 > x_2 = x_3 = x_4$, π_{x_1,x_2,x_3,x_4} is not of Arthur type. It is of critical type when $(x_1, x_2, x_3, x_4) = (\alpha, \alpha-1, \alpha-1, \alpha-1), (\alpha+1, \alpha, \alpha, \alpha)$.*

(8) *When $x_1 = x_2 = x_3 = x_4$, π_{x_1,x_2,x_3,x_4} is not of Arthur type. It is of critical type when $(x_1, x_2, x_3, x_4) = (\alpha, \alpha, \alpha, \alpha)$.*

Proof. This follows directly from Theorem 6.17. □

With this, we are done with the characterization of all non-tempered representations of corank 4 that are of Arthur type. In the next few sections, we will verify Tadic's conjecture and prove that the representations of critical type, good parity, but not of Arthur type are not unitary. Using this, we will write down the unitary dual for representations of corank 4 and show that it is equal to the closure constructed in [HJLLZ25, Section 5]. Before then, we want to give a complete list of all representations of corank 4 that are of Arthur type and of critical type. To do this, we first need to classify all tempered representations of corank 4, which we'll do in the next section.

7. Classification of tempered representations of corank 4 and of good parity

In this section, we will classify all tempered representations of corank 4 and of good parity, following a similar procedure as we did in section 3. Combined with the previous section, this will give us the full list of good parity representations of corank 4. By Theorem 2.7, this allows us to construct the full Arthur dual of corank 4. In particular, we can identify independently within the admissible dual which representations are of Arthur type and which representations are of critical type. This will form the basis of our main results later.

Again, we fix some self-dual $\rho \in \mathcal{C}$. Let $\alpha = \alpha_{\rho,\sigma}$ and suppose that $\pi \in \Pi_{A,gp}(G_n)$ is tempered of corank 4. Then there are four cases to consider.

(A) $\pi \hookrightarrow \rho|\cdot|^{x_1} \rtimes \pi_{temp}$, where π_{temp} is tempered of corank 3. Then π is of the form $T_{I,1}^{x}(\pi_{temp}), T_{IV,3}(\pi_{temp})$ or $T_{V,2}^{\pm}(\pi_{temp})$,

(B) $\pi \hookrightarrow \rho|\cdot|^{x_1} \rtimes |\cdot|^{x_2} \rtimes \pi_{temp}$, where π_{temp} is tempered of corank 2. Then π is of the form $T_{I,2}^{x}(\pi_{temp}), T_{II,3}^{x}(\pi_{temp}), T_{III,2}^{\frac{1}{2}}(\pi_{temp}), T_{IV,5}(\pi_{temp})$ or $T_{V,4}^{\pm}(\pi_{temp})$,

(C) $\pi \hookrightarrow \rho|\cdot|^{x_1} \rtimes |\cdot|^{x_2} \times |\cdot|^{x_3} \rtimes \pi_{temp}$, where π_{temp} is tempered of corank 1. Then π is of the form $T_{I,3}^{x}(\pi_{temp}), T_{III,2}^{1}(\pi_{temp}), T_{IV,5}(\pi_{temp}), T_{V,6}^{\pm}(\pi_{temp})$,

(D) $\pi \hookrightarrow \rho|\cdot|^{x_1} \rtimes |\cdot|^{x_2} \times |\cdot|^{x_3} \times |\cdot|^{x_4} \rtimes \pi_{sc}$, where π_{sc} is supercuspidal. Then π is of the form $T_{I,4}^{x}(\pi_{sc}), T_{II,5}^{x}(\pi_{sc}), T_{III,2}^{\frac{3}{2}}(\pi_{sc}), T_{III,4}^{\frac{1}{2}}(\pi_{sc}), T_{IV,9}(\pi_{sc}), T_{V,8}^{\pm}(\pi_{sc})$.

7.1. **Case** $(A): \pi \hookrightarrow \rho|\cdot|^{x_1} \rtimes \pi_{temp}$**, where** π_{temp} **is tempered of corank** 3**.** We will begin with Case (A). In section 3, we classified all tempered representations of corank 3. These are given below:

$T_{I,1}^{\alpha+2}(T_{I,1}^{\alpha+1}(T_{I,1}^{\alpha}(\pi_{sc}))), T_{I,1}^{\alpha-1}(T_{I,1}^{\alpha+1}(T_{I,1}^{\alpha}(\pi_{sc}))), T_{IV,3}(T_{I,1}^{\alpha+1}(T_{I,1}^{\alpha}(\pi_{sc}))), T_{V,2}^{\pm}(T_{I,1}^{2}(T_{I,1}^{1}(\pi_{sc}))),$
$T_{I,1}^{\alpha-2}(T_{I,1}^{\alpha-1}(T_{I,1}^{\alpha}(\pi_{sc}))), T_{IV,3}(T_{I,1}^{\alpha-1}(T_{I,1}^{\alpha}(\pi_{sc}))), T_{V,2}^{\pm}(T_{I,1}^{1}(T_{I,1}^{2}(\pi_{sc})))), T_{I,1}^{2}(T_{V,2}^{\pm}(T_{I,1}^{1}(\pi_{sc}))),$
$T_{I,1}^{2}(T_{I,1}^{1}(T_{V,2}^{\pm}(\pi_{sc})))T_{IV,3}(T_{I,1}^{1}(T_{V,2}^{\pm}(\pi_{sc}))), T_{I,1}^{\frac{3}{2}}(T_{I,2}^{\frac{1}{2}}(\pi_{sc})), T_{I,1}^{\alpha}(T_{II,3}^{\frac{1}{2}}(\pi_{sc})), T_{I,1}^{\frac{3}{2}}(T_{III,2}^{\frac{1}{2}}(\pi_{sc})),$
$T_{I,1}^{\alpha}(T_{IV,5}(\pi_{sc})), T_{I,1}^{1}(T_{V,4}^{\pm}(\pi_{sc})), T_{V,4}^{\pm}(T_{I,1}^{1}(\pi_{sc})), T_{I,2}^{1}(T_{IV,3}(\pi_{sc})), T_{II,3}^{1}(T_{IV,3}(\pi_{sc})), T_{I,2}^{1}(T_{V,2}^{\pm}(\pi_{sc})),$
$T_{III,2}^{1}(\pi_{sc}), T_{IV,7}(\pi_{sc}), T_{V,6}^{\pm}(\pi_{sc})$

There are 22 possible tempered representations of corank 3, so there are 66 cases to consider in Case (A). We begin with $\pi_{temp} = T_{I,1}^{\alpha+2}(T_{I,1}^{\alpha+1}(T_{I,1}^{\alpha}(\pi_{sc})))$

Proposition 7.1. *Let* $\pi_{temp} = T_{I,1}^{\alpha+2}(T_{I,1}^{\alpha+1}(T_{I,1}^{\alpha}(\pi_{sc})))$ *for* $\alpha > 0$.

(i) The representation $T_{I,1}^{x}(\pi_{temp})$ *is well-defined if and only if* $x = \alpha + 3$ *for* $\alpha > 0$*, or* $x = \alpha - 1$ *for* $\alpha > 1$.

(ii) The representation $T^{x}_{I,1}(\pi_{temp})$ is of critical type when $x = \alpha + 3$ for $\alpha > 0$, or $x = \alpha - 1$ for $\alpha > 1$.

(iii) Let $x \in \{\alpha + 3, \alpha - 1\}$, and define

$$\mathcal{E}_{\alpha+3} := \{([\alpha-2, \epsilon_\rho]_\rho, 0, \eta), ([\alpha+3, \alpha+3]_\rho, 0, (-1)^{\alpha-1-\epsilon_\rho}\eta)\},$$

$$\mathcal{E}_{\alpha-1} := \{([\alpha-3, \epsilon_\rho]_\rho, 0, \eta), ([\alpha-1, \alpha-1]_\rho, 0, (-1)^{\alpha-\epsilon_\rho}\eta), ([\alpha+1, \alpha+1]_\rho, 0, (-1)^{\alpha-\epsilon_\rho}\eta)\}.$$

Then we have $\pi(\mathcal{E}_x) = T^{x}_{I,1}(\pi_{temp})$. Here are the associated symbols. .

$$\mathcal{E}_{\alpha+3} = \begin{array}{c} \begin{array}{cccccccc} \epsilon_\rho & \cdots & \alpha-2 & \alpha-1 & \alpha & \alpha+1 & \alpha+2 & \alpha+3 \end{array} \\ \begin{pmatrix} \odot & \cdots & \odot & & & & & \\ & & & & & & & \odot \end{pmatrix} \end{array},$$

$$\mathcal{E}_{\alpha-1} = \begin{array}{c} \begin{array}{cccccccc} \epsilon_\rho & \cdots & \alpha-3 & \alpha-2 & \alpha-1 & \alpha & \alpha+1 & \alpha+2 \end{array} \\ \begin{pmatrix} \odot & \cdots & \odot & & & & & \\ & & & & \odot & & & \\ & & & & & & & \odot \end{pmatrix} \end{array}.$$

Proof. Part (i) follows from Remark 3.2 and definition of $T^{x}_{I,1}(\pi_{temp})$. Parts (ii) and (iii) follow from definition. □

Proposition 7.2. *Let $\pi_{temp} = T^{\alpha+2}_{I,1}(T^{\alpha+1}_{I,1}(T^{\alpha}_{I,1}(\pi_{sc})))$ for $\alpha > 0$.*

(i) The representation $T_{IV,3}(\pi_{temp})$ is well-defined if and only if $x \in \mathbb{Z}_{>1}$.

(ii) The representation $T_{IV,3}(\pi_{temp})$ is of critical type if and only if $\alpha = 1$.

(iii) Define

$$\mathcal{E} := \{([0,0]_\rho, 0, \eta)^2, ([\alpha-2, 0]_\rho, 0, \eta), ([\alpha+2, \alpha+2]_\rho, 0, (-1)^{\alpha-1}\eta).\}$$

Then $\pi(\mathcal{E}) = T_{IV,3}(\pi_{temp})$. Here is the associated symbol.

$$\mathcal{E} = \begin{array}{c} \begin{array}{ccccccc} 0 & \cdots & \alpha-2 & \alpha-1 & \alpha & \alpha+1 & \alpha+2 \end{array} \\ \begin{pmatrix} \odot & & & & & & \\ \odot & & & & & & \\ \odot & \cdots & \odot & & & & \\ & & & & & & \odot \end{pmatrix} \end{array}.$$

Proof. Part (i) follows from Remark 3.2. Parts (ii) and (iii) follow from definition. □

Proposition 7.3. *Let $\pi_{temp} = T^{\alpha+2}_{I,1}(T^{\alpha+1}_{I,1}(T^{\alpha}_{I,1}(\pi_{sc})))$ for $\alpha > 0$.*

(i) The representation $T^{\pm}_{V,2}(\pi_{temp})$ is well-defined if and only if $\alpha = 1$.

(ii) The representation $T^{\pm}_{V,2}(\pi_{temp})$ is of critical type when $\alpha = 1$.

(iii) When $\alpha = 1$, define

$$\mathcal{E}_{\pm} := \{([0,0]_\rho, 0, \pm)^2, ([3,3]_\rho, 0, \eta)\}.$$

Then $\pi(\mathcal{E}_\pm) = T^{\pm}_{V,2}(\pi_{temp})$. Here are the associated symbols.

$$\mathcal{E}_{+} = \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{pmatrix} \oplus & & & \\ \oplus & & & \\ & & & \odot \end{pmatrix} \end{array}, \quad \mathcal{E}_{-} = \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{pmatrix} \ominus & & & \\ \ominus & & & \\ & & & \odot \end{pmatrix} \end{array}.$$

Proof. Part (i) follow from Remark 3.2. Parts (ii) and (iii) follow from definition. □

We move onto the second case, which is $\pi_{temp} = T^{\alpha-1}_{I,1}(T^{\alpha+1}_{I,1}(T^{\alpha}_{I,1}(\pi_{sc})))$.

Proposition 7.4. *Let $\pi_{temp} = T^{\alpha-1}_{I,1}(T^{\alpha+1}_{I,1}(T^{\alpha}_{I,1}(\pi_{sc})))$ for $\alpha > 1$.*

(i) The representation $T_{I,1}^{x}(\pi_{temp})$ *is well-defined if and only if one of the following holds:*
- $x = \alpha + 2$,
- $x = \alpha$,
- $x = \alpha - 2$ *for* $\alpha > 2$.

(ii) When $x = \alpha + 3$, $x = \alpha$, *or* $x = \alpha - 2$ *for* $\alpha > 2$, *the representation* $T_{I,1}^{x}(\pi_{temp})$ *is of critical type.*

(iii) Let $x \in \{\alpha+2, \alpha, \alpha-2\}$. *Define*

$$\mathcal{E}_{\alpha+2} := \{([\alpha-3,\epsilon_\rho]_\rho, 0, \eta), ([\alpha-1,\alpha-1]_\rho, 0, (-1)^{\alpha-\epsilon_\rho}\eta), ([\alpha+2,\alpha+2]_\rho, 0, (-1)^{\alpha-1-\epsilon_\rho}\eta)\},$$

$$\mathcal{E}_{\alpha} := \{([\alpha-3,\epsilon_\rho]_\rho, 0, \eta), ([\alpha,\alpha]_\rho, 0, (-1)^{\alpha-\epsilon_\rho}\eta), ([\alpha+1,\alpha+1]_\rho, 0, (-1)^{\alpha-1-\epsilon_\rho}\eta)\}.$$

When $\alpha > 2$, *define*

$$\mathcal{E}_{\alpha-2} := \{([\alpha-4,\epsilon_\rho]_\rho, 0, \eta), ([\alpha-2,\alpha-2]_\rho, 0, (-1)^{\alpha-1-\epsilon_\rho}\eta), ([\alpha+1,\alpha+1]_\rho, 0, (-1)^{\alpha-1-\epsilon_\rho}\eta)\}.$$

Then we have $\pi(\mathcal{E}_x) = T_{I,1}^{x}(\pi_{temp})$. *Here are the associated symbols.*

$$\mathcal{E}_{\alpha+2} = \begin{array}{c} \begin{array}{cccccccc} \epsilon_\rho & \cdots & \alpha-3 & \alpha-2 & \alpha-1 & \alpha & \alpha+1 & \alpha+2 \end{array} \\ \begin{pmatrix} \odot & \cdots & \odot & & & & & \\ & & & & \odot & & & \\ & & & & & & & \odot \end{pmatrix} \end{array},$$

$$\mathcal{E}_{\alpha} = \begin{array}{c} \begin{array}{ccccccc} \epsilon_\rho & \cdots & \alpha-3 & \alpha-2 & \alpha-1 & \alpha & \alpha+1 \end{array} \\ \begin{pmatrix} \odot & \cdots & \odot & & & & \\ & & & & & \odot & \\ & & & & & & \odot \end{pmatrix} \end{array},$$

$$\mathcal{E}_{\alpha-2} = \begin{array}{c} \begin{array}{cccccccc} \epsilon_\rho & \cdots & \alpha-4 & \alpha-3 & \alpha-2 & \alpha-1 & \alpha & \alpha+1 \end{array} \\ \begin{pmatrix} \odot & \cdots & \odot & & & & & \\ & & & & \odot & & & \\ & & & & & & & \odot \end{pmatrix} \end{array}.$$

Proof. The proof is similar to Proposition 7.1, which we omit. □

Proposition 7.5. *Let* $\pi_{temp} = T_{I,1}^{\alpha-1}(T_{I,1}^{\alpha+1}(T_{I,1}^{\alpha}(\pi_{sc})))$ *for* $\alpha > 1$.

(i) The representation $T_{IV,3}(\pi_{temp})$ *is well-defined if and only if* $\alpha \in \mathbb{Z}_{>2}$.

(ii) The representation $T_{IV,3}(\pi_{temp})$ *is not of critical type.*

(iii) Define

$$\mathcal{E} := \{([0,0]_\rho, 0, \eta)^2, ([\alpha-3,0]_\rho, 0, \eta), ([\alpha-1,\alpha-1]_\rho, 0, (-1)^{\alpha-1}\eta), ([\alpha+1,\alpha+1]_\rho, 0, (-1)^{\alpha-1}\eta)\}.$$

Then we have $\pi(\mathcal{E}) = T_{IV,5}(\pi_{temp})$. *Here is the associated symbol.*

$$\mathcal{E} = \begin{array}{c} \begin{array}{ccccccc} 0 & \cdots & \alpha-3 & \alpha-2 & \alpha-1 & \alpha & \alpha+1 \end{array} \\ \begin{pmatrix} \odot & & & & & & \\ \odot & & & & & & \\ \odot & \cdots & \odot & & & & \\ & & & & \odot & & \\ & & & & & & \odot \end{pmatrix} \end{array}$$

Proof. The proof is similar to Proposition 7.2, which we omit. □

Proposition 7.6. *Let* $\pi_{temp} = T_{I,1}^{\alpha-1}(T_{I,1}^{\alpha+1}(T_{I,1}^{\alpha}(\pi_{sc})))$ *for* $\alpha > 1$.

(i) The representation $T_{V,2}^{\pm}(\pi_{temp})$ *is well-defined if and only if* $\alpha = 2$.

(ii) When $\alpha = 2$, *the representation* $T_{V,2}^{\pm}(\pi_{temp})$ *is of critical type.*

(iii) When $\alpha = 2$, *define*

$$\mathcal{E}_{\pm} := \{([0,0]_\rho, 0, \pm 1)^2, ([1,1]_\rho, 0, \eta), ([3,3]_\rho, 0, -\eta)\}.$$

. *Then* $\pi(\mathcal{E}_{\pm}) = T^{\pm}_{V,2}(\pi_{temp})$. *Here are the associated symbols.*

$$\mathcal{E}_+ = \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{pmatrix} \oplus & & & \\ \oplus & & & \\ & \odot & & \\ & & & \odot \end{pmatrix} \end{array}, \quad \mathcal{E}_- = \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{pmatrix} \ominus & & & \\ \ominus & & & \\ & \odot & & \\ & & & \odot \end{pmatrix} \end{array}.$$

Proof. The proof is similar to Proposition 7.3, which we omit. $\square$

The next case to consider is $\pi_{temp} = T_{IV,3}(T^{\alpha+1}_{I,1}(T^{\alpha}_{I,1}(\pi_{sc})))$.

Proposition 7.7. *Let* $\pi_{temp} = T_{IV,3}(T^{\alpha+1}_{I,1}(T^{\alpha}_{I,1}(\pi_{sc})))$ *for* $\alpha \in \mathbb{Z}_{>1}$.

(i) The representation $T^x_{I,1}(\pi_{temp})$ *is well-defined if and only if* $x = \alpha - 1$ *or* $\alpha + 2$. *When* $x = \alpha - 1$ *and* $\alpha \in \mathbb{Z}_{>2}$, *then* $T^x_{I,1}(\pi_{temp})$ *is the same as the representation* $T_{IV,3}(T^{\alpha-1}_{I,1}(T^{\alpha+1}_{I,1}(T^{\alpha}_{I,1}(\pi_{sc}))))$ *defined in Proposition 7.5.*
(ii) The representation $T^x_{I,1}(\pi_{temp})$ *is of critical type only when* $(x, \alpha) = (1, 2)$.
(iii) Let $x \in \{\alpha - 1, \alpha + 2\}$. *Define*

$$\mathcal{E}_{\alpha-1} := \{([0,0]_\rho, 0, \eta)^2, ([\alpha-3, 0]_\rho, 0, \eta), ([\alpha-1, \alpha-1]_\rho, 0, (-1)^{\alpha}\eta), ([\alpha+1, \alpha+1]_\rho, 0, (-1)^{\alpha-1}\eta)\},$$

$$\mathcal{E}_{\alpha+2} := \{([0,0]_\rho, 0, \eta)^2, ([\alpha-2, 0]_\rho, 0, \eta), ([\alpha+2, \alpha+2]_\rho, 0, (-1)^{\alpha-1}\eta)\}.$$

Then we have $\pi(\mathcal{E}_x) = T^x_{I,1}(\pi_{temp})$. *Here are the associated symbols.*

$$\mathcal{E}_{\alpha-1} = \begin{array}{c} \begin{array}{ccccccc} 0 & \cdots & \alpha-3 & \alpha-2 & \alpha-1 & \alpha & \alpha+1 \end{array} \\ \begin{pmatrix} \odot & & & & & & \\ \odot & & & & & & \\ \odot & \cdots & \odot & & & & \\ & & & & \odot & & \\ & & & & & & \odot \end{pmatrix} \end{array}$$

$$\mathcal{E}_{\alpha+2} = \begin{array}{c} \begin{array}{ccccccc} 0 & \cdots & \alpha-2 & \alpha-1 & \alpha & \alpha+1 & \alpha+2 \end{array} \\ \begin{pmatrix} \odot & & & & & & \\ \odot & & & & & & \\ \odot & \cdots & \odot & & & & \\ & & & & & & \odot \end{pmatrix} \end{array}$$

(iv) The representations $T_{IV,3}(\pi_{temp})$ *and* $T^{\pm}_{V,2}(\pi_{temp})$ *are not well-defined.*

Proof. The proof of parts (i) through (iv) is similar to Proposition 7.5, which we omit. Part (iv) follows from Remark 3.2, since $m_\phi(\rho \otimes S_1) \geq 2$, where ϕ is the L-parameter associated to π_{temp}. $\square$

Now we move onto the case $\pi_{temp} = T^{\pm}_{V,2}(T^{\alpha+1}_{I,1}(T^{\alpha}_{I,1}(\pi_{sc})))$, which is well-defined only when $\alpha = 1$, by Proposition 3.5.

Proposition 7.8. *Let* $\pi^{\pm}_{temp} = T^{\pm}_{V,2}(T^2_{I,1}(T^1_{I,1}(\pi_{sc})))$ *for* $\alpha = 1$.

(i) The representation $T^x_{I,1}(\pi^{\pm}_{temp})$ *is well-defined if and only if* $x = 1$ *or* $x = 3$.
(ii) When $x = 1$ *or* 3, *the representation* $T^x_{I,1}(\pi^{\pm}_{temp})$ *is of critical type.*

(iii) *Let $x \in \{1, 3\}$. Define*

$$\mathcal{E}_{1,\pm} := \{([0,0]_\rho, 0, \pm 1), ([1,1]_\rho, 0, \pm 1), ([2,2]_\rho, 0, \eta)\},$$

$$\mathcal{E}_{3,\pm} := \{([0,0]_\rho, 0, \pm 1)^2, ([3,3]_\rho, 0, \eta)\}.$$

Then we have $\pi(\mathcal{E}_{x,\pm}) = T_{I,1}^{x}(\pi_{temp}^{\pm})$. Here are the associated symbols.

$$\mathcal{E}_{1,+} = \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{pmatrix} \oplus & & \\ & \oplus & \\ & & \odot \end{pmatrix} \end{array}, \quad \mathcal{E}_{1,-} = \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{pmatrix} \ominus & & \\ & \ominus & \\ & & \odot \end{pmatrix} \end{array}$$

$$\mathcal{E}_{3,+} = \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{pmatrix} \oplus & & & \\ \oplus & & & \\ & & & \odot \end{pmatrix} \end{array}, \quad \mathcal{E}_{3,-} = \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{pmatrix} \ominus & & & \\ \ominus & & & \\ & & & \odot \end{pmatrix} \end{array}.$$

(iv) *The representations $T_{IV,3}(\pi_{temp})$ and $T_{V,2}^{\pm}(\pi_{temp})$ are not well-defined.*

Proof. The proof is similar to Proposition 7.6, which we omit. $\square$

The next case we want to consider is $\pi_{temp} = T_{I,1}^{\alpha-2}(T_{I,1}^{\alpha-1}(T_{I,1}^{\alpha}(\pi_{sc})))$.

Proposition 7.9. *Let $\pi_{temp} = T_{I,1}^{\alpha-2}(T_{I,1}^{\alpha-1}(T_{I,1}^{\alpha}(\pi_{sc})))$ for $\alpha > 2$.*

(i) *The representation $T_{I,1}^{x}(\pi_{temp})$ is well-defined if and only if $x = \alpha - 3$ and $\alpha > 3$, or $x = \alpha + 1$.*

(ii) *When $x = \alpha - 3$ and $\alpha > 3$, or when $x = \alpha + 1$, the representation $T_{I,1}^{x}(\pi_{temp})$ is of critical type.*

(iii) *Let $x \in \{\alpha - 3, \alpha + 1\}$. When $\alpha > 3$, define*

$$\mathcal{E}_{\alpha-3} := \{([\alpha-5, \epsilon_\rho]_\rho, 0, \eta), ([\alpha-3, \alpha-3]_\rho, 0, (-1)^{\alpha-\epsilon_\rho}\eta), ([\alpha, \alpha-2]_\rho, 0, (-1)^{\alpha-1-\epsilon_\rho}\eta)\},$$

$$\mathcal{E}_{\alpha+1} := \{([\alpha-4, \epsilon_\rho]_\rho, 0, \eta), ([\alpha-1, \alpha-2]_\rho, 0, (-1)^{\alpha-1-\epsilon_\rho}\eta), ([\alpha+1, \alpha+1]_\rho, 0, (-1)^{\alpha-1-\epsilon_\rho}\eta)\}.$$

Then we have $\pi(\mathcal{E}_x) = T_{I,1}^{x}(\pi_{temp})$. Here are the associated symbols.

$$\mathcal{E}_{\alpha-3} = \begin{array}{c} \begin{array}{cccccccc} \epsilon_\rho & \cdots & \alpha-5 & \alpha-4 & \alpha-3 & \alpha-2 & \alpha-1 & \alpha \end{array} \\ \begin{pmatrix} \odot & \cdots & \odot & & & & & \\ & & & & \odot & & & \\ & & & & & \odot & \odot & \odot \end{pmatrix} \end{array},$$

$$\mathcal{E}_{\alpha+1} = \begin{array}{c} \begin{array}{cccccccc} \epsilon_\rho & \cdots & \alpha-4 & \alpha-3 & \alpha-2 & \alpha-1 & \alpha & \alpha+1 \end{array} \\ \begin{pmatrix} \odot & \cdots & \odot & & & & & \\ & & & & \odot & \odot & & \\ & & & & & & & \odot \end{pmatrix} \end{array}.$$

Proof. The proof is similar to Proposition 7.1, which we omit. $\square$

Proposition 7.10. *Let $\pi_{temp} = T_{I,1}^{\alpha-2}(T_{I,1}^{\alpha-1}(T_{I,1}^{\alpha}(\pi_{sc})))$ for $\alpha > 2$.*

(i) *The representation $T_{IV,3}(\pi_{temp})$ is well-defined if and only if $\alpha \in \mathbb{Z}_{>3}$.*

(ii) *The representation $T_{IV,3}(\pi_{temp})$ is not of critical type.*

(iii) *When $\alpha \in \mathbb{Z}_{>3}$, define*

$$\mathcal{E} := \{([0,0]_\rho, 0, \eta)^2, [\alpha-4, 0]_\rho, 0, \eta), ([\alpha, \alpha-2]_\rho, 0, (-1)^{\alpha-1}\eta).$$

Then $\pi(\mathcal{E}) = T_{IV,3}(\pi_{temp})$. Here is the associated symbol.

$$\mathcal{E} = \begin{array}{c} \begin{matrix} 0 & \cdots & \alpha-4 & \alpha-3 & \alpha-2 & \alpha-1 & \alpha \end{matrix} \\ \begin{pmatrix} \odot & & & & & & \\ \odot & & & & & & \\ \odot & \cdots & \odot & & & & \\ & & & & \odot & \odot & \odot \end{pmatrix} \end{array}.$$

Proof. The proof is similar to Proposition 7.2, which we omit. □

Proposition 7.11. *Let $\pi_{temp} = T_{I,1}^{\alpha-2}(T_{I,1}^{\alpha-1}(T_{I,1}^{\alpha}(\pi_{sc})))$ for $\alpha > 2$.*

(i) The representation $T_{V,2}^{\pm}(\pi_{temp})$ is well-defined if and only if $\alpha = 3$.
(ii) When $\alpha = 3$, the representation $T_{V,2}^{\pm}(\pi_{temp})$ is of critical type.
(iii) When $\alpha = 3$, define

$$\mathcal{E}_{\pm} := \{([0,0]_\rho, 0, \pm 1)^2, ([3,1]_\rho, 0, -\eta)\}.$$

Then we have $\pi(\mathcal{E}_{\pm}) = T_{V,2}^{\pm}(\pi_{temp})$. Here are the associated symbols.

$$\mathcal{E}_{+} = \begin{array}{c} \begin{matrix} 0 & 1 & 2 & 3 \end{matrix} \\ \begin{pmatrix} \oplus & & & \\ \oplus & & & \\ & \odot & \odot & \odot \end{pmatrix} \end{array}, \quad \mathcal{E}_{-} = \begin{array}{c} \begin{matrix} 0 & 1 & 2 & 3 \end{matrix} \\ \begin{pmatrix} \ominus & & & \\ \ominus & & & \\ & \odot & \odot & \odot \end{pmatrix} \end{array}.$$

Proof. The proof is similar to Proposition 7.3, which we omit. □

Now we move onto the case $\pi_{temp} = T_{IV,3}(T_{I,1}^{\alpha-1}(T_{I,1}^{\alpha}(\pi_{sc})))$. By Proposition 3.7, this is well -defined if and only if $\alpha \in \mathbb{Z}_{>2}$.

Proposition 7.12. *Let $\pi_{temp} = T_{IV,3}(T_{I,1}^{\alpha-1}(T_{I,1}^{\alpha}(\pi_{sc})))$ for $\alpha \in \mathbb{Z}_{>2}$.*

(i) The representation $T_{I,1}^{x}(\pi_{temp})$ is well-defined if and only if $x = \alpha - 2$ or $\alpha + 1$.
(ii) The representation $T_{I,1}^{x}(\pi_{temp})$ is of critical type if and only if $(x, \alpha) = (1, 3)$.
(iii) Let $x \in \{\alpha - 2, \alpha + 1\}$. Define

$\mathcal{E}_{\alpha-2} := \{(([0,0]_\rho, 0, \eta)^2, [\alpha-4, 0]_\rho, 0, \eta), ([\alpha-2, \alpha-2]_\rho, 0, (-1)^{\alpha-1}\eta), ([\alpha, \alpha-1]_\rho, 0, (-1)^{\alpha}\eta)\}$,
$\mathcal{E}_{\alpha+1} := \{(([0,0]_\rho, 0, \eta)^2, [\alpha-3, 0]_\rho, 0, \eta), ([\alpha-1, \alpha-1]_\rho, 0, (-1)^{\alpha}\eta), ([\alpha+1, \alpha+1]_\rho, 0, (-1)^{\alpha-1}\eta)\}$.

Then we have $\pi(\mathcal{E}_x) = T_{I,1}^{x}(\pi_{temp})$. Here are the associated symbols.

$$\mathcal{E}_{\alpha-2} = \begin{array}{c} \begin{matrix} 0 & \cdots & \alpha-4 & \alpha-3 & \alpha-2 & \alpha-1 & \alpha \end{matrix} \\ \begin{pmatrix} \odot & & & & & & \\ \odot & & & & & & \\ \odot & \cdots & \odot & & & & \\ & & & & \odot & & \\ & & & & & \odot & \end{pmatrix} \end{array},$$

$$\mathcal{E}_{\alpha+1} = \begin{array}{c} \begin{matrix} 0 & \cdots & \alpha-3 & \alpha-2 & \alpha-1 & \alpha & \alpha+1 \end{matrix} \\ \begin{pmatrix} \odot & & & & & & \\ \odot & & & & & & \\ \odot & \cdots & \odot & & & & \\ & & & & \odot & & \\ & & & & & & \odot \end{pmatrix} \end{array}.$$

Proof. The proof is similar to Proposition 7.1, which we omit. □

The next case we'll look at is $\pi_{temp} = T_{V,2}^{\pm}(T_{I,1}^{1}(T_{I,1}^{2}(\pi_{sc})))$, which is well-defined only for $\alpha = 2$, by Proposition 3.8.

Proposition 7.13. *Let* $\pi_{temp}^{\pm} = T_{V,2}^{\pm}(T_{I,1}^{1}(T_{I,1}^{2}(\pi_{sc})))$*, and* $\alpha = 2$.

(i) *The representation* $T_{I,1}^{x}(\pi_{temp}^{\pm})$ *is well-defined if and only if* $x = 3$.
(ii) *When* $x = 3$*, the representation* $T_{I,1}^{x}(\pi_{temp}^{\pm})$ *is of critical type. In this case, it is the same as the representations* $T_{V,2}^{\pm}(T_{I,1}^{1}(T_{I,1}^{3}(T_{I,1}^{3}(\pi_{sc}))))$*, as described in Proposition 7.6.*
(iii) *The representations* $T_{IV,3}(\pi_{temp})$ *and* $T_{V,2}^{\pm}(\pi_{temp})$ *are not well-defined.*

Proof. The proof of part (i) is similar to Proposition 7.6, which we omit. Parts (ii) and (iii) follows from definition and by comparing the resulting extended multi-segments to the ones given earlier. □

A similar case to the one above is $\pi_{temp} = T_{I,1}^{2}(T_{V,2}^{\pm}(T_{I,1}^{1}(\pi_{sc})))$, which is well-defined only for $\alpha = 1$.

Proposition 7.14. *Let* $\pi_{temp}^{\pm} = T_{I,1}^{2}((T_{V,2}^{\pm}T_{I,1}^{1}(\pi_{sc})))$ *and* $\alpha = 1$.

(i) *The representation* $T_{I,1}^{x}(\pi_{temp}^{\pm})$ *is well-defined if and only if* $x = 1$ *or* $x = 3$*. When* $x = 1$*, then* $T_{I,1}^{x}(\pi_{temp}^{\pm})$ *is the same as the representation* $T_{I,1}^{x}(T_{V,2}^{\pm}(T_{I,1}^{2}(T_{I,1}^{1}(\pi_{sc}))))$*, as described in Proposition 7.9.*
(ii) *When* $x = 1$ *or* $x = 3$*, the representation* $T_{I,1}^{x}(\pi_{temp}^{\pm})$ *is of critical type.*
(iii) *The representations* $T_{IV,3}(\pi_{temp})$ *and* $T_{V,2}^{\pm}(\pi_{temp})$ *are not well-defined.*

Proof. Part (i) follows from Remark 3.2, and by comparing the resulting extended multi-segments of $T_{I,1}^{x}(\pi_{temp}^{\pm})$ and the ones shown in Proposition 7.9. Part (ii) follows from definition. □

Another similar case would be $\pi_{temp} = T_{I,1}^{2}(T_{I,1}^{1}(T_{V,2}^{\pm}(\pi_{sc})))$, which is well-defined if and only if $\alpha = 0$.

Proposition 7.15. *Let* $\pi_{temp}^{\pm} = T_{I,1}^{2}(T_{I,1}^{1}(T_{V,2}^{\pm}(\pi_{sc})))$ *and* $\alpha = 0$.

(i) *The representation* $T_{I,1}^{x}(\pi_{temp}^{\pm})$ *is well-defined if and only if* $x = 1$ *or* $x = 3$.
(ii) *When* $x = 1$ *or* $x = 3$*, the representation* $T_{I,1}^{x}(\pi_{temp}^{\pm})$ *is of critical type.*
(iii) *Let* $x \in \{1, 3\}$*. Define*

$$\mathcal{E}_{1,\pm} := \{([1,1]_{\rho}, 0, \pm 1), ([2,2]_{\rho}, 0, \pm 1)\},$$
$$\mathcal{E}_{3,\pm} := \{([0,0]_{\rho}, 0, \pm 1), ([3,3]_{\rho}, 0, \pm 1)\}.$$

Then we have that $\pi(\mathcal{E}_{x,\pm}) = T_{I,1}^{x}(\pi_{temp}^{\pm})$*. Here are the associated symbols.*

$$\mathcal{E}_{1,+} = \begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{pmatrix} \oplus & \\ & \oplus \end{pmatrix} \end{array}, \quad \mathcal{E}_{1,-} = \begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{pmatrix} \ominus & \\ & \ominus \end{pmatrix} \end{array},$$

$$\mathcal{E}_{3,+} = \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{pmatrix} \oplus & & & \\ & & & \oplus \end{pmatrix} \end{array}, \quad \mathcal{E}_{3,-} = \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{pmatrix} \ominus & & & \\ & & & \ominus \end{pmatrix} \end{array}.$$

Proof. This follows from Proposition 3.10 and remark 3.2. □

Proposition 7.16. *Let* $\pi_{temp}^{\pm} = T_{I,1}^{2}(T_{I,1}^{1}(T_{V,2}^{\pm}(\pi_{sc})))$ *and* $\alpha = 0$.

(i) *The representation* $T_{IV,3}(\pi_{temp}^{\pm})$ *is well-defined and of critical type.*
(ii) *The representation* $T_{V,2}^{\pm}(\pi_{temp}^{\pm})$ *is not well-defined.*

(iii) Define

$$\mathcal{E}_{\pm} := \{([0,0]_{\rho}, 0, \pm 1)^3, ([2,2]_{\rho}, 0, \pm 1)\}.$$

Then $\pi(\mathcal{E}_{\pm}) = T_{IV,3}(\pi_{temp}^{\pm})$. *Here are the associated symbols.*

$$\mathcal{E}_{+} = \begin{array}{c} \begin{matrix} 0 & 1 & 2 \end{matrix} \\ \begin{pmatrix} \oplus & & \\ \oplus & & \\ \oplus & & \\ & & \oplus \end{pmatrix} \end{array}, \quad \mathcal{E}_{-} = \begin{array}{c} \begin{matrix} 0 & 1 & 2 \end{matrix} \\ \begin{pmatrix} \ominus & & \\ \ominus & & \\ \ominus & & \\ & & \ominus \end{pmatrix} \end{array}.$$

Proof. This follows from Proposition 3.11 and Remark 3.2. □

Another tempered representation of corank 3 is $\pi_{temp} = T_{IV,3}(T_{I,1}^{1}(T_{V,2}^{\pm}(\pi_{sc})))$, which is also well-defined only when $\alpha = 0$.

Proposition 7.17. *Let* $\pi_{temp}^{\pm} = T_{IV,3}(T_{I,1}^{1}(T_{V,2}^{\pm}(\pi_{sc})))$ *and* $\alpha = 0$.

(i) The representation $T_{I,1}^{x}(\pi_{temp}^{\pm})$ *is well-defined if and only if* $x = 2$. *The representations* $T_{I,1}^{2}(\pi_{temp}^{\pm})$ *are the same as the representations* $T_{IV,3}(T_{I,1}^{2}(T_{I,1}^{1}(T_{V,2}^{\pm}(\pi_{sc}))))$, *as described in Proposition 7.16.*

(ii) The representation $T_{I,1}^{2}(\pi_{temp}^{\pm})$ *is of critical type.*

(iii) The representations $T_{IV,3}(\pi_{temp})$ *and* $T_{V,2}^{\pm}(\pi_{temp})$ *are not well-defined.*

Proof. Part (iv) follows from Proposition 3.12 and Remark 3.2. By comparing the resulting extended multi-segments, we get the result in part (i). Part (ii) follows from definition. □

We now proceed to the case $\pi_{temp} = T_{I,1}^{\frac{3}{2}}(T_{I,2}^{\frac{1}{2}}(\pi_{sc}))$, defined only when $\alpha = \frac{1}{2}$.

Proposition 7.18. *Let* $\pi_{temp} = T_{I,1}^{\frac{3}{2}}(T_{I,2}^{\frac{1}{2}}(\pi_{sc}))$ *and* $\alpha = \frac{1}{2}$.

(i) The representation $T_{I,1}^{x}(\pi_{temp})$ *is well-defined if and only if* $x = \frac{5}{2}$.

(ii) The representation $T_{I,1}^{\frac{5}{2}}(\pi_{temp})$ *is of critical type.*

(iii) The set $\Psi(T_{I,1}^{\frac{5}{2}}(\pi_{temp}))$ *is a singleton, and we have* $T_{I,1}^{\frac{5}{2}}(\pi_{temp}) = \pi(\mathcal{E})$, *where*

$$\mathcal{E} = \begin{array}{c} \begin{matrix} \frac{1}{2} & \frac{3}{2} & \frac{5}{2} \end{matrix} \\ \begin{pmatrix} \oplus & & \\ & & \oplus \end{pmatrix} \end{array}.$$

(iv) The representations $T_{IV,3}(\pi_{temp})$ *and* $T_{V,2}^{\pm}(\pi_{temp})$ *are not well-defined.*

Proof. The result follows directly from Proposition 3.13 and Remark 3.2. □

The next case is $\pi_{temp} = T_{I,1}^{\alpha}(T_{II,3}^{\frac{1}{2}}(\pi_{sc}))$, defined for $\alpha \in \frac{1}{2} + \mathbb{Z}_{>0}$.

Proposition 7.19. *Let* $\pi_{temp} = T_{I,1}^{\alpha}(T_{II,3}^{\frac{1}{2}}(\pi_{sc}))$ *for* $\alpha \in \frac{1}{2} + \mathbb{Z}$.

(i) The representation $T_{I,1}^{x}(\pi_{temp})$ *is well-defined if and only if* $x = \alpha + 1$, *or* $x = \alpha - 1$, *and* $\alpha \geq \frac{5}{2}$.

(ii) The representation $T_{I,1}^{\alpha+1}(\pi_{temp})$ *if of critical type when* $\alpha \in \{\frac{3}{2}, \frac{5}{2}\}$. *The representation* $T_{I,1}^{\alpha-1}(\pi_{temp})$ *is of critical type only when* $\alpha = \frac{5}{2}$.

(iii) *Let $x \in \{\alpha+1, \alpha-1\}$. Define*
$$\mathcal{E}_{\alpha+1} := \{([\frac{1}{2}, \frac{1}{2}]_\rho, 0, -1)^2, ([\alpha-2, \frac{1}{2}]_\rho, 0, -1), ([\alpha+1, \alpha+1]_\rho, 0, (-1)^{\alpha-\frac{1}{2}})\}.$$
When $\alpha \in \frac{3}{2} + \mathbb{Z}_{>0}$, define
$$\mathcal{E}_{\alpha-1} := \{([\frac{1}{2}, \frac{1}{2}]_\rho, 0, -1)^2, ([\alpha-3, \frac{1}{2}]_\rho, 0, -1), ([\alpha, \alpha-1]_\rho, 0, (-1)^{\alpha+\frac{1}{2}})\}.$$
Then $\pi(\mathcal{E}_x) = T_{I,1}^{x}(\pi_{temp})$. Here are the associated symbols.
$$\mathcal{E}_{\alpha+1} = \begin{array}{c} \begin{array}{cccccc} \frac{1}{2} & \cdots & \alpha-2 & \alpha-1 & \alpha & \alpha+1 \end{array} \\ \left(\begin{array}{cccccc} \ominus &&&&& \\ \ominus &&&&& \\ \ominus & \cdots & \odot &&& \\ &&&&& \odot \end{array}\right) \end{array},$$
$$\mathcal{E}_{\alpha-1} = \begin{array}{c} \begin{array}{cccccc} \frac{1}{2} & \cdots & \alpha-3 & \alpha-2 & \alpha-1 & \alpha \end{array} \\ \left(\begin{array}{cccccc} \ominus &&&&& \\ \ominus &&&&& \\ \ominus & \cdots & \odot &&& \\ &&&& \odot & \odot \end{array}\right) \end{array}.$$

(iv) *The representations $T_{IV,3}(\pi_{temp})$ and $T_{V,2}^{\pm}(\pi_{temp})$ are not well-defined.*

Proof. The proof of parts (i) through (iii) is similar to Proposition 7.7, which we omit. Part (iv) follows from definition and the good parity condition. □

Another case to consider is $\pi_{temp} = T_{I,1}^{\frac{3}{2}}(T_{III,2}^{\frac{1}{2}})$, which is well-defined only when $\alpha = \frac{1}{2}$, by Proposition 3.15. This will not appear in our Arthur type list later on, since it involves negative supercuspidal support, but we will include it here for completion.

Proposition 7.20. *Let $\pi_{temp} = T_{I,1}^{\frac{3}{2}}(T_{III,2}^{\frac{1}{2}})$ and $\alpha = \frac{1}{2}$.*

(i) *The representation $T_{I,1}^{x}(\pi_{temp})$ is well-defined if and only if $x = \frac{5}{2}$.*

(ii) *The representation $T_{I,1}^{\frac{5}{2}}(\pi_{temp})$ is of critical type.*

(iii) *The following is the set of extended multi-segments $\mathcal{E}$ (up to row exchanges) such that $\pi(\mathcal{E}) = T_{I,1}^{\frac{3}{2}}(\pi_{temp})$,*
$$\left\{ \begin{array}{c} \begin{array}{ccc} \frac{1}{2} & \frac{3}{2} & \frac{5}{2} \end{array} \\ \left(\begin{array}{ccc} \ominus && \\ && \ominus \end{array}\right) \end{array}, \begin{array}{c} \begin{array}{cccc} -\frac{1}{2} & \frac{1}{2} & \frac{3}{2} & \frac{5}{2} \end{array} \\ \left(\begin{array}{cccc} \oplus & \ominus && \\ &&& \ominus \end{array}\right) \end{array} \right\}$$

(iv) *The representations $T_{IV,3}(\pi_{temp})$ and $T_{V,2}^{\pm}(\pi_{temp})$ are not well-defined.*

Proof. Parts (i) through (iii) follows from Proposition 3.15 and Remark 3.2. Part (iv) follows from the good parity condition. □

The next case to look at is $\pi_{temp} = T_{I,1}^{\alpha}(T_{IV,5}(\pi_{sc}))$, which is well-defined if and only if $\alpha \in \mathbb{Z}_{>0}$, by Proposition 3.16.

Proposition 7.21. *Let $\pi_{temp} = T_{I,1}^{\alpha}(T_{IV,5}(\pi_{sc}))$ and $\alpha \in \mathbb{Z}_{>0}$.*

(i) *The representation $T_{I,1}^{x}(\pi_{temp})$ is well-defined if and only if $x = \alpha+1$, or $x = \alpha-1$, and $\alpha > 1$.*

(ii) *The representation $T_{I,1}^{x}(\pi_{temp})$ is of critical type if and only if $(x, \alpha) = (1, 2)$ or $(2, 1)$.*

(iii) *Let $\{x \in \alpha+1, \alpha-1\}$. Define*

$$\mathcal{E}_{\alpha+1} := \{([0,0]_\rho, 0, \eta)^4, ([\alpha-2, 0]_\rho, 0, \eta), ([\alpha+1, \alpha+1]_\rho, 0, (-1)^{\alpha-1}\eta)\}.$$

When $\alpha > 1$, define

$$\mathcal{E}_{\alpha-1} := \{([0,0]_\rho, 0, \eta)^4, ([\alpha-3, 0]_\rho, 0, \eta), ([\alpha-1, \alpha-1]_\rho, 0, (-1)^{\alpha}\eta), ([\alpha, \alpha]_\rho, 0, (-1)^{\alpha-1}\eta)\}.$$

Then we have $\pi(\mathcal{E}_x) = T_{I,1}^x(\pi_{temp})$. Here are the associated symbols.

$$\mathcal{E}_{\alpha+1} = \begin{array}{c} \begin{array}{cccccc} 0 & \cdots & \alpha-2 & \alpha-1 & \alpha & \alpha+1 \end{array} \\ \begin{pmatrix} \odot & & & & & \\ \odot & & & & & \\ \odot & & & & & \\ \odot & & & & & \\ \odot & \cdots & \odot & & & \\ & & & & & \odot \end{pmatrix} \end{array},$$

$$\mathcal{E}_{\alpha-1} = \begin{array}{c} \begin{array}{cccccc} 0 & \cdots & \alpha-3 & \alpha-2 & \alpha-1 & \alpha \end{array} \\ \begin{pmatrix} \odot & & & & & \\ \odot & & & & & \\ \odot & & & & & \\ \odot & & & & & \\ \odot & \cdots & \odot & & & \\ & & & & \odot & \\ & & & & & \odot \end{pmatrix} \end{array}.$$

(iv) *The representations $T_{IV,3}(\pi_{temp})$ and $T_{V,2}^{\pm}(\pi_{temp})$ are not well-defined.*

Proof. The proof is similar to Proposition 7.19, which we omit. □

We now move onto the case $\pi_{temp} = T_{I,1}^1(T_{V,4}^{\pm}(\pi_{sc}))$, which is well-defined if and only if $\alpha = 0$.

Proposition 7.22. *Let $\pi_{temp}^{\pm} = T_{I,1}^1(T_{V,4}^{\pm}(\pi_{sc}))$ and $\alpha = 0$.*

(i) *The representation $T_{I,1}^x(\pi_{temp}^{\pm})$ is well-defined if and only if $x = 2$. In this case, it is the same as the representation $T_{IV,3}(T_{I,1}^2(T_{I,1}^1(T_{V,2}^{\pm}(\pi_{sc}))))$ described in Proposition 7.16.*

(ii) *The representation $T_{I,1}^2(\pi_{temp}^{\pm})$ is of critical type.*

(iii) *The representations $T_{IV,3}(\pi_{temp})$ and $T_{V,2}^{\pm}(\pi_{temp})$ are not well-defined.*

Proof. The result in part (i) follows from comparing the extended multi-segments in Proposition 3.17 and Proposition 7.16. Part (ii) follows from definition and part (iii) follows from Remark 3.2. □

The next case is $\pi_{temp} = T_{V,4}^{\pm}(T_{I,1}^1(\pi_{sc}))$, which is well-defined only for $\alpha = 1$, by Proposition 3.20

Proposition 7.23. *Let $\pi_{temp}^{\pm} = T_{V,4}^{\pm}(T_{I,1}^1(\pi_{sc}))$ and $\alpha = 1$.*

(i) *The representation $T_{I,1}^x(\pi_{temp}^{\pm})$ is well-defined if and only if $x = 2$.*

(ii) *The representation $T_{I,1}^2(\pi_{temp}^{\pm})$ is of critical type.*

(iii) *Define*

$$\mathcal{E}_{\pm} := \{([0,0]_\rho, 0, \pm 1)^4, ([2,2]_\rho, 0, \pm 1)\}.$$

Then we have $\pi(\mathcal{E}_\pm) = T_{I,1}^{2}(\pi_{temp}^{\pm})$*. Here are the associated symbols.*

$$\begin{array}{c} \\ \mathcal{E}_+ = \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{pmatrix} \oplus & & \\ \oplus & & \\ \oplus & & \\ \oplus & & \\ & & \oplus \end{pmatrix} \end{array}, \quad \begin{array}{c} \\ \mathcal{E}_- = \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{pmatrix} \ominus & & \\ \ominus & & \\ \ominus & & \\ \ominus & & \\ & & \ominus \end{pmatrix} \end{array}.$$

(iv) *The representations* $T_{IV,3}(\pi_{temp}^{\pm})$ *and* $T_{V,2}^{\pm}(\pi_{temp}^{\pm})$ *are not well-defined.*

Proof. The proof is similar to Proposition 7.16. which we omit. □

By Proposition 3.22, we have another tempered representation of corank 3 which is only well-defined at $\alpha = 1$, namely $\pi_{temp} = T_{I,2}^{1}(T_{IV,3}(\pi_{sc}))$.

Proposition 7.24. *Let* $\pi_{temp} = T_{I,2}^{1}(T_{IV,3}(\pi_{sc}))$ *and* $\alpha = 1$.

(i) *The representation* $T_{I,1}^{x}(\pi_{temp})$ *is well-defined if and only if* $x = 2$.
(ii) *The representation* $T_{I,1}^{2}(\pi_{temp})$ *is of critical type.*
(iii) *Define*

$$\mathcal{E} := \{([0,0]_\rho, 0, \eta), ([1,1]_\rho, 0, \eta), ([2,2]_\rho, 0, \eta)\}.$$

Then we have $\pi(\mathcal{E}) = T_{I,1}^{2}(\pi_{temp})$*. Here is the associated symbol.*

$$\mathcal{E} = \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{pmatrix} \odot & & \\ & \odot & \\ & & \odot \end{pmatrix} \end{array}.$$

Proof. The proof is similar to Proposition 7.13, which we omit. □

Remark 7.25. *Note that despite their similarities, the extended multi-segment in the proposition above is not the same as the extended multi-segment of* $T_{I,1}^{1}(T_{V,2}^{\pm}(T_{I,1}^{2}(T_{I,1}^{1}(\pi_{sc}))))$*, as shown in Proposition 7.8, due to the sign restrictions.*

Proposition 7.26. *Let* $\pi_{temp} = T_{I,2}^{1}(T_{IV,3}(\pi_{sc}))$ *and* $\alpha = 1$.

(i) *The representation* $T_{IV,3}(\pi_{temp})$ *is well-defined and of critical type.*
(ii) *Define*

$$\mathcal{E} := \{([0,0]_\rho, 0, \eta)^3, ([1,1]_\rho, 0, \eta)^2\}.$$

Then $\pi(\mathcal{E}) = T_{IV,3}(\pi_{temp})$*. Here is the associated symbol.*

$$\mathcal{E} = \begin{array}{c} \begin{array}{cc} 0 & 1 \end{array} \\ \begin{pmatrix} \odot & \\ \odot & \\ \odot & \\ & \odot \\ & \odot \end{pmatrix} \end{array}.$$

(iii) *The representation* $T_{V,2}^{\pm}(\pi_{temp})$ *is not well-defined.*

Proof. The proof is similar to Proposition 7.16, which we omit. □

Five more cases remain to wrap up Case (A). The next one is $\pi_{temp} = T_{II,3}^{1}(T_{IV,3}(\pi_{sc}))$, which is well-defined only when $\alpha \in \mathbb{Z}_{>0}$, by Proposition 3.22.

Proposition 7.27. *Let $\pi_{temp} = T^1_{II,3}(T_{IV,3}(\pi_{sc}))$, and $\alpha \in \mathbb{Z}_{>0}$.*

(i) The representation $T^x_{I,1}(\pi_{temp})$ is well-defined if and only if $x = \alpha$.
(ii) The representation $T^\alpha_{I,1}(\pi_{temp})$ is of critical type only when $\alpha \in \{1, 2\}$.
(iii) Define

$$\mathcal{E} := \{([\alpha-2, 0]_\rho, 0, \eta), ([1,1]_\rho, 0, -\eta)^2, ([\alpha, \alpha]_\rho, 0, *)\}$$

Then we have $\pi(\mathcal{E}) = T^\alpha_{I,1}(\pi_{temp})$. Here is the associated symbol.

$$\mathcal{E} = \begin{array}{c} \begin{array}{cccccc} 0 & 1 & \cdots & \alpha-2 & \alpha-1 & \alpha \end{array} \\ \begin{pmatrix} \odot & \cdots & \cdots & \odot & & \\ & \odot & & & & \\ & \odot & & & & \\ & & & & & \odot \end{pmatrix} \end{array}.$$

Proof. The proof is similar to Proposition 7.24, which we omit. ∎

Proposition 7.28. *Let $\pi_{temp} = T^1_{II,3}(T_{IV,3}(\pi_{sc}))$, and $\alpha \in \mathbb{Z}_{>0}$.*

(i) The representation $T_{IV,3}(\pi_{temp})$ is well-defined.
(ii) The representation $T_{IV,3}(\pi_{temp})$ is of critical type if and only if $\alpha = 1$.
(iii) Define

$$\mathcal{E} := \{([0,0]_\rho, 0, \eta)^2, ([\alpha-1, 0]_\rho, 0, \eta), ([1,1]_\rho, 0, -\eta)^2\}.$$

Then we have $\pi(\mathcal{E}) = T_{IV,3}(\pi_{temp})$. Here is the associated symbol.

$$\mathcal{E} = \begin{array}{c} \begin{array}{ccccc} 0 & 1 & \cdots & \alpha-2 & \alpha-1 \end{array} \\ \begin{pmatrix} \odot & & & & \\ \odot & & & & \\ \odot & \cdots & \cdots & \cdots & \odot \\ & \odot & & & \\ & \odot & & & \end{pmatrix} \end{array}.$$

(iv) The representation $T^\pm_{V,2}(\pi_{temp})$ is not well-defined.

Proof. The proof is similar to Proposition 7.10, which we omit. ∎

The next case we need to consider is $\pi_{temp} = T^1_{I,2}(T^\pm_{V,2}(\pi_{sc}))$. This is well-defined if and only if $\alpha = 0$, by Proposition 3.23.

Proposition 7.29. *Let $\pi^\pm_{temp} = T^1_{I,2}(T^\pm_{V,2}(\pi_{sc}))$ and $\alpha = 0$.*

(1) The representation $T^x_{I,1}(\pi_{temp})$ is well-defined if and only if $x = 2$. When $x = 2$, these are the same as the representations $T^1_{I,1}(T^2_{I,1}(T^1_{I,1}(T^\pm_{V,2}(\pi_{sc}))))$, as described in Proposition 7.15.
(2) The representation $T^2_{I,1}(\pi_{temp})$ is of critical type.
(3) The representations $T_{IV,3}(\pi_{temp})$ and $T^\pm_{V,2}(\pi_{temp})$ are not well-defined.

Proof. The result follows from comparing the extended multi-segments in Propositions 3.23 and 7.15. ∎

Three more cases remain in Case (A). The next one to consider is $\pi_{temp} = T^1_{III,2}(\pi_{sc})$, which is defined only when $\alpha = 1$, by Proposition 3.25.

Proposition 7.30. *Let $\pi_{temp} = T^1_{III,2}(\pi_{sc})$ and $\alpha = 1$.*

(i) *The representation* $T_{I,1}^{x}(\pi_{temp})$ *is well-defined if and only if* $x=2$. *When* $x=2$, *the representation* $T_{I,1}^{2}(\pi_{temp})$ *is of critical type, and is the same as the representation* $T_{I,1}^{2}(T_{I,2}^{1}(T_{IV,3}(\pi_{sc})))$, *as described in Proposition 7.24.*

(ii) *The representation* $T_{IV,3}(\pi_{temp})$ *is well-defined and of critical type. It is the same as the representation* $T_{IV,3}(T_{I,2}^{1}(T_{IV,3}(\pi_{sc})))$, *as described in Proposition 7.26.*

(iii) *The representation* $T_{V,2}^{\pm}(\pi_{temp})$ *is not well-defined.*

Proof. The result follows from Remark 3.2 and a direct comparison of the respective extended multi-segments. ☐

The second last case in Case (A) is $\pi_{temp}=T_{IV,7}(\pi_{sc})$, which is defined only when $\alpha\in\mathbb{Z}_{>0}$, by Proposition 3.26.

Proposition 7.31. *Let* $\pi_{temp}=T_{IV,7}(\pi_{sc})$ *and* $\alpha\in\mathbb{Z}_{>0}$.

(i) *The representation* $T_{I,1}^{x}(\pi_{temp})$ *is well-defined if and only if* $x=\alpha$.

(ii) *The representation* $T_{I,1}^{\alpha}(\pi_{temp})$ *is of critical type if and only if* $\alpha=1$.

(iii) *Define*

$$\mathcal{E}:=\{([0,0]_{\rho},0,\eta)^{6},([\alpha-2,0]_{\rho},0,\eta),([\alpha,\alpha]_{\rho},0,(-1)^{\alpha-1}\eta)\}.$$

Then we have $\pi(\mathcal{E})=T_{I,1}^{\alpha}(\pi_{temp})$. *Here is the associated symbol.*

$$\mathcal{E}=\begin{array}{c}\begin{array}{ccccc} 0 & \cdots & \alpha-2 & \alpha-1 & \alpha\end{array}\\ \begin{pmatrix} \odot &&&& \\ \odot &&&& \\ \odot &&&& \\ \odot &&&& \\ \odot &&&& \\ \odot &&&& \\ \odot & \cdots & \odot && \\ &&&& \odot \end{pmatrix}\end{array}.$$

(iv) *The representations* $T_{IV,3}(\pi_{temp})$ *and* $T_{V,2}^{\pm}(\pi_{temp})$ *are not well-defined.*

Proof. The proof is similar to Proposition 7.23, which we omit. ☐

Finally, the last case in Case (A) is $\pi_{temp}=T_{V,6}^{\pm}(\pi_{sc})$, which is defined only when $\alpha=0$, by Proposition 3.25.

Proposition 7.32. *Let* $\pi_{temp}^{\pm}=T_{V,6}^{\pm}(\pi_{sc})$ *and* $\alpha=0$.

(i) *The representations* $T_{I,1}^{x}(\pi_{temp}^{\pm})$ *are well-defined if and only if* $x=1$.

(ii) *The representations* $T_{I,1}^{1}(\pi_{temp}^{\pm})$ *are of critical type.*

(iii) *Define*

$$\mathcal{E}_{\pm}:=\{([0,0]_{\rho},0,\pm1)^{5},([1,1]_{\rho},0,\pm1)\}.$$

Then $\pi(\mathcal{E}_{\pm})=T_{I,1}^{1}(\pi_{temp}^{\pm})$. *Here are the associated symbols.*

$$\mathcal{E}_{+}=\begin{array}{c}\begin{array}{cc} 0 & 1\end{array}\\ \begin{pmatrix} \oplus & \\ \oplus & \\ \oplus & \\ \oplus & \\ \oplus & \\ & \oplus \end{pmatrix}\end{array},\quad \mathcal{E}_{-}=\begin{array}{c}\begin{array}{cc} 0 & 1\end{array}\\ \begin{pmatrix} \ominus & \\ \ominus & \\ \ominus & \\ \ominus & \\ \ominus & \\ & \ominus \end{pmatrix}\end{array},$$

(iv) The representations $T_{IV,3}(\pi^{\pm}_{temp})$ *and* $T^{\pm}_{V,2}(\pi^{\pm}_{temp})$ *are not well-defined.*

Proof. The proof is similar to Proposition 7.31, which we omit. $\square$

This concludes all our discussion of Case (A). We move onto Case (B).

7.2. **Case** (B) : $\pi \hookrightarrow \rho|\cdot|^{x_1} \times \rho|\cdot|^{x_2} \rtimes \pi_{temp}$, **where** π_{temp} **is tempered of corank** 2. In this section, we consider the possibility when $\pi \hookrightarrow \rho|\cdot|^{x_1} \times \rho|\cdot|^{x_2} \rtimes \pi_{temp}$, where π_{temp} is tempered of corank 2. Here there are five possibilities for the form of π, namely $\pi = T^{x}_{I,2}(\pi_{temp}), T^{x}_{II,3}(\pi_{temp}), T^{\frac12}_{III,2}(\pi_{temp}), T_{IV,5}(\pi_{temp})$, and $T^{\pm}_{V,4}(\pi_{temp})$. We use the same list of tempered, corank 2 representations that is provided at the beginning of Section 3.

We begin with the case $T^{\alpha+1}_{I,1}(T^{\alpha}_{I,1}(\pi_{sc}))$.

Proposition 7.33. *Let* $\pi_{temp} = T^{\alpha+1}_{I,1}(T^{\alpha}_{I,1}(\pi_{sc}))$ *for* $\alpha > 0$

(i) The representation $T^{x}_{I,2}(\pi_{temp})$ *is well-defined if and only if* $x = \frac12$, *and* $\alpha \in \{\frac12, \frac32\}$.

(ii) When $\alpha \in \{\frac12, \frac32\}$, *the representation* $T^{\frac12}_{I,2}(\pi_{temp})$ *is of critical type.*

(iii) When $\alpha = \frac12$, *we have that* $\pi(\mathcal{E}_{I,\frac12}) = T^{\frac12}_{I,2}(\pi_{temp})$, *where*

$$\mathcal{E}_{I,\frac12} = \begin{array}{c} \begin{array}{cc} \frac12 & \frac32 \end{array} \\ \begin{pmatrix} \oplus & \\ \oplus & \\ & \oplus \end{pmatrix} \end{array}.$$

When $\alpha = \frac32$, *we have that* $\pi(\mathcal{E}_{I,\frac32}) = T^{\frac12}_{I,2}(\pi_{temp})$, *where*

$$\mathcal{E}_{I,\frac32} = \begin{array}{c} \begin{array}{ccc} \frac12 & \frac32 & \frac52 \end{array} \\ \begin{pmatrix} \oplus & & \\ \oplus & & \\ & & \ominus \end{pmatrix} \end{array}.$$

(iv) The representation $T^{x}_{II,3}(\pi_{temp})$ *is well-defined if and only if* $x = \frac12$, *and* $\alpha \in \frac12 + \mathbb{Z}_{>1}$. *In this case, it is the same as the representation* $T^{\alpha+1}_{I,1}(T^{\alpha}_{I,1}(T^{\frac12}_{II,3}(\pi_{sc})))$, *as described in Proposition 7.19.*

(v) The representation $T^{\frac12}_{III,2}(\pi_{temp})$ *is well-defined if and only if* $\alpha \in \{\frac12, \frac32\}$.

(vi) When $\alpha \in \{\frac12, \frac32\}$, *the representation* $T^{\frac12}_{III,2}(\pi_{temp})$ *is of critical type.*

(vii) When $\alpha = \frac12$, *we have that* $\pi(\mathcal{E}_{III,\frac12}) = T^{\frac12}_{III,2}(\pi_{temp})$, *where*

$$\mathcal{E}_{III,\frac12} = \begin{array}{c} \begin{array}{cc} \frac12 & \frac32 \end{array} \\ \begin{pmatrix} \ominus & \\ \ominus & \\ & \oplus \end{pmatrix} \end{array}.$$

When $\alpha = \frac32$, *we have that* $\pi(\mathcal{E}_{III,\frac32}) = T^{\frac12}_{I,2}(\pi_{temp})$, *where*

$$\mathcal{E}_{III,\frac32} = \begin{array}{c} \begin{array}{ccc} \frac12 & \frac32 & \frac52 \end{array} \\ \begin{pmatrix} \ominus & & \\ \ominus & & \\ & & \ominus \end{pmatrix} \end{array}.$$

(viii) *The representation $T_{IV,5}(\pi_{temp})$ is well-defined if and only if $\alpha \in \mathbb{Z}_{>1}$. In this case, the representation is the same as the representation $T_{I,1}^{\alpha+1}(T_{I,1}^{\alpha}(T_{IV,5}(\pi_{sc})))$, as described in Proposition 7.21.*

(ix) *The representation $T_{V,4}^{\pm}(\pi_{temp})$ is well-defined if and only if $\alpha = 1$. In this case, the representation is the same as the representation $T_{V,4}^{\pm}(T_{I,1}^{1}(\pi_{sc}))$, as described in Proposition 7.23.*

Proof. Define

$$\mathcal{E} := \{([\alpha-2, \epsilon_\rho]_\rho, 0, \eta), ([\alpha+1, \alpha+1]_\rho, 0, (-1)^{\alpha-1-\epsilon_\rho}\eta)\}$$

Then we have $\pi(\mathcal{E}) = \pi_{temp}$. Here is the associated symbol.

$$\mathcal{E} = \begin{array}{c} \begin{array}{cccccc} \epsilon_\rho & \cdots & \alpha-2 & \alpha-1 & \alpha & \alpha+1 \end{array} \\ \begin{pmatrix} \odot & \cdots & \odot & & & \\ & & & & & \odot \end{pmatrix} \end{array}.$$

As shown, the extended multi-segment is multiplicity free. In order for $T_{I,2}^{x}$ to be well-defined, we need to have $x = \frac{1}{2}$, since by convention the multiplicity $m_\phi(\rho \otimes S_0) = \infty$. We also need that $m_\phi(\rho \otimes S_2) = 0$, which can only happen when $\alpha \in \{\frac{1}{2}, \frac{3}{2}\}$. This proves part (i). Parts (ii) and (iii) follow from definition. Following a similar argument, we can prove parts (v) to (vii).

Parts (iv), $(viii)$ and (ix) follow from comparing the extended multi-segment above to the one given in previous propositions. This completes the proof. □

The second case is $\pi_{temp} = T_{I,1}^{\alpha-1}(T_{I,1}^{\alpha}(\pi_{sc}))$, defined for $\alpha > 1$.

Proposition 7.34. *Let $\pi_{temp} = T_{I,1}^{\alpha-1}(T_{I,1}^{\alpha}(\pi_{sc}))$ and $\alpha > 1$.*

(i) *The representation $T_{I,2}^{x}(\pi_{temp})$ is well-defined if and only if $(x, \alpha) = (\frac{1}{2}, \frac{5}{2})$.*

(ii) *When $\alpha = \frac{5}{2}$, the representation $T_{I,2}^{\frac{1}{2}}(\pi_{temp})$ is of critical type.*

(iii) *Let $\alpha = \frac{5}{2}$. Then we have $\pi(\mathcal{E}_I) = T_{III,2}^{\frac{1}{2}}(\pi_{temp})$, where*

$$\mathcal{E}_I = \begin{array}{c} \begin{array}{ccc} \frac{1}{2} & \frac{3}{2} & \frac{5}{2} \end{array} \\ \begin{pmatrix} \oplus & & \\ \oplus & & \\ & \odot & \odot \end{pmatrix} \end{array}.$$

(iv) *The representation $T_{II,3}^{x}(\pi_{temp})$ is well-defined if and only if $x = \frac{1}{2}$ and $\alpha \in \frac{1}{2} + \mathbb{Z}_{>2}$. In this case, it is the same as the representation $T_{I,1}^{\alpha-1}(T_{I,1}^{\alpha}(T_{II,3}^{\frac{1}{2}}(\pi_{sc})))$, as defined in Proposition 7.19.*

(v) *The representation $T_{III,2}^{\frac{1}{2}}(\pi_{temp})$ is well-defined if and only if $\alpha = \frac{5}{2}$.*

(vi) *When $\alpha = \frac{5}{2}$, the representation $T_{III,2}^{\frac{1}{2}}(\pi_{temp})$ is of critical type.*

(vii) *Let $\alpha = \frac{5}{2}$. Then we have $\pi(\mathcal{E}_{III}) = T_{III,2}^{\frac{1}{2}}(\pi_{temp})$, where*

$$\mathcal{E}_{III} = \begin{array}{c} \begin{array}{ccc} \frac{1}{2} & \frac{3}{2} & \frac{5}{2} \end{array} \\ \begin{pmatrix} \ominus & & \\ \ominus & & \\ & \odot & \odot \end{pmatrix} \end{array}.$$

(viii) *The representation* $T_{IV,5}(\pi_{temp})$ *is well-defined if and only if* $\alpha \in \mathbb{Z}_{\geq 3}$. *In this case, it's the same as the representation* $T_{I,1}^{\alpha-1}(T_{I,1}^{\alpha}(T_{IV,5}(\pi_{sc})))$, *as described in Proposition 7.21.*

(ix) *The representation* $T_{V,4}^{\pm}(\pi_{temp})$ *is well-defined if and only if* $\alpha = 2$. *In this case, define*

$$\mathcal{E}_{\pm} := \{([0,0]_{\rho}, 0, \pm 1)^5, ([2,1]_{\rho}, 0, \eta)\}.$$

Then we have $\pi(\mathcal{E}_{\pm}) = T_{V,4}^{\pm}(\pi_{temp})$. *Here are the associated symbols.*

$$\mathcal{E}_{+} = \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{pmatrix} \oplus & & \\ \oplus & & \\ \oplus & & \\ \oplus & & \\ & \odot & \odot \end{pmatrix} \end{array}, \quad \mathcal{E}_{-} = \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{pmatrix} \ominus & & \\ \ominus & & \\ \ominus & & \\ \ominus & & \\ & \odot & \odot \end{pmatrix} \end{array}.$$

(x) *The representations* $T_{I,2}^{x}(\pi_{temp}), T_{II,3}^{x}(\pi_{temp})$ *are not well-defined.*

Proof. The proof is similar to Proposition 7.33, which we omit. □

The next case to consider is $\pi_{temp} = T_{IV,3}(T_{I,1}^{\alpha}(\pi_{sc}))$, defined for $\alpha \in \mathbb{Z}_{>0}$.

Proposition 7.35. *Let* $\pi_{temp} = T_{IV,3}(T_{I,1}^{\alpha}(\pi_{sc}))$ *and* $\alpha \in \mathbb{Z}_{>0}$.

(i) *The representation* $T_{I,2}^{x}(\pi_{temp})$ *is well-defined if and only if* $(x, \alpha) = (1, 2)$.

(ii) *When* $\alpha = 2$, *the representation* $T_{I,2}^{1}(\pi_{temp})$ *is of critical type.*

(iii) *Define*

$$\mathcal{E} := \{([0,0]_{\rho}, 0, \eta)([1,1]_{\rho}, 0, \eta)^2, ([2,2]_{\rho}, 0, \eta)\}.$$

Then $\pi(\mathcal{E}) = T_{I,1}^{1}(\pi_{temp})$. *Here is the associated symbol.*

$$\mathcal{E} = \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{pmatrix} \odot & & \\ & \odot & \\ & \odot & \\ & & \odot \end{pmatrix} \end{array}.$$

(iv) *The representation* $T_{II,3}^{x}(\pi_{temp})$ *is well-defined if and only if* $x = 1$. *In this case, it is the same as the representation* $T_{I,1}^{\alpha}(T_{II,3}^{1}(T_{IV,3}(\pi_{sc})))$, *as described in Proposition 7.27.*

(v) *The representations* $T_{III,2}^{\frac{1}{2}}(\pi_{temp}), T_{IV,5}(\pi_{temp}), T_{V,4}^{\pm}(\pi_{temp})$ *are not well-defined.*

Proof. Let ϕ be the L-parameter associated to π_{temp}. For $T_{I,2}^{x}(\pi_{temp})$ to be well-defined, we need $m_{\phi}(\rho \otimes S_{2x-1}) \geq 2$ and $S(\rho \otimes S_{2x+1}) = 0$. By examining the extended multi-segment, we see that this is only possible when $(x, \alpha) = (1, 2)$. This proves part (i). Parts $(ii), (iii)$ and (v) follows from definition. For part (iv), we see that $T_{II,3}^{x}$ is only well-defined when $m_{\phi}(\rho \otimes S_{2x-1}) \geq 3$, which means that $x = 1$. The result follows by comparing the corresponding extended multi-segments. This concludes the proof. □

We now move onto the case $\pi_{temp} = T_{V,2}^{\pm}(T_{I,1}^{1}(\pi_{sc}))$, which is well-defined only for $\alpha = 1$.

Proposition 7.36. *Let* $\pi_{temp} = T_{V,2}^{\pm}(T_{I,1}^{1}(\pi_{sc}))$ *and* $\alpha = 1$, *then the representations* $T_{I,2}^{x}(\pi_{temp})$, $T_{II,3}^{x}(\pi_{temp}), T_{III,2}^{\frac{1}{2}}(\pi_{temp}), T_{IV,5}(\pi_{temp})$ *and* $T_{V,4}^{\pm}(\pi_{temp})$ *are all not well-defined.*

Proof. Let ϕ be the L-parameter corresponding to π_{temp}, then we have that $m_\phi(\rho\otimes S_1) = 2, m_\phi(\rho\otimes S_3)=1$, and $m_\phi(\rho\otimes S_a)=0$ otherwise for $a\in\mathbb{Z}$. This proves that

$$T^x_{I,2}(\pi_{temp}), T^x_{II,3}(\pi_{temp}), T_{IV,5}(\pi_{temp}), T^{\pm}_{V,2}(\pi_{temp})$$

are not well-defined.

Finally, $T^{\frac{1}{2}}_{III,2}(\pi_{temp})$ is not well-defined by the good parity condition. □

A similar case to the one above is $\pi_{temp} = T^1_{I,1}(T^{\pm}_{V,2}(\pi_{sc}))$, which is well-defined if and only if $\alpha = 0$.

Proposition 7.37. *Let $\pi_{temp} = T^1_{I,1}(T^{\pm}_{V,2}(\pi_{sc}))$ and $\alpha = 0$.*

(i) The representation $T_{IV,5}(\pi_{temp})$ is well-defined and of critical type. It is the same as the representation $T^1_{I,1}(T^{\pm}_{V,6}(\pi_{sc}))$, as described in Proposition 7.32.

(ii) The representations $T^x_{I,2}(\pi_{temp}), T^x_{II,3}(\pi_{temp}), T^{\frac{1}{2}}_{III,2}(\pi_{temp})$ and $T^{\pm}_{V,4}(\pi_{temp})$ are not well-defined.

Proof. Part (i) follows from a comparison of the resulting extended multi-segments. The proof of part (ii) is similar to Proposition 7.36, which we omit. □

The next case we have is $\pi_{temp} = T^{\frac{1}{2}}_{I,2}(\pi_{sc})$, which is defined only when $\alpha = \frac{1}{2}$.

Proposition 7.38. *Let $\pi_{temp} = T^{\frac{1}{2}}_{I,2}(\pi_{sc})$ and $\alpha = \frac{1}{2}$.*

(i) The representation $T^x_{I,2}(\pi_{temp})$ is well-defined if and only if $x = \frac{3}{2}$.

(ii) The representation $T^{\frac{3}{2}}_{I,2}(\pi_{temp})$ is of critical type.

(iii) We have $\pi(\mathcal{E}) = T^{\frac{3}{2}}_{I,2}(\pi_{temp})$, where

$$\mathcal{E} = \begin{pmatrix} \frac{3}{2} \\ \oplus \\ \oplus \end{pmatrix}.$$

(iv) The representations $T^x_{II,3}(\pi_{temp}), T^{\frac{1}{2}}_{III,2}(\pi_{temp}), T_{IV,5}(\pi_{temp})$, and $T^{\pm}_{V,4}(\pi_{temp})$ are all not well-defined.

Proof. The proof is similar to Proposition 7.35, which we omit. □

The next case to consider is $\pi_{temp} = T^{\frac{1}{2}}_{II,3}(\pi_{sc})$, which is well-defined only when $\alpha \in \frac{1}{2}+\mathbb{Z}_{>0}$.

Proposition 7.39. *Let $\pi_{temp} = T^{\frac{1}{2}}_{II,3}(\pi_{sc})$, and $\alpha \in \frac{1}{2}+\mathbb{Z}_{>0}$.*

(i) The representation $T^x_{II,3}(\pi_{temp})$ is well-defined if and only if $x = \frac{3}{2}$.

(ii) The representation $T^{\frac{3}{2}}_{II,3}(\pi_{temp})$ is of critical type.

(iii) Define

$$\mathcal{E} := \{([\alpha-1,\frac{1}{2}]_\rho, 0, -1), [\frac{3}{2},\frac{3}{2}]_\rho, 0, 1)^2\}.$$

Then $\pi(\mathcal{E}) = T^{\frac{1}{2}}_{II,3}(\pi_{temp})$. Here is the associated symbol.

$$\mathcal{E} = \begin{pmatrix} \frac{1}{2} & \frac{3}{2} & \cdots & \alpha-1 \\ \ominus & \oplus & \cdots & \odot \\ & \oplus & & \\ & \oplus & & \end{pmatrix}.$$

(iv) The representations $T_{I,2}^{x}(\pi_{temp}), T_{III,2}^{\frac{1}{2}}(\pi_{temp}), T_{IV,5}(\pi_{temp})$ *and* $T_{V,4}^{\pm}(\pi_{temp})$ *are all not well-defined.*

Proof. The proof is similar to Proposition 7.38, which we omit. □

There are three more cases to examine in Case (B). The next one is $\pi_{temp} = T_{III,2}^{\frac{1}{2}}(\pi_{sc})$, which is well-defined only when $\alpha = \frac{1}{2}$.

Proposition 7.40. *Let* $\pi_{temp} = T_{III,2}^{\frac{1}{2}}(\pi_{sc})$ *and* $\alpha = \frac{1}{2}$.

(i) The representation $T_{I,2}^{x}(\pi_{temp})$ *is well-defined if and only if* $x = \frac{3}{2}$.

(ii) The representation $T_{I,2}^{\frac{3}{2}}(\pi_{temp})$ *is of critical type.*

(iii) We have that $\pi(\mathcal{E}) = T_{I,2}^{\frac{3}{2}}(\pi_{temp})$, *where*

$$\mathcal{E} = \begin{array}{c} \frac{3}{2} \\ \begin{pmatrix} \ominus \\ \ominus \end{pmatrix} \end{array}.$$

(iv) The representations $T_{II,3}^{x}(\pi_{temp}), T_{III,2}^{\frac{1}{2}}(\pi_{temp}), T_{IV,5}(\pi_{temp})$ *and* $T_{V,4}^{\pm}(\pi_{temp})$ *are all not well-defined.*

Proof. The proof is similar to Proposition 7.38, which we omit. □

The second last case in Case (B) is $\pi_{temp} = T_{IV,5}(\pi_{sc})$, which is well-defined if and only if $\alpha \in \mathbb{Z}_{>0}$.

Proposition 7.41. *Let* $\pi_{temp} = T_{IV,5}(\pi_{sc})$, *and* $\alpha \in \mathbb{Z}_{>0}$.

(i) The representation $T_{I,2}^{x}(\pi_{temp})$ *is well-defined if and only if* $(x, \alpha) = (1, 1)$. *In this case,* $T_{I,2}^{1}(\pi_{temp})$ *is of critical type and is the same as the representation* $T_{I,2}^{1}(T_{IV,3}(\pi_{sc}))$, *as described in Proposition 7.26.*

(ii) The representation $T_{II,3}^{x}(\pi_{temp})$ *is well-defined if and only if* $x = 1$, *and* $\alpha \in \mathbb{Z}_{>1}$. *In this case, the representation* $T_{II,3}^{1}(\pi_{temp})$ *is not of critical type, and it is the same as the representation* $T_{IV,3}(T_{II,3}^{1}(T_{IV,3}(\pi_{sc})))$, *as described in Proposition 7.28.*

(iii) The representations $T_{III,2}^{\frac{1}{2}}(\pi_{temp}), T_{IV,5}(\pi_{temp})$, *and* $T_{V,4}^{\pm}(\pi_{temp})$ *are all not well-defined.*

Proof. The proof is similar to Proposition 7.38, which we omit. □

The final case in Case (B) is $\pi_{temp} = T_{V,4}^{\pm}(\pi_{sc})$, which is well-defined if and only if $\alpha = 0$.

Proposition 7.42. *Let* $\pi_{temp}^{\pm} = T_{V,4}^{\pm}(\pi_{sc})$ *and* $\alpha = 0$.

(i) The representations $T_{I,2}^{x}(\pi_{temp}^{\pm})$ *are well-defined if and only if* $x = 1$.

(ii) The representations $T_{I,2}^{1}(\pi_{temp}^{\pm})$ *are of critical type.*

(iii) Define

$$\mathcal{E}_{\pm} := \{([0,0]_{\rho}, 0, \pm 1)^2, ([1,1]_{\rho}, 0, \pm 1)^2\}.$$

Then $\pi(\mathcal{E}_{\pm}) = T_{I,2}^{1}(\pi_{temp}^{\pm})$. *Here are the associated symbols.*

$$\mathcal{E}_{+} = \begin{array}{c} \begin{array}{cc} 0 & 1 \end{array} \\ \begin{pmatrix} \oplus & \\ \oplus & \\ & \oplus \\ & \oplus \end{pmatrix} \end{array}, \quad \mathcal{E}_{-} = \begin{array}{c} \begin{array}{cc} 0 & 1 \end{array} \\ \begin{pmatrix} \ominus & \\ \ominus & \\ & \ominus \\ & \ominus \end{pmatrix} \end{array}.$$

Proof. The proof is similar to Proposition 7.38, which we omit. $\square$

This concludes our discussion of Case (B). We will now move onto Case (C).

7.3. **Case** $(C) : \pi \hookrightarrow \rho|\cdot|^{x_1} \times \rho|\cdot|^{x_1} \times \rho|\cdot|^{x_3} \rtimes \pi_{temp}$**, where** π_{temp} **is tempered of corank 1.** In this subsection, we consider representations π of the form

$$T_{I,3}^{x}(\pi_{temp}), T_{III,2}^{1}(\pi_{temp}), T_{IV,7}(\pi_{temp}), T_{V,6}^{\pm}(\pi_{temp})$$

where π_{temp} is tempered of corank 1. Since there are three tempered representations of corank 1, namely $T_{I,1}^{\alpha}(\pi_{sc}), T_{IV,5}(\pi_{sc})$ and $T_{V,2}^{\pm}(\pi_{sc})$. There are 12 total cases to consider. We begin with $\pi_{temp} = T_{I,1}^{\alpha}(\pi_{sc})$, which is defined if and only when $\alpha > 0$.

Proposition 7.43. *Let $\pi_{temp} = T_{I,1}^{\alpha}(\pi_{sc})$, for $\alpha > 0$.*

- *(i) The representation $T_{III,2}^{1}(\pi_{temp})$ is well-defined if and only if $\alpha = 2$. When $\alpha = 2$, the representation $T_{III,2}^{1}(\pi_{temp})$ is of critical type. It is the same as the representation $T_{I,1}^{1}(T_{IV,3}(T_{I,1}^{2}(\pi_{sc})))$, as described in Proposition 7.35.*
- *(ii) The representation $T_{IV,7}(\pi_{temp})$ is well-defined if and only if $\alpha \in \mathbb{Z}_{>0}$. It is the same as the representation $T_{I,1}^{\alpha}(T_{IV,7}(\pi_{sc}))$, as described in Proposition 7.31, and it is of critical type only when $\alpha = 1$.*
- *(iii) The representations $T_{I,3}^{x}(\pi_{temp}), T_{V,6}^{\pm}(\pi_{temp})$ are not well-defined.*

Proof. Parts (i) and (ii) follows from definition, and by comparing the resulting extended multi-segments to the ones given in previous propositions. Part (iii) follows from the fact that $\alpha > 0$, and ϕ is multiplicity free, where ϕ is the L-parameter corresponding to π_{temp}. $\square$

The second tempered representation of corank 1 is $\pi_{temp} = T_{IV,3}(\pi_{sc})$, which is well-defined only when $\alpha \in \mathbb{Z}_{>0}$.

Proposition 7.44. *Let $\pi_{temp} = T_{IV,3}(\pi_{sc})$, and $\alpha \in \mathbb{Z}_{>0}$.*

- *(i) The representation $T_{I,3}^{x}(\pi_{temp})$ is well-defined if and only if $(x, \alpha) = (1, 1)$.*
- *(ii) When $\alpha = 1$, the representation $T_{I,3}^{1}(\pi_{temp})$ is of critical type.*
- *(iii) When $\alpha = 1$, we have $\pi(\mathcal{E}) = T_{I,3}^{1}(\pi_{temp})$, where*

$$\mathcal{E}_{=} \begin{array}{c} 1 \\ \begin{pmatrix} \odot \\ \odot \\ \odot \end{pmatrix} \end{array}.$$

- *(iv) The representation $T_{III,2}^{1}(\pi_{temp})$ is well-defined if and only if $\alpha = 1$. In this case, it is of critical type, and is the same as the representation $T_{IV,3}(T_{I,2}^{1}(T_{IV,3}(\pi_{sc})))$, as described in Proposition 7.26.*
- *(v) The representations $T_{IV,7}(\pi_{temp})$ and $T_{V,6}^{\pm}(\pi_{temp})$ are not well-defined.*

Proof. Let ϕ be the L-parameter corresponding to π_{temp}. Then $m_{\phi}(\rho \otimes S_{2x+1}) \geq 3$ only when $x = 0$. If $T_{I,3}^{x}(\pi_{temp})$ is well-defined then we also need $m_{\phi}(\rho \otimes S_{2x+1}) = 0$, which is only possible when $\alpha = 1$. This proves part (i). Parts (ii) and (iii) follows from definition.

For $T_{III,2}^{1}(\pi_{temp})$ to be well-defined, we need $m_{\phi}(\rho \otimes S_3) = 0$, which can only happen when $\alpha = 1$. By comparing the corresponding extended multi-segments in this case, we prove (iv). Part (v) follows from definition and the good parity condition. $\square$

The last tempered representation of corank 1 is $\pi_{temp} = T_{V,2}^{\pm}(\pi_{sc})$, which is well-defined if and only if $\alpha = 0$.

Proposition 7.45. *Let $\pi_{temp}^{\pm} = T_{V,2}^{\pm}(\pi_{sc})$ and $\alpha = 0$.*

(i) The representations $T_{III,2}^{1}(\pi_{temp}^{\pm})$ are well-defined and of critical type.

(ii) Define

$$\mathcal{E}_{\pm} := \{([0,0]_\rho, 0, \pm 1)^2, ([1,1]_\rho, 0, \mp 1)^2\}.$$

Then we have that $\pi(\mathcal{E}_{\pm}) = T_{III,2}^{1}(\pi_{temp}^{\pm})$, where

$$\mathcal{E}_+ = \begin{array}{c} \begin{array}{cc} 0 & 1 \end{array} \\ \begin{pmatrix} \oplus & \\ \oplus & \\ & \ominus \\ & \ominus \end{pmatrix} \end{array}, \quad \mathcal{E}_- = \begin{array}{c} \begin{array}{cc} 0 & 1 \end{array} \\ \begin{pmatrix} \ominus & \\ \ominus & \\ & \oplus \\ & \oplus \end{pmatrix} \end{array}.$$

(iii) The representations $T_{I,3}^{x}(\pi_{temp}), T_{IV,7}(\pi_{temp})$ and $T_{V,6}^{\pm}(\pi_{temp})$ are not well-defined.

Proof. The proof is similar to Proposition 7.44, which we omit. □

This concludes Case (C). We now move onto the final case in our classification of tempered representations of corank 4.

7.4. **Case** $(D) : \pi \hookrightarrow \rho|\cdot|^{x_1} \times \rho|\cdot|^{x_1} \times \rho|\cdot|^{x_3} \times \rho|\cdot|^{x_4} \rtimes \pi_{sc}$**, where** π_{sc} **is supercuspidal.** By Theorem 3.1, there are 6 total possibilities to consider in this case, as listed in the beginning of the section. Let $\pi_{sc} = \pi(\phi, \epsilon)$, then ϕ must be multiplicity free.

The first case is $T_{I,4}^{x}(\pi_{sc})$.

Proposition 7.46. *Let $\pi_{sc} = \pi(\phi, \epsilon)$ and $\alpha = \alpha_{\rho,\epsilon}$.*

(i) The representation $T_{I,4}^{x}(\pi_{sc})$ is well-defined if and only if $(x, \alpha) = (\frac{1}{2}, \frac{1}{2})$.

(ii) When $\alpha = \frac{1}{2}$, the representation $T_{I,4}^{x}(\pi_{sc})$ is of critical type.

(iii) When $\alpha = \frac{1}{2}$, we have that $\pi(\mathcal{E}) = T_{I,4}^{\frac{1}{2}}(\pi_{sc})$, where

$$\mathcal{E} = \begin{array}{c} \frac{1}{2} \\ \begin{pmatrix} \oplus \\ \oplus \\ \oplus \\ \oplus \end{pmatrix} \end{array}.$$

Proof. $T_{I,4}^{x}(\pi_{temp})$ is only well-defined when $m_\phi(\rho \otimes S_{2x-1}) \geq 4$. Since ϕ is multiplicity free, this can only occur when $x = \frac{1}{2}$. Additionally we need that $m_\phi(\rho \otimes S_{2x+1}) = 0$, which means that α must be $\frac{1}{2}$. This proves part (i). Parts (ii) and (iii) follow from definition. □

The second case is $\pi_{temp} = T_{II,5}^{x}(\pi_{sc})$.

Proposition 7.47. *Let $\pi_{sc} = \pi(\phi, \epsilon)$ and $\alpha = \alpha_{\rho,\epsilon}$.*

(i) The representation $T_{II,5}^{x}(\pi_{sc})$ is well-defined if and only if $x = \frac{1}{2}$ and $\alpha \in \frac{1}{2} + \mathbb{Z}_{>0}$.

(ii) When $\alpha \in \frac{1}{2} + \mathbb{Z}_{>0}$, the representation $T_{II,5}^{\frac{1}{2}}(\pi_{sc})$ is not of critical type.

(iii) When $\alpha \in \frac{1}{2} + \mathbb{Z}_{>0}$, define

$$\mathcal{E} := \{([\frac{1}{2}, \frac{1}{2}]_\rho, 0, -1)^4, ([\alpha - 1, \frac{1}{2}]_\rho, 0, -1)\}.$$

Then we have $\pi(\mathcal{E}) = T^{\frac{1}{2}}_{II,5}(\pi_{sc})$*. Here is the associated symbol.*

$$\mathcal{E} = \begin{array}{c} \begin{array}{ccc} \frac{1}{2} & \cdots & \alpha-1 \end{array} \\ \begin{pmatrix} \ominus & & \\ \ominus & & \\ \ominus & & \\ \ominus & & \\ \ominus & \cdots & \odot \end{pmatrix} \end{array}$$

Proof. The proof is similar to Proposition 7.46, which we omit. □

The third case is $\pi_{temp} = T^{\frac{3}{2}}_{III,2}(\pi_{sc})$.

Proposition 7.48. *Let* $\pi_{sc} = \pi(\phi, \epsilon)$ *and* $\alpha = \alpha_{\rho,\epsilon}$*.*

(i) The representation $T^{\frac{3}{2}}_{III,2}(\pi_{sc})$ *is well-defined if and only if* $\alpha = \frac{3}{2}$*.*

(ii) When $\alpha = \frac{3}{2}$*, the representation* $T^{\frac{3}{2}}_{III,2}(\pi_{temp})$ *is of critical type.*

(iii) Define

$$\mathcal{E} := \{([\frac{1}{2}, \frac{1}{2}]_\rho, 0, \eta)^2, ([\frac{3}{2}, \frac{3}{2}]_\rho, 0, -\eta)^2\}.$$

Then we have $\pi(\mathcal{E}) = T^{\frac{3}{2}}_{III,2}(\pi_{sc})$*. Here is the associated symbol.*

$$\mathcal{E} = \begin{array}{c} \begin{array}{cc} \frac{1}{2} & \frac{3}{2} \end{array} \\ \begin{pmatrix} \odot & \\ & \odot \\ & \odot \end{pmatrix} \end{array}.$$

Proof. By Remark 3.2, $T^{\frac{3}{2}}_{III,2}(\pi_{sc})$ is well-defined if and only if $m_\phi(\rho \otimes S_4) = 0$, and $m_\phi(\rho \otimes S_2) \geq 1$. This can only occur when $\alpha = \frac{3}{2}$, which proves part (i). Parts (ii) and (iii) follow from definition. □

We move onto the case $T^{\frac{1}{2}}_{III,4}(\pi_{sc})$.

Proposition 7.49. *Let* $\pi_{sc} = \pi(\phi, \epsilon)$ *and* $\alpha = \alpha_{\rho,\epsilon}$*.*

(1) The representation $T^{\frac{1}{2}}_{III,4}(\pi_{sc})$ *is well-defined if and only if* $\alpha = \frac{1}{2}$*.*

(2) When $\alpha = \frac{1}{2}$*, the representation* $T^{\frac{1}{2}}_{III,4}(\pi_{sc})$ *is of critical type.*

(3) We have that $\pi(\mathcal{E}) = T^{\frac{1}{2}}_{III,4}(\pi_{sc})$*, where*

$$\mathcal{E} = \begin{array}{c} \frac{1}{2} \\ \begin{pmatrix} \ominus \\ \ominus \\ \ominus \\ \ominus \end{pmatrix} \end{array}.$$

Proof. The proof is similar to Proposition 7.46, which we omit. □

The next case to consider is the representation $\pi_{temp} = T_{IV,9}(\pi_{sc})$.

Proposition 7.50. *Let* $\pi_{sc} = \pi(\phi, \epsilon)$ *and* $\alpha = \alpha_{\rho,\epsilon}$*.*

(i) The representation $T_{IV,9}(\pi_{sc})$ *is well-defined if and only if* $\alpha \in \mathbb{Z}_{>0}$*.*

(ii) The representation $T_{IV,9}(\pi_{sc})$ *is not of critical type.*

(iii) Define

$$\mathcal{E} := \{([0,0]_\rho, 0, \eta)^8, ([\alpha-1,0]_\rho, 0, \eta)\}.$$

Then we have $\pi(\mathcal{E}) = T_{IV,9}(\pi_{sc})$. *Here is the associated symbol.*

$$\mathcal{E} = \begin{array}{c} \begin{array}{ccc} 0 & \cdots & \alpha-1 \end{array} \\ \begin{pmatrix} \odot & & \\ \odot & & \\ \odot & & \\ \odot & & \\ \odot & & \\ \odot & & \\ \odot & & \\ \odot & & \\ \odot & \cdots & \odot \end{pmatrix} \end{array}.$$

Proof. This follows directly from definition and Remark 3.2. □

The final case to consider is the representation $\pi_{temp} = T^{\pm}_{V,8}(\pi_{sc})$.

Proposition 7.51. *Let* $\pi_{sc} = \pi(\phi, \epsilon)$ *and* $\alpha = \alpha_{\rho,\epsilon}$.

(i) The representation $\pi^{\pm}_{temp} = T^{\pm}_{V,8}(\pi_{sc})$ *is well-defined if and only if* $\alpha = 0$

(ii) When $\alpha = 0$, *the representation* $\pi^{\pm}_{temp}$ *is of critical type.*

(iii) We have that $\pi(\mathcal{E}_\pm) = \pi^{\pm}_{temp}$, *where*

$$\mathcal{E}_+ = \begin{array}{c} 0 \\ \begin{pmatrix} \oplus \\ \oplus \\ \oplus \\ \oplus \\ \oplus \\ \oplus \\ \oplus \\ \oplus \end{pmatrix} \end{array}, \quad \mathcal{E}_- = \begin{array}{c} 0 \\ \begin{pmatrix} \ominus \\ \ominus \\ \ominus \\ \ominus \\ \ominus \\ \ominus \\ \ominus \\ \ominus \end{pmatrix} \end{array}.$$

Proof. This follows directly from definition and Remark 3.2. □

With this we have classified all the tempered representations of corank 4. Combined with the non-tempered representations of corank 4 which we classified in Section **??**, we can give a complete list of all representations of corank 4, which are both of Arthur type and of critical type. This will be given in the next section.

By Theorem 2.5, we can conclude that that all of the representations listed in Proposition A.1 are unitarizable. Furthermore, the list contains all representations of $G(F)$ of corank 4 that are unitarizable and of critical type.

8. Open connected components in the unitary dual of corank 4

In this section, we will use the lists given in §A and §B to give the full list of unitary open connected components in corank 4. This will be the first step to constructing the full unitary dual. To this end, we will use the algorithm to compute $\Pi_{\overline{A}}(G_n)$ introduced in [HJLLZ25, §8] to determine inductively the unitarity of a given open connected component.

Using the technique of unitary reduction in step 3 of algorithm 8.1, we will later on prove that certain connected components are non-unitary. In fact, we highlight that there are five main methods to prove that a certain connected component in $C \subset \mathbb{R}^n$ is non-unitary. Four of them are given in steps (3-1) to (3-4) of Algorithm 8.1, respectively. The last method is to show directly that there exists a point on the boundary of C which contains non-unitarizable subquotients.

The steps to construct the full unitary dual of corank 4 is as follows: First, in Proposition 8.3, we give a list of open connected components, which we prove to be unitary. Then, in subsections 8.2 to 8.4, we show that all other connected components are non-unitary, using Algorithm 8.1 together with Tadic's techniques of dimensionality reduction in [Tad23]. Explicitly, this means that we characterize all possible connected components of dimension 3 that can possibly appear on the boundary of a unitary open connected component of dimension 4. This provides a necessary condition the open unitary connected components must satisfy, which allows us to show that the list of open connected components in Proposition 8.3 is complete.

Later, in §9 and §10, we append to the unitary dual all representations which appear as part of a lower-dimensional complementary series (in this case those with one or two parameters), as described in Step 2 of Algorithm 8.1. This will give us the list of all possible unitarizable representations of corank 4.

8.1. Algorithm for computing $\Pi_{\overline{A}}(G_n)$.

Algorithm 8.1. [HJLLZ25, Algorithm 8.5] *Let π_{sc} be an irreducible supercuspidal representation of G_n, ρ be an irreducible self-dual supercuspidal representation of $\mathrm{GL}_d(F)$, and let $r \in \mathbb{Z}_{\geq 0}$. In this algorithm, we output the set*

$$\Pi_{\overline{A}}(\pi_{sc}, \rho, r) := \Pi_{\overline{A}}(G_{n+rd}) \cap \mathrm{Irr}(X_\rho; \pi_{sc}).$$

Write $\alpha = \alpha_{\pi_{sc},\rho}$ for short.

Step 1. Compute $\Pi_{A,gp}(\pi_{sc}, \rho, s)$ for $0 \leq s \leq r$. Let $\Omega_{A,gp} := \cup_{0\leq s\leq r}\Pi_{A,gp}(\pi_{sc}, \rho, s)$.

Step 2. For $0 \leq s \leq r' \leq r$, initiate $R(\pi_{sc}, \rho, r', s)$ to be the empty set. Repeat the following steps for each $\pi_A \in \Pi_{A,gp}(\pi_{sc}, \rho, s)$:

- *(2-1) Let $\Psi(r'-s)$ be the collection of ordered tuples of pairs $\psi = ((a_1, b_1), \ldots, (a_{l(\psi)}, b_{l(\psi)})) \in (\mathbb{Z}^2_{\geq 1})^{l(\psi)}$, $l(\psi) \in \mathbb{Z}_{>0}$, such that $\sum_{i=1}^{l(\psi)} a_i b_i = r' - s$. By convention, let $\Psi(0) := \{\emptyset\}$.*
- *(2-2) Suppose that $r' - s > 0$. For each $\psi \in \Psi(r' - s)$, let $\mathcal{H}(\psi, \pi_A)$ denote the set consisting of the following reducibility hyperplanes in $\mathbb{R}^{l(\psi)}$:*
  - *$\{x_i = t \mid u_\rho(a_i, b_i)|\cdot|^t \rtimes \pi_A$ is reducible$\}$,*
  - *$\{x_i \pm x_j = t \mid u_\rho(a_i, b_i)|\cdot|^t \times u_\rho(a_j, b_j)$ is reducible$, 1 \leq i < j \leq l(\psi)\}$.*
- *(2-3) If $r' - s > 0$, let $R(\psi, \pi_A)$ denote the (finite) set of connected components of $\mathbb{R}^{l(\psi)} \setminus (\cup_{H\in\mathcal{H}(\psi,\pi_A)}H)$. For each $C \in R(\psi, \pi_A)$, append the triple (C, ψ, π_A) into $R(\pi_{sc}, \rho, r', s)$. If $r' - s = 0$, then append $\{(\mathbb{R}^0, \emptyset, \pi_A)\}$ into $R(\pi_{sc}, \rho, r', s)$.*

Step 3. Let $R := \cup_{0\leq s\leq r'\leq r}R(\pi_{sc}, \rho, r', s) \sqcup \{-1\}$. Define an equivalence relation $\sim$ on R as follows. Let $(C, \psi, \pi_A) \in R(\pi_{sc}, \rho, r', s)$ where $C \subseteq \mathbb{R}^{l(\psi)}$.

- *(3-1) Suppose that C is unbounded. Then we define $(C, \psi, \pi_A) \sim -1$.*
- *(3-2) Suppose that $C \cap \{x_i = 0\} \neq \emptyset$. Define ψ^- by removing (a_i, b_i) from ψ. Take any point $\underline{y} \in C \cap \{x_i = 0\}$ and define $\underline{y}^- \in \mathbb{R}^{l(\psi)-1}$ by removing the i-th coordinate (if $l(\psi) = 1$, then set $\underline{y}^- = 0$). Let (C^-, ψ^-, π_A) be the unique element in R such that $\underline{y}^- \in C^-$. Then we define $(C, \psi, \pi_A) \sim (C^-, \psi^-, \pi_A)$.*
- *(3-3) Suppose that $C \cap \{x_i = t\} \neq \emptyset$ for some $t \in (\alpha + \frac{a_i+b_i}{2}) + \mathbb{Z}$. Let $\pi_A^+ := u_\rho(a_i, b_i)|\cdot|^t \rtimes \pi_A$, which is irreducible and of good parity. If π_A^+ is not in $\Omega_{A,gp}$,*

then define $(C,\psi,\pi_A) \sim -1$. *If* $\pi_A^+ \in \Omega_{A,gp}$, *then define* ψ^- *and take a point* $\underline{y}^- \in \mathbb{R}^{l(\psi)-1}$ *as in the previous case. Let* (C^-,ψ^-,π_A^+) *be the unique element in* R *such that* $\underline{y}^- \in C^-$. *Then we define* $(C,\psi,\pi_A) \sim (C^-,\psi^-,\pi_A^+)$.

(3-4) Suppose that $(a_i,b_i) = (a_j,b_j)$ *with* $i \neq j$ *and* $C \cap \{x_i = \pm x_j\} \neq \emptyset$. *Define* ψ^- *by removing both* (a_i,b_i) *and* (a_j,b_j) *from* ψ. *Take any point* $\underline{y} = (y_1,\ldots,y_l) \in C \cap \{x_i = \pm x_j\}$. *If* $|y_i| > \frac{1}{2}$, *then define* $(C,\psi,\pi_A) \sim -1$. *If* $|y_i| < 1/2$, *then define* $\underline{y}^- \in \mathbb{R}^{l(\psi)-2}$ *by removing the* i*-th and* j*-th coordinates (if* $l(\psi) = 2$, *then set* $\underline{y}^- = 0$*). Let* (C^-,ψ^-,π_A) *be the unique element in* R *such that* $\underline{y}^- \in C^-$. *Then we define* $(C,\psi,\pi_A) \sim (C^-,\psi^-,\pi_A)$.

Let $R_{\overline{A}}$ *be the collection of* $(C,\psi,\pi_A) \in R$ *such that* $(C,\psi,\pi_A) \sim (\mathbb{R}^0,\emptyset,\pi_A')$ *for some* $\pi_A' \in \Omega_{A,gp}$.

Step 4. For each $(C,\psi = ((a_1,b_1),\ldots,(a_{l(\psi)},b_{l(\psi)})),\pi_A) \in R$, *let*

$$\Pi(C,\psi,\pi_A) := \left\{ \bigtimes_{i=1}^{l(\psi)} u_\rho(a_i,b_i)|\cdot|^{y_i} \rtimes \pi_A \mid (y_1,\ldots,y_{l(\psi)}) \in C \right\},$$

if $l(\psi) > 0$, *and* $\Pi(\mathbb{R}^0,\emptyset,\pi_A) := \{\pi_A\}$. *Then*

$$\Pi_{\overline{A}}(\pi_{sc},\rho,r) = \left(\bigcup_{(C,\psi,\pi_A)\in R_{\overline{A}}} \Pi(C,\psi,\pi_A) \right) \cap \Pi(G_{n+rd}).$$

Remark 8.2. *Note that to compute the set* $\Pi_{A,gp}(\pi_{sc},\rho,s)$, *we repeat the steps from Sections 3 to 7, which can be done by induction for any arbitrary corank* r.

8.2. **Unitarizability for the regular components.** For the rest of this section, we will follow Tadic's notation in [Tad23, Chapter 8]. For brevity, let $\underline{x} = (x_1,x_2,x_3,x_4)$ and $\Pi_{\underline{x}} = \Pi_{x_1,x_2,x_3,x_4}$. Let

$$\mathbb{R}^4_{++} = \{\underline{x} \in \mathbb{R}^4 : 0 \leq x_1 \leq x_2 \leq x_3 \leq x_4\}.$$

Then $\Pi_{\underline{x}}$ is reducible if and only if $\underline{x}$ lies on one of the following singular affine hyperplanes:

$$\begin{aligned} x_i &= \pm\alpha, \quad i = 1,2,3,4, \\ x_i \pm x_j &= \pm 1, \quad 1 \leq i < j \leq 4. \end{aligned} \tag{8.1}$$

We say that $\underline{x} \in \mathbb{R}^4$ is regular if it does not lie on any of these planes, and denote the set of such elements as $\mathbb{R}^4_{\mathrm{reg}}$. Define

$$\mathbb{R}^4_{\mathrm{reg},++} = \mathbb{R}^4_{\mathrm{reg}} \cap \mathbb{R}^4_{++}.$$

We also say that a point $\underline{x} \in \mathbb{R}^4$ is *strongly unitary* (resp. *strongly non-unitary*) if all irreducible subquotients of $\Pi_{\underline{x}}$ are unitarizable (resp. non-unitarizable).

Similarly, let $\mathbb{R}^3_{\mathrm{reg}}$ be the set of $(x_1,x_2,x_3) \in \mathbb{R}^3$ that lies in the complement of the hyperplanes

$$\begin{aligned} x_i &= \pm\alpha, \quad i = 1,2,3, \\ x_i \pm x_j &= \pm 1, \quad 1 \leq i < j \leq 3, \end{aligned} \tag{8.2}$$

and $\mathbb{R}^3_{\mathrm{reg},++} = \mathbb{R}^3_{++} \cap \mathbb{R}^3_{\mathrm{reg}}$.

We wish to classify all unitary representations of corank 4. To begin, we will consider the case where $\underline{x}$ is regular, in which case $\Pi_{\underline{x}}$ is either strongly unitary or strongly non-unitary. Denote the group of signed permutations by W. Since the property of being strongly unitary/non-unitary depends only on the W-orbit of $\underline{x}$, it suffices to consider the region $\mathbb{R}^4_{++}$.

We say that an open connected component $\Omega \subset \mathbb{R}^4_{\mathrm{reg},++}$ is unitary if for all points $\underline{x} \in \Omega$, $\Pi_{\underline{x}}$ is strongly unitary. For convenience, we do not distinguish between the connected component and the inequalities defining the connected component.

Proposition 8.3. *The following open connected components of $\mathbb{R}^4_{reg,++}$ are unitary. For $\alpha \geq 1$,*

$$x_3 + x_4 < 1. \tag{8.1a}$$

$$x_2 + x_3 < 1, \quad x_4 - x_3 > 1, \quad x_4 < \alpha \quad (\alpha > 1). \tag{8.1b}$$

$$x_2 + x_3 < 1, \quad x_4 - x_2 < 1, \quad x_4 - x_1 > 1, \quad x_4 < \alpha \quad (\alpha > 1). \tag{8.1c}$$

$$x_2 - x_1 > 1, \quad x_3 - x_2 > 1, \quad x_4 - x_3 > 1, \quad x_4 < \alpha, \quad (\alpha > 3). \tag{8.1d}$$

$$x_1 + x_2 < 1, \quad x_3 - x_2 > 1, \quad x_4 - x_3 > 1, \quad x_4 < \alpha, \quad (\alpha > 2). \tag{8.1e}$$

$$x_1 + x_2 < 1, \quad x_1 + x_3 > 1, \quad x_3 - x_1 < 1, \quad x_4 - x_3 > 1, \quad x_4 < \alpha, \quad (\alpha > \frac{3}{2}). \tag{8.1f}$$

$$x_2 + x_3 < 1, \quad x_1 + x_4 < 1, \quad x_2 + x_4 > 1, \quad x_4 < \alpha. \tag{8.1g}$$

(8.3)

For $\alpha = \frac{1}{2}$:

$$x_4 < \frac{1}{2}. \tag{8.1i}$$

For $\alpha = 0$:

$$x_3 + x_4 < 1. \tag{8.1j}$$

Consequently, any point $\underline{x} \in \mathbb{R}^4_{++}$ that lies in the closure of the above region is also strongly unitary.

Proof. For $\alpha \geq 1$, note that the non-empty connected components defined by (8.1a) to (8.1g) are all mutually disjoint. When $\alpha \neq 0$, $\Pi_{\vec{0}}$ is unitarizable and thus so is its connected component, which is given by (8.1a) when $\alpha \geq 1$ and (8.1i) when $\alpha = \frac{1}{2}$.

For $\alpha = 0$, we want to show that the connected component (8.1j) is unitary. In this case, note that the condition

$$x_3 + x_4 < 1$$

implies $0 \leq x_1 \leq x_2 \leq x_3 \leq x_4 \leq \frac{1}{2}$. We take the intersection of (8.1j) with the hyperplane $x_3 = x_4$, which is nonempty. Applying step 3-4 of Algorithm 8.1, we reduce to the connected component

$$0 \leq x_1 \leq x_2 < \frac{1}{2}$$

in corank 2, and using Theorem 8.5, one can conclude that (8.1j) is indeed unitary.

Assume now that $\alpha \geq 1$. Let us consider the complementary series

$$\pi_{x_4} = \Delta_\rho[-x_4, -x_4] \rtimes \pi_{sc},$$

which is irreducible and unitarizable for $0 \le x_4 < \alpha$. We will use it to prove that the connected components (8.1b) to (8.1g) are unitary one by one.

(1) Assume $\alpha > 1$ and fix $1 < x_4 < \alpha$. Then $(0,0,0,x_4) \in \mathbb{R}^4_{\text{reg},++}$ is strongly unitary by considering π_{x_4}. Since (8.1b) is the connected component of $(0,0,0,x_4)$, it must be unitary.

(2) Let $\alpha > 1$. Fix $1 < x_4 < \frac{3}{2}$. Then π_{x_4} is unitarizable. Now fix $x_4 - 1 < x_3 < \frac{1}{2}$, then the complementary series

$$\Delta_\rho[-x_3,-x_3] \times \Delta_\rho[-x_3,-x_3] \rtimes \pi_{x_4}$$

is irreducible and unitarizable. Finally, if we take $x_1 < x_4 - 1$, then the complementary series

$$\Delta_\rho[-x_1,-x_1] \times \Delta_\rho[-x_3,-x_3] \times \Delta_\rho[-x_3,-x_3] \rtimes \pi_{x_4}$$

is irreducible and unitarizable. The set of such points (x_1,x_3,x_3,x_4) lies in the connected component (8.1c), so the component must be unitary.

(3) Let $\alpha > 3$. Fix $3 < x_4 < \alpha$ and construct the complementary series

$$\Delta_\rho[-x_3,-x_3] \rtimes \pi_{x_4},$$

for $0 \le x_3 < x_4 - 1$. Fixing $2 \le x_3 < x_4 - 1$, we can construct the complementary series

$$\Delta_\rho[-x_2,-x_2] \times \Delta_\rho[-x_3,-x_3] \rtimes \pi_{x_4},$$

for $0 \le x_2 < x_3 - 1$. Now fix $1 < x_2 < 1 - x_3$ and construct the complementary series

$$\Delta_\rho[-x_1,-x_1] \times \Delta_\rho[-x_2,-x_2] \times \Delta_\rho[-x_3,-x_3] \rtimes \pi_{x_4},$$

for $0 \le x_1 < x_2 - 1$. The corresponding (x_1,x_2,x_3,x_4) is unitary and is contained in the connected component (8.1d). Therefore (8.1d) must be unitary.

(4) Let $\alpha > 2$. Fixing $2 < x_4 < \alpha$, we can construct the complementary series

$$\Delta_\rho[-x_3,-x_3] \rtimes \pi_{x_4},$$

which is irreducible and unitarizable for $0 \le x_3 < x_4 - 1$. Now fix $1 < x_3 < x_4 - 1$ and consider the complementary series

$$\Delta_\rho[-x_2,-x_2] \times \Delta_\rho[-x_3,-x_3] \rtimes \pi_{x_4}.$$

This is irreducible and unitarizable for $0 \le x_2 < x_3 - 1$. Subsequently, fix $0 < x_2 < \min(\frac{1}{2}, x_3 - 1)$, the complementary series

$$\Delta_\rho[-x_1,-x_1] \times \Delta_\rho[-x_2,-x_2] \times \Delta_\rho[-x_3,-x_3] \rtimes \pi_{x_4},$$

is irreducible and unitarizable for $0 \le x_1 < x_2$. The connected component containing such (x_1,x_2,x_3,x_4) is (8.1e), so (8.1e) must be unitary.

(5) Let $\alpha > \frac{3}{2}$. Fixing $\frac{3}{2} < x_4 < \alpha$ we can construct the complementary series

$$\Delta_\rho[-x_3,-x_3] \rtimes \pi_{x_4},$$

which is irreducible and unitarizable for $0 \le x_3 < x_4 - 1$. Now fix $\frac{1}{2} < x_3 < x_4 - 1$. The point $(0,0,x_3,x_4)$ is now strongly unitary. This point lies in the connected component (8.1f), so (8.1f) must be unitary.

(6) Let $\alpha \geq 1$. Then the connected component (8.1g) has nontrivial 3-dimensional intersection with the hyperplane $x_1 = 0$. The intersection can be described by the inequalities
$$x_2 + x_3 < 1, \quad x_4 < 1, \quad x_2 + x_4 > 1.$$
For $(x_1, x_2, x_3, x_4) \in \mathbb{R}^4_{\mathrm{reg},++}$, the representation
$$\Pi_{0,x_2,x_3,x_4} = \rho \times \rho|\cdot|^{x_2} \times \rho|\cdot|^{x_3} \times \rho|\cdot|^{x_4}$$
is unitarizable if and only if the representation
$$\Pi_{x_2,x_3,x_4} = |\cdot|^{x_2} \times \rho|\cdot|^{x_3} \times \rho|\cdot|^{x_4}$$
is unitarizable. Since above 3-dimensional region is contained in one of the connected components listed in Proposition 8.9, it follows that (8.1g) is unitary.

□

In §8.5 below, we show that the other open connected components in $\mathbb{R}^4_{++}$ are not unitary. To this end, we need to consider the cases where $\underline{x}$ is not regular, beginning with the singular affine hyperplanes.

8.3. **Unitarizability for the irregular components - slanted hyperplanes.** As described in step (3-3) and (3-4) of Algorithm 8.1, there are two types of reducibility hyperplanes, which we will classify in this and the following subsection. In this subsection, we aim to to perform the inductive process described in step (3-4) of Algorithm 8.1. To begin, let us consider the hyperplane
$$H_{\mathrm{sla}} = \{\underline{x} \in \mathbb{R}^4 : x_2 - x_1 = 1\}.$$
We call the W-orbit of H_{sla} the set of *slanted hyperplanes*, which is precisely the set of hyperplanes defined by equations of the form
$$x_i \pm x_j = \pm 1, \quad 1 \leq i < j \leq 4.$$
Let $x = x_1 + \frac{1}{2}, y = x_3, z = x_4$, then clearly $\iota(x,y,z) : \mathbb{R}^3 \to H_{\mathrm{sla}}$ defined by
$$\iota(x,y,z) = (x - \frac{1}{2}, x + \frac{1}{2}, y, z)$$
is an affine isomorphism.

Consider all reducibility hyperplanes which intersect H_{sla} non-trivially. These are precisely the set of hyperplanes described in (8.1) other than $x_2 - x_1 = \pm 1$. We define $\mathbb{R}^3_{\upsilon\mathrm{reg,sla}} \subset \mathbb{R}^3$ to be the complement of the hyperplanes
$$x \pm y = \pm\tfrac{1}{2}, \quad x \pm z = \pm\tfrac{1}{2}, \quad x = \pm\tfrac{1}{2},$$
$$x \pm y = \pm\tfrac{3}{2}, \quad x \pm z = \pm\tfrac{3}{2}, \quad y \pm z = \pm 1,$$
$$y = \pm\alpha, \quad z = \pm\alpha, \quad x = \pm(\alpha - \tfrac{1}{2}), \quad x = \pm(\alpha + \tfrac{1}{2}).$$
Note that $\mathbb{R}^3_{\upsilon\mathrm{reg,sla},++} := \{(x,y,z) \in \mathbb{R}^3_{\upsilon\mathrm{reg,sla}} \,|\, 0 \leq x \leq y \leq z\}$ is the complement of the following hyperplanes:
$$y \pm x = \tfrac{1}{2}, \quad z \pm x = \tfrac{1}{2}, \quad x = \tfrac{1}{2},$$
$$y \pm x = \tfrac{3}{2}, \quad z \pm x = \tfrac{3}{2}, \quad z \pm y = 1,$$
$$y = \alpha, \quad z = \alpha, \quad x = |\alpha - \tfrac{1}{2}|, \quad x = \alpha + \tfrac{1}{2}.$$

Then $\iota(\mathbb{R}^3_{vreg,sla}) = H_{sla} \cap \mathbb{R}^4_{reg}$.

For $(x, y, z) \in \mathbb{R}^3$, we can decompose $\Pi_{\iota(x,y,z)}$ in the Grothendieck group as

$$\Pi_{\iota(x,y,z)} = \pi^+_{(x,y,z)} + \pi^-_{(x,y,z)},$$

where

$$\pi^+_{(x,y,z)} = |\cdot|^x \Delta_\rho[\frac{1}{2}, -\frac{1}{2}] \times \Delta_\rho[-y, -y] \times \Delta_\rho[-z, -z] \rtimes \pi_{sc},$$
$$\pi^-_{(x,y,z)} = |\cdot|^x L(\Delta_\rho[-\frac{1}{2}, -\frac{1}{2}], \Delta_\rho[\frac{1}{2}, \frac{1}{2}]) \times \Delta_\rho[-y, -y] \times \Delta_\rho[-z, -z] \rtimes \pi_{sc}.$$

Now, let $\mathbb{R}^3_{reg,sla} \subset \mathbb{R}^3$ be the complement of the hyperplanes

$$x \pm y = \pm\tfrac{3}{2}, \quad x \pm z = \pm\tfrac{3}{2}, \quad y \pm z = \pm 1,$$
$$y = \pm\alpha, \quad z = \pm\alpha, \quad x = \pm(\alpha - \frac{1}{2}), \quad x = \pm(\alpha + \frac{1}{2}).$$

Then, $\pi^+_{(x,y,z)}$ and $\pi^-_{(x,y,z)}$ are both irreducible if and only if $(x, y, z) \in \mathbb{R}^3_{reg,sla}$. In other words, for $(x, y, z) \in \mathbb{R}^3_{reg,sla}$, $\iota(x, y, z)$ is strongly unitary (resp., strongly non-unitary) if and only if both $\pi^+_{(x,y,z)}$ and $\pi^-_{(x,y,z)}$ are unitarizable (resp., non-unitarizable).

Denote

$$W_{sla} = \{w \in W : w(H_{sla}) = H_{sla}\}.$$

Then $W_{sla} \cong \{\pm 1\}^3$, and W_{sla} can be generated by the signed permutations

$$(x_1, x_2, x_3, x_4) \mapsto (-x_2, -x_1, x_3, x_4),$$
$$(x_1, x_2, x_3, x_4) \mapsto (x_1, x_2, -x_3, x_4),$$
$$(x_1, x_2, x_3, x_4) \mapsto (x_1, x_2, x_3, -x_4).$$

We say that a point $(x, y, z) \in \mathbb{R}^3_{reg,sla}$ is unitary$^+$ (resp., unitary $^-$) if $\pi^+_{(x,y,z)}$ (resp., $\pi^-_{(x,y,z)}$) is unitarizable. We say that a point $(x, y, z) \in \mathbb{R}^3_{reg,sla}$ is unitary$^\pm$ if it is both unitary$^+$ and unitary$^-$. We say that a connected component is unitary$^+$ (resp., unitary$^-$, unitary$^\pm$) if every point in it is unitary$^+$ (resp., unitary$^-$, unitary$^\pm$).

Recall that $\mathbb{R}^3_+ = \{(x, y, z) \in \mathbb{R}^3 : x, y, z \geq 0\}$, $\mathbb{R}^3_{++} = \{(x, y, z) \in \mathbb{R}^3 : 0 \leq x \leq y \leq z\}$, and let $\mathbb{R}^3_{reg,sla,+} = \mathbb{R}^3_+ \cap \mathbb{R}^3_{reg,sla}, \mathbb{R}^3_{reg,sla,++} = \mathbb{R}^3_{++} \cap \mathbb{R}^3_{reg,sla}$. We start by looking at the preimage of the slanted hyperplane boundaries of the regions described in Proposition 8.3, under the isomorphism ι. In the proposition below, we denote the connected components (8.1a) to (8.1g) defined in Proposition 8.3 by $C_a, C_b, C_c, C_d, C_e, C_f, C_g, C_h$.

Proposition 8.4. *For $\alpha \geq 1$, any connected component of $\mathbb{R}^3_{vreg,sla,++}$ is either fully contained in one of the following regions in $\mathbb{R}^3_{++}$, or has empty intersection with each of them. Moreover, in the former case, that component is unitary$^\pm$:*

$$x + z < \frac{1}{2}, \tag{L'_1}$$

$$x + y > \frac{1}{2}, \quad z - x < \frac{1}{2}, \quad y + z < 1, \quad x < \frac{1}{2}. \tag{L'_2}$$

For $\alpha > 1$:

$$x + y < \frac{1}{2}, \quad z - x > \frac{3}{2}, \quad z < \alpha. \tag{L'_3}$$

For $\alpha > \frac{3}{2}$:

$$x+y<\frac{3}{2},\quad z-y>1,\quad x<\frac{1}{2},\quad z<\alpha, \tag{L_4'}$$

$$x+y>\frac{3}{2},\quad z-x>\frac{3}{2},\quad x<\frac{1}{2},\quad z<\alpha, \tag{L_5'}$$

$$x+y<\frac{3}{2},\quad z-x>\frac{3}{2},\quad x>\frac{1}{2},\quad z<\alpha. \tag{L_6'}$$

For $\alpha > 3$:

$$y-x>\frac{3}{2},\quad z-y>1,\quad z<\alpha. \tag{L_7'}$$

Furthermore, for each of the connected components $L_i \in \{L_1', \ldots, L_7'\}$, there exists some $w_i \in W$ and $X_i \in \{C_a, C_b, C_c, C_d, C_e, C_f, C_g, C_h\}$ such that $\iota(L_i')$ is contained in $\partial(w_i(X_i))$.

Proof. For $\alpha \geq 1$, consider the region L_1', defined by

$$x+z<\frac{1}{2},\quad 0\leq x\leq y\leq z.$$

Take $X_1 = C_a$, which is defined by the equation

$$x_3+x_4<1,\quad 0\leq x_1\leq x_2\leq x_3\leq x_4.$$

The boundary of the connected component X_1 is the region defined by

$$x_3+x_4=1,\quad 0\leq x_1\leq x_2\leq x_3\leq x_4.$$

Now let w_1 be the map defined by

$$w_1(x_1,x_2,x_3,x_4)=(-x_3,x_4,x_1,x_2):=(x_1',x_2',x_3',x_4').$$

Then the boundary $\partial(w_1(X_1))$ becomes

$$x_2'-x_1'=1,\quad 0\leq x_3'\leq x_4'\leq -x_1'\leq x_2'.$$

Introducing the transformation $(x_1',x_2',x_3',x_4')=(x-\frac{1}{2},x+\frac{1}{2},y,z)$ denoted as ι, the condition on the coordinates translates to

$$0\leq y\leq z\leq -x+\frac{1}{2}\leq x+\frac{1}{2}.$$

We see then clearly the region L_1' satisfies these constraints, and thus is contained in the boundary $\partial(w_1(X_1))$. It follows that any connected component of $\mathbb{R}^3_{\upsilon\mathrm{reg,sla}}$ that is fully contained in L_1' is unitary $^{\pm}$. This proves the statement for $i=1$. Now consider

$$\begin{aligned}
X_2 &= C_g,\quad w_2:(x_1,x_2,x_3,x_4)\mapsto(-x_1,x_3,x_4,x_2),\\
X_3 &= C_b,\quad w_3:(x_1,x_2,x_3,x_4)\mapsto(-x_1,x_3,x_1,x_4),\\
X_4 &= C_f,\quad w_4:(x_1,x_2,x_3,x_4)\mapsto(-x_1,x_3,x_2,x_4),\\
X_5 &= C_f,\quad w_5:(x_1,x_2,x_3,x_4)\mapsto(-x_1,x_3,-x_2,x_4),\\
X_6 &= C_f,\quad w_6:(x_1,x_2,x_3,x_4)\mapsto(x_1,x_3,x_2,x_4),\\
X_7 &= C_d,\quad w_7:(x_1,x_2,x_3,x_4)\mapsto(x_1,x_2,x_3,x_4).
\end{aligned}$$

Using these X_is and w_is, we can prove the statement for $i = 2, \dots 9$ in an analogous way. $\square$

Using this result, we can find all of the unitary connected components of $\mathbb{R}^3_{\text{reg,sla},++}$. To do this, we need to invoke our classification of critical points in the previous sections.

Let us first recall Tadic's result on the unitary dual of corank 2:

Proposition 8.5. [Tad23, Proposition 7.2] *The irreducible unitarizable subquotients of $\Pi_{\underline{x}}$, where $\underline{x} = (x_1, x_2) \in \mathbb{R}^2_{++}$, are the following:*

(1) $(\alpha > 1)$ *All irreducible subquotients when $x_1 + 1 \leq x_2 \leq \alpha$.*
(2) $(\alpha \neq \frac{1}{2})$ *All irreducible subquotients when $x_1 + x_2 \leq 1$.*
(3) $(\alpha = \frac{1}{2})$ *All irreducible subquotients when $x_2 \leq \frac{1}{2}$.*
(4) $(\alpha > 0)$ *The irreducible representations $T^{\alpha+1}_{I,1}(T^{\alpha}_{I,1}(\pi_{sc}))$ and $L([-\alpha, -\alpha], [-\alpha-1, -\alpha-1]; \pi_{sc})$*

Let us now give the list of the unitary components inside $\mathbb{R}^3_{\text{reg,sla},++}$, which we will prove to be the complete list using a method of exhaustion. Note that one can use various computer algorithms, such as Sage, to determine the possible bounded connected components in a region, given a set of hyperplanes separating the components.

Proposition 8.6. *The unitary$^{\pm}$ connected components of $\mathbb{R}^3_{reg,sla,++}$ are precisely given by*

$$y + z < 1, \quad (\alpha \geq 1, \alpha = 0) \tag{8.6a}$$

$$x + z < \frac{3}{2}, \quad z - y > 1 \quad z < \alpha, \quad (\alpha > 1) \tag{8.6b}$$

$$x + y < \frac{3}{2}, \quad x + z > \frac{3}{2}, \quad z - x < \frac{3}{2}, \quad z - y > 1, \quad (\alpha > 1) \tag{8.6c}$$

$$x + y < \frac{3}{2}, \quad z - x > \frac{3}{2}, \quad z - y > 1, \quad z < \alpha, \quad (\alpha > \frac{3}{2}) \tag{8.6d}$$

$$y - x > \frac{3}{2}, \quad z - y > 1, \quad z < \alpha, \quad (\alpha > \frac{5}{2}) \tag{8.6e}$$

$$z < \frac{1}{2} \quad (\alpha = \frac{1}{2}). \tag{8.6f}$$

Proof. For $\alpha > 3$, the following are all the possible bounded connected components of $\mathbb{R}^3_{\text{reg,sla},++}$:

$$y + z < 1, \tag{$C_{>3,1}$}$$

$$x + z < \frac{3}{2}, \quad y + z > 1, \quad z - y < 1, \tag{$C_{>3,2}$}$$

$$x + z < \frac{3}{2}, \quad z - y > 1, \tag{$C_{>3,3}$}$$

$$x + y < \frac{3}{2}, \quad x + z > \frac{3}{2}, \quad z - x < \frac{3}{2}, \quad z - y < 1, \tag{$C_{>3,4}$}$$

$$x + y < \frac{3}{2}, \quad x + z > \frac{3}{2}, \quad z - x < \frac{3}{2}, \quad z - y > 1, \tag{$C_{>3,5}$}$$

$$x + y < \frac{3}{2}, \quad z - x > \frac{3}{2}, \quad z - y < 1, \tag{$C_{>3,6}$}$$

$$x + y < \frac{3}{2}, \quad z - x > \frac{3}{2}, \quad z - y > 1, \quad z < \alpha, \tag{$C_{>3,7}$}$$

$$x < \alpha - \frac{1}{2}, \quad z < \alpha, \quad x + y > \frac{3}{2}, \quad z - y < 1, \quad z - x < \frac{3}{2}, \tag{$C_{>3,8}$}$$

$$x < \alpha - \frac{1}{2}, \quad y < \alpha, \quad z - x < \frac{3}{2}, \quad z > \alpha, \quad z - y < 1, \tag{$C_{>3,9}$}$$

$$x > \alpha - \frac{1}{2}, \quad z < \alpha, \tag{$C_{>3,10}$}$$

$$x > \alpha - \frac{1}{2}, \quad y < \alpha, \quad z > \alpha, \quad z - y < 1, \tag{$C_{>3,11}$}$$

$$\alpha - \frac{1}{2} < x < \alpha + \frac{1}{2}, \quad z - x < \frac{3}{2}, \quad z - y < 1, \quad y > \alpha, \tag{$C_{>3,12}$}$$

$$z - y > 1, \quad z < \alpha, \quad x + y > \frac{3}{2}, \quad z - x < \frac{3}{2}, \tag{$C_{>3,13}$}$$

$$y < \alpha, \quad z - y > 1, \quad z - x < \frac{3}{2}, \quad x > \alpha - \frac{1}{2}, \tag{$C_{>3,14}$}$$

$$z - x > \frac{3}{2}, \quad z < \alpha, \quad x + y > \frac{3}{2}, \quad z - y > 1, \quad y - x < \frac{3}{2}, \tag{$C_{>3,15}$}$$

$$z - x > \frac{3}{2}, \quad y < \alpha, \quad y - x < \frac{3}{2}, \quad z > \alpha, \quad z - y < 1, \tag{$C_{>3,16}$}$$

$$z - x > \frac{3}{2}, \quad x < \alpha - \frac{1}{2}, \quad y - x < \frac{3}{2}, \quad z - y < 1, \quad y > \alpha, \tag{$C_{>3,17}$}$$

$$z - x > \frac{3}{2}, \quad \alpha - \frac{1}{2} < x < \alpha + \frac{1}{2}, \quad y - x < \frac{3}{2}, \quad z - y < 1, \tag{$C_{>3,18}$}$$

$$z - x > \frac{3}{2}, \quad y - x < \frac{3}{2}, \quad z - y < 1, \quad x > \alpha + \frac{1}{2}, \tag{$C_{>3,19}$}$$

$$z - x > \frac{3}{2}, \quad z - y > 1, \quad z < \alpha, \quad x + y > \frac{3}{2}, \quad y - x < \frac{3}{2}, \tag{$C_{>3,20}$}$$

$$y - x > \frac{3}{2}, \quad z < \alpha, \quad z - y < 1, \tag{$C_{>3,21}$}$$

$$y - x > \frac{3}{2}, \quad y < \alpha, \quad z > \alpha, \quad z - y < 1, \tag{$C_{>3,22}$}$$

$$y - x > \frac{3}{2}, \quad z - y > 1, \quad z < \alpha. \tag{$C_{>3,23}$}$$

One can verify through direct computation that $C_{>3,i} \subset \cup_{j=1}^{7} L'_j$ for $i = 1, 3, 5, 7, 23$. We now show that the other connected components are not unitary. In the case $\alpha > 3$, there are 6 points that gets mapped to not strongly unitary points in $\mathbb{R}^4_{++}$ under ι:

$$(\alpha + \frac{1}{2}, \alpha + 2, \alpha + 3), \quad (\alpha - \frac{1}{2}, \alpha + 1, \alpha + 2), \quad (\alpha - \frac{1}{2}, \alpha, \alpha + 1),$$

$$(\alpha - \frac{3}{2}, \alpha, \alpha + 1), \quad (\alpha - \frac{3}{2}, \alpha, \alpha), \quad (\alpha - \frac{3}{2}, \alpha - 1, \alpha).$$

One can easily verify that $C_{>3,i}$ contains at least one of the above critical points on its boundary for $i = 8, 9, 11, 12, 13 \ldots, 23$. The only components remaining are $C_{>3,i}$ for $i = 2, 4, 6, 10$.

For $i = 2, 4, 6$, the connected component $C_{>3,i}$ has nonempty 3-dimensional intersection with the hyperplane $x = 0$. Applying step (3-2) of Algorithm 8.1, one can see that the representations

$$\pi^{+}_{(0,y,z)} = \Delta_\rho[\frac{1}{2}, -\frac{1}{2}] \times \Delta_\rho[-y,-y] \times \Delta_\rho[-z,-z] \rtimes \pi_{sc},$$
$$\pi^{-}_{(0,y,z)} = L(\Delta_\rho[-\frac{1}{2}, -\frac{1}{2}], \Delta_\rho[\frac{1}{2}, \frac{1}{2}]) \times \Delta_\rho[-y,-y] \times \Delta_\rho[-z,-z] \rtimes \pi_{sc}.$$

are unitarizable if and only if $\Pi_{(x,y)} = \Delta_\rho[-x,-x] \times \Delta_\rho[-y,-y] \rtimes \pi_{sc}$ contains only unitarizable subquotients. By Proposition 8.5, the representations $\pi_{(0,y,z)}$ is not unitary for $(y,z) = (\frac{2}{3}, \frac{2}{3})$ or $(\frac{4}{3}, \frac{5}{3})$. However, the points $(0, \frac{2}{3}, \frac{2}{3})$ lies in $C_{>3,2}$ and the point $(0, \frac{4}{3}, \frac{5}{3})$ lies in $C_{>3,6}$. Therefore these two connected components cannot be unitary$^{\pm}$.

For $C_{>3,10}$, consider the critical point $(\alpha - \frac{1}{2}, \alpha, \alpha)$, which lies on the boundary. By Proposition B.1, the point

$$\iota(\alpha - \frac{1}{2}, \alpha, \alpha) = (\alpha - 1, \alpha, \alpha, \alpha)$$

is not strongly unitary for $\alpha > 3$. Thus the connected component $C_{>3,11}$ is not unitary$^{\pm}$.

Finally, for $C_{>3,4}$, we split into two cases. When $\alpha \in \frac{1}{2} + \mathbb{Z}$, the point $(0, \frac{3}{2}, \frac{3}{2})$ is a critical point that lies on the boundary of $C_{>3,4}$. By Proposition 8.5, $(\frac{3}{2}, \frac{3}{2})$ is not in the unitary dual of corank 2, so the representations $(0, \frac{3}{2}, \frac{3}{2})$ is not strongly unitary, which means that $C_{>3,4}$ is not unitary$^{\pm}$.

Now consider the case $\alpha \in \mathbb{Z}_{>3}$. In this case, the connected component $C_{>3,4}$ contains no points of good parity. Instead, we consider the point $(\frac{1}{2}, 1, 1)$ which lies its boundary. The representations

$$\pi^{+}_{(\frac{1}{2},1,1)} = |\cdot|^{\frac{1}{2}} \Delta_\rho[\frac{1}{2}, -\frac{1}{2}] \times \rho|\cdot|^1 \times \rho|\cdot|^1 \rtimes \pi_{sc},$$
$$\pi^{-}_{(\frac{1}{2},1,1)} = |\cdot|^{\frac{1}{2}} L(\Delta_\rho[-\frac{1}{2}, -\frac{1}{2}], \Delta_\rho[-\frac{1}{2}, \frac{1}{2}]) \times \rho|\cdot|^1 \times \rho|\cdot|^1 \rtimes \pi_{sc},$$

are irreducible. Consider now the complementary series

$$\tau^{+}_{(x,1,1)} = |\cdot|^{x} \Delta_\rho[\frac{1}{2}, -\frac{1}{2}] \times \rho|\cdot|^1 \times \rho|\cdot|^1 \rtimes \pi_{sc},$$
$$\tau^{-}_{(x,1,1)} = |\cdot|^{x} (\Delta_\rho[-\frac{1}{2}, -\frac{1}{2}], \Delta_\rho[-\frac{1}{2}, \frac{1}{2}]) \times \rho|\cdot|^1 \times \rho|\cdot|^1 \rtimes \pi_{sc},$$

which is irreducible for $0 \leq x \leq \frac{1}{2}$. When $x = 0$, we can apply step (3-2) of Algorithm 8.1 again to conclude that $\tau_{(0,1,1)}$ is non-unitarizable, since $(1,1)$ does not lie in the unitary dual of corank 2. Therefore it follows that $\tau^{\pm}_{\frac{1}{2},1,1}$ is non-unitarizable, which means that $C_{>3,4}$ is not unitary $^{\pm}$. This proves the claim for $\alpha > 3$. When $\alpha = 3$, the number of connected components and their descriptions are completely identical to the case $\alpha > 3$. The critical points for $\alpha = 3$ also matches the description for the critical points for $\alpha > 3$. The proof in this case is exactly the same as above, so we omit it.

Now consider the case $\alpha = \frac{5}{2}$. In this case the list of possible bounded connected components in $\mathbb{R}^3_{\mathrm{reg,sla},++}$ is as follows:

$(C_{\frac{5}{2},1})$ $$y + z < 1,$$

$(C_{\frac{5}{2},2})$ $$x + z < \frac{3}{2}, \quad y + z > 1, \quad z - y < 1,$$

$$x+z<\frac{3}{2},\quad z-y>1, \tag{$C_{\frac{5}{2},3}$}$$

$$x+y<\frac{3}{2},\quad x+z>\frac{3}{2},\quad z-y<1,\quad z-x<\frac{3}{2}, \tag{$C_{\frac{5}{2},4}$}$$

$$x+y<\frac{3}{2},\quad x+z>\frac{3}{2},\quad z-y>1,\quad z-x<\frac{3}{2}, \tag{$C_{\frac{5}{2},5}$}$$

$$x+y<\frac{3}{2},\quad z-x>\frac{3}{2},\quad z-y<1, \tag{$C_{\frac{5}{2},6}$}$$

$$x+y<\frac{3}{2},\quad z-x>\frac{3}{2},\quad z-y>1,\quad z<\alpha, \tag{$C_{\frac{5}{2},7}$}$$

$$x<\alpha-\frac{1}{2},\quad z<\alpha,\quad x+y>\frac{3}{2},\quad z-y<1,\quad z-x<\frac{3}{2}, \tag{$C_{\frac{5}{2},8}$}$$

$$x<\alpha-\frac{1}{2},\quad y<\alpha,\quad z-x<\frac{3}{2},\quad z>\alpha,\quad z-y<1, \tag{$C_{\frac{5}{2},9}$}$$

$$x>\alpha-\frac{1}{2},\quad z<\alpha, \tag{$C_{\frac{5}{2},10}$}$$

$$x>\alpha-\frac{1}{2},\quad y<\alpha,\quad z>\alpha\quad z-y<1, \tag{$C_{\frac{5}{2},11}$}$$

$$\alpha-\frac{1}{2}<x<\alpha+\frac{1}{2},\quad z-x<\frac{3}{2},\quad z-y<1,\quad y>\alpha, \tag{$C_{\frac{5}{2},12}$}$$

$$z-y>1,\quad z<\alpha\quad x+y>\frac{3}{2},\quad z-x<\frac{3}{2}, \tag{$C_{\frac{5}{2},13}$}$$

$$z-y>1,\quad y<\alpha,\quad z-x<\frac{3}{2},\quad x>\alpha-\frac{1}{2} \tag{$C_{\frac{5}{2},14}$}$$

$$z-x>\frac{3}{2},\quad z<\alpha,\quad x+y>\frac{3}{2},\quad z-y<1,\quad y-x<\frac{3}{2}, \tag{$C_{\frac{5}{2},15}$}$$

$$z-x>\frac{3}{2},\quad y<\alpha,\quad y-x<\frac{3}{2},\quad z>\alpha,\quad z-y<1, \tag{$C_{\frac{5}{2},16}$}$$

$$z-x>\frac{3}{2},\quad x<\alpha-\frac{1}{2},\quad y-x<\frac{3}{2},\quad z-y<1,\quad y>\alpha, \tag{$C_{\frac{5}{2},17}$}$$

$$z-x>\frac{3}{2},\quad \alpha-\frac{1}{2}<x<\alpha+\frac{1}{2},\quad y-x<\frac{3}{2},\quad z-y<1, \tag{$C_{\frac{5}{2},18}$}$$

$$z-x>\frac{3}{2},\quad z-y>1,\quad z<\alpha,\quad x+y>\frac{3}{2}, \tag{$C_{\frac{5}{2},19}$}$$

$$y-x>\frac{3}{2},\quad z<\alpha, \tag{$C_{\frac{5}{2},20}$}$$

$$y-x>\frac{3}{2},\quad y<\alpha,\quad z>\alpha,\quad z-y<1. \tag{$C_{\frac{5}{2},21}$}$$

In this case, one can verify that $C_{\frac{5}{2},i}\subset\cup_{j=1}^{6}L_{j'}$ for $i=1,3,5,7$. By Proposition B.1, there are 6 points that get mapped to not strongly unitary points in $\mathbb{R}^4_{++}$ under ι, which are

$$(\alpha+\frac{1}{2},\alpha+2,\alpha+3),\quad(\alpha-\frac{1}{2},\alpha+1,\alpha+2),\quad(\alpha-\frac{1}{2},\alpha,\alpha+1),$$
$$(\alpha-\frac{3}{2},\alpha,\alpha+1),\quad(\alpha-\frac{3}{2},\alpha,\alpha),\quad(\alpha-\frac{3}{2},\alpha-1,\alpha).$$

The proof of non-unitarity for the other regions is identical to the case $\alpha > 3$, which we omit.

The next case to consider is $\alpha = 2$. In this case, the list of possible bounded connected components is as follows:

$$y + z < 1, \tag{$C_{2,1}$}$$

$$x + z < \frac{3}{2}, \quad z - y < 1, \quad y + z > 1, \tag{$C_{2,2}$}$$

$$x + z < \frac{3}{2}, \quad z - y > 1, \tag{$C_{2,3}$}$$

$$x + y < \frac{3}{2}, \quad x + z > \frac{3}{2}, \quad z - y < 1, \quad z - x < \frac{3}{2}, \tag{$C_{2,4}$}$$

$$x + y < \frac{3}{2}, \quad z - y > 1, \quad z < \alpha, \quad x + z > \frac{3}{2}, \quad z - x < \frac{3}{2}, \tag{$C_{2,5}$}$$

$$x + y < \frac{3}{2}, \quad z > \alpha, \quad z - y < 1, \tag{$C_{2,6}$}$$

$$z - x > \frac{3}{2}, \quad z - y > 1, \quad z < \alpha, \tag{$C_{2,7}$}$$

$$x < \alpha - \frac{1}{2}, \quad z < \alpha, \quad x + y > \frac{3}{2}, \quad z - y < 1, \quad z - x < \frac{3}{2}, \tag{$C_{2,8}$}$$

$$x < \alpha - \frac{1}{2}, \quad y < \alpha, \quad z - x < \frac{3}{2}, \quad z > \alpha, \quad z - y < 1, \tag{$C_{2,9}$}$$

$$x > \alpha - \frac{1}{2}, \quad z < \alpha, \tag{$C_{2,10}$}$$

$$x > \alpha - \frac{1}{2}, \quad y < \alpha, \quad z > \alpha, \quad z - y < 1, \tag{$C_{2,11}$}$$

$$\alpha - \frac{1}{2} < x < \alpha + \frac{1}{2}, \quad z - y < 1, \quad y > \alpha, \quad z - x < \frac{3}{2}, \tag{$C_{2,12}$}$$

$$z - y > 1, \quad z < \alpha, \quad x + y > \frac{3}{2}, \tag{$C_{2,13}$}$$

$$x > \alpha - \frac{1}{2}, \quad y < \alpha, \quad z - y > 1, \quad z - x < \frac{3}{2}, \tag{$C_{2,14}$}$$

$$z - x > \frac{3}{2}, \quad z < \alpha, \quad x + y > \frac{3}{2}, \quad y - x < \frac{3}{2}, \tag{$C_{2,15}$}$$

$$z - x > \frac{3}{2}, \quad y < \alpha, \quad x + y > \frac{3}{2}, \quad z > \alpha, \quad z - y < 1, \quad y - x < \frac{3}{2}, \tag{$C_{2,16}$}$$

$$x < \alpha - \frac{1}{2}, \quad z - x > \frac{3}{2}, \quad y - x < \frac{3}{2}, \quad z - y < 1, \quad y > \alpha, \tag{$C_{2,17}$}$$

$$\alpha - \frac{1}{2} < x < \alpha + \frac{1}{2}, \quad z - x > \frac{1}{2}, \quad z - y < 1, \quad y - x < \frac{3}{2}, \tag{$C_{2,18}$}$$

$$y - x > \frac{3}{2}, \quad z < \alpha, \tag{$C_{2,19}$}$$

$$y - x > \frac{3}{2}, \quad y < \alpha, \quad z > \alpha, \quad z - y < 1. \tag{$C_{2,20}$}$$

One can verify that $C_{2,i} \subset \cup_{j=1}^{6} L'_j$ for $i = 1, 3, 5, 7$. Note that ($C_{2,7}$) is exactly the same as (8.6d) when $\alpha = 2$. By considering the critical points described in Proposition B.1 and using step (3-2) of Algorithm 8.1, we can prove the non-unitarity for all regions $C_{2,i}$, except when $i = 4$.

For $i = 4$, we consider the point $(\frac{1}{2}, 1, 1)$, which lies on the boundary of $(C_{2,4})$. When $\alpha = 2$, the point

$$\iota(\frac{1}{2}, 1, 1) = (0, 1, 1, 1) \in \mathbb{R}^4_{++}$$

is not strongly unitary, since it contains the representation

$$L(\Delta_\rho[-1, -1], \Delta_\rho[-1, -1], \Delta_\rho[0, -1]; \pi_{sc}),$$

which is not of Arthur type and hence not unitarizable by Proposition 2.5. Suppose $(C_{2,4})$ is a unitary connected component in $\mathbb{R}^3_{\text{reg,sla},++}$, then $\iota(C_{2,4})$ must be lie in the W-orbit of a unitary connected component in R^4. However, this is impossible since the point $(0, 1, 1, 1)$ lies on the boundary of $(C_{2,4})$. Therefore $(C_{2,4})$ is non-unitary. This proves the Proposition for $\alpha = 2$.

Two cases remain. Let us now consider the case $\alpha = \frac{3}{2}$. The list of possible bounded connected components in $\mathbb{R}^3_{\text{reg,sla},++}$ in this case is as follows:

$(C_{\frac{3}{2},1})$ $$y + z < 1,$$

$(C_{\frac{3}{2},2})$ $$x + z < \frac{3}{2}, \quad z - y < 1, \quad y + z > 1,$$

$(C_{\frac{3}{2},3})$ $$x + z < \frac{3}{2}, \quad z - y > 1,$$

$(C_{\frac{3}{2},4})$ $$x + y < \frac{3}{2}, \quad z < \alpha, \quad x + z > \frac{3}{2}, \quad z - y < 1,$$

$(C_{\frac{3}{2},5})$ $$x + y < \frac{3}{2}, \quad z - x < \frac{3}{2}, \quad z > \alpha, \quad z - y < 1,$$

$(C_{\frac{3}{2},6})$ $$z - y > 1, \quad z < \alpha, \quad x + z > \frac{3}{2},$$

$(C_{\frac{3}{2},7})$ $$x + y < \frac{3}{2}, \quad z - y > 1, \quad z - x < \frac{3}{2}, \quad z > \alpha,$$

$(C_{\frac{3}{2},8})$ $$x + y < \frac{3}{2}, \quad z - x > \frac{3}{2}, \quad z - y < 1,$$

$(C_{\frac{3}{2},9})$ $$x < \alpha - \frac{1}{2}, \quad z < \alpha, \quad x + y > \frac{3}{2},$$

$(C_{\frac{3}{2},10})$ $$x < \alpha - \frac{1}{2}, \quad y < \alpha, \quad x + y > \frac{3}{2}, \quad z > \alpha, \quad z - y < 1, \quad z - x < \frac{3}{2},$$

$(C_{\frac{3}{2},11})$ $$x > \alpha - \frac{1}{2}, \quad z < \alpha,$$

$(C_{\frac{3}{2},12})$ $$x > \alpha - \frac{1}{2}, \quad y < \alpha, \quad z > \alpha, \quad z - y < 1,$$

$(C_{\frac{3}{2},13})$ $$\alpha - \frac{1}{2} < x < \alpha + \frac{1}{2}, \quad z - x < \frac{3}{2}, \quad z - y < 1, \quad y > \alpha,$$

$(C_{\frac{3}{2},14})$ $$x > \alpha - \frac{1}{2}, \quad y < \alpha, \quad z - y > 1, \quad z - x < \frac{3}{2},$$

$(C_{\frac{3}{2},15})$ $$z - x > \frac{3}{2}, \quad y < \alpha, \quad x + y > \frac{3}{2}, \quad z - y < 1,$$

$$(C_{\frac{3}{2},16}) \qquad z-x>\frac{3}{2},\quad x<\alpha-\frac{1}{2},\quad y-x<\frac{3}{2},\quad z-y<1,\quad y>\alpha,$$

$$(C_{\frac{3}{2},17}) \qquad z-x>\frac{3}{2},\quad \alpha-\frac{1}{2}<x<\alpha+\frac{1}{2},\quad y-x<\frac{3}{2},\quad z-y<1.$$

In this case, we have $C_{\frac{3}{2},i}\in\cup_{j=1}^{3}L'_j$ if and only if $i=1$. From Proposition B.1, there are 3 points that get mapped to not strongly unitary points in $\mathbb{R}^4_{++}$ under ι:

$$(\alpha+\frac{1}{2},\alpha+2,\alpha+3),\quad(\alpha-\frac{1}{2},\alpha+1,\alpha+2),\quad(\alpha-\frac{1}{2},\alpha,\alpha+1)$$

These points lie on the boundary of $C_{\frac{3}{2},i}$ for $i=10,12,13,\dots 17$. For $i=3,6$, $C_{\frac{3}{2},i}$ has nonempty intersection with the hyperplane $x=0$. Applying unitary reduction, one can conclude that ($C_{\frac{3}{2},3}$) and ($C_{\frac{3}{2},6}$) are unitary$^{\pm}$. Similarly one can also show that ($C_{\frac{3}{2},2}$)),($C_{\frac{3}{2},4}$), ($C_{\frac{3}{2},8}$) are not unitary$^{\pm}$. It remains to show that ($C_{\frac{3}{2},4}$)), ($C_{\frac{3}{2},7}$)), ($C_{\frac{3}{2},9}$)) are not unitary$^{\pm}$.

For ($C_{\frac{3}{2},4}$)), consider the point $(0,\frac{3}{2},\frac{3}{2})$ which lies on its boundary. We have that

$$\iota(0,\frac{3}{2},\frac{3}{2})=(-\frac{1}{2},\frac{1}{2},\frac{3}{2},\frac{3}{2}),$$

and

$$\Pi_{(-\frac{1}{2},\frac{1}{2},\frac{3}{2},\frac{3}{2})}\cong\Pi_{(\frac{1}{2},\frac{1}{2},\frac{3}{2},\frac{3}{2})}.$$

contains non-unitarizable subquotients by Proposition B.1 and 2.5. Therefore ($C_{\frac{3}{2},4}$) must be non-unitary.

Similarly, for ($C_{\frac{3}{2},7}$)), consider the point $(0,\frac{1}{2},\frac{3}{2})$ which lies on its boundary and gets mapped to $(-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{3}{2})$ under ι. We have that

$$\Pi_{(-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{3}{2})}\cong\Pi_{(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{3}{2})},$$

which contains non-unitarizable subquotients by Proposition B.1 and 2.5. Therefore ($C_{\frac{3}{2},7}$) must be non-unitary.

Now consider the point $(1,\frac{3}{2},\frac{3}{2})$. This point lies on the boundary of ($C_{\frac{3}{2},9}$) but

$$\iota(1,\frac{3}{2},\frac{3}{2})=(\frac{1}{2},\frac{3}{2},\frac{3}{2},\frac{3}{2})$$

contains non-unitarizable subquotients by Proposition B.1 and 2.5. Therefore ($C_{\frac{3}{2},9}$) is not unitary$^{\pm}$.

This proves the claim for $\alpha=\frac{3}{2}$. Now suppose $\alpha=1$. In this case, the list of possible bounded connected components of $\mathbb{R}^3_{\text{reg,sla},++}$ is as follows:

$$(C_{1,1}) \qquad y+z<1,$$

$$(C_{1,2}) \qquad x<\alpha-\frac{1}{2},\quad z<\alpha,\quad y+z>1,$$

$$(C_{1,3}) \qquad x+z<\frac{3}{2},\quad y<\alpha,\quad z>\alpha,\quad z-y<1,$$

$$(C_{1,4}) \qquad x+z<\frac{3}{2},\quad y>\alpha,$$

$$(C_{1,5}) \qquad x+z<\frac{3}{2},\quad x>\alpha-\frac{1}{2},$$

$$x+z<\frac{3}{2},\quad z-y>1, \tag{$C_{1,6}$}$$

$$x<\alpha-\frac{1}{2},\quad y<a,\quad x+z>\frac{3}{2},\quad z-y<1,\quad z-x<\frac{3}{2}, \tag{$C_{1,7}$}$$

$$x+y<\frac{3}{2},\quad x+z>\frac{3}{2},\quad z-x<\frac{3}{2}, \tag{$C_{1,8}$}$$

$$x+y<\frac{3}{2},\quad z<\alpha,\quad x+z>\frac{3}{2}, \tag{$C_{1,9}$}$$

$$x+y<\frac{3}{2},\quad x>\alpha-\frac{1}{2},\quad z>\alpha,\quad z-y<1, \tag{$C_{1,10}$}$$

$$x+y<\frac{3}{2},\quad z-y>1,\quad x>\alpha-\frac{1}{2},\quad z-x<\frac{3}{2}, \tag{$C_{1,11}$}$$

$$z-x>\frac{3}{2},\quad y<\alpha,\quad z-y<1, \tag{$C_{1,12}$}$$

$$x+y<\frac{3}{2},\quad z-x>\frac{3}{2},\quad z-y<1,\quad y>\alpha, \tag{$C_{1,13}$}$$

$$x+y>\frac{3}{2},\quad z<\alpha, \tag{$C_{1,14}$}$$

$$y<\alpha,\quad x+y>\frac{3}{2},\quad z>\alpha,\quad z-y<1, \tag{$C_{1,15}$}$$

$$\alpha-\frac{1}{2}<x<\alpha+\frac{1}{2},\quad z-y<1,\quad y>\alpha,\quad z-x<\frac{3}{2}, \tag{$C_{1,16}$}$$

$$y<\alpha,\quad z-y>1,\quad x+y>\frac{3}{2},\quad z-x<\frac{3}{2}, \tag{$C_{1,17}$}$$

$$x<\alpha-\frac{1}{2},\quad z-x>\frac{3}{2},\quad x+y>\frac{3}{2},\quad z-y<1,\quad y-x<\frac{3}{2}, \tag{$C_{1,18}$}$$

$$\alpha-\frac{1}{2}<x<\alpha+\frac{1}{2},\quad z-x>\frac{3}{2},\quad z-y<1,\quad y-x<\frac{3}{2}, \tag{$C_{1,19}$}$$

$$z-x>\frac{3}{2},\quad x>\alpha+\frac{1}{2},\quad z-y<1,\quad y-x<\frac{3}{2}. \tag{$C_{1,20}$}$$

By Proposition B.1, the following critical points are not strongly unitary:

$$(\alpha+\frac{1}{2},\alpha+2,\alpha+3),(\alpha-\frac{1}{2},\alpha+1,\alpha+2),(\alpha-\frac{1}{2},\alpha,\alpha+1),(\alpha-\frac{1}{2},\alpha,\alpha).$$

One can easily verify that all the other regions contain at least one of the critical points above in their boundaries except for the region ($C_{1,6}$).

This region has nonempty 3-dimensional intersection with the hyperplane $x=0$ inside $\mathbb{R}^3_{\mathrm{reg,sla},++}$. Applying step (3-2) of Algorithm 8.1, we can conclude that the region is not unitary $^{\pm}$, since for example, the point $(0,\frac{4}{3})$ does not lie in the unitary dual of corank 2, but $(0,0,\frac{4}{3})\in(C_{1,6})$.

Now let $\alpha=\frac{1}{2}$. By Proposition B.1, the following points are not strongly unitary:

$$(\alpha+\frac{1}{2},\alpha+1,\alpha+1),(\alpha+\frac{1}{2},\alpha+1,\alpha+2),(\alpha+\frac{1}{2},\alpha+1,\alpha+2),(\alpha+\frac{1}{2},\alpha+2,\alpha+3).$$

Using these points, by the same method as before, we can eliminate all connected components except for the following:

$$z < \frac{1}{2}, \tag{$C_{1/2,1}$}$$

$$y + z < 1, \quad z > \frac{1}{2}, \tag{$C_{1/2,2}$}$$

$$x + z < \frac{3}{2}, \quad y < \frac{1}{2}, \quad z - y < 1, \quad y + z > 1, \tag{$C_{1/2,3}$}$$

$$x + z < \frac{3}{2}, \quad y > \frac{1}{2}, \tag{$C_{1/2,4}$}$$

$$x + z < \frac{3}{2}, \quad z - y > 1, \tag{$C_{1/2,5}$}$$

$$y < \frac{1}{2}, \quad x + z > \frac{3}{2}, \quad z - y < 1, \tag{$C_{1/2,6}$}$$

$$x + y < \frac{3}{2}, \quad z - y < 1, \quad y > \frac{1}{2}, \quad z - x > \frac{3}{2} \tag{$C_{1/2,7}$}$$

$$y < \frac{1}{2}, \quad z - y > 1, \quad x + z > \frac{3}{2}, \quad z - x < \frac{3}{2}, \tag{$C_{1/2,8}$}$$

$$x + y < \frac{3}{2}, \quad z - y > 1, \quad z - x < \frac{3}{2}, \quad y > \frac{1}{2}, \tag{$C_{1/2,9}$}$$

$$x + y < \frac{3}{2}, \quad z - x > \frac{3}{2}, \quad z - y < 1. \tag{$C_{1/2,10}$}$$

Using step 3-2 of Algorithm 8.1 and Proposition 8.5, we can conclude that $(C_{1/2,i})$ is not unitary $^{\pm}$ for $i = 2, 3, 4, 5, 7, 10$.

For $i = 6, 8, 9$, consider the point $(0, \frac{1}{2}, \frac{3}{2})$ which lies on its boundary of $(C_{1/2,i})$. Under ι, this gets mapped to

$$\Pi_{(-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{3}{2})} \cong \Pi_{(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{3}{2})}.$$

By Proposition B.1, this point is not strongly unitary, and thus $(C_{1/2,i})$ are not unitary$^{\pm}$ for $i = 6, 8, 9$. One can easily prove that $(C_{1/2,1})$ is unitary by mapping it to the boundary of (8.1i).

Finally, let $\alpha = 0$. In this case, by Proposition B.1, the points

$$(\alpha + \frac{1}{2}, \alpha + 1, \alpha + 1), (\alpha + \frac{1}{2}, \alpha + 1, \alpha + 2), (\alpha + \frac{1}{2}, \alpha + 2, \alpha + 2), (\alpha + \frac{1}{2}, \alpha + 2, \alpha + 3)$$

are not strongly unitary. Using these points, we can eliminate all connected components other than

$$y + z < 1, \tag{$C_{0,1}$}$$

$$x + z < \frac{3}{2}, \quad z - y > 1 \tag{$C_{0,2}$}$$

By applying step 3-2 of Algorithm 8.1 and using Proposition 8.5, one can show that $(C_{0,2})$ is not unitary$^{\pm}$ and $(C_{0,1})$ is unitary$^{\pm}$. This concludes the proof.

$\square$

Corollary 8.7. *For $\alpha \geq 1$, the unitary$^{\pm}$ connected components of $\mathbb{R}^3_{vreg,sla,++}$ are exactly as follows:*

$$x+z<\frac{1}{2}, \tag{L_1}$$

$$y+z<1, \quad x+z>\frac{1}{2}, \quad z-x<\frac{1}{2}, \tag{L_2}$$

$$y+z<1, \quad z-x>\frac{1}{2} \tag{L_3}$$

$$x+z<\frac{3}{2}, \quad z-y>1, \quad (\alpha>1) \tag{L_4}$$

$$x<\frac{1}{2}, \quad z-y>1, \quad x+z>\frac{3}{2}, \quad z-x<\frac{3}{2}, \quad (\alpha>1) \tag{L_5}$$

$$x>\frac{1}{2}, \quad x+y<\frac{3}{2}, \quad z-y>1, \quad z<\alpha, \quad (\alpha>\frac{3}{2}) \tag{L_6}$$

$$x<\frac{1}{2}, \quad z-x>\frac{3}{2}, \quad y-x<\frac{1}{2}, \quad z<\alpha, \quad (\alpha>\frac{3}{2}) \tag{L_7}$$

$$x+y<\frac{3}{2}, \quad y-x>\frac{1}{2}, \quad z-y>1, \quad z<\alpha, \quad (\alpha>\frac{3}{2}) \tag{L_8}$$

$$x+y<\frac{3}{2}, \quad z-x>\frac{3}{2}, \quad z<\alpha, \quad x>\frac{1}{2}, \quad (\alpha>2) \tag{L_9}$$

$$x<\frac{1}{2}, \quad y-x>\frac{3}{2}, \quad z-y>1, \quad z<\alpha, \quad (\alpha>\frac{5}{2}) \tag{L_{10}}$$

$$x>\frac{1}{2}, \quad y-x>\frac{3}{2}, \quad z-y>1, \quad z<\alpha \quad (\alpha>3). \tag{L_{11}}$$

Proof. The regions above are obtained by slicing up the regions (8.6a) to (8.6e) using the reducibility hyperplanes inside $R^3_{\text{vreg,sla},++}$. The result follows directly from Proposition 8.6. ∎

8.4. **Unitarizability for the irregular components - level hyperplanes.** In this subsection we will perform the inductive process described in step (3-3) of Algorithm 8.1. In particular, we will consider reducibility hyperplanes which are contained in the W-orbit of H_{lev}, where

$$H_{\text{lev}} = \{\underline{x} \in \mathbb{R}^4 : x_4 = \alpha\}.$$

We call these the set of *level hyperplanes*. Similar to the slanted hyperplanes case, for any $\underline{x} \in H_{\text{lev}}$, we can decompose $\Pi_{(x,y,z,\alpha)}$ in the Grothendieck group as $\tau^+_{(x,y,z)} + \tau^-_{(x,y,z)}$, where

$$\begin{aligned}\tau^+_{(x,y,z)} &= \Delta_\rho[-x,-x] \times \Delta_\rho[-y,-y] \times \Delta_\rho[-z,-z] \rtimes T^{\alpha}_{I,1}(\pi_{sc}),\\ \tau^-_{(x,y,z)} &= \Delta_\rho[-x,-x] \times \Delta_\rho[-y,-y] \times \Delta_\rho[-z,-z] \rtimes L(\Delta_\rho[-\alpha,-\alpha];\pi_{sc}),\end{aligned}$$

when $\alpha \neq 0$, and

$$\tau^{\pm}_{(x,y,z)} = \Delta_\rho[-x,-x] \times \Delta_\rho[-y,-y] \times \Delta_\rho[-z,-z] \rtimes T^{\pm}_{V,2}(\pi_{sc})$$

when $\alpha = 0$.

Similar to before, define $\mathbb{R}^3_{\text{reg,lev}}$ to be the complement of the singular affine hyperplanes other than H_{lev}. In other words, if we identify (x,y,z) with (x_1,x_2,x_3), it is the complement of the hyperplanes

$$x_i \pm x_j = \epsilon,\ x_i = \pm(\alpha+\epsilon),\quad \epsilon = \pm 1,\quad i = 1,2,3,\quad i<j.$$

We say that a point $(x,y,z) \in \mathbb{R}^3_{\mathrm{reg,lev}}$ is unitary$^+$ (resp., unitary $^-$) if $\pi^+_{(x,y,z)}$ (resp., $\pi^-_{(x,y,z)}$ is unitarizable). We say that a point $(x,y,z) \in \mathbb{R}^3_{\mathrm{reg,lev}}$ is unitary$^\pm$ if it is both unitary$^+$ and unitary$^-$. We say that a connected component is unitary$^+$ (resp., unitary$^-$, unitary$^\pm$) if every point in it is unitary$^+$ (resp., unitary$^-$, unitary$^\pm$). Then, $\tau^+_{(x,y,z)}$ and $\tau^-_{(x,y,z)}$ are irreducible precisely when $(x,y,z) \in \mathbb{R}^3_{\mathrm{reg,lev}}$. Therefore, for all $(x,y,z) \in \mathbb{R}^3_{\mathrm{reg},++}$, $\iota(x,y,z)$ is strongly unitary (resp., strongly non-unitary) if and only if both $\tau^+_{(x,y,z)}$ and $\tau^-_{(x,y,z)}$) are unitarizable (resp., non-unitarizable).

Let $\mathbb{R}^3_{\mathrm{reg,lev},+} = \mathbb{R}^3_+ \cap \mathbb{R}^3_{\mathrm{reg,lev}}, \mathbb{R}^3_{\mathrm{reg,lev},++} = \mathbb{R}^3_{++} \cap \mathbb{R}^3_{\mathrm{reg,lev}}$. The following proposition describes the unitary$^\pm$ connected components of $\mathbb{R}^3_{\mathrm{reg,lev},++}$.

Proposition 8.8. *The unitary$^\pm$ connected components of $\mathbb{R}^3_{reg,lev,++}$ are as follows:*

$$(\alpha > 3)\ \ y-x>1,\quad z-y>1,\quad z<\alpha-1, \tag{8.4}$$

$$(\alpha > 2)\ \ x+y<1,\quad z-y>1,\quad z<\alpha-1, \tag{8.5}$$

$$(\alpha \geq 1)\ \ y+z<1,\quad z<\alpha-1, \tag{8.6}$$

$$(\alpha > \frac{3}{2})\ \ x+y<1,\quad x+z>1,\quad z-x<1,\quad z<\alpha-1, \tag{8.7}$$

$$(\alpha = \frac{1}{2})\ \ z<\frac{1}{2}, \tag{8.8}$$

$$(\alpha = 2)\ \ x+y<1,\quad z-x<1,\quad z>\alpha-1, \tag{8.9}$$

$$(\alpha = \frac{3}{2})\ \ y+z<1,\quad z>\alpha-1. \tag{8.10}$$

$$(\alpha = 0)\ \ y+z<1 \tag{8.11}$$

Proof. The connected components (8.6), (8.8), (8.11) are precisely the connected components that contain the origin, in the cases $\alpha \geq 1, \alpha = \frac{1}{2}, \alpha = 0$ respectively, so they are unitary$^\pm$.

Now let us assume $\alpha > \frac{3}{2}$. Define the representations

$$\sigma^+ = T^{\alpha}_{I,1}(\pi_{sc}),$$
$$\sigma^- = L(\Delta_\rho[-\alpha,-\alpha];\pi_{sc}).$$

Consider the complementary series

$$\pi_z = \Delta_\rho[-z,-z] \rtimes \sigma^\pm,$$

which is irreducible and unitary for $0 \leq z < \alpha - 1$. First let us look at the complementary series

$$\Delta_\rho[-x,-x] \times \Delta_\rho[x,x] \rtimes \pi_z,$$

for $\frac{1}{2} < z < \min(\frac{3}{2}, \alpha-1)$ and $|1-z| < x < \frac{1}{2}$. This gives us an irreducible unitarizable representation $\Pi_{(x,-x,z)} \cong \Pi_{(x,x,z)}$. Since (8.7) is the connected component of

$$\{(x,x,z) : |1-z| < x < \frac{1}{2},\ \ \frac{1}{2} < z < \min(\frac{3}{2}, \alpha-1)\}.$$

It must be unitary$^\pm$.

Now assume $\alpha > 2$. Fix $1 < z < \alpha - 1$. Clearly the representation $\rho \times \rho \rtimes \pi_z$ is irreducible and unitary. It follows that $(0, 0, z)$ is unitary$^{\pm}$. Since (8.5) is the connected component of $(0, 0, z)$ for $1 < z < \alpha - 1$, it must be unitary $^{\pm}$.

Now assume $\alpha > 3$. First fix $2 < z < \alpha - 1$. Construct the complementary series

$$\Delta_\rho[-y, -y] \rtimes \pi_z,$$

for $0 \leq y < z - 1$, which is irreducible and unitarizable. Now fix $1 < y < z - 1$ and construct the complementary series

$$\Delta_\rho[-x, -x] \rtimes \Delta_\rho[-y, -y] \rtimes \pi_z,$$

which is irreducible and unitarizable for $0 \leq x < y - 1$. Since (8.4) is the connected component containing such (x, y, z), we may conclude that it is unitary$^{\pm}$.

Let $\alpha = 2$. Fix $1 < z < \frac{3}{2}$. Then the complementary series π_z is irreducible and unitary. Now pick $z - 1 < x < \frac{1}{2}$. Then the complementary series

$$\rho|\cdot|^x \times \rho|\cdot|^x \rtimes \pi_z$$

is irreducible and unitarizable. Since such (x, x, z) is contained in (8.9), the connected component must be unitary$^{\pm}$.

Let $\alpha = \frac{3}{2}$. By applying step (3-2) of Algorithm 8.1 and using Proposition 8.5, one sees that (8.10) is unitary$^{\pm}$. This proves that all connected components listed in the proposition are unitary$^{\pm}$.

The proof that these are the only unitary$^{\pm}$ components is similar to Proposition 8.6, which we omit. $\square$

8.5. **Final list of open unitary connected components.** In the previous two subsections, we've completed Step 3 of Algorithm 8.1, which means we are now ready to prove the list of unitary connected components in Proposition 8.3 is in fact the full list of connected unitary components. First, let us recall Tadic's result on the unitary dual of corank 3:

Proposition 8.9. [Tad23, Proposition 8.3] *The following connected components of $\mathbb{R}^3_{reg,++}$ are unitary.*

For $\alpha \geq 1$:

$$x_2 + x_3 < 1, \tag{8.12a}$$

$$x_1 + x_2 < 1, \quad , x_3 - x_2 > 1, \quad , x_3 < \alpha, \quad (\alpha > 1) \tag{8.12b}$$

$$x_1 + x_2 < 1, \quad , x_1 + x_3 > 1, \ \ x_3 - x_1 < 1, \quad , x_3 < \alpha \tag{8.12c}$$

$$x_2 - x_1 > 1, \quad , x_3 - x_2 > 1, \ \ x_3 < \alpha \quad (\alpha > 2). \tag{8.12d}$$

(The constraint $x_3 < \alpha$ in (8.12c) is redundant unless $\alpha = 1$.)

For $\alpha = \frac{1}{2}$:

$$x_3 < \tfrac{1}{2}. \tag{8.13}$$

Consequently, any $\mathbf{x} \in \mathbb{R}^3_{++}$ in the closure of the above regions (i.e., changing strict inequalities to non-strict ones) is strongly unitary.

By [Tad23, Proposition 8.12], this list is exhaustive. Now we are ready to prove the same for our list in corank 4.

Proposition 8.10. *The unitary open connected components of $\mathbb{R}^4_{++}$ are exactly those listed in Proposition 8.3. In particular, there are no unitary connected components for $\alpha = 0$.*

Proof. Let C be a unitary connected component of $\mathbb{R}^4_{\text{reg},++}$. Let $\mathscr{C}$ be the unitary connected component of $\mathbb{R}^4_{++}$ containing $\mathscr{C}$. C and $\mathscr{C}$ must be bounded, and their boundary must be contained in the union of all reducibility hyperplanes. In other words, any three dimensional volume in the boundary of $\mathscr{C}$ must be contained in a W-translate of H_{lev} or H_{sla}, since all slanted/level hyperplanes are in the same W-orbit of $H_{\text{sla}}/H_{\text{lev}}$.

Consider the case $\alpha = 0$. If the boundary of $\mathscr{C}$ is contained entirely in the union of level hyperplanes, then $\mathscr{C}$ cannot be bounded. Therefore there exists some unitary$^{\pm}$ 3-dimensional volume in the boundary of $\mathscr{C}$ contained in a slanted hyperplane. By Proposition 8.6, the boundary must lie inside W-orbit of the component

$$x_2 - x_1 = 1, \quad x_3 + x_4 < 1.$$

We take the unique $w \in W$ such that the image of the above component under w has nonempty intersection with $\mathbb{R}^4_{++}$, which is

$$w : (x_1, x_2, x_3, x_4) \to (x_3, x_4, -x_1, x_2) = (y_1, y_2, y_3, y_4),$$

under which the transformed boundary becomes the region

$$y_1 + y_2 < 1, \quad y_3 + y_4 = 1.$$

The two connected components in $\mathbb{R}^4$ sharing this boundary are (8.1j) and some unbounded region. This proves that when $\alpha = 0$, the only unitary connected component in $\mathbb{R}^4_{++}$ is (8.1j).

For all $\alpha > 0$, using Proposition 8.6 and 8.8, we can conclude that

$$\mathscr{C} \subset \{x \in \mathbb{R}^4_{\text{reg}} : |x_i| < \alpha, i = 1, 2, 3, 4\}$$

and

$$C \subset \{x \in \mathbb{R}^4_{\text{reg},++} : x_4 < \alpha\},$$

by passing them to the boundary. For $\alpha = \frac{1}{2}$, we can conclude that C is exactly the component given in Proposition 8.3.

For the rest of the proof we consider the case $\alpha \geq 1$. Now suppose some 3-dimensional volume on the boundary of $\mathscr{C}$ lies in a level hyperplane. Up to W-translation, we may assume this to be H_{lev}. It suffices to consider the unitary$^{\pm}$ components of $R^3_{\text{reg,lev},++}$.

By Proposition 8.8, there are 6 possible unitary$^{\pm}$ components in this case. Let us first consider the case $\alpha > 2$. The first possibility is the region

$$y - x > 1, \quad z - y > 1, \quad z < \alpha - 1,$$

for $\alpha > 3$. One can easily show that this is contained in the closure of the region (8.1d), by considering the affine isomorphism $(x_1, x_2, x_3) \mapsto (x_1, x_2, x_3, \alpha)$. Moreover, (8.1d) is the only connected component that contains (8.4) and is contained in the region $\{\underline{x} \in \mathbb{R}^4_{++} : x_4 < \alpha\}$. Thus we can conclude that C is (8.1d). Similarly, the region (8.5) described by

$$x + y < 1, \quad z - y > 1, \quad z < \alpha - 1,$$

for $\alpha > 2$, is contained in the closure of (8.1e). By the same reasoning, we may conclude that C is (8.1e).

Now let's look at the region (8.6), described by

$$y + z < 1, \quad z < \alpha - 1,$$

for $\alpha \geq 1$. For $\alpha > 1$, this is contained in the closure of (8.1b).

Lastly, we consider the region (8.7), described by

$$x + y < 1, \quad x + z > 1, \quad z - x < 1, \quad z < \alpha - 1,$$

for $\alpha > \frac{3}{2}$. This is contained in the closure of the connected component (8.1f). By the same reasoning as above, we can conclude that C can only be the connected component (8.1f).

When $\alpha = 2$, (8.9) is contained entirely in the boundary of (8.1g). When $\alpha = \frac{3}{2}$, (8.10) is contained entirely in the boundary of (8.1a). In both cases, we can conclude that they are the only possibilities for C.

It remains to consider the case where the boundary of C is contained entirely in slanted hyperplanes. Take H to be a slanted hyperplane that contains a 3-dimensional volume in the boundary of C. Fix $w_C \in W$ such that $w_C(H) = H_{\mathrm{sla}}$. It follows that $w_C(\partial(C)) \in H_{\mathrm{sla}}$ and

$$w_C\partial(C) = \partial(w_C(C)) \supset \iota(w_iL_i),$$

for some $1 \leq i \leq 11$ and $w_i \in W$. This implies that

$$\partial w_C(C) \cap \mathbb{R}^4_{++} \supset \iota(w_iL_i) \cap \mathbb{R}^4_{++}.$$

It suffices now to show the following: For all $1 \leq i \leq 11$, for all $w_i \in W$ such that $\iota(w_iL_i) \cap \mathbb{R}^4_{++}$ is nonempty, out of the two connected components in $\mathbb{R}^4_{++}$ which share the common boundary that contains $\iota(w_iL_i) \cap \mathbb{R}^4_{++}$, one is unitary if and only if it is listed in Proposition 8.3. In fact, by definition of $\mathbb{R}^4_{++}$, one can show that such w_i is unique.

We begin with the case $\alpha \geq 1$. For the component (L_1), the only possible $w_1 \in W$ such that

$$\iota(w_1L_1) \cap \mathbb{R}^4_{++} \neq \varnothing,$$

is the map

$$w_1 : (x_1, x_2, x_3, x_4) \mapsto (x_3, x_4, -x_1, x_2) = (y_1, y_2, y_3, y_4).$$

Under this map, the resulting boundary can be described by

$$y_3 + y_4 = 1, \quad 0 \leq \frac{1}{2} - y_3 \leq y_1 \leq y_2.$$

Clearly this is the subset of the boundary of the connected component (8.1a). Denote the only other connected component in $\mathbb{R}^4_{++}$ sharing this boundary by $C_{a,opp}$. Then $C_{a,opp}$ can be described by

$$x_2 + x_4 < 1, \quad x_3 + x_4 > 1.$$

This connected component has 4-dimensional intersection with the hyperplanes $x_1 = 0$ and $x_2 = 0$. Applying step (3-2) of Algorithm 8.1 twice, one can conclude that $C_{a,opp}$ is non-unitary, since, for example $(0, 0, \frac{2}{3}, \frac{2}{3})$ is contained in $C_{a,opp}$, but $(\frac{2}{3}, \frac{2}{3})$ is not contained in the unitary dual of corank 2.

The only possible w_2 is

$$w_2 : (x_1, x_2, x_3, x_4) \mapsto (x_1, x_3, x_4, x_2) = (y_1, y_2, y_3, y_4).$$

In this case, the transformed region in $\mathbb{R}^4_{++}$ can be described by

$$y_2 + y_3 < 1, \quad y_4 - y_1 = 1, \quad y_3 - y_1 < 1, \quad 0 \leq y_1 + \frac{1}{2} \leq y_2 \leq y_3.$$

This region has empty intersection with $\mathbb{R}^4_{++}$, so we may ignore it.

For $i = 3$, we must have:

$$w_3 : (x_1, x_2, x_3, x_4) \mapsto (x_3, -x_1, x_2, x_4) = (y_1, y_2, y_3, y_4).$$

Then the transformed region in $\mathbb{R}^4_{++}$ is

$$y_2 + y_3 = 1, \quad y_1 + y_4 < 1, \quad y_2 + y_4 > 1, \quad 0 \le y_3 - \frac{1}{2} \le y_1 \le y_4.$$

There are two possible connected components with this boundary. Call them $C_{3,1}$ and $C_{3,2}$, defined by

$$C_{3,1} : y_2 + y_3 < 1, \quad y_1 + y_4 < 1, \quad y_2 + y_4 > 1,$$
$$C_{3,2} : y_2 + y_3 > 1, \quad y_1 + y_4 < 1.$$

$C_{3,1}$ is the same as (8.1g), which is unitary. $C_{3,2}$ has nontrivial 3-dimensional intersection with the hyperplane $y_1 = 0$. The intersection can be described by

$$y_2 + y_3 > 1, \quad y_4 < 1.$$

One can easily verify that this region is not contained in any of the connected components listed in Proposition 8.9, so $C_{3,2}$ is non-unitary by step (3-2) of Algorithm 8.1.

Now let $\alpha > 1$. For $i = 4$, the only possible w_4 is

$$w_4 : (x_1, x_2, x_3, x_4) \mapsto (x_3, -x_1, x_2, x_4) = (y_1, y_2, y_3, y_4).$$

The transformed region in $\mathbb{R}^4_{++}$ is now

$$y_2 + y_3 = 1, \quad y_4 - y_2 < 1, \quad y_4 - y_1 > 1 \quad 0 \le y_3 - \frac{1}{2} \le y_1 \le y_4.$$

This is contained in the boundary of (8.1b). The other possible component containing this region in its boundary is

$$x_1 + x_3 < 1, \quad x_4 - x_1 > 1, \quad x_4 - x_2 < 1, \quad x_2 + x_3 > 1, \quad x_4 < \alpha.$$

Using step (3-2) of Algorithm 8.1 and Proposition 8.9, this connected component is non-unitary.

For $i = 5$, the only possibility is

$$w_5 : (x_1, x_2, x_3, x_4) \mapsto (x_3, -x_1, x_2, x_4) = (y_1, y_2, y_3, y_4).$$

The transformed region in $\mathbb{R}^4_{++}$ is

$$y_2 + y_3 = 1, \quad y_3 - y_2 < 1, \quad y_4 - y_2 > 1, \quad y_4 - y_3 < 1.$$

This is contained in the boundary of (8.1g). The other possible component can be described by

$$x_2 + x_3 > 1, \quad x_3 - x_2 < 1, \quad x_4 - x_2 > 1, \quad x_4 - x_3 < 1.$$

By applying step (3-2) of Algorithm 8.1, one can conclude that this connected component is non-unitary.

Now consider $\alpha > \frac{3}{2}$. We must have

$$w_6 : (x_1, x_2, x_3, x_4) \mapsto (x_1, x_3, x_2, x_4) = (y_1, y_2, y_3, y_4).$$

The transformed region in $\mathbb{R}^4_{++}$ is

$$y_3 - y_1 = 1, \quad y_1 + y_2 < 1, \quad y_4 - y_2 > 1, \quad y_4 < \alpha, \quad 0 \le y_3 - \frac{1}{2} \le y_1 \le y_4.$$

This is contained in the boundary of (8.1f). The other possible component can be described by

$$x_1 + x_2 < 1, \quad x_3 - x_1 > 1, \quad x_4 - x_2 > 1, \quad x_3 - x_2 < 1, \quad x_4 < \alpha.$$

This is non-unitary by step (3-2) of Algorithm 8.1 and Proposition 8.9.

For $i = 7$, we have

$$w_7 : (x_1, x_2, x_3, x_4) \mapsto (-x_1, x_3, x_2, x_4) = (y_1, y_2, y_3, y_4).$$

The transformed region in $\mathbb{R}^4_{++}$ is

$$y_1 + y_3 = 1, \quad y_4 - y_3 > 1, \quad y_1 + y_2 < 1 \quad y_4 < \alpha, 0 \leq y_3 - \frac{1}{2} \leq y_2 \leq y_4.$$

This is contained in the boundary of (8.1f). The other component containing with boundary containing the above region is

$$x_1 + x_3 < 1, \quad x_4 - x_3 > 1, \quad x_2 + x_3 > 1, \quad x_4 < \alpha.$$

which is non-unitary by the same reasoning as before.

For $i = 8$, we have

$$w_8 : (x_1, x_2, x_3, x_4) \mapsto (-x_1, x_2, x_3, x_4) = (y_1, y_2, y_3, y_4).$$

The transformed region in $\mathbb{R}^4_{++}$ is

$$y_1 + y_2 = 1, \quad y_3 - y_1 < 1, \quad y_1 + y_3 > 1, \quad y_4 - y_3 > 1 \quad y_4 < \alpha, \quad 0 \leq y_2 - \frac{1}{2} \leq y_3 \leq y_4.$$

This is contained in the boundary of (8.1f). The other component with boundary containing the above region is

$$x_1 + x_2 > 1, \quad x_3 - x_1 < 1, \quad x_1 + x_3 > 1, \quad x_4 - x_3 > 1, \quad x_4 < \alpha.$$

When $\alpha \in \mathbb{Z}$, then $(0, 1, 1, 2)$ is a point of good parity contained in the boundary of the connected component above. By Proposition 6.18, this point contains subquotients that are not of Arthur type and hence it is not strongly unitary by Proposition 2.5. When $\alpha \in \frac{1}{2} + \mathbb{Z}$, then $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{3}{2})$ is a point of good parity in the boundary of the above component. It also contains subquotients that are not of Arthur type by Proposition 6.18. Thus,in both cases, we can conclude that the above component is non-unitary.

Now we move onto the case $\alpha > 2$. For $i = 9$, we have

$$w_9 : (x_1, x_2, x_3, x_4) \mapsto (x_1, x_3, x_2, x_4) = (y_1, y_2, y_3, y_4).$$

The transformed region in $\mathbb{R}^4_{++}$ is

$$y_3 - y_1 = 1, \quad y_1 + y_2 < 1, \quad y_4 - y_3 > 1, \quad y_1 + y_3 > 1, \quad y_4 < \alpha, \quad 0 \leq y_3 - \frac{1}{2} \leq y_2 \leq y_4.$$

This is contained in the boundary of (8.1f). The other component with boundary containing the above region is

$$x_1 + x_2 < 1, \quad x_3 - x_1 > 1, \quad x_3 - x_2 < 1 \quad x_4 - x_2 > 1, \quad x_4 - x_3 < 1, \quad x_4 < \alpha.$$

Using step (3-2) of Algorithm 8.1 and Proposition 8.9, we can conclude that the above component is non-unitary.

Now let $\alpha > \frac{5}{2}$. For $i = 10$, we have

$$w_{10} : (x_1, x_2, x_3, x_4) \mapsto (-x_1, x_2, x_3, x_4) = (y_1, y_2, y_3, y_4).$$

The transformed region in $\mathbb{R}^4_{++}$ is

$$y_1 + y_2 = 1, \quad y_2 - y_1 < 1, \quad y_3 - y_2 > 1, \quad y_4 - y_3 > 1, \quad 0 \leq y_2 - \frac{1}{2} \leq y_3 \leq y_4.$$

This is contained in the boundary of (8.1e). The other component with boundary containing the above region is

$$x_4 - x_3 > 1, \quad x_1 + x_2 > 1, \quad x_3 - x_1 < 1, \quad x_4 < \alpha.$$

Using the same method as the case $i = 8$, we can prove show that the above component is non-unitary.

Finally, let $\alpha > 3$. For $i = 11$, we have

$$w_{10} : (x_1, x_2, x_3, x_4) \mapsto (x_1, x_2, x_3, x_4).$$

In this case, the transformed region in $\mathbb{R}^4_{++}$ is

$$x_2 - x_1 = 1, \quad x_1 + x_2 > 1, \quad x_3 - x_2 > 1, \quad x_4 - x_3 > 1, \quad x_4 < \alpha, \quad 0 \leq x_2 - \frac{1}{2} \leq x_3 \leq x_4.$$

This is contained in the boundary of (8.1d)). The other component with boundary containing the above region is

$$x_1 + x_2 > 1, \quad x_2 - x_1 < 1, \quad x_3 - x_2 > 1, \quad x_4 - x_3 > 1, \quad x_4 < \alpha.$$

When $\alpha \in \mathbb{Z}$, the point $(1, 1, 2, 3)$ is a point of good parity contained in the boundary of the above component. By Proposition 6.18, the point contains subquotients that are not of Arthur type and hence, by Proposition 2.5, it is not strongly unitary. Similarly, when $\alpha \in \frac{1}{2} + \mathbb{Z}$, the point $(\frac{3}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2})$ is a point of good parity contained in the boundary. It contains subquotients that are not of Arthur type and hence is not strongly unitary. In both cases, we can conclude that the above component is non-unitary. This concludes the proof. ☐

Remark 8.11. *It is clear from our proof that any 3-parameter complementary series within the corank* 4 *unitary dual is contained in the boundary of some unitary open connected component in* $\mathbb{R}^4$*. We expect this to hold for any arbitrary corank* n*, i.e. within the unitary dual of corank* n*, any* $n-1$*-dimensional complementary series should be fully contained in the boundary of some unitary open connected component in* $\mathbb{R}^n$*.*

9. One-parameter complementary series

The proof of exhaustion above gives all unitarizable representations of corank 4 which appear as part of a 4 or 3-dimensional complementary series. To construct the full unitary dual, we will also need to include unitarizable representations of corank 4 that appear as part of a complementary series with one or two parameters, as described in step 2 of Algorithm 8.1. In this section, we will classify all one-parameter complementary series that contains a continuous family of irreducible, unitarizable representations of corank 4.

9.1. **One-parameter complementary series induced from unitarizable representations.** We begin by listing representations inside a one-parameter complementary series, which emanate from a critical type, unitarizable, irreducible representation of corank 3. Using Tadic's results in [Tad23], we have an exhaustive list of such representations. From now on, we say that an induced representation is unitary if all of its irreducible subquotients are unitary. Our results can be summarized as follows:

Proposition 9.1. *In Table 9.1, we list all possible one parameter complementary series of the form*

$$\pi_x = u_\rho(a, b)|\cdot|^x \rtimes \pi_A,$$

where $\pi_A \in \Pi_{A,gp}$ is a unitarizable, irreducible representation of the maximal Levi subgroup, which is of critical type. For each pair of such representation and its dual (denoted by $'$), we list all its complementary series π_x for $x \geq 0$ and all its reducibility points. When irreducible, π_x, $x \geq 0$ is unitarizable up to the first nonzero reducibility point. When reducibility occurs at 0, there is no complementary series.

Proof. The statement is clear when there is only one reducibility point for π_x. When there are 2 or 3 reducibility points, then it suffices to show that π_{x_0} contains a non-unitarizable subquotient, when x_0 is the second nonzero reducibility point. When there are 4 reducibility points, then it suffices to show that π_x contains a non-unitarizable subquotient at the second and third nonzero reducibility point, or the second and fourth nonzero reducibility point. All the relevant information is given in Table 2, where the particular non-unitarizable subquotient is denoted by π. By Aubert-Zelevinsky duality, we only need to consider one complementary series for each pair. This proves the statement for all cases except for when $N^\circ = 21$.

When $N^\circ = 21$, all irreducible subquotients are unitarizable at both the first and second reducibility point. In this case, when $\frac{1}{2} < x < \frac{3}{2}$, the one-parameter complementary series falls within a non-unitary connected component of the two-parameter complementary series

$$\rho|\cdot|^x \times \rho|\cdot|^y \rtimes L(\Delta_\rho[-\frac{1}{2}, -\frac{1}{2}], T_{I,1}^{\frac{1}{2}}(\pi_{sc}))$$

with $y = \frac{1}{2}$. This is proved in Proposition 10.5. This proves the claim for $N^\circ = 21$.

The fact that the list is exhaustive follows from [Tad23]. It suffices now to show that the unitarizable representations we provide in Table 2 are indeed non-unitarizable subquotients of the fully induced representation in Table 9.1. By exploiting Aubert duality and Remark 9.2, it suffices to show this when $N^\circ = 30$.

When $N^\circ = 30$, it suffices to consider the case $\alpha \geq \frac{1}{2}$. In this case, we need to find a non-unitarizable subquotient of

$$\Pi_\alpha = L(\Delta_\rho[0,-1], \Delta_\rho[1,0])|\cdot|^\alpha \rtimes \pi_{sc} = L(\Delta_\rho[\alpha, |\alpha-1|], \Delta_\rho[\alpha+1, \alpha]) \rtimes \pi_{sc}.$$

First, note that

$$D^{(k_\alpha)}_{\rho|\cdot|^\alpha} D^{(k_{(|\alpha-1|)})}_{\rho|\cdot||^{\alpha-1|}}(\Pi_\alpha) = \Delta_\rho[\alpha+1, \alpha] \rtimes \pi_{sc},$$

where k_i is the highest order derivative of Π_α with respect to $\rho|\cdot|^i$. In particular, we have $k_\alpha = k_{(|\alpha-1|)} = 1$. The above fully induced representation has a subquotient

$$L(\Delta_\rho[-\alpha+1, -\alpha+1]; T^\alpha_{I,1}(\pi_{sc})),$$

which is of Arthur type and unitarizable. Further more, we have

$$S^{(k_\alpha)}_{\rho|\cdot|^\alpha} S^{(k_{(|\alpha-1|)})}_{\rho|\cdot||^{\alpha-1|}}(L(\Delta_\rho[-\alpha+1, -\alpha+1]; T^\alpha_{I,1}(\pi_{sc}))) = L(\Delta_\rho[-\alpha, -\alpha-1], \Delta_\rho[-|\alpha-1|, -\alpha]; \pi_{sc}).$$

This implies that $L(\Delta_\rho[-\alpha, -\alpha-1], \Delta_\rho[-|\alpha-1|, -\alpha]; \pi_{sc})$ is a subquotient of Π_α, as desired. It is non-unitarizable since it is not of Arthur type.

$\square$

Table 1. Reducibility data for one parameter complementary series induced from unitarizable representation

N°	π_x	Reducibility points	Cases
1.	$\rho\lvert\cdot\rvert^x \rtimes T_{I,1}^{\alpha+2}(T_{I,1}^{\alpha+1}(T_{I,1}^{\alpha}(\pi_{sc})))$	$\alpha-1, \alpha+3$	$\alpha \geq \frac{1}{2}$
1'.	$\rho\lvert\cdot\rvert^x \rtimes L(\Delta_\rho[-\alpha-2,-\alpha-2], \Delta_\rho[-\alpha-1,-\alpha-1], \Delta_\rho[-\alpha,-\alpha]; \pi_{sc})$	$\alpha-1, \alpha+3$	$\alpha \geq \frac{1}{2}$
2.	$\rho\lvert\cdot\rvert^x \rtimes L(\Delta_\rho[-\alpha-1,-\alpha-1], \Delta_\rho[-\alpha+1,-\alpha]; \pi_{sc})$	$\lvert\alpha-2\rvert, \alpha, \alpha+2$	$\alpha > 1$
2'.	$\rho\lvert\cdot\rvert^x \rtimes T_{I,1}^{\alpha-1}(T_{I,1}^{\alpha+1}(T_{I,1}^{\alpha}(\pi_{sc})))$	$\lvert\alpha-2\rvert, \alpha, \alpha+2$	$\alpha > 1$
3.	$\rho\lvert\cdot\rvert^x \rtimes L(\Delta_\rho[-\alpha+1,-\alpha+1]; T_{I,1}^{\alpha+1}(T_{I,1}^{\alpha}(\pi_{sc})))$	$\lvert\alpha-2\rvert, \alpha, \alpha+2$	$\alpha > 1$
3'.	$\rho\lvert\cdot\rvert^x \rtimes L(\Delta_\rho[-\alpha-1,-\alpha-1], \Delta_\rho[-\alpha,-\alpha], \Delta_\rho[-\alpha+1,-\alpha+1]; \pi_{sc})$	$\lvert\alpha-2\rvert, \alpha, \alpha+2$	$\alpha > 1$
4.	$\rho\lvert\cdot\rvert^x \rtimes T_{I,1}^{\alpha-2}(T_{I,1}^{\alpha-1}(T_{I,1}^{\alpha}(\pi_{sc})))$	$\lvert\alpha-3\rvert, \alpha+1$	$\alpha \geq 2$
4'.	$\rho\lvert\cdot\rvert^x \rtimes L(\Delta_\rho[-\alpha+2,-\alpha]; \pi_{sc})$	$\lvert\alpha-3\rvert, \alpha+1$	$\alpha \geq 2$
5.	$\rho\lvert\cdot\rvert^x \rtimes L(\Delta_\rho[-\alpha+2,-\alpha+2]; T_{I,1}^{\alpha-1}(T_{I,1}^{\alpha}(\pi_{sc})))$	$\lvert\alpha-3\rvert, \alpha-1, \alpha+1$	$\alpha \geq 2$
5'.	$\rho\lvert\cdot\rvert^x \rtimes L(\Delta_\rho[-\alpha+1,-\alpha], \Delta_\rho[-\alpha+2,-\alpha+2]; \pi_{sc})$	$\lvert\alpha-3\rvert, \alpha-1, \alpha+1$	$\alpha \geq 2$
6.	$\rho\lvert\cdot\rvert^x \rtimes L(\Delta_\rho[-\alpha+1,-\alpha+1], \Delta_\rho[-\alpha+2,-\alpha+2]; T_{I,1}^{\alpha}(\pi_{sc}))$	$\lvert\alpha-3\rvert, \alpha-1, \alpha, \alpha+1$	$\alpha \geq 2$
6'.	$\rho\lvert\cdot\rvert^x \rtimes L(\Delta_\rho[-\alpha,-\alpha]; \Delta_\rho[-\alpha+2,-\alpha+1]; \pi_{sc})$	$\lvert\alpha-3\rvert, \alpha-1, \alpha, \alpha+1$	$\alpha \geq 2$
7.	$\rho\lvert\cdot\rvert^x \rtimes L(\Delta_\rho[-\alpha+2,-\alpha+1]; T_{I,1}^{\alpha}(\pi_{sc}))$	$\lvert\alpha-3\rvert, \alpha, \alpha+1$	$\alpha \geq 2$
7'.	$\rho\lvert\cdot\rvert^x \rtimes L(\Delta_\rho[-\alpha,-\alpha], \Delta_\rho[-\alpha+1,-\alpha+1], \Delta_\rho[-\alpha+2,-\alpha+2]; \pi_{sc})$	$\lvert\alpha-3\rvert, \alpha, \alpha+1$	$\alpha \geq 2$
8.	$\rho\lvert\cdot\rvert^x \rtimes L(\Delta_\rho[-2,-2], \Delta_\rho[-1,-1]; T_{IV,3}(\pi_{sc}))$	$0, 1, 3$	$\alpha = 1$
8'.	$\rho\lvert\cdot\rvert^x \rtimes T_{V,2}^{+}(T_{I,1}^{2}(T_{I,1}^{1}(\pi_{sc})))$	$0, 1, 3$	$\alpha = 1$
9.	$\rho\lvert\cdot\rvert^x \rtimes L(\Delta_\rho[-2,-2], \Delta_\rho[0,-1]; \pi_{sc})$	$0, 1, 3$	$\alpha = 1$
9'.	$\rho\lvert\cdot\rvert^x \rtimes T_{V,2}^{-}(T_{I,1}^{2}(T_{I,1}^{1}(\pi_{sc})))$	$0, 1, 3$	$\alpha = 1$
10.	$\rho\lvert\cdot\rvert^x \rtimes L(\Delta_\rho[-1,-1], \Delta_\rho[0,-1]; \pi_{sc})$	$0, 2$	$\alpha = 1$
10'.	$\rho\lvert\cdot\rvert^x \rtimes L(\Delta_\rho[0,-1]; T_{I,1}^{1}(\pi_{sc}))$	$0, 2$	$\alpha = 1$
11.	$\rho\lvert\cdot\rvert^x \rtimes L(\Delta_\rho[-1,-1], \Delta_\rho[-1,-1]; T_{IV,3}(\pi_{sc}))$	$0, 1, 2$	$\alpha = 1$
11'.	$\rho\lvert\cdot\rvert^x \rtimes T_{I,2}^{1}(T_{IV,3}(\pi_{sc}))$	$0, 1, 2$	$\alpha = 1$
12.	$\rho\lvert\cdot\rvert^x \rtimes L(\Delta_\rho[-1,-1]; T_{V,2}^{-}(T_{I,1}^{1}(\pi_{sc})))$	$0, 2$	$\alpha = 1$
12'.	$\rho\lvert\cdot\rvert^x \rtimes T_{III,2}^{1}(T_{IV,3}(\pi_{sc}))$	$0, 2$	$\alpha = 1$
13.	$\rho\lvert\cdot\rvert^x \rtimes L(\Delta_\rho[-1,-1]; T_{V,2}^{+}(T_{I,1}^{1}(\pi_{sc})))$	$0, 1, 2$	$\alpha = 1$
14.	$\rho\lvert\cdot\rvert^x \rtimes L(\Delta_\rho[-1,-1]; T_{IV,5}(\pi_{sc}))$	$0, 1, 2$	$\alpha = 1$
14'.	$\rho\lvert\cdot\rvert^x \rtimes T_{V,4}^{+}(T_{I,1}^{1}(\pi_{sc}))$	$0, 1, 2$	$\alpha = 1$
15.	$\rho\lvert\cdot\rvert^x \rtimes L(\Delta_\rho[0,-1]; T_{IV,3}(\pi_{sc}))$	$0, 1, 2$	$\alpha = 1$
15'.	$\rho\lvert\cdot\rvert^x \rtimes T_{V,4}^{-}(T_{I,1}^{1}(\pi_{sc}))$	$0, 1, 2$	$\alpha = 1$
16.	$\rho\lvert\cdot\rvert^x \rtimes L(\Delta_\rho[-\frac{3}{2},-\frac{3}{2}], \Delta_\rho[-\frac{1}{2},-\frac{1}{2}], \Delta_\rho[-\frac{1}{2},-\frac{1}{2}]; \pi_{sc})$	$\frac{1}{2}, \frac{3}{2}, \frac{5}{2}$	$\alpha = \frac{1}{2}$
16'.	$\rho\lvert\cdot\rvert^x \rtimes T_{I,1}^{\frac{3}{2}}(T_{II,3}^{\frac{1}{2}}(\pi_{sc}))$	$\frac{1}{2}, \frac{3}{2}, \frac{5}{2}$	$\alpha = \frac{1}{2}$
17.	$\rho\lvert\cdot\rvert^x \rtimes L(\Delta_\rho[\frac{1}{2},-\frac{3}{2}]; \pi_{sc})$	$\frac{1}{2}$	$\alpha = \frac{1}{2}$
17'.	$\rho\lvert\cdot\rvert^x \rtimes L(\Delta_\rho[-\frac{1}{2},-\frac{3}{2}]; T_{I,1}^{\frac{1}{2}}(\pi_{sc}))$	$\frac{1}{2}$	$\alpha = \frac{1}{2}$
18.	$\rho\lvert\cdot\rvert^x \rtimes L(\Delta_\rho[-\frac{3}{2},-\frac{3}{2}], \Delta_\rho[-\frac{1}{2},-\frac{1}{2}]; T_{I,1}^{\frac{1}{2}}(\pi_{sc}))$	$\frac{3}{2}, \frac{5}{2}$	$\alpha = \frac{1}{2}$
18'.	$\rho\lvert\cdot\rvert^x \rtimes T_{I,1}^{\frac{3}{2}}(T_{I,2}^{\frac{1}{2}}(\pi_{sc}))$	$\frac{3}{2}, \frac{5}{2}$	$\alpha = \frac{1}{2}$
19.	$\rho\lvert\cdot\rvert^x \rtimes L(\Delta_\rho[-\frac{1}{2},-\frac{1}{2}]; T_{I,1}^{\frac{3}{2}}(T_{I,1}^{\frac{1}{2}}(\pi_{sc})))$	$\frac{1}{2}, \frac{3}{2}, \frac{5}{2}$	$\alpha = \frac{1}{2}$
19'.	$\rho\lvert\cdot\rvert^x \rtimes L(\Delta_\rho[-\frac{3}{2},-\frac{3}{2}]; T_{I,2}^{\frac{1}{2}}(\pi_{sc}))$	$\frac{1}{2}, \frac{3}{2}, \frac{5}{2}$	$\alpha = \frac{1}{2}$
20.	$\rho\lvert\cdot\rvert^x \rtimes T_{I,3}^{\frac{1}{2}}(\pi_{sc})$	$\frac{1}{2}, \frac{3}{2}$	$\alpha = \frac{1}{2}$
20'.	$\rho\lvert\cdot\rvert^x \rtimes L(\Delta_\rho[-\frac{1}{2},-\frac{1}{2}], \Delta_\rho[-\frac{1}{2},-\frac{1}{2}], \Delta_\rho[-\frac{1}{2},-\frac{1}{2}]; \pi_{sc})$	$\frac{1}{2}, \frac{3}{2}$	$\alpha = \frac{1}{2}$

N°	π_x	Reducibility points	Cases
21.	$\rho\lvert\cdot\rvert^x \rtimes L(\Delta_\rho[-\frac{1}{2},-\frac{1}{2}],\Delta_\rho[-\frac{1}{2},-\frac{1}{2}];T_{I,1}^{\frac{1}{2}}(\pi_{sc}))$	$\frac{1}{2},\frac{3}{2}$	$\alpha=\frac{1}{2}$
21'.	$\rho\lvert\cdot\rvert^x \rtimes L(\Delta_\rho[-\frac{1}{2},-\frac{1}{2}];T_{II,3}^{\frac{1}{2}}(\pi_{sc}))$	$\frac{1}{2},\frac{3}{2}$	$\alpha=\frac{1}{2}$
22.	$\rho\lvert\cdot\rvert^x \rtimes L(\Delta_\rho[-\frac{1}{2},-\frac{1}{2}];T_{I,2}^{\frac{1}{2}}(\pi_{sc}))$	$\frac{3}{2}$	$\alpha=\frac{1}{2}$
23.	$\rho\lvert\cdot\rvert^x \rtimes L(\Delta_\rho[-2,-2],\Delta_\rho[-1,-1];T_{V,2}^{+}(\pi_{sc}))$	$0,1,3$	$\alpha=0$
23'.	$\rho\lvert\cdot\rvert^x \rtimes T_{I,1}^{2}(T_{I,1}^{1}(T_{V,2}^{-}(\pi_{sc})))$	$0,1,3$	$\alpha=0$
24.	$\rho\lvert\cdot\rvert^x \rtimes L(\Delta_\rho[-2,-2],\Delta_\rho[-1,-1];T_{V,2}^{-}(\pi_{sc}))$	$0,1,3$	$\alpha=0$
24'.	$\rho\lvert\cdot\rvert^x \rtimes T_{I,1}^{2}(T_{I,1}^{1}(T_{V,2}^{+}(\pi_{sc})))$	$0,1,3$	$\alpha=0$
25.	$\rho\lvert\cdot\rvert^x \rtimes L(\Delta_\rho[-1,-1],\Delta_\rho[-1,-1];T_{V,2}^{\pm}(\pi_{sc}))$	$0,2$	$\alpha=0$
25'.	$\rho\lvert\cdot\rvert^x \rtimes T_{I,2}^{\pm}(T_{V,2}^{\pm}(\pi_{sc}))$	$0,2$	$\alpha=0$
26.	$\rho\lvert\cdot\rvert^x \rtimes L(\Delta_\rho[-1,-1],\Delta_\rho[-1,-1];T_{I,1}^{1}(T_{V,2}^{\pm}(\pi_{sc})))$	$0,2$	$\alpha=0$
27.	$\rho\lvert\cdot\rvert^x \rtimes L(\Delta_\rho[-1,-1];T_{V,4}^{\pm}(\pi_{sc}))$	$0,1,2$	$\alpha=0$
27'.	$\rho\lvert\cdot\rvert^x \rtimes T_{I,1}^{1}(T_{V,4}^{\pm}(\pi_{sc}))$	$0,1,2$	$\alpha=0$
28.	$\rho\lvert\cdot\rvert^x \rtimes L(\Delta_\rho[0,-1];T_{V,2}^{\pm}(\pi_{sc}))$	$0,1,2$	$\alpha=0$
29.	$\rho\lvert\cdot\rvert^x \rtimes T_{V,6}^{\pm}(\pi_{sc})$	$0,1$	$\alpha=0$
30.	$L(\Delta_\rho[0,-1],\Delta_\rho[1,0])\lvert\cdot\rvert^x \rtimes \pi_{sc}$	$\lvert\alpha-1\rvert,\alpha,\alpha+1$	all
31.	$\Delta_\rho[1,-1]\lvert\cdot\rvert^x \rtimes T_{I,1}^{\alpha}(\pi_{sc})$	$\lvert\alpha-2\rvert,\alpha,\alpha+1,\alpha+2$	$\alpha\neq 0$
31'.	$L(\Delta_\rho[-1,-1],\Delta_\rho[0,0],\Delta_\rho[1,1])\lvert\cdot\rvert^x \rtimes L(\Delta_\rho[-\alpha,-\alpha];\pi_{sc})$	$\lvert\alpha-2\rvert,\alpha,\alpha+1,\alpha+2$	$\alpha\neq 0$
32.	$\Delta_\rho[1,-1]\lvert\cdot\rvert^x \rtimes L(\Delta_\rho[-\alpha,-\alpha];\pi_{sc})$	$\lvert\alpha-2\rvert,\lvert\alpha-1\rvert,\alpha,\alpha+2$	$\alpha\neq 0$
32'.	$L(\Delta_\rho[-1,-1],\Delta_\rho[0,0],\Delta_\rho[1,1])\lvert\cdot\rvert^x \rtimes T_{I,1}^{\alpha}(\pi_{sc})$	$\lvert\alpha-2\rvert,\lvert\alpha-1\rvert,\alpha,\alpha+2$	$\alpha\neq 0$
33.	$\Delta_\rho[1,-1]\lvert\cdot\rvert^x \rtimes T_{V,2}^{\pm}(\pi_{sc})$	$0,1,2$	$\alpha=0$
33'.	$L(\Delta_\rho[-1,-1],\Delta_\rho[0,0],\Delta_\rho[1,1])\lvert\cdot\rvert^x \rtimes T_{V,2}^{\mp}(\pi_{sc})$	$0,1,2$	$\alpha=0$
34.	$\Delta_\rho[\frac{1}{2},-\frac{1}{2}]\lvert\cdot\rvert^x \rtimes L(\Delta_\rho[-\alpha-1,-\alpha-1],\Delta_\rho[-\alpha,-\alpha];\pi_{sc})$	$\lvert\alpha-\frac{3}{2}\rvert,\alpha-\frac{1}{2},\alpha+\frac{5}{2}$	$\alpha\geq\frac{1}{2}$
34'.	$L(\Delta_\rho[-\frac{1}{2},-\frac{1}{2}],\Delta_\rho[\frac{1}{2},\frac{1}{2}])\lvert\cdot\rvert^x \rtimes T_{I,1}^{\alpha+1}(T_{I,1}^{\alpha}(\pi_{sc}))$	$\lvert\alpha-\frac{3}{2}\rvert,\alpha-\frac{1}{2},\alpha+\frac{5}{2}$	$\alpha\geq\frac{1}{2}$
35.	$\Delta_\rho[\frac{1}{2},-\frac{1}{2}]\lvert\cdot\rvert^x \rtimes T_{I,1}^{\alpha+1}(T_{I,1}^{\alpha}(\pi_{sc}))$	$\lvert\alpha-\frac{3}{2}\rvert,\alpha-\frac{1}{2},\alpha+\frac{3}{2},\alpha+\frac{5}{2}$	$\alpha\geq\frac{1}{2}$
35'.	$L(\Delta_\rho[-\frac{1}{2},-\frac{1}{2}],\Delta_\rho[\frac{1}{2},\frac{1}{2}])\lvert\cdot\rvert^x \rtimes L(\Delta_\rho[-\alpha-1,-\alpha-1],\Delta_\rho[-\alpha,-\alpha];\pi_{sc})$	$\lvert\alpha-\frac{3}{2}\rvert,\alpha-\frac{1}{2},\alpha+\frac{3}{2},\alpha+\frac{5}{2}$	$\alpha\geq\frac{1}{2}$
36.	$\Delta_\rho[\frac{1}{2},-\frac{1}{2}]\lvert\cdot\rvert^x \rtimes L(\Delta_\rho[-\alpha+1,-\alpha];\pi_{sc})$	$\lvert\alpha-\frac{5}{2}\rvert,\alpha-\frac{3}{2},\alpha+\frac{1}{2},\alpha+\frac{3}{2}$	$\alpha\geq\frac{3}{2}$
36'.	$L(\Delta_\rho[-\frac{1}{2},-\frac{1}{2}],\Delta_\rho[\frac{1}{2},\frac{1}{2}])\lvert\cdot\rvert^x \rtimes T_{I,1}^{\alpha-1}(T_{I,1}^{\alpha}(\pi_{sc}))$	$\lvert\alpha-\frac{5}{2}\rvert,\alpha-\frac{3}{2},\alpha+\frac{1}{2},\alpha+\frac{3}{2}$	$\alpha\geq\frac{3}{2}$
37.	$\Delta_\rho[\frac{1}{2},-\frac{1}{2}]\lvert\cdot\rvert^x \rtimes T_{I,1}^{\alpha-1}(T_{I,1}^{\alpha}(\pi_{sc}))$	$\lvert\alpha-\frac{5}{2}\rvert,\alpha+\frac{1}{2},\alpha+\frac{3}{2}$	$\alpha\geq\frac{3}{2}$
37'.	$L(\Delta_\rho[-\frac{1}{2},-\frac{1}{2}],\Delta_\rho[\frac{1}{2},\frac{1}{2}])\lvert\cdot\rvert^x \rtimes L(\Delta_\rho[-\alpha+1,-\alpha];\pi_{sc})$	$\lvert\alpha-\frac{5}{2}\rvert,\alpha+\frac{1}{2},\alpha+\frac{3}{2}$	$\alpha\geq\frac{3}{2}$
38.	$\Delta_\rho[\frac{1}{2},-\frac{1}{2}]\lvert\cdot\rvert^x \rtimes L(\Delta_\rho[-\alpha+1,-\alpha+1];T_{I,1}^{\alpha}(\pi_{sc}))$	$\lvert\alpha-\frac{5}{2}\rvert,\alpha-\frac{3}{2},\alpha+\frac{1}{2},\alpha+\frac{3}{2}$	$\alpha\geq\frac{3}{2}$
38'.	$L(\Delta_\rho[-\frac{1}{2},-\frac{1}{2}],\Delta_\rho[\frac{1}{2},\frac{1}{2}])\lvert\cdot\rvert^x \rtimes L(\Delta_\rho[-\alpha+1,-\alpha+1],\Delta_\rho[-\alpha,-\alpha];\pi_{sc})$	$\lvert\alpha-\frac{5}{2}\rvert,\alpha-\frac{3}{2},\alpha+\frac{1}{2},\alpha+\frac{3}{2}$	$\alpha\geq\frac{3}{2}$
39.	$\Delta_\rho[\frac{1}{2},-\frac{1}{2}]\lvert\cdot\rvert^x \rtimes L(\Delta_\rho[-\alpha+1,-\alpha+1],\Delta_\rho[-\alpha,-\alpha];\pi_{sc})$	$\lvert\alpha-\frac{5}{2}\rvert,\alpha-\frac{1}{2},\alpha+\frac{1}{2},\alpha+\frac{3}{2}$	$\alpha\geq\frac{3}{2}$
39'.	$L(\Delta_\rho[-\frac{1}{2},-\frac{1}{2}],\Delta_\rho[\frac{1}{2},\frac{1}{2}])\lvert\cdot\rvert^x \rtimes L(\Delta_\rho[-\alpha+1,-\alpha+1];T_{I,1}^{\alpha}(\pi_{sc}))$	$\lvert\alpha-\frac{5}{2}\rvert,\alpha-\frac{1}{2},\alpha+\frac{1}{2},\alpha+\frac{3}{2}$	$\alpha\geq\frac{3}{2}$
40.	$\Delta_\rho[\frac{1}{2},-\frac{1}{2}]\lvert\cdot\rvert^x \rtimes L(\Delta_\rho[-1,-1];T_{IV,3}(\pi_{sc}))$	$\frac{1}{2},\frac{3}{2},\frac{5}{2}$	$\alpha=1$
40'.	$L(\Delta_\rho[-\frac{1}{2},-\frac{1}{2}],\Delta_\rho[\frac{1}{2},\frac{1}{2}])\lvert\cdot\rvert^x \rtimes T_{V,2}^{+}(T_{I,1}^{1}(\pi_{sc}))$	$\frac{1}{2},\frac{3}{2},\frac{5}{2}$	$\alpha=1$
41.	$\Delta_\rho[\frac{1}{2},-\frac{1}{2}]\lvert\cdot\rvert^x \rtimes T_{V,2}^{+}(T_{I,1}^{1}(\pi_{sc}))$	$\frac{1}{2},\frac{3}{2},\frac{5}{2}$	$\alpha=1$
41'.	$L(\Delta_\rho[-\frac{1}{2},-\frac{1}{2}],\Delta_\rho[\frac{1}{2},\frac{1}{2}])\lvert\cdot\rvert^x \rtimes L(\Delta_\rho[-1,-1];T_{IV,3}(\pi_{sc}))$	$\frac{1}{2},\frac{3}{2},\frac{5}{2}$	$\alpha=1$
42.	$\Delta_\rho[\frac{1}{2},-\frac{1}{2}]\lvert\cdot\rvert^x \rtimes L(\Delta_\rho[0,-1];\pi_{sc})$	$\frac{1}{2},\frac{3}{2},\frac{5}{2}$	$\alpha=1$
42'.	$L(\Delta_\rho[-\frac{1}{2},-\frac{1}{2}],\Delta_\rho[\frac{1}{2},\frac{1}{2}])\lvert\cdot\rvert^x \rtimes T_{V,2}^{-}(T_{I,1}^{1}(\pi_{sc}))$	$\frac{1}{2},\frac{3}{2},\frac{5}{2}$	$\alpha=1$
43.	$\Delta_\rho[\frac{1}{2},-\frac{1}{2}]\lvert\cdot\rvert^x \rtimes T_{V,2}^{-}(T_{I,1}^{1}(\pi_{sc}))$	$\frac{1}{2},\frac{3}{2},\frac{5}{2}$	$\alpha=1$
43'.	$L(\Delta_\rho[-\frac{1}{2},-\frac{1}{2}],\Delta_\rho[\frac{1}{2},\frac{1}{2}])\lvert\cdot\rvert^x \rtimes L(\Delta_\rho[0,-1];\pi_{sc})$	$\frac{1}{2},\frac{3}{2},\frac{5}{2}$	$\alpha=1$
44.	$\Delta_\rho[\frac{1}{2},-\frac{1}{2}]\lvert\cdot\rvert^x \rtimes L(\Delta_\rho[-\frac{1}{2},-\frac{1}{2}],\Delta_\rho[-\frac{1}{2},-\frac{1}{2}];\pi_{sc})$	$0,1,2$	$\alpha=\frac{1}{2}$
44'.	$L(\Delta_\rho[-\frac{1}{2},-\frac{1}{2}],\Delta_\rho[\frac{1}{2},\frac{1}{2}])\lvert\cdot\rvert^x \rtimes T_{I,2}^{\frac{1}{2}}(\pi_{sc})$	$0,1,2$	$\alpha=\frac{1}{2}$

N°	π_x	Reducibility points	Cases
45.	$\Delta_\rho[\frac{1}{2},-\frac{1}{2}]\lvert\cdot\rvert^x \rtimes T^{\frac{1}{2}}_{I,2}(\pi_{sc})$	$1,2$	$\alpha=\frac{1}{2}$
45'.	$L(\Delta_\rho[-\frac{1}{2},-\frac{1}{2}],\Delta_\rho[\frac{1}{2},\frac{1}{2}])\lvert\cdot\rvert^x \rtimes L(\Delta_\rho[-\frac{1}{2},-\frac{1}{2}],\Delta_\rho[-\frac{1}{2},-\frac{1}{2}];\pi_{sc})$	$1,2$	$\alpha=\frac{1}{2}$
46.	$\Delta_\rho[\frac{1}{2},-\frac{1}{2}]\lvert\cdot\rvert^x \rtimes L(\Delta_\rho[-\frac{1}{2},-\frac{1}{2}];T^{\frac{1}{2}}_{I,1}(\pi_{sc}))$	$1,2$	$\alpha=\frac{1}{2}$
46'.	$L(\Delta_\rho[-\frac{1}{2},-\frac{1}{2}],\Delta_\rho[\frac{1}{2},\frac{1}{2}])\lvert\cdot\rvert^x \rtimes T^{\frac{1}{2}}_{III,2}(\pi_{sc})$	$1,2$	$\alpha=\frac{1}{2}$
47.	$\Delta_\rho[\frac{1}{2},-\frac{1}{2}]\lvert\cdot\rvert^x \rtimes T^{\frac{1}{2}}_{III,2}(\pi_{sc})$	$1,2$	$\alpha=\frac{1}{2}$
47'.	$L(\Delta_\rho[-\frac{1}{2},-\frac{1}{2}],\Delta_\rho[\frac{1}{2},\frac{1}{2}])\lvert\cdot\rvert^x \rtimes T^{\frac{1}{2}}_{I,1}(\pi_{sc}))$	$1,2$	$\alpha=\frac{1}{2}$
48.	$\Delta_\rho[\frac{1}{2},-\frac{1}{2}]\lvert\cdot\rvert^x \rtimes L(\Delta_\rho[0,-1];\pi_{sc})$	$\frac{1}{2},\frac{3}{2},\frac{5}{2}$	$\alpha=0$
48'.	$L(\Delta_\rho[-\frac{1}{2},-\frac{1}{2}],\Delta_\rho[\frac{1}{2},\frac{1}{2}])\lvert\cdot\rvert^x \rtimes L(\Delta_\rho[0,-1];\pi_{sc})$	$\frac{1}{2},\frac{3}{2},\frac{5}{2}$-	$\alpha=0$
49.	$\Delta_\rho[\frac{1}{2},-\frac{1}{2}]\lvert\cdot\rvert^x \rtimes L(\Delta_\rho[-1,-1];T^{\pm}_{V,2}(\pi_{sc}))$	$\frac{1}{2},\frac{3}{2},\frac{5}{2}$	$\alpha=0$
49'.	$L(\Delta_\rho[-\frac{1}{2},-\frac{1}{2}],\Delta_\rho[\frac{1}{2},\frac{1}{2}])\lvert\cdot\rvert^x \rtimes T^{1}_{I,1}(T^{\mp}_{V,2}(\pi_{sc}))$	$\frac{1}{2},\frac{3}{2},\frac{5}{2}$	$\alpha=0$
50.	$\Delta_\rho[\frac{1}{2},-\frac{1}{2}]\lvert\cdot\rvert^x \rtimes T^{1}_{I,1}(T^{\mp}_{V,2}(\pi_{sc}))$	$\frac{1}{2},\frac{3}{2},\frac{5}{2}$	$\alpha=0$
50'.	$L(\Delta_\rho[-\frac{1}{2},-\frac{1}{2}],\Delta_\rho[\frac{1}{2},\frac{1}{2}])\lvert\cdot\rvert^x \rtimes L(\Delta_\rho[-1,-1];T^{\pm}_{V,2}(\pi_{sc}))$	$\frac{1}{2},\frac{3}{2},\frac{5}{2}$	$\alpha=0$

Remark 9.2. *In general, when computing subquotients of a fully induced representation of the form*

$$u_\rho(a,b)\lvert\cdot\rvert^x \rtimes \pi_A,$$

where $\pi_A \in \Pi_{A,gp}$ and x is a reducibility point of the parabolic induction, the difficulty arises when there are more than 1 Steinberg segment in the L-data of $u_\rho(a,b)$. When there is exactly 1 segment, the induced representation is of the form

$$\begin{aligned}\Delta_\rho[x_1,y_1]\lvert\cdot\rvert^x \rtimes \pi_u &= \Delta_\rho[x_1+x,y_1+x] \rtimes \pi_u\\ &= \rho\lvert\cdot\rvert^{x_1+x}\times\ldots\times\rho\lvert\cdot\rvert^{y_1+x}\rtimes\pi_u\\ &= \rho\lvert\cdot\rvert^{x_1+x}\rtimes\ldots\rtimes\rho\lvert\cdot\rvert^{y_1+x}\rtimes\pi_u.\end{aligned}$$

All subquotients of the induced representation above have a fixed supercuspidal support. By induction, one can compute all subquotients of the above representation if one can compute all subquotients of $\rho\lvert\cdot\rvert^{x'}\rtimes\pi_u$ for a given point x'. This can be done using the process illustrated in Sections 3 to 7, and identifying those representations with the given supercuspidal support. After that, by Theorem 2.5, one can easily verify if a given subquotient is unitarizable by verifying if it is of Arthur type.

When there are more than 1 segments which are linked, however, this method is insufficient as the supercuspidal support of the possible subquotients may not be unique. Such cases must be dealt with individually.

Table 2. Non-unitarizable irreducible subquotients at the reducibility point x_0

N°	π	x_0	Cases
1.	$L(\Delta_\rho[-\alpha-3,-\alpha-3],T^{\alpha+2}_{I,1}(T^{\alpha+1}_{I,1}(T^{\alpha}_{I,1}(\pi_{sc}))))$	$\alpha+3$	$\alpha\geq\frac{1}{2}$

N°	π	x_0	Cases
2.	$L(\Delta_\rho[-\alpha-1,-\alpha-1],\Delta_\rho[-\alpha,-\alpha],\Delta_\rho[-\alpha+1,-\alpha];\pi_{sc})$	α	$\alpha>1$
3.	$L(\Delta_\rho[-\alpha+1,-\alpha];T_{I,1}^{\alpha+1}(T_{I,1}^{\alpha}(\pi_{sc})))$	α	$\alpha>1$
4.	$L(\Delta_\rho[-\alpha-1,-\alpha-1];T_{I,1}^{\alpha-2}(T_{I,1}^{\alpha-1}(T_{I,1}^{\alpha}(\pi_{sc}))))$	$\alpha+1$	$\alpha\geq 2$
5.	$L(\Delta_\rho[-\alpha-2,-\alpha-1];T_{I,1}^{\alpha-1}(T_{I,1}^{\alpha}(\pi_{sc})))$	$\alpha-1$	$\alpha\geq 2$
6.	$L(\Delta_\rho[-\alpha+1,-\alpha+1],\Delta_\rho[-\alpha+1,-\alpha+1],\Delta_\rho[-\alpha+2,-\alpha+2];T_{I,1}^{\alpha}(\pi_{sc}))$	$\alpha-1$	$\alpha\geq 2$
6.	$L(\Delta_\rho[-\alpha-1,-\alpha-1],\Delta_\rho[-\alpha+1,-\alpha+1],\Delta_\rho[-\alpha+2,-\alpha+2];T_{I,1}^{\alpha}(\pi_{sc}))$	$\alpha+1$	$\alpha\geq 2$
7.	$L(\Delta_\rho[-\alpha+2,-\alpha];T_{I,1}^{\alpha}(\pi_{sc}))$	α	$\alpha\geq 2$
8.	$L(\Delta_\rho[-2,-2];T_{V,4}^{\pm}(T_{I,1}^{1}(\pi_{sc}))$	0	$\alpha=1$
8.	$L(\Delta_\rho[-2,-2],\Delta_\rho[-2,-2],\Delta_\rho[-1,-1];T_{IV,3}(\pi_{sc})$	3	$\alpha=1$
9.	$L(\Delta_\rho[-2,-2];T_{V,4}^{\pm}(T_{I,1}^{1}(\pi_{sc}))$	0	$\alpha=1$
9.	$L(\Delta_\rho[-2,-3],\Delta_\rho[0,-1];\pi_{sc})$	3	$\alpha=1$
10.	$L(\Delta_\rho[-1,-1],\Delta_\rho[0,-2];\pi_{sc})$	2	$\alpha=1$
11.	$L(\Delta_\rho[-1,-2],\Delta_\rho[-1,-1];T_{IV,3}(\pi_{sc}))$	2	$\alpha=1$
13.	$L(\Delta_\rho[-1,-2];T_{I,1}^{1}(T_{IV,3}(\pi_{sc}))$	2	$\alpha=1$
14.	$L(\Delta_\rho[-1,-2];T_{I,1}^{1}(T_{IV,3}(\pi_{sc}))$	2	$\alpha=1$
15.	$L(\Delta_\rho[0,-2];T_{IV,3}(\pi_{sc}))$	2	$\alpha=1$
16.	$L(\Delta_\rho[-\frac{3}{2},-\frac{3}{2}],\Delta_\rho[-\frac{1}{2},-\frac{3}{2}],\Delta_\rho[-\frac{1}{2},-\frac{1}{2}];\pi_{sc})$	$\frac{3}{2}$	$\alpha=\frac{1}{2}$
18.	$L(\Delta_\rho[-\frac{3}{2},-\frac{5}{2}],\Delta_\rho[-\frac{1}{2},-\frac{1}{2}];T_{I,1}^{\frac{1}{2}}(\pi_{sc}))$	$\frac{5}{2}$	$\alpha=\frac{1}{2}$
19.	$L(\Delta_\rho[-\frac{1}{2},-\frac{3}{2}];T_{I,1}^{\frac{3}{2}}(T_{I,1}^{\frac{1}{2}}(\pi_{sc})))$	$\frac{3}{2}$	$\alpha=\frac{1}{2}$
20.	$L(\Delta_\rho[-\frac{3}{2},-\frac{3}{2}];T_{I,3}^{\frac{1}{2}}(\pi_{sc}))$	$\frac{3}{2}$	$\alpha=\frac{1}{2}$
23.	$L(\Delta_\rho[-2,-3],\Delta_\rho[-1,-1];T_{V,2}^{+}(\pi_{sc}))$	3	$\alpha=0$
24.	$L(\Delta_\rho[-2,-3],\Delta_\rho[-1,-1];T_{V,2}^{-}(\pi_{sc}))$	3	$\alpha=0$
27.	$L(\Delta_\rho[-1,-2];T_{V,4}^{\pm}(\pi_{sc}))$	2	$\alpha=0$
28.	$L(\Delta_\rho[-2,-2],\Delta_\rho[0,-1];T_{V,2}^{\pm}(\pi_{sc}))$	2	$\alpha=0$
30.	$L(\Delta_\rho[-\alpha-1,-\alpha],\Delta_\rho[-\lvert\alpha-1\rvert,-\lvert\alpha-1\rvert];T_{I,1}^{\alpha}(\pi_{sc}))$	α	$\alpha\neq 0$
31.	$L(\Delta_\rho[-\alpha+1,-\alpha-1];T_{I,1}^{\alpha}(\pi_{sc}))$	α	$\alpha>1$
31.	$L(\Delta_\rho[-\alpha,-\alpha-2];T_{I,1}^{\alpha}(\pi_{sc}))$	$\alpha+1$	$\alpha\neq 0$
32.	$L(\Delta_\rho[-\alpha,-\alpha-2],\Delta_\rho[-\alpha,-\alpha];\pi_{sc})$	$\alpha-1$	$\alpha\geq\frac{3}{2}$
32.	$L(\Delta_\rho[-\alpha+1,-\alpha-1],\Delta_\rho[-\alpha,-\alpha];\pi_{sc})$	α	$\alpha\geq 1$
32.	$L(\Delta_\rho[-\frac{1}{2},-\frac{5}{2}],\Delta_\rho[-\frac{1}{2},-\frac{1}{2}];\pi_{sc})$	$\frac{3}{2}$	$\alpha=\frac{1}{2}$
33.	$L(\Delta_\rho[-1,-3];T_{V,2}^{\pm}(\pi_{sc}))$	2	$\alpha=0$
34.	$L(\Delta_\rho[-\alpha-1,-\alpha-1],\Delta_\rho[-\alpha,-\alpha],\Delta_\rho[-\alpha+1,-\alpha];\pi_{sc})$	$\alpha-\frac{1}{2}$	$\alpha\geq\frac{3}{2}$
34.	$L(\Delta_\rho[-\alpha-2,-\alpha-3],\Delta_\rho[-\alpha-1,-\alpha-1],\Delta_\rho[-\alpha,-\alpha];\pi_{sc})$	$\alpha+\frac{5}{2}$	$\alpha=\frac{1}{2},1$
35.	$L(\Delta_\rho[-\alpha+1,-\alpha];T_{I,1}^{\alpha+1}(T_{I,1}^{\alpha}(\pi_{sc})))$	$\alpha-\frac{1}{2}$	$\alpha\geq 1$
35.	$L(\Delta_\rho[-\alpha-1,-\alpha-2];T_{I,1}^{\alpha+1}(T_{I,1}^{\alpha}(\pi_{sc})))$	$\alpha+\frac{3}{2}$	$\alpha\geq\frac{1}{2}$
36.	$L(\Delta_\rho[-\alpha-2,-\alpha-2],\Delta_\rho[-\alpha+1,-\alpha];T_{I,1}^{\alpha}(\pi_{sc}))$	$\alpha-\frac{3}{2}$	$\alpha\geq 2$

N°	π	x_0	Cases
36.	$L(\Delta_\rho[-\alpha-1,-\alpha-1],\Delta_\rho[-\alpha+1,-\alpha];T^{\alpha}_{I,1}(\pi_{sc}))$	$\alpha+\frac{1}{2}$	$\alpha\geq\frac{3}{2}$
37.	$L(\Delta_\rho[-\alpha,-\alpha-1];T^{\alpha-1}_{I,1}(T^{\alpha}_{I,1}(\pi_{sc})))$	$\alpha+\frac{1}{2}$	$\alpha\geq\frac{3}{2}$
38.	$L(\Delta_\rho[-\alpha+2,-\alpha+1],\Delta_\rho-\alpha+1,-\alpha+1];T^{\alpha}_{I,1}(\pi_{sc}))$	$\alpha-\frac{3}{2}$	$\alpha\geq 2$
38.	$L(\Delta_\rho[-\alpha+1,-\alpha+1],\Delta_\rho[-\alpha,-\alpha-1];T^{\alpha}_{I,1}(\pi_{sc}))$	$\alpha+\frac{1}{2}$	$\alpha\geq\frac{3}{2}$
39.	$L(\Delta_\rho[-\alpha+1,-\alpha+1],\Delta_\rho[-\alpha+1,-\alpha],\Delta_\rho[-\alpha,-\alpha];\pi_{sc})$	$\alpha-\frac{1}{2}$	$\alpha\geq\frac{3}{2}$
39.	$L(\Delta_\rho[-\alpha+1,-\alpha+1],\Delta_\rho[-\alpha,-\alpha],\Delta_\rho[-\alpha,-\alpha-1];\pi_{sc})$	$\alpha+\frac{1}{2}$	$\alpha\geq\frac{3}{2}$
40.	$L(\Delta_\rho[-1,-1],\Delta_\rho[-1,-2];T_{IV,3}(\pi_{sc}))$	$\frac{3}{2}$	$\alpha=1$
41.	$L(\Delta_\rho[-1,-2];T^{+}_{V,2}(T^{1}_{I,1}(\pi_{sc})))$	$\frac{3}{2}$	$\alpha=1$
42.	$L(\Delta_\rho[0,-1],\Delta_\rho[-1,-2];\pi_{sc})$	$\frac{3}{2}$	$\alpha=1$
43.	$L(\Delta_\rho[-1,-2];T^{-}_{V,2}(T^{1}_{I,1}(\pi_{sc})))$	$\frac{3}{2}$	$\alpha=1$
44.	$L(\Delta_\rho[-\frac{3}{2},-\frac{5}{2}],\Delta_\rho[-\frac{1}{2},-\frac{1}{2}],\Delta_\rho[-\frac{1}{2},-\frac{1}{2}];\pi_{sc})$	2	$\alpha=\frac{1}{2}$
45.	$L(\Delta_\rho[-\frac{3}{2},-\frac{5}{2}];T^{\frac{1}{2}}_{I,2}(\pi_{sc}))$	2	$\alpha=\frac{1}{2}$
46.	$L(\Delta_\rho[-\frac{1}{2},-\frac{1}{2}],\Delta_\rho[-\frac{3}{2},-\frac{5}{2}];T^{\frac{1}{2}}_{I,1}(\pi_{sc}))$	2	$\alpha=\frac{1}{2}$
47.	$L(\Delta_\rho[-\frac{3}{2},-\frac{5}{2}];T^{\frac{1}{2}}_{III,2}(\pi_{sc})$	2	$\alpha=\frac{1}{2}$
48.	$L(\Delta_\rho[0,-1],\Delta_\rho[-1,-2];\pi_{sc})$	$\frac{3}{2}$	$\alpha=0$
49.	$L(\Delta_\rho[-1,-1],\Delta_\rho[-1,-2];T^{\pm}_{V,2}(\pi_{sc}))$	$\frac{3}{2}$	$\alpha=0$
50.	$L(\Delta_\rho[-1,-2];T^{1}_{I,1}(T^{\mp}_{V,2}(\pi_{sc})))$	$\frac{3}{2}$	$\alpha=0$

Proposition 9.1 and the two tables above summarize all possible unitarizable one-parameter complementary series emanated from a unitarizable representation. To finish up our classification of one-parameter complementary series, we now consider all other families that are not of the form $u_\rho(a,b)|\cdot|^x\rtimes\pi_A$, where $\pi_A\in\Pi_{A,gp}$. To begin, we consider all other families induced from a supercuspidal representation.

Proposition 9.3. *For all $x\in\mathbb{R}$, let*

$$\pi_{1,x}=L(\Delta_\rho[0,0],\Delta_\rho[3,1])|\cdot|^x\rtimes\pi_{sc},$$

$$\pi_{2,x}=L(\Delta_\rho[1,0],\Delta_\rho[3,2])|\cdot|^x\rtimes\pi_{sc},$$

$$\pi_{3,x}=L(\Delta_\rho[2,0],\Delta_\rho[3,3])|\cdot|^x\rtimes\pi_{sc}.$$

Then for $i=1,2,3$, $\pi_{i,x}$ is reducible if and only if $x\in\{\pm(\alpha-3),\pm(\alpha-2),\pm(\alpha-1),\pm\alpha\}$. When $\pi_{i,x}$ is irreducible, $\pi_{i,x}$ is unitarizable if and only if $-\alpha<x<\alpha-3$.

Proof. First let $i=1$. The reducibility statement follows from Theorem 2.18. By Proposition B.1, for $\alpha>2$, the representation $L(\Delta_\rho[-\alpha+1,-\alpha-1],\Delta_\rho[-\alpha+2,-\alpha+2];\pi_{sc})$ is a non-unitarizable subquotient of $\pi_{1,\alpha-2}$ and

$$L(\Delta_\rho[-\alpha,-\alpha-2],\Delta_\rho[-\alpha+1,-\alpha+1];\pi_{sc})$$

is a non-unitarizable subquotient of $\pi_{1,\alpha-1}$.

This proves that there is no complementary series when $x>\alpha-3$. By Proposition A.1 and B.1, all subquotients of $\Pi_{|\alpha-3|,\alpha-2,\alpha-1,\alpha}$ are of Arthur type and hence unitarizable for

$\alpha > 2$. Similarly, one can show that all subquotients of $\Pi_{x,x+1,x+2,x+3}$ are unitarizable, when $x = -\alpha, -(\alpha-1), -(\alpha-2), -(\alpha-3)$. This proves the statement for $\alpha > 2$.

For $\alpha = 2$, reducibility occurs at 0 so there is no unitary complementary series for $x \geq 0$. When $x < 0$, the region $-2 < x < -1$ is unitarizable since when $x = -1$, all subquotients of $\Pi_{(-1,0,1,2)} \cong \Pi_{(0,1,1,2)}$ are unitarizable by Proposition B.1.

For $\alpha = 0, 1$, reducibility occurs at 0 so there is no complementary series beyond 0. When $\alpha = 1$,

$$L(\Delta_\rho[-1,-1], \Delta_\rho[0,-2]; \pi_{sc})$$

is a non-unitarizable subquotient of $\pi_{1,-1}$. When $\alpha = 0$,

$$L(\Delta_\rho[-3,-3], \Delta_\rho[0,-2]; \pi_{sc})$$

is a non-unitarizable subquotient of $\pi_{1,-3}$, and

$$L(\Delta_\rho[-1,-1], \Delta_\rho[0,-2]; \pi_{sc}$$

is a non-unitarizable subquotient of $\pi_{1,-1}$.

Now it remains to consider the cases $\alpha = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}$. When $\alpha = \frac{1}{2}$, the representation

$$L(\Delta_\rho[-\frac{1}{2}, -\frac{5}{2}], \Delta_\rho[-\frac{1}{2}, -\frac{1}{2}]; \pi_{sc})$$

is a non-unitarizable subquotient of $\pi_{1,\frac{3}{2}}$ and $\pi_{1,-\frac{1}{2}}$. and

$$L(\Delta_\rho[-\frac{3}{2}, -\frac{3}{2}], \Delta_\rho[\frac{1}{2}, -\frac{3}{2}]; \pi_{sc})$$

is a non-unitarizable subquotient of $\pi_{1,\frac{1}{2}}$ and $\pi_{1,-\frac{3}{2}}$.

For $\alpha = \frac{3}{2}$,

$$L(\Delta_\rho[-\frac{1}{2}, -\frac{5}{2}], \Delta_\rho[-\frac{1}{2}, -\frac{1}{2}]; \pi_{sc})$$

is a non-unitarizable subquotient of $\pi_{1,-\frac{1}{2}}$ and

$$L(\Delta_\rho[-\frac{3}{2}, -\frac{7}{2}], \Delta_\rho[-\frac{1}{2}, -\frac{1}{2}]; \pi_{sc})$$

is a non-unitarizable subquotient of $\pi_{1,\frac{3}{2}}$.

For $\alpha = \frac{5}{2}$, the representation

$$L(\Delta_\rho[-\frac{5}{2}, -\frac{9}{2}], \Delta_\rho[-\frac{3}{2}, -\frac{3}{2}]; \pi_{sc})$$

is a non-unitarizable subquotient of $\pi_{1,\frac{3}{2}}$; the representation

$$L(\Delta_\rho[-\frac{3}{2}, -\frac{7}{2}], \Delta_\rho[-\frac{1}{2}, -\frac{1}{2}]; \pi_{sc})$$

is a non-unitarizable subquotient of $\pi_{1,\frac{1}{2}}$. Same as before, all subquotients of $\Pi_{x,x+1,x+2,x+3}$ are unitarizable for $x = -\frac{5}{2}, -\frac{3}{2}, -\frac{1}{2}$, by Proposition B.1 and A.1. This proves the claim for $i = 1$. The proof for $i = 2, 3$ are similar. $\square$

Proposition 9.4. *For $x \in \mathbb{R}$, let*

$$\pi_{1,x} = L(\Delta_\rho[0,0], \Delta_\rho[1,1], \Delta_\rho[3,2])|\cdot|^x \rtimes \pi_{sc},$$
$$\pi_{2,x} = L(\Delta_\rho[0,0], \Delta_\rho[2,1], \Delta_\rho[3,3])|\cdot|^x \rtimes \pi_{sc},$$
$$\pi_{3,x} = L(\Delta_\rho[1,0], \Delta_\rho[2,2], \Delta_\rho[3,3])|\cdot|^x \rtimes \pi_{sc}.$$

Then for $i = 1, 2, 3$, $\pi_{i,x}$ is reducible if and only if $x \in \{\pm(\alpha-2), \pm(\alpha-1), \pm\alpha, \pm(\alpha+1)\}$. When $\pi_{i,x}$ is irreducible, $\pi_{i,x}$ is unitarizable if and only if $-\alpha \leq x < \alpha - 3$.

Proof. The proof is similar to Proposition 9.3, which we omit. □

Proposition 9.5. *For $x \geq 0$, let*

$$\pi_{1,x} = L(\Delta_\rho[0,0], \Delta_\rho[1,0], \Delta_\rho[2,2])|\cdot|^x \rtimes \pi_{sc},$$
$$\pi_{2,x} = L(\Delta_\rho[0,0], \Delta_\rho[2,0])|\cdot|^x \rtimes \pi_{sc}.$$
$$\pi_{3,x} = L(\Delta_\rho[0,0], \Delta_\rho[0,0], \Delta_\rho[2,1])|\cdot|^x \rtimes \pi_{sc}.$$

Then for $i = 1, 2, 3$, $\pi_{i,x}$ are irreducible when $x \in \{|\alpha-2|, |\alpha-1|, \alpha\}$. When irreducible, $\pi_{1,x}, \pi_{2,x}$ are non-unitarizable for any x.

Proof. The reducibility statement follows from Theorem 2.18. Let $i = 1$. For all α, the representation

$$L(\Delta_\rho[-\alpha-4, -\alpha-4], \Delta_\rho[-\alpha-2, -\alpha-3], \Delta_\rho[-\alpha-2, -\alpha-2]; \pi_{sc})$$

is a non-unitarizable subquotient of $\pi_{1,|\alpha-2|}$, and

$$L(\Delta_\rho[-\alpha-3, -\alpha-3], \Delta_\rho[-\alpha-1, -\alpha-2], \Delta_\rho[-\alpha-1, -\alpha-1]; \pi_{sc})$$

is a non-unitarizable subquotient of $\pi_{1,|\alpha-1|}$. This proves the claim for $\pi_{1,x}$. Similarly, we have that for all α, the representation

$$L(\Delta_\rho[-\alpha-2, -\alpha-4], \Delta_\rho[-\alpha-2, -\alpha-2]; \pi_{sc})$$

is a non-unitarizable subquotient of $\pi_{2,|\alpha-1|}$ and

$$L(\Delta_\rho[-\alpha, -\alpha-2], \Delta_\rho[-\alpha, -\alpha]; \pi_{sc})$$

is a non-unitarizable subquotient of $\pi_{2,\alpha}$. This proves the claim for $i = 1$. The proof is similar for $i = 2$. □

Proposition 9.6. *For $x \geq 0$, let*

$$\pi_{1,x} = L(\Delta_\rho[0,0], \Delta_\rho[2,1], \Delta_\rho[2,2])|\cdot|^x \rtimes \pi_{sc},$$
$$\pi_{2,x} = L(\Delta_\rho[2,0], \Delta_\rho[2,2]) \rtimes \pi_{sc}.$$
$$\pi_{3,x} = L(\Delta_\rho[1,0], \Delta_\rho[2,2], \Delta_\rho[2,2]) \rtimes \pi_{sc},$$

Then $\pi_{1,x}$ and $\pi_{2,x}$ are irreducible when $x \in \{|\alpha-2|, |\alpha-1|, \alpha\}$. When irreducible, $\pi_{1,x}, \pi_{2,x}$ are non-unitarizable for any x.

Proof. The proof is similar to Proposition 9.5, which we omit. □

Proposition 9.7. *For $x \geq 0$, let*

$$\pi_{1,x} = L(\Delta_\rho[1,0], \Delta_\rho[1,1], \Delta_\rho[2,2])|\cdot|^x \rtimes \pi_{sc},$$
$$\pi_{2,x} = L(\Delta_\rho[2,0], \Delta_\rho[1,1])|\cdot|^x \rtimes \pi_{sc},$$
$$\pi_{3,x} = L(\Delta_\rho[0,0], \Delta_\rho[1,1], \Delta_\rho[2,1])|\cdot|^x \rtimes \pi_{sc}.$$

Then for $i = 1, 2, 3$, $\pi_{i,x}$ are irreducible if and only if $x \in \{|\alpha-2|, |\alpha-1|, \alpha\}$, and when irreducible, $\pi_{i,x}$ are non-unitarizable for any x.

Proof. The proof is similar to Proposition 9.5, which we omit. □

Proposition 9.8. *For $\alpha > 0$, let*
$$\pi = T^{\alpha}_{I,1}(\pi_{sc}),$$
$$\hat{\pi} = L(\Delta_\rho[-\alpha, -\alpha]; \pi_{sc}).$$
For $x \geq 0$, consider the complementary series
$$\tau_x = L(\Delta_\rho[x, x], \Delta_\rho[x+2, x+1]) \rtimes \pi,$$
$$\hat{\tau}_x = L(\Delta_\rho[x, x], \Delta_\rho[x+2, x+1]) \rtimes \hat{\pi}.$$
Then τ_x and $\hat{\tau}_x$ are not unitarizable for any $x \geq 0$.

Proof. By Aubert duality it suffices to show it for τ_x. By Proposition 5.21, τ_x always contain a non-unitarizable subquotient, unless $\epsilon_\rho \leq x \leq \alpha - 4$. Therefore it suffices to show that for $\epsilon_\rho \leq x \leq \alpha - 4$, τ_x is irreducible.

Fix $\epsilon_\rho \leq x \leq \alpha - 4$, and let
$$X_1 = \{\pm\alpha\},$$
$$X_2 = \{\pm x, \pm(x+1), \pm(x+2)\}.$$
Then $X_1 \sqcup X_2$ forms a regular partition of the set
$$X = \{\pm\alpha, \pm x, \pm(x+1), \pm(x+2)\}.$$
Let
$$\beta_1 = \rho|\cdot|^{\alpha}, \quad \beta_2 = L(\Delta_\rho[x, x], \Delta_\rho[x+2, x+1]),$$
$$\gamma_1 = T^{\alpha}_{I,1}(\pi_{sc}), \quad \gamma_2 = L(\Delta_\rho[-x-1, -x-2], \Delta_\rho[-x, -x]; \pi_{sc}),$$
where 0_F denotes the trivial representation of $\mathrm{GL}_0(F)$. Then by Theorem 2.20, we have that τ_x is irreducible if and only if both $\beta_1 \rtimes \gamma_2$ and $\beta_2 \rtimes \gamma_1$ are irreducible. For $\epsilon + 1 \leq x \leq \alpha + 1$, both representations above are irreducible by Theorem 2.18, so we are done. $\square$

Proposition 9.9. *For $\alpha > 0$, let*
$$\pi = T^{\alpha}_{I,1}(\pi_{sc}),$$
$$\hat{\pi} = L(\Delta_\rho[-\alpha, -\alpha]; \pi_{sc}).$$
For $x \geq 0$, consider the complementary series
$$\tau_x = L(\Delta_\rho[x+1, x], \Delta_\rho[x+2, x+2]) \rtimes \pi,$$
$$\hat{\tau}_x = L(\Delta_\rho[x+1, x], \Delta_\rho[x+2, x+2]) \rtimes \hat{\pi}.$$
Then τ_x and $\hat{\tau}_x$ are not unitarizable for any $x \geq 0$.

Proof. By Aubert duality, we only need to prove it for τ_x a Let $\alpha > 2$, then the representation
$$L(\Delta_\rho[-\alpha+2, -\alpha]; T^{\alpha}_{I,1}(\pi_{sc}))$$
is a non-unitarizable subquotient of $\tau_{1,\alpha-2}$ (this is proved using the same method as in the proof of Proposition 9.1).
$$L(\Delta_\rho[-\alpha, -\alpha], \Delta_\rho[-\alpha, -\alpha], \Delta_\rho[-\alpha+1, -\alpha+1]; T^{\alpha}_{I,1}(\pi_{sc}))$$
is a non-unitarizable subquotient of $\tau_{1,\alpha-1}$, and
$$L(\Delta_\rho[-\alpha-1, -\alpha-1], \Delta_\rho[-\alpha, -\alpha], \Delta_\rho[-\alpha, -\alpha]; T^{\alpha}_{I,1}(\pi_{sc}))$$
is a non-unitarizable subquotient of $\tau_{1,\alpha}$. For $\alpha > 2$, this shows that there is no unitarizable complementary series.

For $\alpha = 0, 1, 2$ we get reducibility at 0 so there is no complementary series to consider. Now let $\alpha = \frac{1}{2}$, using the same method as before, one can show that the representation

$$L(\Delta_\rho[-\frac{3}{2}, -\frac{3}{2}]; T_{I,3}^{\frac{1}{2}}(\pi_{sc}))$$

is a non-unitarizable subquotient of $\tau_{\frac{1}{2}}$. Let $\alpha = \frac{3}{2}$. The representation

$$L(\Delta_\rho[-\frac{3}{2}, -\frac{3}{2}], \Delta_\rho[-\frac{3}{2}, -\frac{3}{2}], \Delta_\rho[-\frac{1}{2}, -\frac{1}{2}]; T_{I,1}^{\frac{3}{2}}(\pi_{sc}))$$

is a non-unitarizable subquotient of $\tau_{\frac{1}{2}}$, and

$$L(\Delta_\rho[-\frac{5}{2}, -\frac{5}{2}], \Delta_\rho[-\frac{3}{2}, -\frac{3}{2}], \Delta_\rho[-\frac{3}{2}, -\frac{3}{2}]; T_{I,1}^{\frac{3}{2}}(\pi_{sc}))$$

is a non-unitarizable subquotient of $\tau_{\frac{3}{2}}$. This proves the claim. □

Proposition 9.10. *For $\alpha = 0$, let*

$$\pi^{\pm} = T_{V,2}^{\pm}(\pi_{sc}),$$

and let

$$\tau_{1,x}^{\pm} = L(\Delta_\rho[x, x], \Delta_\rho[x+2, x+1]) \rtimes \pi^{\pm},$$
$$\tau_{2,x}^{\pm} = L(\Delta_\rho[x+1, x], \Delta_\rho[x+2, x+2]) \rtimes \pi^{\pm}.$$

Then, $\tau_{1,x}^{\pm}$ and $\tau_{2,x}^{\pm}$ are not unitarizable for any $x \geq 0$.

Proof. This follows directly from Propositions 5.23 and 5.26. □

Finally, we will consider the one-parameter families induced from non-unitarizable representations.

Proposition 9.11. *Let*

$$\pi = L(\Delta_\rho[-\alpha, -\alpha]; T_{I,1}^{\alpha}(\pi_{sc})),$$
$$\hat{\pi} = L(\Delta_\rho[-\alpha, -\alpha], \Delta_\rho[-\alpha, -\alpha]; \pi_{sc}).$$

For $x \geq 0$, consider the complementary series

$$\tau_x = \Delta_\rho[x, x+1] \rtimes \pi,$$
$$\hat{\tau}_x = \Delta_\rho[x, x+1] \rtimes \hat{\pi}.$$

Then $\tau_{i,x}$ and $\hat{\tau}_{i,x}$ are reducible if and only if $x \in \{|\alpha - 2|, |\alpha - 1|, \alpha, \alpha + 1\}$. When they are irreducible, τ_x and $\hat{\tau}_x$ are always non-unitarizable.

Proof. The proof is identical to Proposition 9.8, which we omit. □

Proposition 9.12. *Let*

$$\pi = L(\Delta_\rho[-\alpha - 1, -\alpha - 1]; T_{I,1}^{\alpha}(\pi_{sc})),$$
$$\hat{\pi} = L(\Delta_\rho[-\alpha, -\alpha - 1]; \pi_{sc}).$$

For $x \geq 0$, consider the complementary series

$$\tau_x = \Delta_\rho[x, x+1] \rtimes \pi,$$
$$\hat{\tau}_x = \Delta_\rho[x, x+1] \rtimes \hat{\pi}.$$

Then $\tau_{i,x}$ and $\hat{\tau}_{i,x}$ are reducible if and only if $x \in \{|\alpha - 2|, |\alpha - 1|, \alpha, \alpha + 1, \alpha + 2\}$. When they are irreducible, τ_x and $\hat{\tau}_x$ are always non-unitarizable.

Proof. The proof is similar to Proposition 9.11, which we omit. □

The previous two propositions resolve the case of a complementary series induced from a non-unitarizable representation of corank 2. To conclude this subsection, we turn our attention to complementary series induced from a non-unitarizable representation of corank 3.

Proposition 9.13. *Let*

$$\pi_x = \rho|\cdot|^x \rtimes \pi_{nu},$$

where π_{nu} is non-unitarizable representation of critical type and of corank 3. Then π_x is non-unitarizable when it is irreducible.

Proof. The proof is similar to Proposition 9.1. The relevant data are summarized in Table 9.13 and Table 4. □

TABLE 3. Reducibility data for one parameter complementary series induced from non-unitarizable representation

N°	π_x	Reducibility points	Cases
1.	$\rho\lvert\cdot\rvert^x \rtimes L(\Delta_\rho[-\alpha-2,-\alpha-2],\Delta_\rho[-\alpha-1,-\alpha-1];T_{I,1}^{\alpha}(\pi_{sc}))$	$\lvert\alpha-1\rvert,\alpha,\alpha+3$	$\alpha\geq\frac{1}{2}$
1'.	$\rho\lvert\cdot\rvert^x \rtimes L(\Delta_\rho[-\alpha,-\alpha-2];\pi_{sc})$	$\lvert\alpha-1\rvert,\alpha,\alpha+3$	$\alpha\geq\frac{1}{2}$
2.	$\rho\lvert\cdot\rvert^x \rtimes L(\Delta_\rho[-\alpha-2,-\alpha-2],\Delta_\rho[-\alpha,-\alpha-1];\pi_{sc})$	$\lvert\alpha-1\rvert,\alpha,\alpha+1,\alpha+3$	$\alpha\geq\frac{1}{2}$
2'.	$\rho\lvert\cdot\rvert^x \rtimes L(\Delta_\rho[-\alpha-1,-\alpha-2],T_{I,1}^{\alpha}(\pi_{sc}))$	$\lvert\alpha-1\rvert,\alpha,\alpha+1,\alpha+3$	$\alpha\geq\frac{1}{2}$
3.	$\rho\lvert\cdot\rvert^x \rtimes L(\Delta_\rho[-\alpha-2,-\alpha-2];T_{I,1}^{\alpha+1}(T_{I,1}^{\alpha}(\pi_{sc}))))$	$\lvert\alpha-1\rvert,\alpha+1,\alpha+3$	$\alpha\geq\frac{1}{2}$
3'.	$\rho\lvert\cdot\rvert^x \rtimes L(\Delta_\rho[-\alpha-1,-\alpha-2],\Delta_\rho[-\alpha,-\alpha];\pi_{sc})$	$\lvert\alpha-1\rvert,\alpha+1,\alpha+3$	$\alpha\geq\frac{1}{2}$
4.	$\rho\lvert\cdot\rvert^x \rtimes L(\Delta_\rho[-\alpha-1,-\alpha-1],\Delta_\rho[-\alpha-1,-\alpha-1],\Delta_\rho[-\alpha,-\alpha];\pi_{sc})$	$\lvert\alpha-1\rvert,\alpha,\alpha+2$	$\alpha\geq\frac{1}{2}$
4'.	$\rho\lvert\cdot\rvert^x \rtimes L(\Delta_\rho[-\alpha-1,-\alpha-1],T_{I,1}^{\alpha+1}(T_{I,1}^{\alpha}(\pi_{sc})))$	$\lvert\alpha-1\rvert,\alpha,\alpha+2$	$\alpha\geq\frac{1}{2}$
5.	$\rho\lvert\cdot\rvert^x \rtimes L(\Delta_\rho[-\alpha-1,-\alpha-1],\Delta_\rho[-\alpha-1,-\alpha-1];T_{I,1}^{\alpha}(\pi_{sc}))$	$\lvert\alpha-1\rvert,\alpha,\alpha+2$	$\alpha\geq\frac{1}{2}$
5'.	$\rho\lvert\cdot\rvert^x \rtimes L(\Delta_\rho[-\alpha-1,-\alpha-1],\Delta_\rho[-\alpha,-\alpha-1];\pi_{sc})$	$\lvert\alpha-1\rvert,\alpha,\alpha+2$	$\alpha\geq\frac{1}{2}$
6.	$\rho\lvert\cdot\rvert^x \rtimes L(\Delta_\rho[-\alpha-1,-\alpha-1],\Delta_\rho[-\alpha,-\alpha],\Delta_\rho[-\alpha,-\alpha];\pi_{sc})$	$\alpha-1,\alpha+1,\alpha+2$	$\alpha\geq1$
6'.	$\rho\lvert\cdot\rvert^x \rtimes L(\Delta_\rho[-\alpha,-\alpha];T_{I,1}^{\alpha+1}(T_{I,1}^{\alpha}(\pi_{sc})))$	$\alpha-1,\alpha+1,\alpha+2$	$\alpha\geq1$
7.	$\rho\lvert\cdot\rvert^x \rtimes L(\Delta_\rho[-\alpha-1,-\alpha-1],\Delta_\rho[-\alpha,-\alpha];T_{I,1}^{\alpha}(\pi_{sc}))$	$\alpha-1,\alpha+1,\alpha+2$	$\alpha\geq1$
7'.	$\rho\lvert\cdot\rvert^x \rtimes L(\Delta_\rho[-\alpha,-\alpha-1],\Delta_\rho[-\alpha,-\alpha];\pi_{sc})$	$\alpha-1,\alpha+1,\alpha+2$	$\alpha\geq1$
8.	$\rho\lvert\cdot\rvert^x \rtimes L(\Delta_\rho[-\alpha,-\alpha-1];T_{I,1}^{\alpha}(\pi_{sc}))$	$\alpha-1,\alpha+2$	$\alpha\geq1$
9.	$\rho\lvert\cdot\rvert^x \rtimes L(\Delta_\rho[-\alpha,-\alpha],\Delta_\rho[-\alpha,-\alpha],\Delta_\rho[-\alpha,-\alpha];\pi_{sc})$	$\alpha-1,\alpha+1$	$\alpha\geq1$
9'.	$\rho\lvert\cdot\rvert^x \rtimes L(\Delta_\rho[-\alpha,-\alpha],\Delta_\rho[-\alpha,-\alpha];T_{I,1}^{\alpha}(\pi_{sc}))$	$\alpha-1,\alpha+1$	$\alpha\geq1$
10.	$\rho\lvert\cdot\rvert^x \rtimes L(\Delta_\rho[-\alpha-1,-\alpha-1],\Delta_\rho[-\alpha+1,-\alpha+1];T_{I,1}^{\alpha}(\pi_{sc}))$	$\lvert\alpha-2\rvert,\alpha,\alpha+2$	$\alpha>1$
10'.	$\rho\lvert\cdot\rvert^x \rtimes L(\Delta_\rho[-\alpha,-\alpha-1],\Delta_\rho[-\alpha+1,-\alpha+1];\pi_{sc})$	$\lvert\alpha-2\rvert,\alpha,\alpha+2$	$\alpha>1$
11.	$\rho\lvert\cdot\rvert^x \rtimes L(\Delta_\rho[-\alpha-1,-\alpha-1],T_{I,1}^{\alpha-1}(T_{I,1}^{\alpha}(\pi_{sc})))$	$\lvert\alpha-2\rvert,\alpha,\alpha+2$	$\alpha>1$
11'.	$\rho\lvert\cdot\rvert^x \rtimes L(\Delta_\rho[-\alpha+1,-\alpha-1];\pi_{sc})$	$\lvert\alpha-2\rvert,\alpha,\alpha+2$	$\alpha>1$
12.	$\rho\lvert\cdot\rvert^x \rtimes L(\Delta_\rho[-\alpha,-\alpha],\Delta_\rho[-\alpha,-\alpha],\Delta_\rho[-\alpha+1,-\alpha+1];\pi_{sc})$	$\lvert\alpha-2\rvert,\alpha-1,\alpha+1$	$\alpha>1$
12'.	$\rho\lvert\cdot\rvert^x \rtimes L(\Delta_\rho[-\alpha+1,-\alpha];T_{I,1}^{\alpha}(\pi_{sc}))$	$\lvert\alpha-2\rvert,\alpha-1,\alpha+1$	$\alpha>1$
13.	$\rho\lvert\cdot\rvert^x \rtimes L(\Delta_\rho[-\alpha,-\alpha],\Delta_\rho[-\alpha+1,-\alpha];\pi_{sc})$	$\lvert\alpha-2\rvert,\alpha-1,\alpha+1$	$\alpha>1$
13'.	$\rho\lvert\cdot\rvert^x \rtimes L(\Delta_\rho[-\alpha,-\alpha];T_{I,1}^{\alpha-1}(T_{I,1}^{\alpha}(\pi_{sc})))$	$\lvert\alpha-2\rvert,\alpha-1,\alpha+1$	$\alpha>1$

N°	π_x	Reducibility points	Cases
14.	$\rho\lvert\cdot\rvert^x \rtimes L(\Delta_\rho[-\alpha,-\alpha],\Delta_\rho[-\alpha+1,-\alpha+1],\Delta_\rho[-\alpha+1,-\alpha+1];\pi_{sc})$	$\lvert\alpha-2\rvert,\alpha,\alpha+1$	$\alpha>1$
14'.	$\rho\lvert\cdot\rvert^x \rtimes L(\Delta_\rho[-\alpha+1,-\alpha+1],\Delta_\rho[-\alpha+1,-\alpha+1];T^{\alpha}_{I,1}(\pi_{sc}))$	$\lvert\alpha-2\rvert,\alpha,\alpha+1$	$\alpha>1$
15.	$\rho\lvert\cdot\rvert^x \rtimes L(\Delta_\rho[-\alpha+1,-\alpha],\Delta_\rho[-\alpha+1,-\alpha+1];\pi_{sc})$	$\lvert\alpha-2\rvert,\alpha,\alpha+1$	$\alpha>1$
15'.	$\rho\lvert\cdot\rvert^x \rtimes L(\Delta_\rho[-\alpha+1,-\alpha+1],T^{\alpha-1}_{I,1}(T^{\alpha}_{I,1}(\pi_{sc})))$	$\lvert\alpha-2\rvert,\alpha,\alpha+1$	$\alpha>1$
16.	$\rho\lvert\cdot\rvert^x \rtimes L(\Delta_\rho[-2,-2];T^{\pm}_{V,2}(T^{1}_{I,1}(\pi_{sc})))$	$0,1,3$	$\alpha=1$
17.	$\rho\lvert\cdot\rvert^x \rtimes L(\Delta_\rho[-1,-2];T_{IV,3}(\pi_{sc}))$	$0,1,3$	$\alpha=1$
18.	$\rho\lvert\cdot\rvert^x \rtimes L(\Delta_\rho[0,-2];\pi_{sc})$	$0,1,3$	$\alpha=1$
19.	$\rho\lvert\cdot\rvert^x \rtimes L(\Delta_\rho[-\frac{3}{2},-\frac{3}{2}];T^{\frac{1}{2}}_{I,2}(\pi_{sc}))$	$\frac{1}{2},\frac{3}{2},\frac{5}{2}$	$\alpha=\frac{1}{2}$
19'.	$\rho\lvert\cdot\rvert^x \rtimes L(\Delta_\rho[-\frac{1}{2},-\frac{3}{2}],\Delta_\rho[-\frac{1}{2},-\frac{1}{2}];\pi_{sc})$	$\frac{1}{2},\frac{3}{2},\frac{5}{2}$	$\alpha=\frac{1}{2}$
20.	$\rho\lvert\cdot\rvert^x \rtimes L(\Delta_\rho[-2,-2];T^{1}_{I,1}(T^{\pm}_{V,2}(\pi_{sc})))$	$0,1,3$	$\alpha=0$
20'.	$\rho\lvert\cdot\rvert^x \rtimes L(\Delta_\rho[-1,-2],T^{\mp}_{V,2}(\pi_{sc}))$	$0,1,3$	$\alpha=0$
21.	$\rho\lvert\cdot\rvert^x \rtimes L(\Delta_\rho[-2,-2],\Delta_\rho[0,-1];\pi_{sc})$	$0,1,3$	$\alpha=0$
21.'	$\rho\lvert\cdot\rvert^x \rtimes L(\Delta_\rho[0,-2],\pi_{sc})$	$0,1,3$	$\alpha=0$
22.	$\rho\lvert\cdot\rvert^x \rtimes L(\Delta_\rho[-1,-1],\Delta_\rho[0,-1];\pi_{sc})$	$0,2$	$\alpha=0$

TABLE 4. Non-unitarizable irreducible subquotients at the reducibility point x_1

N°	π	x_1	Cases
1.	$L(\Delta_\rho[-\alpha-2,-\alpha-2],\Delta_\rho[-\alpha,-\alpha],\Delta_\rho[-\alpha+1,-\alpha+1];T^{\alpha}_{I,1}(\pi_{sc}))$	$\lvert\alpha-1\rvert$	$\alpha\geq\frac{1}{2}$
1.	$L(\Delta_\rho[-\alpha-2,-\alpha-2],\Delta_\rho[-\alpha,-\alpha-1];T^{\alpha}_{I,1}(\pi_{sc}))$	α	$\alpha\geq\frac{1}{2}$
2.	$L(\Delta_\rho[-\alpha-2,-\alpha-2],\Delta_\rho[-\alpha,-\alpha-1],\Delta_\rho[-\alpha+1,-\alpha+1];\pi_{sc})$	$\lvert\alpha-1\rvert$	$\alpha\geq\frac{1}{2}$
2.	$L(\Delta_\rho[-\alpha-2,-\alpha-2],\Delta_\rho[-\alpha-1,-\alpha-1],\Delta_\rho[-\alpha,-\alpha-1];\pi_{sc})$	$\alpha+1$	$\alpha\geq\frac{1}{2}$
3.	$L(\Delta_\rho[-\alpha-2,-\alpha-2],\Delta_\rho[-\alpha+1,-\alpha+1],T^{\alpha+1}_{I,1}(T^{\alpha}_{I,1}(\pi_{sc})))$	$\lvert\alpha-1\rvert$	$\alpha\geq\frac{1}{2}$
3.	$L(\Delta_\rho[-\alpha-2,-\alpha-2],\Delta_\rho[-\alpha-1,-\alpha-1],T^{\alpha+1}_{I,1}(T^{\alpha}_{I,1}(\pi_{sc})))$	$\alpha+1$	$\alpha\geq\frac{1}{2}$
4.	$L(\Delta_\rho[-\alpha-2,-\alpha-2],\Delta_\rho[-\alpha-1,-\alpha-1],\Delta_\rho[-\alpha,-\alpha],\Delta_\rho[-\alpha+1,-\alpha+1];\pi_{sc})$	$\lvert\alpha-1\rvert$	$\alpha\geq\frac{1}{2}$
4.	$L(\Delta_\rho[-\alpha-2,-\alpha-2],\Delta_\rho[-\alpha-1,-\alpha-1],\Delta_\rho[-\alpha,-\alpha],\Delta_\rho[-\alpha,-\alpha];\pi_{sc})$	α	$\alpha\geq\frac{1}{2}$
5.	$L(\Delta_\rho[-\alpha-1,-\alpha-1],\Delta_\rho[-\alpha-1,-\alpha-1],\Delta_\rho[-\alpha+1,-\alpha+1];T^{\alpha}_{I,1}(\pi_{sc}))$	$\lvert\alpha-1\rvert$	$\alpha\geq\frac{1}{2}$
5.	$L(\Delta_\rho[-\alpha-1,-\alpha-1],\Delta_\rho[-\alpha-1,-\alpha-1],\Delta_\rho[-\alpha,-\alpha];T^{\alpha}_{I,1}(\pi_{sc}))$	α	$\alpha\geq\frac{1}{2}$
6.	$L(\Delta_\rho[-\alpha-1,-\alpha-1],\Delta_\rho[-\alpha,-\alpha],\Delta_\rho[-\alpha,-\alpha],\Delta_\rho[-\alpha+1,-\alpha+1];\pi_{sc}))$	$\lvert\alpha-1\rvert$	$\alpha\geq1$
6.	$L(\Delta_\rho[-\alpha-1,-\alpha-1],\Delta_\rho[-\alpha-1,-\alpha-1],\Delta_\rho[-\alpha,-\alpha],\Delta_\rho[-\alpha,-\alpha];\pi_{sc}))$	$\alpha+1$	$\alpha\geq1$
7.	$L(\Delta_\rho[-\alpha-1,-\alpha-1],\Delta_\rho[-\alpha,-\alpha];T^{\alpha-1}_{I,1}(T^{\alpha}_{I,1}(\pi_{sc})))$	$\alpha-1$	$\alpha\geq1$
7.	$L(\Delta_\rho[-\alpha-1,-\alpha-1],\Delta_\rho[-\alpha-1,-\alpha-1],\Delta_\rho[-\alpha,-\alpha];T^{\alpha}_{I,1}(\pi_{sc}))$	$\alpha+1$	$\alpha\geq1$
8.	$L(\Delta_\rho[-\alpha,-\alpha-1],\Delta_\rho[-\alpha+1,-\alpha+1];T^{\alpha}_{I,1}(\pi_{sc}))$	$\alpha-1$	$\alpha\geq1$
9.	$L(\Delta_\rho[-\alpha,-\alpha],\Delta_\rho[-\alpha,-\alpha],\Delta_\rho[-\alpha,-\alpha]\Delta_\rho[-\alpha+1,-\alpha+1];\pi_{sc})$	$\alpha-1$	$\alpha\geq1$
10.	$L(\Delta_\rho[-\alpha-1,-\alpha-1],\Delta_\rho[-\alpha+1,-\alpha+1],\Delta_\rho[-\alpha+2,-\alpha+2];T^{\alpha}_{I,1}(\pi_{sc}))$	$\lvert\alpha-2\rvert$	$\alpha>1$
10.	$L(\Delta_\rho[-\alpha-1,-\alpha-1],\Delta_\rho[-\alpha+1,-\alpha];T^{\alpha}_{I,1}(\pi_{sc}))$	α	$\alpha>1$

N°	π	x_1	Cases
11.	$L(\Delta_\rho[-\alpha-1,-\alpha-1],\Delta_\rho[-\alpha+2,-\alpha+2];T_{I,1}^{\alpha-1}(T_{I,1}^{\alpha}(\pi_{sc})))$	$\lvert\alpha-2\rvert$	$\alpha>1$
11.	$L(\Delta_\rho[-\alpha-1,-\alpha-1],\Delta_\rho[-\alpha,-\alpha];T_{I,1}^{\alpha-1}(T_{I,1}^{\alpha}(\pi_{sc})))$	α	$\alpha>1$
12.	$L(\Delta_\rho[-\alpha,-\alpha],\Delta_\rho[-\alpha,-\alpha],\Delta_\rho[-\alpha+1,-\alpha+1],\Delta_\rho[-\alpha+2,-\alpha+2];\pi_{sc})$	$\lvert\alpha-2\rvert$	$\alpha>1$
12.	$L(\Delta_\rho[-\alpha,-\alpha],\Delta_\rho[-\alpha,-\alpha],\Delta_\rho[-\alpha+1,-\alpha+1],\Delta_\rho[-\alpha+1,-\alpha+1];\pi_{sc})$	$\alpha-1$	$\alpha>1$
13.	$L(\Delta_\rho[-\alpha,-\alpha],\Delta_\rho[-\alpha+1,-\alpha],\Delta_\rho[-\alpha+2,-\alpha+2];\pi_{sc})$	$\lvert\alpha-2\rvert$	$\alpha>1$
13.	$L(\Delta_\rho[-\alpha,-\alpha],\Delta_\rho[-\alpha+1,-\alpha],\Delta_\rho[-\alpha+1,-\alpha+1];\pi_{sc})$	$\alpha-1$	$\alpha>1$
14.	$L(\Delta_\rho[-\alpha,-\alpha],\Delta_\rho[-\alpha+1,-\alpha+1],\Delta_\rho[-\alpha+1,-\alpha+1],\Delta_\rho[-\alpha+2,-\alpha+2];\pi_{sc})$	$\lvert\alpha-2\rvert$	$\alpha>1$
14.	$L(\Delta_\rho[-\alpha,-\alpha],\Delta_\rho[-\alpha,-\alpha],\Delta_\rho[-\alpha+1,-\alpha+1],\Delta_\rho[-\alpha+1,-\alpha+1];\pi_{sc})$	α	$\alpha>1$
15.	$L(\Delta_\rho[-\alpha+1,-\alpha],\Delta_\rho[-\alpha+1,-\alpha+1],\Delta_\rho[-\alpha+2,-\alpha+2];\pi_{sc})$	$\lvert\alpha-2\rvert$	$\alpha>1$
15.	$L(\Delta_\rho[-\alpha,-\alpha],\Delta_\rho[-\alpha+1,-\alpha],\Delta_\rho[-\alpha+1,-\alpha+1];\pi_{sc})$	α	$\alpha>1$
16.	$L(\Delta_\rho[-2,-2];T_{I,1}^{2}(T_{V,2}^{\pm}(\pi_{sc})))$	1	$\alpha=1$
17.	$L(\Delta_\rho[-1,-2],\Delta_\rho[-1,-1];T_{IV,3}(\pi_{sc}))$	1	$\alpha=1$
18.	$L(\Delta_\rho[-1,-1],\Delta_\rho[0,-2];\pi_{sc})$	1	$\alpha=1$
19.	$L(\Delta_\rho[-\frac{3}{2},-\frac{3}{2}];T_{I,3}^{\frac{1}{2}}(\pi_{sc}))$	$\frac{1}{2}$	$\alpha=\frac{1}{2}$
19.	$L(\Delta_\rho[-\frac{3}{2},-\frac{3}{2}],\Delta_\rho[-\frac{3}{2},-\frac{3}{2}];T_{I,2}^{\frac{1}{2}}(\pi_{sc}))$	$\frac{3}{2}$	$\alpha=\frac{1}{2}$
20.	$L(\Delta_\rho[-2,-2];T_{I,2}^{1}(T_{V,2}^{\pm}(\pi_{sc})))$	1	$\alpha=0$
21.	$L(\Delta_\rho[-2,-2],\Delta_\rho[-1,-1],\Delta_\rho[0,-1];\pi_{sc})$	1	$\alpha=0$

Now it remains to do the same for two-parameter complementary series. For this, we'll need the exhaustive list of all critical type, unitarizable, irreducible representations of corank 2. This can be found in [Tad23].

10. Two-parameter complementary series

In this section, we will classify all possible two-parameter complementary series that contains a continuous family of irreducible, unitarizable representations of corank 4. To begin our classification of two-parameter complementary series, we consider complementary series of the form

$$\Pi_{(x,y)}=\rho|\cdot|^x\times\rho|\cdot|^y\rtimes\pi_u,$$

where π_u is unitarizable of corank 2.

Proposition 10.1. *Let $\alpha\geq\frac{1}{2}$, and let*

$$\pi=L(\Delta_\rho[-\alpha-1,-\alpha-1],\Delta_\rho[-\alpha,-\alpha];\pi_{sc}),\quad \hat{\pi}=T_{I,1}^{\alpha+1}(T_{I,1}^{\alpha}(\pi_{sc})).$$

Then for $0\leq x\leq y$, the complementary series

$$\Pi_{(x,y)}=\rho|\cdot|^x\times\rho|\cdot|^y\rtimes\pi,$$

and

$$\hat{\Pi}_{(x,y)}=\rho|\cdot|^x\times\rho|\cdot|^y\rtimes\hat{\pi}$$

is unitarizable in the following regions:

$$x+y<1,\quad y<\alpha-1,\quad (\alpha>\frac{3}{2})$$

$$x + 1 < y < \alpha - 1, \quad (\alpha > 2)$$
$$y < \frac{1}{2} \quad (\alpha = \frac{1}{2}).$$

Proof. The complementary series $\rho|\cdot|^x \rtimes \pi$ and $\rho|\cdot|^x \rtimes \hat{\pi}$ are reducible at $|\alpha - 1|$ and $\alpha + 2$. The other reducibility hyperplanes are

$$x + y = 1, \quad |x - y| = 1.$$

By Aubert duality we only need to consider the complementary series $\Pi_{(x,y)}$. The Figures 1, 2, 3, 4, 5 show the unitarizability of the two complementary series in the region $\mathbb{R}^2_+ = \{(x, y) \in \mathbb{R}^2 : x, y \geq 0\}$ for the cases $\alpha = 3, \alpha = 2, \alpha = \frac{3}{2}, \alpha = 1, \alpha = \frac{1}{2}$ respectively.

By repeatedly applying the decomposition algorithm to the critical points, we are able to classify them as either strongly unitary, strongly non-unitary, or neither (denoted as black, white, and light gray balls respectively). This proves that list of unitary connected components we gave is exhaustive for $\alpha > 2$.

If a connected component contains a non-strongly unitary point on its boundary, then it's non-unitary. To prove our claim, it suffices to show that the region C_1 in Figure 1 is unitary and the region C_2 in Figure 5 is non-unitary.

First we prove that the region C_1 is unitary. Consider the line $L : \{(x, y) \in C_1 : x = 0\}$, which is nonempty and one-dimensional. By applying step (3-2) of Algorithm 8.1, one can see that the region C_1 is unitary if and only if any point on L is strongly unitary. This reduces the problem to the unitarizability of the one-dimensional complementary series

$$\rho|\cdot|^y \rtimes \pi.$$

By [Tad23, Proposition 8.1], this is unitarizable for $1 \leq y < \alpha - 1$. The conclusion follows.

Similarly, when $\alpha = \frac{1}{2}$, the slanted side of C_2 is contains either the complementary series

$$\Delta_\rho[\frac{1}{2}, -\frac{1}{2}]|\cdot|^x \rtimes \pi,$$

or

$$L(\Delta_\rho[\frac{1}{2}, -\frac{1}{2}], \Delta_\rho[-\frac{1}{2}, \frac{1}{2}])|\cdot|^x \rtimes \pi,$$

for $0 < x < \frac{1}{2}$. By Proposition 9.1, this is non-unitarizable. This proves that C_2 is non-unitary.

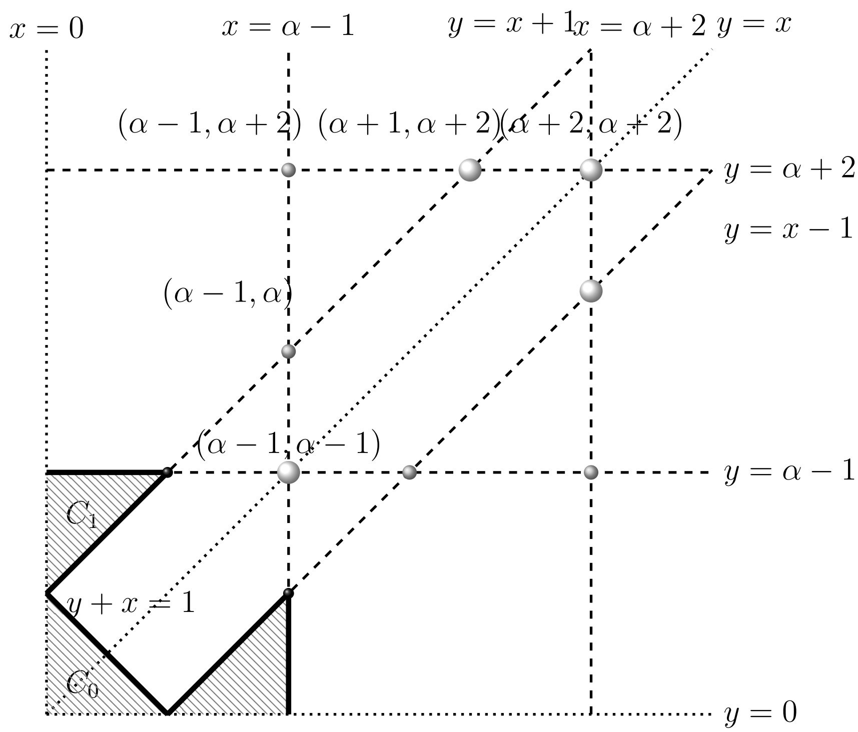

FIGURE 1. Unitarizability for $\Pi_{(x,y)}$ (case $\alpha = 3$)

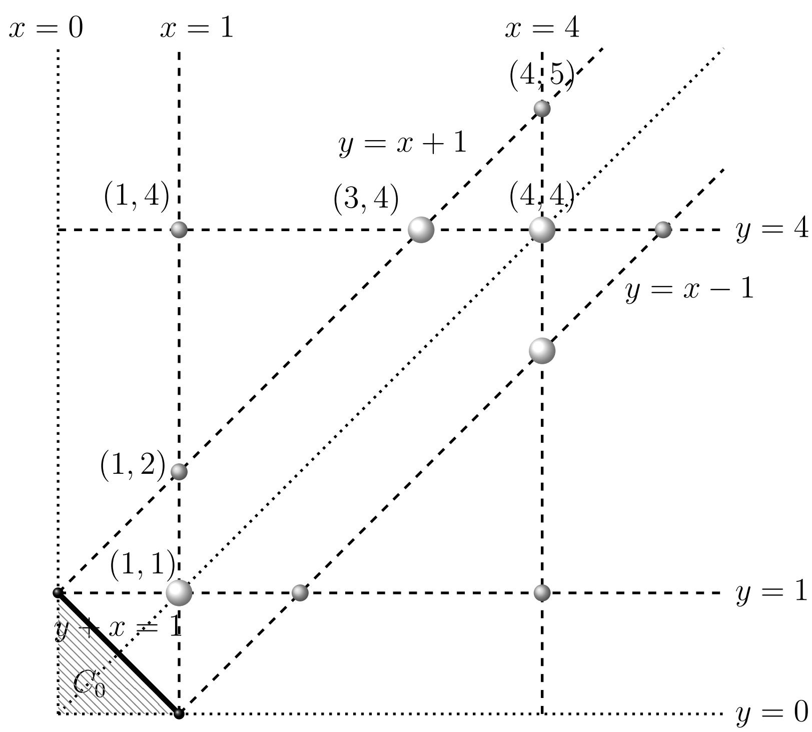

FIGURE 2. Unitarizability for $\Pi_{(x,y)}$ (case $\alpha = 2$)

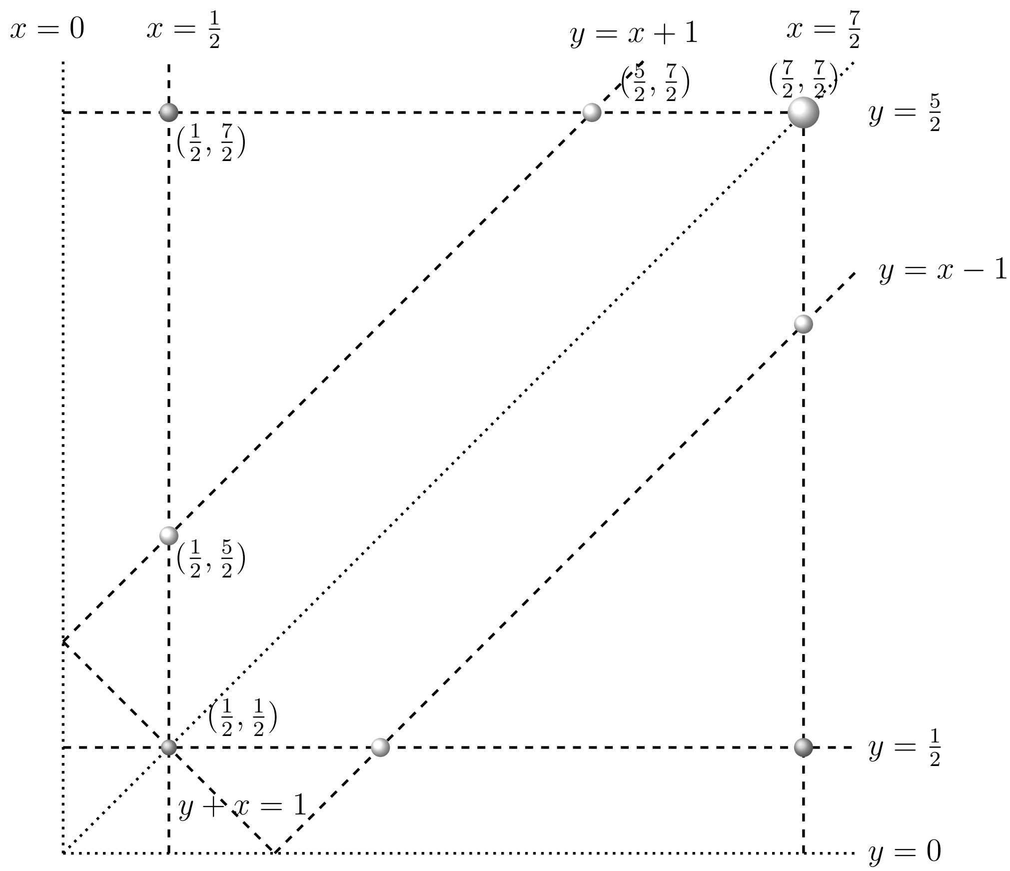

FIGURE 3. Unitarizability for $\Pi_{(x,y)}$ (case $\alpha=\frac{3}{2}$)

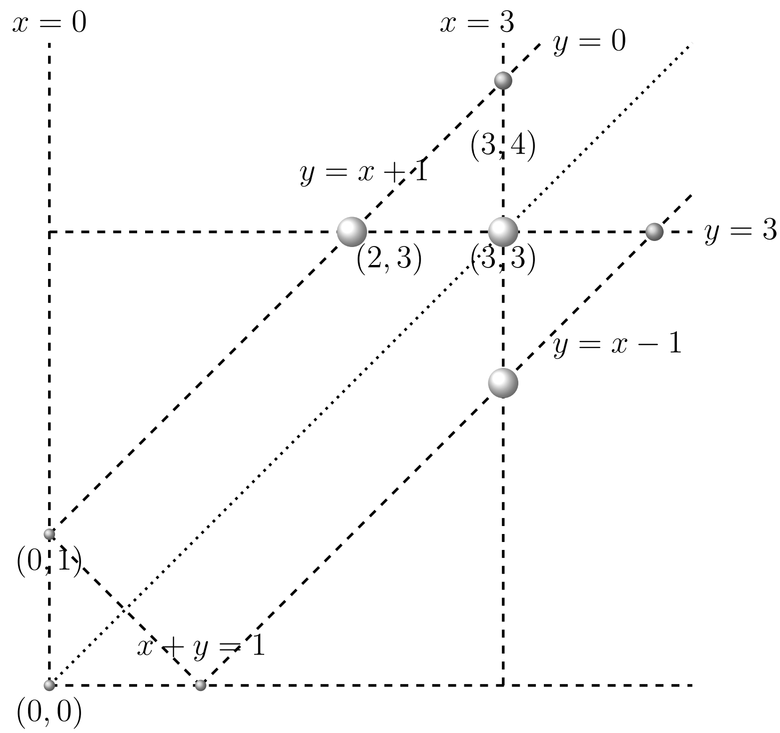

FIGURE 4. Unitarizability for $\Pi_{(x,y)}$ (case $\alpha = 1$)

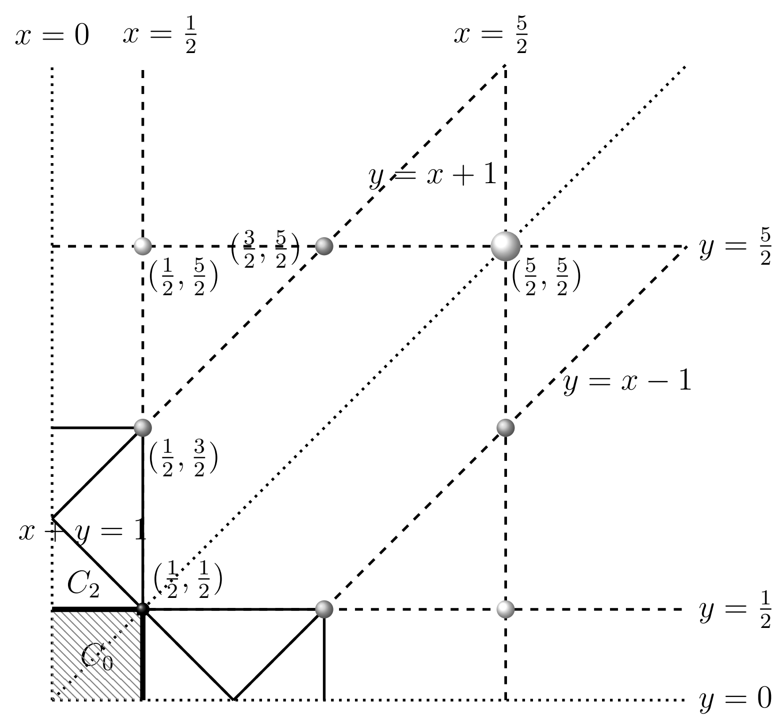

FIGURE 5. Unitarizability for $\Pi_{(x,y)}$ (case $\alpha = \frac{1}{2}$)

$\square$

Proposition 10.2. *Let* $\alpha \geq \frac{3}{2}$*, and let*

$$\pi = L(\Delta_\rho[-\alpha+1,-\alpha];\pi_{sc}), \quad \hat{\pi} = T_{I,1}^{\alpha-1}(T_{I,1}^{\alpha}(\pi_{sc})).$$

Then for $0 \leq x \leq y$*, the complementary series*

$$\Pi_{(x,y)} = \rho|\cdot|^x \times \rho|\cdot|^y \rtimes \pi,$$

and

$$\hat{\Pi}_{(x,y)} = \rho|\cdot|^x \times \rho|\cdot|^y \rtimes \hat{\pi}$$

is unitary in the following regions:

$$x+y<1, \quad (\alpha \geq \frac{3}{2}).$$

Proof. The proof is similar to Proposition 10.1, which we omit. □

Proposition 10.3. *Let* $\alpha \geq \frac{3}{2}$*, and let*

$$\pi = L(\Delta_\rho[-\alpha+1,-\alpha+1];T_{I,1}^{\alpha}(\pi_{sc})), \quad \hat{\pi} = L(\Delta_\rho[-\alpha+1,-\alpha+1],\Delta_\rho[-\alpha,-\alpha];\pi_{sc})$$

Then for $0 \leq x \leq y$*, the complementary series*

$$\Pi_{(x,y)} = \rho|\cdot|^x \times \rho|\cdot|^y \rtimes \pi,$$

and

$$\hat{\Pi}_{(x,y)} = \rho|\cdot|^x \times \rho|\cdot|^y \rtimes \hat{\pi},$$

is unitary in the following regions:

$$y<\frac{1}{2}, \quad (\alpha = \frac{3}{2})$$
$$x+y<1, \quad (\alpha > 2).$$

Proof. The proof is similar to Proposition 10.1, which we omit. Note that when $\alpha = 2$, reducibility occurs at the origin so there is no complementary series. □

Proposition 10.4. *Let* $\alpha = \frac{1}{2}$*, and let*

$$\pi = L(\Delta_\rho[-\frac{1}{2},-\frac{1}{2}],\Delta_\rho[-\frac{1}{2},-\frac{1}{2}];\pi_{sc}), \quad \hat{\pi} = T_{I,2}^{\frac{1}{2}}(\pi_{sc}).$$

Then for $0 \leq x \leq y$*, the complementary series*

$$\Pi_{(x,y)} = \rho|\cdot|^x \times \rho|\cdot|^y \rtimes \pi,$$

and

$$\hat{\Pi}_{(x,y)} = \rho|\cdot|^x \times \rho|\cdot|^y \rtimes \hat{\pi},$$

is unitary in the following regions:

$$y<\frac{1}{2}.$$

Proof. The proof is similar to Proposition 10.1, which we omit. □

Proposition 10.5. *Let $\alpha = \frac{1}{2}$, and let*
$$\pi = L(\Delta_\rho[-\frac{1}{2}, -\frac{1}{2}], T_{I,1}^{\frac{1}{2}}(\pi_{sc})), \quad \hat{\pi} = T_{III,2}^{\frac{1}{2}}(\pi_{sc})).$$
Then for $0 \leq x \leq y$, the complementary series
$$\Pi_{(x,y)} = \rho|\cdot|^x \times \rho|\cdot|^y \rtimes \pi,$$
and
$$\hat{\Pi}_{(x,y)} = \rho|\cdot|^x \times \rho|\cdot|^y \rtimes \hat{\pi}$$
is unitary in the following regions:
$$y < \frac{1}{2}.$$

Proof. The proof is similar to Proposition 10.4, which we omit. □

Proposition 10.6. *Let $\alpha = 0$, and let*
$$\pi^{\pm} = L(\Delta_\rho[-1, -1]; T_{V,2}^{\pm}(\pi_{sc})), \quad \hat{\pi}^{\pm} = T_{I,1}^{1}(T_{V,2}^{\mp}(\pi_{sc})).$$
Then for $0 \leq x \leq y$, the complementary series
$$\Pi_{(x,y)}^{\pm} = \rho|\cdot|^x \times \rho|\cdot|^y \rtimes \pi^{\pm}$$
and
$$\hat{\Pi}_{(x,y)}^{\pm} = \rho|\cdot|^x \times \rho|\cdot|^y \rtimes \hat{\pi}^{\pm}$$
is unitary in the following regions:
$$x + y < 1.$$

Proof. The proof is similar to Proposition 10.5, which we omit. □

Proposition 10.7. *Let $\alpha = 0$, and let*
$$\pi^{\pm} = L(\Delta_\rho[0, -1]; \pi_{sc}).$$
Then for $0 \leq x \leq y$, the complementary series
$$\Pi_{(x,y)} = \rho|\cdot|^x \times \rho|\cdot|^y \rtimes \pi$$
is unitary in the following regions:
$$x + y < 1.$$

Proof. The proof is similar to Proposition 10.6, which we omit. □

Propositions 10.1 to 10.7 gives all possible two-parameter complementary series of the form $\rho|\cdot|^x \times \rho|\cdot|^y \rtimes \pi_u$, where π_u is unitarizable of corank 2. Now we turn our attention to complementary series of the form
$$u_\rho(a_1, b_1)|\cdot|^x \times u_\rho(a_2, b_2)|\cdot|^y \rtimes \pi_{sc}.$$

Proposition 10.8. *Let*
$$\Pi_{(x,y)} = \Delta_\rho[\frac{1}{2}, -\frac{1}{2}]|\cdot|^x \times \Delta_\rho[\frac{1}{2}, -\frac{1}{2}]|\cdot|^y \rtimes \pi_{sc},$$
and
$$\hat{\Pi}_{(x,y)} = L(\Delta_\rho[-\frac{1}{2}, -\frac{1}{2}], \Delta_\rho[\frac{1}{2}, \frac{1}{2}])|\cdot|^x \times L(\Delta_\rho[-\frac{1}{2}, -\frac{1}{2}], \Delta_\rho[\frac{1}{2}, \frac{1}{2}])|\cdot|^y \rtimes \pi_{sc}.$$

Then for $0 \leq x \leq y$, $\Pi_{(x,y)}$ *and* $\hat{\Pi}_{(x,y)}$ *are unitarizable in the following regions:*

$$y - x > 2, \quad y < \alpha - \frac{1}{2}, \quad (\alpha \geq 3)$$
$$1 < y - x < 2, \quad y < \alpha - \frac{1}{2}, \quad (\alpha \geq \frac{5}{2})$$
$$x + y < 1, \quad (\alpha \geq \frac{3}{2})$$
$$y < \frac{1}{2}, \quad (\alpha = 0, 1).$$

Proof. By Aubert duality, it suffices to show this for $\Pi_{(x,y)}$. First note that in the region $0 \leq x \leq y$, the complementary series

$$\Delta_\rho[\frac{1}{2}, -\frac{1}{2}]|\cdot|^x \times \Delta_\rho[\frac{1}{2}, -\frac{1}{2}]|\cdot|^y$$

is reducible precisely when $|x - y| = 1, 2$ or $x + y = 1$. Let $\pi_x = \Delta_\rho[\frac{1}{2}, -\frac{1}{2}]|\cdot|^x \rtimes \pi_{sc}$, then π_x is reducible when $x = |\alpha - \frac{1}{2}|, \alpha + \frac{1}{2}$. When irreducible, the one-dimensional complementary series π_x is unitarizable if and only if $0 \leq x < |\alpha - \frac{1}{2}|$. By unitary induction, one can easily show that the regions above are unitary, by applying Step 3-2 of Algorithm 8.1. To show that the remaining regions are not unitary, we can simply consider the subquotient

$$L(\Delta_\rho[-\alpha + 1, -\alpha], \Delta_\rho[-\alpha + 1, -\alpha]; \pi_{sc}),$$

which is non-unitarizable for $\alpha \geq \frac{3}{2}$. Since

$$\begin{aligned} L(\Delta_\rho[-\alpha + 1, -\alpha], \Delta_\rho[-\alpha + 1, -\alpha]; \pi_{sc}) &\leq S^{(2)}_{\rho|\cdot|^{\alpha-1}}(S^{(2)}_{\rho|\cdot|^{\alpha}}(D^{(2)}_{\rho|\cdot|^{\alpha}}(D^{(2)}_{\rho|\cdot|^{\alpha-1}})(\Delta_\rho[\alpha, \alpha - 1] \\ &\quad \times \Delta_\rho[\alpha, \alpha - 1] \rtimes \pi_{sc}))) \\ &\cong \Pi_{(\alpha-1,\alpha-1)}, \end{aligned}$$

the conclusion follows. For $\alpha = \frac{1}{2}$ no regions are unitary since reducibility occurs at 0. Finally for $\alpha = 0, 1$, one can use Proposition 9.1 to show that the region

$$x + y < 1, \quad y > \frac{1}{2}$$

is non-unitarizable (use case $N^\circ = 30$ in table 9.1). For $\alpha = 0$, This concludes the proof. □

Proposition 10.9. *Let*

$$\Pi_{(x,y)} = \Delta_\rho[\frac{1}{2}, -\frac{1}{2}]|\cdot|^x \times L(\Delta_\rho[-\frac{1}{2}, -\frac{1}{2}], \Delta_\rho[\frac{1}{2}, \frac{1}{2}])|\cdot|^y \rtimes \pi_{sc},$$

and

$$\hat{\Pi}_{(x,y)} = L(\Delta_\rho[-\frac{1}{2}, -\frac{1}{2}], \Delta_\rho[\frac{1}{2}, \frac{1}{2}])|\cdot|^x \times \Delta_\rho[\frac{1}{2}, -\frac{1}{2}]|\cdot|^y \rtimes \pi_{sc}.$$

Then for $0 \leq x \leq y$, $\Pi_{(x,y)}$ *and* $\hat{\Pi}_{(x,y)}$ *are unitarizable in the following regions:*

$$y - x > 2, \quad y < \alpha - \frac{1}{2}, \quad (\alpha \geq 3)$$
$$1 < y - x < 2, \quad y < \alpha - \frac{1}{2}, \quad (\alpha \geq \frac{5}{2})$$
$$x + y < 1, \quad (\alpha \geq \frac{3}{2})$$

$$y < \frac{1}{2}, \quad (\alpha = 0, 1).$$

Proof. The proof is similar to Proposition 10.9, which we omit. □

Proposition 10.10. *Let*

$$\Pi_{(x,y)} = L(\Delta_\rho[-1,-1], \Delta_\rho[0,0], \Delta_\rho[1,1])|\cdot|^x \times \rho|\cdot|^y \rtimes \pi_{sc},$$

and

$$\hat{\Pi}_{(x,y)} = \Delta_\rho[1,-1]|\cdot|^x \times \rho|\cdot|^y \rtimes \pi_{sc}.$$

Then for $x, y \geq 0$, $\Pi_{(x,y)}$ and $\hat{\Pi}_{(x,y)}$ are unitary in the following regions:

$$x - y > 2, \quad x < \alpha - 1, \quad (\alpha \geq \frac{7}{2})$$
$$x + y < 2, \quad x < \alpha - 1, \quad (\alpha \geq 2)$$
$$y - x > 2, \quad y < \alpha, \quad (\alpha \geq \frac{5}{2})$$
$$y < \frac{3}{2}, \quad x < \frac{1}{2}, \quad (\alpha = \frac{3}{2})$$
$$x, y < \frac{1}{2}, \quad (\alpha = \frac{1}{2})$$

Proof. Same as before, it suffices to consider $\Pi_{(x,y)}$. First, note that in this case, the reducibility lines in the region $x, y \geq 0$ are

$$|x - y| = 2, \quad x + y = 2, \quad x = |\alpha - 1|, \alpha, \alpha + 1, \quad y = \alpha.$$

In particular, the connected components are asymmetric in the two coordinates. In the one-parameter complementary series

$$L(\Delta_\rho[-1,-1], \Delta_\rho[0,0], \Delta_\rho[1,1])|\cdot|^x \rtimes \pi_{sc},$$

and

$$\rho|\cdot|^y \rtimes \pi_{sc}$$

respectively, there are no unitarizable points beyond the first reducibility point.

Thus, by applying step 3-2 of Algorithm 8.1, it is straightforward to show that the regions listed are unitary. To show that other regions are not unitarizable, it suffices to consider the subquotient

$$L(\Delta_\rho[-\alpha, -\alpha], \Delta_\rho[-\alpha + 2, -\alpha]; \pi_{sc}),$$

which is a non-unitarizable subquotient of $\Pi_{(\alpha-1,\alpha)}$ for $\alpha \geq \frac{3}{2}$. This proves the claim for $\alpha \geq \frac{3}{2}$.

When $\alpha = 1$ or $\alpha = 0$, reducibility occurs at $x = 0$, so there are no unitary regions. When $\alpha = \frac{1}{2}$, the first reducibility point in x and y are both $\frac{1}{2}$, and the component

$$x, y < \frac{1}{2}$$

contains the origin. This proves the claim. □

Propositions 10.8 to 10.10 characterizes all complementary series of the form

$$u_\rho(a_1, b_1)|\cdot|^x \times u_\rho(a_2, b_2)|\cdot|^y \rtimes \pi_{sc}.$$

Next we consider two-parameter families of the form

$$u_\rho(a_1,b_1)|\cdot|^x \times u_\rho(a_2,b_2)|\cdot|^y \rtimes \pi_A,$$

where π_A is unitarizable and of critical type. For the corank of the induced representation to be 4, the only possibility remaining is to have π_A to be of corank 1. We will summarize the results in the following two propositions.

Proposition 10.11. *For $x, y \geq 0$, let*

$$\Pi_{(x,y)} = \Delta_\rho[\frac{1}{2}, -\frac{1}{2}]|\cdot|^x \times \rho|\cdot|^y \rtimes T^{\alpha}_{I,1}(\pi_{sc}),$$

and

$$\hat{\Pi}_{(x,y)} = L(\Delta_\rho[-\frac{1}{2}, -\frac{1}{2}], \Delta_\rho[\frac{1}{2}, \frac{1}{2}])|\cdot|^x \times \rho|\cdot|^y \rtimes L(\Delta_\rho[-\alpha, -\alpha]; \pi_{sc}).$$

Then for $x, y \geq 0$, $\Pi_{(x,y)}$ and $\hat{\Pi}_{(x,y)}$ are unitary in the following regions:

$$x + y < \frac{3}{2}, \quad x < \alpha - \frac{3}{2}, \quad y < \alpha - 1, \quad (\alpha \geq 2)$$

$$y - x > 2, \quad y < \alpha - 1, \quad (\alpha \geq \frac{7}{2})$$

$$x - y > \frac{3}{2}, \quad x < \alpha - \frac{3}{2}, \quad (\alpha \geq \frac{7}{2})$$

$$x < 1, \quad y < \frac{1}{2}, \quad (\alpha = \frac{1}{2})$$

$$x < \frac{1}{2}, \quad y < 1, \quad (\alpha = 0)$$

Proof. In this case, the reducibility lines are

$$|x - y| = \frac{3}{2}, \quad x + y = \frac{3}{2}, \quad y = |\alpha - 1|, \alpha + 1, \quad x = |\alpha \pm \frac{3}{2}|, \alpha + \frac{1}{2}.$$

Similar to before, it suffices to consider the regions bounded by $y < |\alpha - 1|, x < \min(|\alpha - \frac{3}{2}|, \alpha + \frac{1}{2})$. The rest of the proof is similar to Proposition 10.10, which we omit. □

Proposition 10.12. *For $x, y \geq 0$, let*

$$\Pi_{(x,y)} = \Delta_\rho[\frac{1}{2}, -\frac{1}{2}]|\cdot|^x \times \rho|\cdot|^y \rtimes L(\Delta_\rho[-\alpha, -\alpha]; \pi_{sc}),$$

and

$$\hat{\Pi}_{(x,y)} = L(\Delta_\rho[-\frac{1}{2}, -\frac{1}{2}], \Delta_\rho[\frac{1}{2}, \frac{1}{2}])|\cdot|^x \times \rho|\cdot|^y \rtimes T^{\alpha}_{I,1}(\pi_{sc}).$$

Then for $x, y \geq 0$, $\Pi_{(x,y)}$ and $\hat{\Pi}_{(x,y)}$ are unitary in the following regions:

$$x + y < \frac{3}{2}, \quad x < |\alpha - \frac{3}{2}|, \quad y < |\alpha - 1|, \quad (\alpha \geq 2)$$

$$y - x > 2, \quad y < \alpha - 1, \quad (\alpha \geq 3)$$

$$x - y > \frac{3}{2}, \quad x < \alpha - \frac{3}{2}, \quad (\alpha \geq \frac{7}{2})$$

$$x < \frac{1}{2}, \quad y < 1, \quad (\alpha = 0)$$

Proof. The proof is similar to Proposition 10.11, which we omit. □

Finally, we conclude by showing that there are no unitary regions in the 2-dimensional complementary series induced from non-unitarizable representations.

Proposition 10.13. *Let*

$$\Pi_{(x,y)} = \rho|\cdot|^x \times \rho|\cdot|^y \rtimes \pi_{nu},$$

where π_{nu} is non-unitarizable, of critical type, and of corank 2. Then when $\Pi_{(x,y)}$ is irreducible, it is non-unitarizable.

Proof. This follows directly by considering the unitarizability of the one-parameter family

$$\rho|\cdot|^x \rtimes \pi_{nu},$$

when π_{nu} is non-unitarizable, of critical type, and of corank 2. It is proven by Tadic in [Tad23, Lemma 8.2] that such families are non-unitarizable whenever they are irreducible. When $\Pi_{(x,y)}$ is irreducible, (x, y) lies in an irreducible component. If the component has nontrivial intersection with $x = 0$ or $y = 0$, then we can use Step 3-2 of Algorithm 8.1 to reduce the problem of unitarizability to the one-parameter case, which is known. Otherwise, the connected component will have a boundary point of the form (x_0, y_0), where x_0, y_0 is the first nonzero reducibility point of $\rho|\cdot|^x \rtimes \pi_{nu}$ and $\rho|\cdot|^y \rtimes \pi_{nu}$ respectively. In this case, one can first show that $\rho|\cdot|^x \rtimes \pi_{nu}$ has a non-unitarizable subquotient π_{x_0} of corank 3 at $x = x_0$ by the result of Tadic. Then, using Proposition 9.13, one can show that a non-unitarizable subquotient $\rho|\cdot|^{y_0} \rtimes \pi_{x_0}$ exists. This proves the claim. □

Since the lowest possible corank of a non-unitarizable representation of critical type is 2, the above proposition gives the only possible two-parameter complementary series induced from a non-unitarizable representation of critical type. Therefore, this concludes step 2 of Algorithm 8.1, and we are now ready to state our final result.

11. Conclusion

In this section, we will finally compile and state the main results of this paper. In Proposition 11.1, we give the full unitary dual of corank 4 (up to a W-orbit), and in Proposition 11.2, we show that the full unitary dual is indeed equal to the set $\Pi_{\overline{A}}^{\lim}(G_n)$, when restricted to representations of corank 4. Note that for presentation purposes, some unitarizable representations may be listed in Proposition 11.1 more than once.

Theorem 11.1. *The irreducible unitarizable subquotients of $\Pi_{\underline{x}}$, when $\underline{x} = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4_{++}$, are precisely the following:*

(1) $(\alpha \geq 1)$ *All irreducible subquotients of $\Pi_{\underline{x}}$ when $\underline{x}$ lies in the closure of the connected components (8.1a) to (8.1g).*
(2) $(\alpha \geq 0)$ *All irreducible subquotients of critical points $\underline{x}$ listed in Proposition A.1.*
(3) $(\alpha \geq 0)$ *All one parameter complementary series of the form $u_\rho(a, b)|\cdot|^{x_1} \rtimes \pi_A$ listed in Table 9.1, for $0 \leq x_1 < FRP$, where FRP is the first nonzero reducibility point, given in the table.*
(4) $(\alpha = \frac{1}{2})$ *All irreducible subquotients when $x_4 \leq \frac{1}{2}$.*
(5) $(\alpha \geq \frac{1}{2})$
   - $\underline{x} = (x_1, x_2, \alpha, \alpha + 1)$. *The complementary series*

$$\rho|\cdot|^{x_1} \times \rho|\cdot|^{x_2} \times L(\Delta_\rho[-\alpha - 1, -\alpha - 1], \Delta_\rho[-\alpha, -\alpha]; \pi_{sc}),$$

and

$$\rho|\cdot|^{x_1} \times \rho|\cdot|^{x_2} \rtimes T_{I,1}^{\alpha+1}(T_{I,1}^{\alpha}(\pi_{sc})),$$

for

$$x_1 + x_2 < 1, \quad x_2 < \alpha - 1, \quad (\alpha > \frac{3}{2})$$
$$x_1 + 1 < x_2 < \alpha - 1, \quad (\alpha > 2)$$
$$x_2 < \frac{1}{2}, \quad (\alpha = \frac{1}{2}).$$

- $(\alpha \geq \frac{3}{2}), \underline{x} = (x_1, x_2, \alpha - 1, \alpha)$. *The complementary series*
$$\rho|\cdot|^{x_1} \times \rho|\cdot|^{x_2} \times L(\Delta_\rho[-\alpha+1, -\alpha]; \pi_{sc}),$$
and
$$\rho|\cdot|^{x_1} \times \rho|\cdot|^{x_2} \rtimes T_{I,1}^{\alpha-1}(T_{I,1}^{\alpha}(\pi_{sc})),$$
for $x_1 + x_2 < 1$.
- $(\alpha \geq \frac{3}{2}), \underline{x} = (x_1, x_2, \alpha - 1, \alpha)$. *The complementary series*
$$\rho|\cdot|^{x_1} \times \rho|\cdot|^{x_2} \times L(\Delta_\rho[-\alpha+1, -\alpha+1]; T_{I,1}^{\alpha}(\pi_{sc})),$$
and
$$\rho|\cdot|^{x_1} \times \rho|\cdot|^{x_2} \rtimes L(\Delta_\rho[-\alpha+1, -\alpha+1], \Delta_\rho[-\alpha, -\alpha]; \pi_{sc}),$$
for
$$x_2 < \frac{1}{2}, \quad (\alpha = \frac{3}{2})$$
$$x_1 + x_2 < 1, \quad (\alpha > 2).$$

(6) $(\alpha \geq 0)$*:*
- *The complementary series*
$$\Delta_\rho[\frac{1}{2}, -\frac{1}{2}]|\cdot|^{x_1} \times \Delta_\rho[\frac{1}{2}, -\frac{1}{2}]|\cdot|^{x_2} \rtimes \pi_{sc},$$
or its Aubert dual, when
$$x_2 - x_1 > 2, \quad x_2 < \alpha - \frac{1}{2}, \quad (\alpha \geq 3)$$
$$1 < x_2 - x_1 < 2, \quad x_2 < \alpha - \frac{1}{2}, \quad (\alpha \geq \frac{5}{2})$$
$$x_1 + x_2 < 1, \quad (\alpha \geq \frac{3}{2})$$
$$x_2 < \frac{1}{2}, \quad (\alpha = 0, 1).$$
- *The complementary series*
$$\Delta_\rho[\frac{1}{2}, -\frac{1}{2}]|\cdot|^{x_1} \times L(\Delta_\rho[-\frac{1}{2}, -\frac{1}{2}], \Delta_\rho[\frac{1}{2}, \frac{1}{2}])|\cdot|^{x_2} \rtimes \pi_{sc},$$
or its Aubert dual, when
$$x_2 - x_1 > 2, \quad x_2 < \alpha - \frac{1}{2}, \quad (\alpha \geq 3)$$
$$1 < x_2 - x_1 < 2, \quad x_2 < \alpha - \frac{1}{2}, \quad (\alpha \geq \frac{5}{2})$$

$$x_1 + x_2 < 1, \quad (\alpha \geq \frac{3}{2})$$
$$x_2 < \frac{1}{2}, \quad (\alpha = 0, 1).$$

- *The complementary series*
$$L(\Delta_\rho[-1,-1], \Delta_\rho[0,0], \Delta_\rho[1,1])|\cdot|^{x_1} \times \rho|\cdot|^{x_2} \rtimes \pi_{sc}$$
or its Aubert dual, when
$$x_1 - x_2 > 2, \quad x_1 < \alpha - 1, \quad (\alpha \geq \frac{7}{2})$$
$$x_1 + x_2 < 2, \quad x_1 < \alpha - 1, \quad (\alpha \geq 2)$$
$$x_2 - x_1 > 2, \quad x_2 < \alpha, \quad (\alpha \geq 3)$$
$$x_2 < \frac{3}{2}, \quad x_1 < \frac{1}{2}, \quad (\alpha = \frac{3}{2})$$
$$x_1, x_2 < \frac{1}{2}, \quad (\alpha = \frac{1}{2}).$$
- *The complementary series*
$$\Delta_\rho[\frac{1}{2}, -\frac{1}{2}]|\cdot|^{x_1} \times \rho|\cdot|^{x_2} \rtimes T^{\alpha}_{I,1}(\pi_{sc}),$$
when
$$x_1 + x_2 < \frac{3}{2}, \quad x_1 < \alpha - \frac{3}{2}, \quad x_2 < \alpha - 1, \quad (\alpha \geq 2)$$
$$x_2 - x_1 > 2, \quad x_2 < \alpha - 1, \quad (\alpha \geq 3)$$
$$x_2 - x_1 > \frac{3}{2}, \quad x_1 < \alpha - \frac{3}{2}, \quad (\alpha \geq \frac{7}{2})$$
$$x_1 < 1, \quad x_2 < \frac{1}{2}, \quad (\alpha = \frac{1}{2})$$
$$x_1 < \frac{1}{2}, \quad x_2 < 1, \quad (\alpha = 0)$$
- *The complementary series*
$$\Delta_\rho[\frac{1}{2}, -\frac{1}{2}]|\cdot|^{x_1} \times \rho|\cdot|^{x_2} \rtimes L(\Delta_\rho[-\alpha, -\alpha]; \pi_{sc}),$$
or its Aubert dual, when
$$x_1 + x_2 < \frac{3}{2}, \quad x_1 < |\alpha - \frac{3}{2}|, \quad x_2 < |\alpha - 1|, \quad (\alpha \geq 2)$$
$$x_2 - x_1 > 2, \quad x_2 < \alpha - 1, \quad (\alpha \geq 3)$$
$$x_2 - x_1 > \frac{3}{2}, \quad x_1 < \alpha - \frac{3}{2}, \quad (\alpha \geq \frac{7}{2})$$
$$x_1 < \frac{1}{2}, \quad x_2 < 1, \quad (\alpha = 0).$$

(7) $(\alpha = \frac{1}{2}), \underline{x} = (x_1, x_2, \frac{1}{2}, \frac{1}{2})$. *The complementary series*
$$\rho|\cdot|^{x_1} \times \rho|\cdot|^{x_2} \rtimes \pi,$$

where $\pi = L(\Delta_\rho[-\frac{1}{2},-\frac{1}{2}],\Delta_\rho[-\frac{1}{2},-\frac{1}{2}];\pi_{sc}), T^{\frac{1}{2}}_{I,2}(\pi_{sc}), L(\Delta_\rho[-\frac{1}{2},-\frac{1}{2}];T^{\frac{1}{2}}_{I,1}(\pi_{sc})), T^{\frac{1}{2}}_{III,2}(\pi_{sc})$, *when*

$$0 \leq x \leq y \leq \frac{1}{2}.$$

(8) $(\alpha = \frac{1}{2})$

- $\underline{x} = (x_1, \frac{1}{2}, \frac{1}{2}, \frac{3}{2})$ *The complementary series*
$$\rho|\cdot|^{x_1} \rtimes \pi,$$
where $\pi = L(\Delta_\rho[-\frac{3}{2},-\frac{3}{2}],\Delta_\rho[-\frac{1}{2},-\frac{1}{2}],\Delta_\rho[-\frac{1}{2},-\frac{1}{2}];\pi_{sc})$, $L(\Delta_\rho[\frac{1}{2},-\frac{3}{2}];\pi_{sc}), L(\Delta_\rho[-\frac{1}{2},-\frac{1}{2}];T^{\frac{3}{2}}_{I,1}(T^{\frac{1}{2}}_{I,1}(\pi_{sc}))), L(\Delta_\rho[-\frac{1}{2},-\frac{3}{2}];T^{\frac{1}{2}}_{I,1}(\pi_{sc}))$ *or their Aubert duals.*
- $\underline{x} = (\frac{1}{2}, \frac{1}{2}, x_3, \frac{3}{2})$*. The complementary series*
$$\rho|\cdot|^{x_3} \rtimes \pi,$$
for $\frac{1}{2} < x_3 < \frac{3}{2}$*, where* $\pi = L(\Delta_\rho[-\frac{3}{2},-\frac{3}{2}],\Delta_\rho[-\frac{1}{2},-\frac{1}{2}];T^{\frac{1}{2}}_{I,1}(\pi_{sc}))$ *or its Aubert dual.*
- $\underline{x} = (x_1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ *The complementary series*
$$\rho|\cdot|^{x_1} \rtimes \pi,$$
where $\pi = T^{\frac{1}{2}}_{I,3}(\pi_{sc}), L(\Delta_\rho[-\frac{1}{2},-\frac{1}{2}],\Delta_\rho[-\frac{1}{2},-\frac{1}{2}];T^{\frac{1}{2}}_{I,1}(\pi_{sc}))$ *or their Aubert duals.*
- $\underline{x} = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, x_4)$*. The complementary series*
$$\rho|\cdot|^{x_4} \rtimes L(\Delta_\rho[-\frac{1}{2},-\frac{1}{2}];T^{\frac{1}{2}}_{I,2}(\pi_{sc})),$$
for $\frac{1}{2} < x_4 < \frac{3}{2}$.

(9) $(\alpha = 0)$ *All irreducible subquotients when* $\underline{x} = (0,0,x_3,x_4)$ *with* $x_3 + x_4 \leq 1$.

(10) $(\alpha = 0), \underline{x} = (0, x_2, x_3, 1)$*. The complementary series*

$$\rho|\cdot|^{x_2} \times \rho|\cdot|^{x_3} \rtimes \pi,$$

for $x_2 + x_3 < 1$*, where* $\pi = L(\Delta_\rho[-1,-1];T^{\pm}_{V,2}(\pi_{sc})), L(\Delta_\rho[0,-1];\pi_{sc})$ *or their Aubert duals.*

Proof. The unitarity of the above representations follow from previous propositions. It suffices to prove that the list is exhaustive, and by symmetry under the W-action, it suffices to consider the subquotients of $\Pi_{\underline{x}}$ when $\underline{x} \in \mathbb{R}^4_{++}$. If $\underline{x} \in \mathbb{R}^4_{++}$ is regular and unitarizable, then it must be contained in one of the connected components (8.1a) to (8.1g), as proven in Proposition 8.10.

Otherwise, if $\underline{x} \in \mathbb{R}^4_{++}$ is not regular, then there are two possibilities. The first possibility occurs when the W-orbit of $\underline{x}$ intersects the image of a unitary$^{\pm}$ connected component of $\mathbb{R}^3_{\text{reg,sla},++}$ (resp. $R^3_{\text{reg,lev},++}$), under the affine isomorphism ι defined in Sections 8.3 (resp. 8.4). In this case, as stated in the proof of Proposition 8.10, $\underline{x}$ will lie in the closure of the connected components (8.1a) to (8.1g).

From now on, we assume the second possibility, which is when the W-orbit of $\underline{x}$ intersects neither $\iota(R^3_{\text{reg,sla},++})$, nor $\iota(\mathbb{R}^3_{\text{reg,lev},++})$. In this case, if an irreducible representation π is unitarizable of corank 4, it must either be a subquotient of $\Pi_{\underline{x}}$ when $\underline{x}$ is an isolated point, or it lies in a continuous family of representations induced from a representation on the maximal Levi of G_n which is of critical type. Any point $\underline{x} \in \mathbb{R}^4_{++}$ is either a regular point, or it is connected to a point of critical type by a one-parameter family which lies completely in a reducibility hyperplane. This implies that all isolated points are of critical type. Therefore, when $\underline{x}$ is isolated, the unitarizable subquotients of $\Pi_{\underline{x}}$ are listed in Proposition A.1.

When $\underline{x}$ is not an isolated point, it must be part of a continuous family of hermitian representations induced from a representation of critical type. By Proposition 8.10 and Remark 8.11, it suffices to consider families with one or two parameters.

Suppose π appears as part of a one-parameter complementary induced from a unitarizable representation of critical type, then there are a few possibilities. First, suppose π lies in a complementary series of the form $u_\rho(a,b)|\cdot|^x \rtimes \pi_A$, where π_A is of critical type and of Arthur type, then it must be in one of the cases in table 9.1, which are all unitarizable from 0 up to the first nonzero reducibility point. The endpoints of such complementary series which are unitarizable are already included in Proposition A.1. One can verify that all of the relevant complementary series are listed above. Otherwise, we have one of the cases below:

(1) π lies in a continuous family of the form $\tau|\cdot|^x \rtimes \pi_{sc}$, where $\tau \neq u_\rho(a,b)$ for any a, b.
(2) $\pi \leq \tau|\cdot|^x \rtimes \pi_A$, where π_A is of critical type and unitarizable of corank 1, and $\tau \neq u_\rho(a,b)$ for any a, b.

Case (1) is dealt with in Propositions 9.3 to 9.7, and the relevant unitarizable subquotients are included. Case (2) is dealt with in Propositions 9.8 to 9.10, where we proved that in this case none of the subquotients are unitarizable.

Otherwise, π is part of a hermitian family induced from a non-unitarizable representation of critical type. Since there are no non-unitarizable, critical type subquotients of corank 1, the following two possibilities remain:

(1) $\pi \leq \rho|\cdot|^x \rtimes \pi_{nu}$, where π_{nu} is non-unitarizable of corank 3.
(2) $\pi \leq \Delta_\rho[x+1,x] \rtimes \pi_{nu}$, where π_{nu} is non-unitarizable of corank 2.

Case (1) is covered in Proposition 9.13, and case (2) is covered by Propositions 9.11 and 9.12. In each case, we proved all subquotients are non-unitarizable.

Now, we consider the two-parameter families. First, consider those 2-parameter complementary series induced from unitarizable representations. There are three possibilities:

(1) $\pi \leq \rho|\cdot|^x \times \rho|\cdot|^y \rtimes \pi_A$, where π_A is unitarizable, of critical type, and of corank 2.
(2) $\pi \leq u_\rho(a,b)|\cdot|^x \times \rho|\cdot|^y \rtimes \pi_A$, where π_A is unitarizable, of critical type, and of corank 1.
(3) $\pi \leq u_\rho(a_1,b_1)|\cdot|^x \times u_\rho(a_2,b_2)\rho|\cdot|^y \rtimes \pi_{sc}$.

Case (1) is covered by Propositions 10.1 to 10.7. When $\alpha > 3$, one can check that the unitarizable regions in these complementary series lie in the closure of (8.1d), and when $2 < \alpha \leq 3$, they lie in the closure of (8.1e). Case (2) is covered by Propositions 10.11 and 10.12, and case (3) is covered by Propositions 10.8 to 10.10.

Finally, consider those two-parameter hermitian families induced from non-unitarizable representations. Since there are no non-unitarizable, critical type representations of corank 1, the only possibility we have are families of the form

$$\rho|\cdot|^x \times \rho|\cdot|^y \rtimes \pi_{nu},$$

where π_{nu} is critical type, non-unitarizable of corank 2. By Proposition 10.13, none of the subquotients in families of this form are unitarizable. This concludes the proof. $\square$

Now that we've constructed the full unitary dual for corank 4 representations, we will show that it is indeed the same as $\Pi_{\overline{A}}^{\lim}(G_n)$, as we conjectured.

Proposition 11.2. *If $\pi \in \Pi_u(G_n)$ is a unitary representation of corank 4, then $\pi \in \Pi_{\overline{A}}^{\lim}(G_n)$. In particular, this implies that $\Pi_u(G_n) = \Pi_{\overline{A}}^{\lim}(G_n)$ when restricted to corank 4 representations.*

Proof. It suffices to show that all representations given in Proposition 11.1 lie in $\Pi_{\overline{A}}^{\lim}(G_n)$. To prove this, we first note that the closure of all irreducible subquotients of $\Pi_{\underline{x}}$, where $\underline{x}$ lie in the closure of a unitary connected component in $\mathbb{R}^4_{++}$, lie in $\Pi_{\overline{A}}(G_n)$.

By definition, since all one-parameter complementary series listed in Proposition 9.1 are of the form $u_\rho(a,b)|\cdot|^{x_1} \rtimes \Pi_A$, where x_1 lies in an irreducible region, they are contained in $\Pi_{\overline{A}}^{\lim}(G_n)$.

Similarly, the two-parameter families listed in Proposition 11.1 are all of the form

$$u_\rho(a_1,b_1)|\cdot|^{x_1} \times u_\rho(a_2,b_2)|\cdot|^{x_2} \rtimes \pi_A,$$

where (x_1,x_2) lies in an irreducible component $\mathbb{R}^2$ for the given parabolic induction. Therefore, they all lie in $\Pi_{\overline{A}}^{\lim}(G_n)$. This finishes the proof. $\square$

We note that in the definition of $\Pi_{\overline{A}}^{\lim}(G_n)$, we take the closure of all unitary connected components in the last step. An alternative approach would be to take the closure of the complementary series at each inductive step. That is, we can define $\Pi_{\overline{A}}^{\lim}(G_n)$ inductively as follows:

Definition′ 11.3. *(a) For $k \in \mathbb{Z}_{\geq 0}$, we define subsets $\Pi_{\overline{A}}^{\lim,(k)}(G_n)$ of $\Pi_u(G_n)$ inductively as follows. Set $\Pi_{\overline{A}}^{\lim,(0)}(G_n) := \Pi_A(G_n)$. For $k \geq 1$, and any $\pi \in \Pi(G_n)$, we say that $\pi \in \Pi_{\overline{A}}^{\lim,(k)}(G_n)$ if there exists a triple*

$$(\Pi_{\underline{x}} = \Pi_{x_1,\ldots x_s} = u_{\rho_1}(a_1,b_1)|\cdot|^{x_1} \times \cdots \times u_{\rho_s}(a_s,b_s)|\cdot|^{x_s} \rtimes \pi_A, \underline{y} \in \mathbb{R}^s, \underline{z} \in \mathbb{R}^s) \tag{11.1}$$

satisfying the following conditions

(1) $\pi_A \in \pi_{A,gp}(G_m)$ for some $m < n$, and the ρ_i's are irreducible unitary supercuspidal representations of the general linear groups (not necessarily self-dual).

(2) The point $\underline{y}$ lies in the open set $U := \{\underline{x} \in \mathbb{R}^s | \Pi_{\underline{x}}$ is irreducible$\}$ and $\Pi_{\underline{y}} = \tau \rtimes \pi^{(k-1)}$ for some unitary representation τ of $GL_d(F)$ and $\pi^{(k-1)} \in \Pi_{\overline{A}}^{\lim,(k-1)}(G_{n-d})$.

(3) The points $\underline{z}$ lies in the ***closure*** *of the unique connected component of U containing $\underline{y}$ and $\pi \cong \Pi_{\underline{z}}$.*

Finally, we let $\Pi_{\overline{A}}^{\lim}(G_n) = \bigcup_{k \in \mathbb{Z}_{\geq 0}} \Pi_{\overline{A}}^{\lim,(k)}(G_n)$.

We refer to this definition as "Definition′"to distinguish it from Definition 2.22. This would create a set that contains the $\Pi_{\overline{A}}^{\lim}$ as it is currently defined. In the case of corank 4 representations, the two definitions are actually identical. Indeed, under both definitions, the set $\Pi_{\overline{A}}^{\lim}(G_n)$ is contained in $\Pi_u(G_n)$ and we proved that $\Pi_u(G_n) \subset \Pi_{\overline{A}}^{\lim}(G_n)$. Therefore if we take the enlarged definition of $\Pi_{\overline{A}}^{\lim}(G_n)$, we would still have

$$\Pi_{\overline{A}}^{\lim}(G_n) = \Pi_u(G_n)$$

as desired, for representations of corank 4.

Appendix A.

List of representations of corank 4 that are of Arthur type and critical type, sorted by cuspidal support

Let us first consider the induced representation $\Pi_{x_1,x_2,x_3,x_4} = \rho|\cdot|^{x_1} \times \rho|\cdot|^{x_2} \times \rho|\cdot|^{x_3} \times \rho|\cdot|^{x_4} \rtimes \pi_{sc}$ for $0 \leq x_1 \leq x_2 \leq x_3 \leq x_4$. To explicitly construct the unitary dual for corank 4 representations, we would like a list that tells us exactly which points (x_1, x_2, x_3, x_4) correspond to the induced representations which only contain unitarizable subquotients. In Proposition A.1, we will do so for the critical points. Once this is constructed, we will give an alternative list in the next section that gives us those representations which are of critical type but not of Arthur type. These two lists will tell us exactly which critical point contains non-unitarizable subquotients, which will help us determine whether a connected component is unitary when we construct the unitary dual later on.

Proposition A.1. *Let $\Pi_{x_1,x_2,x_3,x_4} = \rho|\cdot|^{x_1} \times \rho|\cdot|^{x_2} \times \rho|\cdot|^{x_3} \times \rho|\cdot|^{x_4} \rtimes \pi_{sc}$ for $0 \leq x_1 \leq x_2 \leq x_3 \leq x_4$. Then for a given quadruple (x_1, x_2, x_3, x_4), the following list contains all of the irreducible subquotients of Π_{x_1,x_2,x_3,x_4} which are of Arthur type and critical type.*

(1) $(\alpha = 0)$:
- (a) $(0, 0, 0, 1)$:
  - $T^1_{I,1}(T^{\pm}_{V,6}(\pi_{sc}))$
  - $L(\Delta_\rho[-1, -1]; T^{\pm}_{V,6}(\pi_{sc}))$
  - $L(\Delta_\rho[0, -1]; T^{\pm}_{V,4}(\pi_{sc}))$
- (b) $(0, 0, 1, 1)$:
  - $T^1_{I,2}(T^{\pm}_{V,4}(\pi_{sc}))$
  - $T^1_{II,3}(T^{\pm}_{V,4}(\pi_{sc}))$
  - $L(\Delta_\rho[-1, -1]; T^1_{I,1}(T^{\pm}_{V,4}(\pi_{sc})))$
  - $L(\Delta_\rho[0, -1]; T^1_{I,1}(T^{\pm}_{V,2}(\pi_{sc})))$
  - $L(\Delta_\rho[0, -1], \Delta_\rho[0, -1]; \pi_{sc})$
  - $L(\Delta_\rho[-1, -1], \Delta_\rho[-1, -1]; T^{\pm}_{V,4}(\pi_{sc}))$
  - $L(\Delta_\rho[-1, -1], \Delta_\rho[0, -1]; T^{\pm}_{V,2}(\pi_{sc}))$
- (c) $(0, 0, 1, 2)$:
  - $T^2_{I,1}(T^1_{I,1}(T^{\pm}_{V,4}(\pi_{sc})))$
  - $L(\Delta_\rho[-2, -2], \Delta_\rho[-1, -1]; T^{\pm}_{V,4}(\pi_{sc}))$
- (d) $(0, 1, 1, 2)$:
  - $T^1_{I,1}(T^2_{I,1}(T^1_{I,1}(T^{\pm}_{V,2}(\pi_{sc}))))$
  - $L(\Delta_\rho[-1, -1]; T^2_{I,1}(T^1_{I,1}(T^{\pm}_{V,2}(\pi_{sc}))))$
  - $L(\Delta_\rho[1, -2]; \pi_{sc})$
  - $L(\Delta_\rho[-1, -2], \Delta_\rho[0, -1]; \pi_{sc})$
  - $L(\Delta_\rho[-2, -2], \Delta_\rho[-1, -1]; T^{\pm}_{V,2}(T^1_{I,1}(\pi_{sc})))$
  - $L(\Delta_\rho[-2, -2], \Delta_\rho[-1, -1], \Delta_\rho[-1, -1]; T^{\pm}_{V,2}(\pi_{sc}))$
- (e) $(0, 1, 2, 3)$:
  - $T^3_{I,1}(T^2_{I,1}(T^1_{I,1}(T^{\pm}_{V,2}(\pi_{sc}))))$
  - $L(\Delta_\rho[-3, -3], \Delta_\rho[-2, -2], \Delta_\rho[-1, -1]; T^{\pm}_{V,2}(\pi_{sc}))$

(2) $(\alpha = \frac{1}{2})$:
- (a) $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$:
  - $T^{\frac{1}{2}}_{I,4}(\pi_{sc})$

- $T^{\frac{1}{2}}_{II,5}(\pi_{sc})$
- $L(\Delta_\rho[-\frac{1}{2},-\frac{1}{2}];T^{\frac{1}{2}}_{I,3}(\pi_{sc})$
- $L(\Delta_\rho[-\frac{1}{2},-\frac{1}{2}],\Delta_\rho[-\frac{1}{2},-\frac{1}{2}];T^{\frac{1}{2}}_{II,3}(\pi_{sc}))$
- $L(\Delta_\rho[-\frac{1}{2},-\frac{1}{2}],\Delta_\rho[-\frac{1}{2},-\frac{1}{2}];T^{\frac{1}{2}}_{I,2}(\pi_{sc}))$
- $L(\Delta_\rho[-\frac{1}{2},-\frac{1}{2}],\Delta_\rho[-\frac{1}{2},-\frac{1}{2}],\Delta_\rho[-\frac{1}{2},-\frac{1}{2}];T^{\frac{1}{2}}_{I,1}(\pi_{sc}))$
- $L(\Delta_\rho[-\frac{1}{2},-\frac{1}{2}],\Delta_\rho[-\frac{1}{2},-\frac{1}{2}],\Delta_\rho[-\frac{1}{2},-\frac{1}{2}],\Delta_\rho[-\frac{1}{2},-\frac{1}{2}];\pi_{sc})$

(b) $(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{3}{2})$*:*

- $T^{\frac{1}{2}}_{I,2}(T^{\frac{3}{2}}_{I,1}(T^{\frac{1}{2}}_{I,1}(\pi_{sc})))$
- $T^{\frac{1}{2}}_{III,2}(T^{\frac{3}{2}}_{I,1}(T^{\frac{1}{2}}_{I,1}(\pi_{sc})))$
- $L(\Delta_\rho[-\frac{1}{2},-\frac{1}{2}];T^{\frac{1}{2}}_{I,1}(T^{\frac{3}{2}}_{I,1}(T^{\frac{1}{2}}_{I,1}(\pi_{sc}))))$
- $L(\Delta_\rho[-\frac{1}{2},-\frac{1}{2}];T^{\frac{3}{2}}_{I,1}(T^{\frac{1}{2}}_{II,3}(\pi_{sc}))))$
- $L(\Delta_\rho[\frac{1}{2},-\frac{3}{2}];T^{\frac{1}{2}}_{I,1}(\pi_{sc}))$
- $L(\Delta_\rho[-\frac{1}{2},-\frac{3}{2}];T^{\frac{1}{2}}_{I,2}(\pi_{sc}))$
- $L(\Delta_\rho[-\frac{1}{2},-\frac{3}{2}];T^{\frac{1}{2}}_{III,2}(\pi_{sc}))$
- $L(\Delta_\rho[-\frac{1}{2},-\frac{1}{2}],\Delta_\rho[-\frac{1}{2},-\frac{1}{2}];T^{\frac{3}{2}}_{I,1}(T^{\frac{1}{2}}_{I,1}(\pi_{sc})))$
- $L(\Delta_\rho[-\frac{1}{2},-\frac{1}{2}],\Delta_\rho[\frac{1}{2},-\frac{3}{2}];\pi_{sc})$
- $L(\Delta_\rho[-\frac{1}{2},-\frac{3}{2}],\Delta_\rho[-\frac{1}{2},-\frac{1}{2}];T^{\frac{1}{2}}_{I,1}(\pi_{sc}))$
- $L(\Delta_\rho[-\frac{3}{2},-\frac{3}{2}],\Delta_\rho[-\frac{1}{2},-\frac{1}{2}];T^{\frac{1}{2}}_{I,2}(\pi_{sc}))$
- $L(\Delta_\rho[-\frac{3}{2},-\frac{3}{2}],\Delta_\rho[-\frac{1}{2},-\frac{1}{2}];T^{\frac{1}{2}}_{III,2}(\pi_{sc}))$
- $L(\Delta_\rho[-\frac{3}{2},-\frac{3}{2}],\Delta_\rho[-\frac{1}{2},-\frac{1}{2}],\Delta_\rho[-\frac{1}{2},-\frac{1}{2}];T^{\frac{1}{2}}_{I,1}(\pi_{sc}))$
- $L(\Delta_\rho[-\frac{3}{2},-\frac{3}{2}],\Delta_\rho[-\frac{1}{2},-\frac{1}{2}],\Delta_\rho[-\frac{1}{2},-\frac{1}{2}],\Delta_\rho[-\frac{1}{2},-\frac{1}{2}];\pi_{sc})$

(c) $(\frac{1}{2},\frac{1}{2},\frac{3}{2},\frac{3}{2})$*:*

- $T^{\frac{3}{2}}_{I,2}(T^{\frac{1}{2}}_{I,2}(\pi_{sc}))$
- $T^{\frac{3}{2}}_{I,2}(T^{\frac{1}{2}}_{III,2}(\pi_{sc}))$
- $L(\Delta_\rho[-\frac{3}{2},-\frac{3}{2}];T^{\frac{3}{2}}_{I,1}(T^{\frac{1}{2}}_{III,2}(\pi_{sc})))$
- $L(\Delta_\rho[-\frac{3}{2},-\frac{3}{2}],\Delta_\rho[-\frac{1}{2},-\frac{1}{2}];T^{\frac{3}{2}}_{I,1}(T^{\frac{1}{2}}_{I,1}(\pi_{sc})))$
- $L(\Delta_\rho[-\frac{3}{2},-\frac{3}{2}],\Delta_\rho[-\frac{3}{2},-\frac{3}{2}],\Delta_\rho[-\frac{1}{2},-\frac{1}{2}];T^{\frac{1}{2}}_{I,1}(\pi_{sc}))$
- $L(\Delta_\rho[-\frac{3}{2},-\frac{3}{2}],\Delta_\rho[-\frac{3}{2},-\frac{3}{2}],\Delta_\rho[-\frac{1}{2},-\frac{1}{2}],\Delta_\rho[-\frac{1}{2},-\frac{1}{2}];\pi_{sc})$

(d) $(\frac{1}{2},\frac{1}{2},\frac{3}{2},\frac{5}{2})$*:*

- $T^{\frac{5}{2}}_{I,1}(T^{\frac{3}{2}}_{I,1}(T^{\frac{1}{2}}_{I,2}(\pi_{sc}))$
- $T^{\frac{5}{2}}_{I,1}(T^{\frac{3}{2}}_{I,1}(T^{\frac{1}{2}}_{III,2}(\pi_{sc}))$
- $L(\Delta_\rho[-\frac{1}{2},-\frac{1}{2}];T^{\frac{5}{2}}_{I,1}(T^{\frac{3}{2}}_{I,1}(T^{\frac{1}{2}}_{I,1}(\pi_{sc}))))$
- $L(\Delta_\rho[-\frac{5}{2},-\frac{5}{2}],\Delta_\rho[-\frac{3}{2},-\frac{3}{2}],\Delta_\rho[-\frac{1}{2},-\frac{1}{2}];T^{\frac{1}{2}}_{I,1}(\pi_{sc}))$
- $L(\Delta_\rho[-\frac{3}{2},-\frac{5}{2}],\Delta_\rho[-\frac{1}{2},-\frac{1}{2}],\Delta_\rho[-\frac{1}{2},-\frac{1}{2}];\pi_{sc})$

- $L(\Delta_\rho[-\frac{5}{2},-\frac{5}{2}],\Delta_\rho[-\frac{3}{2},-\frac{3}{2}],\Delta_\rho[-\frac{1}{2},-\frac{1}{2}],\Delta_\rho[-\frac{1}{2},-\frac{1}{2}];\pi_{sc})$

(e) $(\frac{1}{2},\frac{3}{2},\frac{5}{2},\frac{7}{2})$*:*
- $T_{I,1}^{\frac{7}{2}}(T_{I,1}^{\frac{5}{2}}(T_{I,1}^{\frac{3}{2}}(T_{I,1}^{\frac{1}{2}}(\pi_{sc}))))$
- $L(\Delta_\rho[-\frac{7}{2},-\frac{7}{2}],\Delta_\rho[-\frac{5}{2},-\frac{5}{2}],\Delta_\rho[-\frac{3}{2},-\frac{3}{2}],\Delta_\rho[-\frac{1}{2},-\frac{1}{2}];\pi_{sc})$

(3) $(\alpha=1)$*:*

(a) $(0,0,0,1)$ *:*
- $T_{V,6}^{\pm}(T_{I,1}^{1}(\pi_{sc}))$
- $L(\Delta_\rho[-1,-1];T_{IV,7}(\pi_{sc}))$
- $L(\Delta_\rho[0,-1];T_{IV,5}(\pi_{sc}))$

(b) $(0,0,1,1)$*:*
- $T_{I,2}^{1}(T_{IV,5}(\pi_{sc}))$
- $T_{II,3}^{1}(T_{IV,5}(\pi_{sc}))$
- $L(\Delta_\rho[-1,-1];T_{V,4}^{\pm}(T_{I,1}^{1}(\pi_{sc})))$
- $L(\Delta_\rho[0,-1];T_{V,2}^{\pm}(T_{I,1}^{1}(\pi_{sc})))$
- $L(\Delta_\rho[0,-1],\Delta_\rho[0,-1];\pi_{sc})$
- $L(\Delta_\rho[-1,-1],\Delta_\rho[-1,-1];T_{IV,5}(\pi_{sc}))$
- $L(\Delta_\rho[-1,-1],\Delta_\rho[0,-1];T_{IV,3}(\pi_{sc}))$

(c) $(0,0,1,2)$*:*
- $T_{V,4}^{\pm}(T_{I,1}^{2}(T_{I,1}^{1}(\pi_{sc})))$
- $L(\Delta_\rho[-2,-2],\Delta_\rho[-1,-1];T_{IV,5}(\pi_{sc}))$
- $L(\Delta_\rho[-2,-2],\Delta_\rho[0,-1];T_{IV,3}(\pi_{sc}))$

(d) $(0,1,1,1)$*:*
- $T_{I,3}^{1}(T_{IV,3}(\pi_{sc}))$
- $L(\Delta_\rho[-1,-1];T_{I,2}^{1}(T_{IV,3}(\pi_{sc})))$
- $L(\Delta_\rho[-1,-1];T_{III,3}^{1}(T_{IV,3}(\pi_{sc})))$
- $L(\Delta_\rho[-1,-1],\Delta_\rho[-1,-1];T_{V,2}^{\pm}(T_{I,1}^{1}(\pi_{sc})))$
- $L(\Delta_\rho[-1,-1],\Delta_\rho[-1,-1],\Delta_\rho[-1,-1];T_{IV,3}(\pi_{sc}))$

(e) $(0,1,1,2)$*:*
- $T_{V,2}^{\pm}(T_{I,1}^{1}(T_{I,1}^{2}(T_{I,1}^{1}(\pi_{sc}))))$
- $T_{I,1}^{2}(T_{III,2}^{1}(\pi_{sc}))$
- $L(\Delta_\rho[-1,-1];T_{V,2}^{\pm}(T_{I,1}^{2}(T_{I,1}^{1}(\pi_{sc}))))$
- $L(\Delta_\rho[0,-1];T_{I,1}^{2}(T_{I,1}^{1}(\pi_{sc})))$
- $L(\Delta_\rho[-1,-2];T_{V,2}^{\pm}(T_{I,1}^{1}(\pi_{sc})))$ *with* $\epsilon_{sc}(\rho\otimes S_3)=\mp 1.$
- $L(\Delta_\rho[0,-2];T_{I,1}^{1}(\pi_{sc}))$
- $L(\Delta_\rho[1,-2];\pi_{sc})$
- $L(\Delta_\rho[-1,-2],\Delta_\rho[0,-1];\pi_{sc})$
- $L(\Delta_\rho[-2,-2];T_{II,3}^{1}(T_{IV,3}(\pi_{sc})))$
- $L(\Delta_\rho[-2,-2],\Delta_\rho[-1,-1];T_{V,4}^{\pm}(T_{I,1}^{1}(\pi_{sc})))$
- $L(\Delta_\rho[-2,-2],\Delta_\rho[-1,-1],\Delta_\rho[0,-1];\pi_{sc})$
- $L(\Delta_\rho[-2,-2],\Delta_\rho[-1,-1],\Delta_\rho[-1,-1];T_{IV,3}(\pi_{sc}))$

(f) $(0,1,2,3)$*:*
- $T_{V,2}^{\pm}(T_{I,1}^{3}(T_{I,1}^{2}(T_{I,1}^{1}(\pi_{sc}))))$
- $L(\Delta_\rho[-3,-3],\Delta_\rho[-2,-2],\Delta_\rho[-1,-1];T_{IV,3}(\pi_{sc}))$
- $L(\Delta_\rho[-3,-3],\Delta_\rho[-2,-2],\Delta_\rho[0,-1];\pi_{sc})$

(g) $(1,2,3,4)$:
- $T^4_{I,1}(T^3_{I,1}(T^2_{I,1}(T^1_{I,1}(\pi_{sc}))))$
- $L(\Delta_\rho[-4,-4],\Delta_\rho[-3,-3],\Delta_\rho[-2,-2],\Delta_\rho[-1,-1];\pi_{sc})$

(4) ($\alpha=\frac{3}{2}$):

(a) $(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{3}{2})$:
- $T^{\frac{1}{2}}_{I,3}(T^{\frac{3}{2}}_{I,1}(\pi_{sc}))$
- $L(\Delta_\rho[-\frac{1}{2},-\frac{1}{2}];T^{\frac{3}{2}}_{I,1}(T^{\frac{1}{2}}_{II,3}(\pi_{sc})))$
- $L(\Delta_\rho[-\frac{1}{2},-\frac{1}{2}];T^{\frac{1}{2}}_{I,2}(T^{\frac{3}{2}}_{I,1}(\pi_{sc})))$
- $L(\Delta_\rho[-\frac{1}{2},-\frac{3}{2}];T^{\frac{1}{2}}_{II,3}(\pi_{sc}))$
- $L(\Delta_\rho[-\frac{1}{2},-\frac{1}{2}],\Delta_\rho[\frac{1}{2},-\frac{3}{2}];\pi_{sc})$
- $L(\Delta_\rho[-\frac{3}{2},-\frac{3}{2}],\Delta_\rho[-\frac{1}{2},-\frac{1}{2}];T^{\frac{1}{2}}_{II,3}(\pi_{sc}))$
- $L(\Delta_\rho[-\frac{1}{2},-\frac{1}{2}],\Delta_\rho[-\frac{1}{2},-\frac{1}{2}];T^{\frac{1}{2}}_{I,1}(T^{\frac{3}{2}}_{I,1}(\pi_{sc})))$
- $L(\Delta_\rho[-\frac{1}{2},-\frac{1}{2}],\Delta_\rho[-\frac{1}{2},-\frac{1}{2}],\Delta_\rho[-\frac{1}{2},-\frac{1}{2}];T^{\frac{3}{2}}_{I,1}(\pi_{sc}))$
- $L(\Delta_\rho[-\frac{1}{2},-\frac{3}{2}],\Delta_\rho[-\frac{1}{2},-\frac{1}{2}],\Delta_\rho[-\frac{1}{2},-\frac{1}{2}];\pi_{sc})$
- $L(\Delta_\rho[-\frac{3}{2},-\frac{3}{2}],\Delta_\rho[-\frac{1}{2},-\frac{1}{2}],\Delta_\rho[-\frac{1}{2},-\frac{1}{2}],\Delta_\rho[-\frac{1}{2},-\frac{1}{2}];\pi_{sc})$

(b) $(\frac{1}{2},\frac{1}{2},\frac{3}{2},\frac{3}{2})$:
- $T^{\frac{3}{2}}_{I,2}(T^{\frac{1}{2}}_{II,3}(\pi_{sc}))$
- $T^{\frac{3}{2}}_{III,2}(\pi_{sc})$
- $L(\Delta_\rho[\frac{1}{2},-\frac{3}{2}];T^{\frac{3}{2}}_{I,1}(\pi_{sc}))$
- $L(\Delta_\rho[-\frac{1}{2},-\frac{3}{2}];T^{\frac{1}{2}}_{I,1}(T^{\frac{3}{2}}_{I,1}(\pi_{sc})))$
- $L(\Delta_\rho[-\frac{3}{2},-\frac{3}{2}];T^{\frac{3}{2}}_{I,1}(T^{\frac{1}{2}}_{II,3}(\pi_{sc})))$
- $L(\Delta_\rho[-\frac{3}{2},-\frac{3}{2}],\Delta_\rho[-\frac{1}{2},-\frac{1}{2}];T^{\frac{1}{2}}_{I,1}(T^{\frac{3}{2}}_{I,1}(\pi_{sc})))$
- $L(\Delta_\rho[-\frac{3}{2},-\frac{3}{2}],\Delta_\rho[\frac{1}{2},-\frac{3}{2}];\pi_{sc})$
- $L(\Delta_\rho[-\frac{3}{2},-\frac{3}{2}],\Delta_\rho[-\frac{1}{2},-\frac{1}{2}],\Delta_\rho[-\frac{1}{2},-\frac{1}{2}];T^{\frac{3}{2}}_{I,1}(\pi_{sc}))$
- $L(\Delta_\rho[-\frac{3}{2},-\frac{3}{2}],\Delta_\rho[-\frac{1}{2},-\frac{3}{2}],\Delta_\rho[-\frac{1}{2},-\frac{1}{2}];\pi_{sc})$
- $L(\Delta_\rho[-\frac{3}{2},-\frac{3}{2}],\Delta_\rho[-\frac{1}{2},-\frac{1}{2}],\Delta_\rho[-\frac{1}{2},-\frac{1}{2}];T^{\frac{3}{2}}_{I,1}(\pi_{sc}))$

(c) $(\frac{1}{2},\frac{1}{2},\frac{3}{2},\frac{5}{2})$:
- $T^{\frac{1}{2}}_{I,2}(T^{\frac{5}{2}}_{I,1}(T^{\frac{3}{2}}_{I,1}(\pi_{sc})))$
- $T^{\frac{1}{2}}_{II,3}(T^{\frac{5}{2}}_{I,1}(T^{\frac{3}{2}}_{I,1}(\pi_{sc})))$
- $L(\Delta_\rho[\frac{1}{2},-\frac{5}{2}];\pi_{sc})\ +$
- $L(\Delta_\rho[-\frac{5}{2},-\frac{5}{2}],\Delta_\rho[-\frac{3}{2},-\frac{3}{2}];T^{\frac{1}{2}}_{II,3}(\pi_{sc}))$
- $L(\Delta_\rho[-\frac{5}{2},-\frac{5}{2}],\Delta_\rho[\frac{1}{2},-\frac{3}{2}];\pi_{sc})$
- $L(\Delta_\rho[-\frac{1}{2},-\frac{1}{2}],\Delta_\rho[-\frac{1}{2},-\frac{1}{2}];T^{\frac{5}{2}}_{I,1}(T^{\frac{3}{2}}_{I,1}(\pi_{sc})))$
- $L(\Delta_\rho[-\frac{3}{2},-\frac{5}{2}],\Delta_\rho[-\frac{1}{2},-\frac{1}{2}],\Delta_\rho[-\frac{1}{2},-\frac{1}{2}];\pi_{sc})$
- $L(\Delta_\rho[-\frac{5}{2},-\frac{5}{2}],\Delta_\rho[-\frac{1}{2},-\frac{3}{2}],\Delta_\rho[-\frac{1}{2},-\frac{1}{2}];\pi_{sc})$

(d) $(\frac{1}{2},\frac{3}{2},\frac{3}{2},\frac{5}{2})$:

- $T_{I,1}^{\frac{3}{2}}(T_{I,1}^{\frac{1}{2}}(T_{I,1}^{\frac{5}{2}}(T_{I,1}^{\frac{3}{2}}(\pi_{sc}))))$
- $L(\Delta_\rho[-\frac{3}{2},-\frac{3}{2}],\Delta_\rho[-\frac{1}{2},-\frac{1}{2}];T_{I,1}^{\frac{5}{2}}(T_{I,1}^{\frac{3}{2}}(\pi_{sc})))$
- $L(\Delta_\rho[-\frac{3}{2},-\frac{5}{2}],\Delta_\rho[-\frac{1}{2},-\frac{3}{2}];\pi_{sc})$
- $L(\Delta_\rho[-\frac{5}{2},-\frac{5}{2}],\Delta_\rho[-\frac{3}{2},-\frac{3}{2}],\Delta_\rho[-\frac{1}{2},-\frac{1}{2}];T_{I,1}^{\frac{3}{2}}(\pi_{sc}))$

(e) $(\frac{1}{2},\frac{3}{2},\frac{5}{2},\frac{7}{2})$:
- $T_{I,1}^{\frac{1}{2}}(T_{I,1}^{\frac{7}{2}}(T_{I,1}^{\frac{5}{2}}(T_{I,1}^{\frac{3}{2}}(\pi_{sc}))))$
- $L(\Delta_\rho[-\frac{1}{2},-\frac{1}{2}];T_{I,1}^{\frac{7}{2}}(T_{I,1}^{\frac{5}{2}}(T_{I,1}^{\frac{3}{2}}(\pi_{sc}))))$
- $L(\Delta_\rho[-\frac{7}{2},-\frac{7}{2}],\Delta_\rho[-\frac{5}{2},-\frac{5}{2}],\Delta_\rho[-\frac{1}{2},-\frac{3}{2}];\pi_{sc})$
- $L(\Delta_\rho[-\frac{7}{2},-\frac{7}{2}],\Delta_\rho[-\frac{5}{2},-\frac{5}{2}],\Delta_\rho[-\frac{3}{2},-\frac{3}{2}],\Delta_\rho[-\frac{1}{2},-\frac{1}{2}];\pi_{sc})$

(f) $(\frac{3}{2},\frac{5}{2},\frac{7}{2},\frac{9}{2})$:
- $T_{I,1}^{\frac{9}{2}}(T_{I,1}^{\frac{7}{2}}(T_{I,1}^{\frac{5}{2}}(T_{I,1}^{\frac{3}{2}}(\pi_{sc}))))$
- $L(\Delta_\rho[-\frac{9}{2},-\frac{9}{2}],\Delta_\rho[-\frac{7}{2},-\frac{7}{2}],\Delta_\rho[-\frac{5}{2},-\frac{5}{2}],\Delta_\rho[-\frac{3}{2},-\frac{3}{2}];\pi_{sc})$

(5) $(\alpha=2)$:

(a) $(0,0,1,2)$:
- $T_{V,4}^{\pm}(T_{I,1}^{2}(T_{I,1}^{1}(\pi_{sc})))$
- $L(\Delta_\rho[-1,-1];T_{I,1}^{2}(T_{IV,5}(\pi_{sc})))$
- $L(\Delta_\rho[0,-1];T_{IV,3}(T_{I,1}^{2}(\pi_{sc})))$
- $L(\Delta_\rho[0,-2];T_{IV,3}(\pi_{sc}))$
- $L(\Delta_\rho[-1,-2];T_{IV,5}(\pi_{sc}))$
- $L(\Delta_\rho[-2,-1];T_{IV,5}(\pi_{sc}))$
- $L(\Delta_\rho[-2,-2],\Delta_\rho[0,-1];T_{IV,3}(\pi_{sc}))$
- $L(\Delta_\rho[-2,-2],\Delta_\rho[-1,-1];T_{IV,5}(\pi_{sc}))$

(b) $(0,1,1,2)$:
- $T_{I,2}^{1}(T_{IV,3}(T_{I,1}^{2}(\pi_{sc})))$
- $T_{II,3}^{1}(T_{IV,3}(T_{I,1}^{2}(\pi_{sc})))$
- $L(\Delta_\rho[0,-1];T_{I,1}^{1}(T_{I,1}^{2}(\pi_{sc})))$
- $L(\Delta_\rho[-1,-1];T_{V,2}^{\pm}(T_{I,1}^{1}(T_{I,1}^{2}(\pi_{sc}))))$
- $L(\Delta_\rho[-1,-1];T_{I,1}^{1}(T_{IV,3}(T_{I,1}^{2}(\pi_{sc}))))$
- $L(\Delta_\rho[1,-2];\pi_{sc})$
- $L(\Delta_\rho[-2,-2];T_{II,3}^{1}(T_{IV,3}(\pi_{sc})))$
- $L(\Delta_\rho[-1,-1],\Delta_\rho[-1,-1];T_{V,2}^{\pm}(T_{I,1}^{1}(\pi_{sc})))$
- $L(\Delta_\rho[-1,-1],\Delta_\rho[0,-1];T_{I,1}^{2}(\pi_{sc}))$
- $L(\Delta_\rho[-1,-1],\Delta_\rho[0,-2];\pi_{sc})$
- $L(\Delta_\rho[-2,-2],\Delta_\rho[-1,-1],T_{IV,3}(T_{I,1}^{1}(\pi_{sc})))$
- $L(\Delta_\rho[-2,-2],\Delta_\rho[-1,-1],\Delta_\rho[0,-1];\pi_{sc})$
- $L(\Delta_\rho[-2,-2],\Delta_\rho[-1,-1],\Delta_\rho[-1,-1];T_{IV,3}(\pi_{sc}))$

(c) $(0,1,2,2)$:
- $L(\Delta_\rho[-2,-2],\Delta_\rho[-1,-1];T_{IV,3}(T_{I,1}^{2}(\pi_{sc})))$
- $L(\Delta_\rho[-2,-2],\Delta_\rho[0,-1];T_{I,1}^{2}(\pi_{sc}))$

(d) $(0,1,2,3)$:
- $T_{V,2}^{\pm}(T_{I,1}^{1}(T_{I,1}^{3}(T_{I,1}^{2}(\pi_{sc})))$
- $L(\Delta_\rho[-1,-1];T_{IV,3}(T_{I,1}^{3}(T_{I,1}^{2}(\pi_{sc}))))$

- $L(\Delta_\rho[0,-1]; T^3_{I,1}(T^2_{I,1}(\pi_{sc})))$
- $L(\Delta_\rho[-3,-3], \Delta_\rho[0,-2]; \pi_{sc})$
- $L(\Delta_\rho[-3,-3], \Delta_\rho[-1,-2]; T_{IV,3}(\pi_{sc}))$
- $L(\Delta_\rho[-3,-3], \Delta_\rho[-2,-2], \Delta_\rho[-1,-1]; T_{IV,3}(\pi_{sc}))$
- $L(\Delta_\rho[-3,-3], \Delta_\rho[-2,-2], \Delta_\rho[0,-1]; \pi_{sc})$

(e) $(1,2,2,3)$*:*
- $T^2_{I,1}(T^1_{I,1}(T^3_{I,1}(T^2_{I,1}(\pi_{sc}))))$
- $L(\Delta_\rho[-2,-2], \Delta_\rho[-1,-1]; T^3_{I,1}(T^2_{I,1}(\pi_{sc})))$
- $L(\Delta_\rho[-2,-3], \Delta_\rho[-1,-2]; \pi_{sc})$
- $L(\Delta_\rho[-3,-3], \Delta_\rho[-2,-2], \Delta_\rho[-1,-1]; T^2_{I,1}(\pi_{sc}))$

(f) $(1,2,3,4)$*:*
- $T^1_{I,1}(T^4_{I,1}(T^3_{I,1}(T^2_{I,1}(\pi_{sc}))))$
- $L(\Delta_\rho[-1,-1]; T^4_{I,1}(T^3_{I,1}(T^2_{I,1}(\pi_{sc}))))$
- $L(\Delta_\rho[-4,-4], \Delta_\rho[-3,-3], \Delta_\rho[-1,-2]; \pi_{sc})$
- $L(\Delta_\rho[-4,-4], \Delta_\rho[-3,-3], \Delta_\rho[-2,-2], \Delta_\rho[-1,-1]; \pi_{sc})$

(g) $(2,3,4,5)$*:*
- $T^5_{I,1}(T^4_{I,1}(T^3_{I,1}(T^2_{I,1}(\pi_{sc}))))$
- $L(\Delta_\rho[-5,-5], \Delta_\rho[-4,-4], \Delta_\rho[-3,-3], \Delta_\rho[-2,-2]; \pi_{sc})$

(6) $(\alpha = \frac{5}{2})$*:*

(a) $(\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{5}{2})$*:*
- $T^{\frac{1}{2}}_{III,2}(T^{\frac{3}{2}}_{I,1}(T^{\frac{5}{2}}_{I,1}(\pi_{sc})))$
- $T^{\frac{1}{2}}_{I,2}(T^{\frac{3}{2}}_{I,1}(T^{\frac{5}{2}}_{I,1}(\pi_{sc})))$
- $L(\Delta_\rho[\frac{1}{2}, -\frac{3}{2}]; T^{\frac{5}{2}}_{I,1}(\pi_{sc}))$
- $L(\Delta_\rho[\frac{1}{2}, -\frac{5}{2}]; \pi_{sc})$
- $L(\Delta_\rho[-\frac{1}{2}, -\frac{1}{2}]; T^{\frac{1}{2}}_{I,1}(T^{\frac{3}{2}}_{I,1}(T^{\frac{5}{2}}_{I,1}(\pi_{sc}))))$
- $L(\Delta_\rho[-\frac{3}{2}, -\frac{3}{2}]; T^{\frac{5}{2}}_{I,1}(T^{\frac{1}{2}}_{II,3}(\pi_{sc})))$
- $L(\Delta_\rho[-\frac{3}{2}, -\frac{5}{2}]; T^{\frac{1}{2}}_{II,3}(\pi_{sc}))$
- $L(\Delta_\rho[-\frac{5}{2}, -\frac{5}{2}], \Delta_\rho[-\frac{3}{2}, -\frac{3}{2}]; T^{\frac{1}{2}}_{II,3}(\pi_{sc}))$
- $L(\Delta_\rho[-\frac{1}{2}, -\frac{1}{2}], \Delta_\rho[-\frac{1}{2}, -\frac{1}{2}]; T^{\frac{3}{2}}_{I,1}(T^{\frac{5}{2}}_{I,1}(\pi_{sc})))$
- $L(\Delta_\rho[-\frac{1}{2}, -\frac{3}{2}], \Delta_\rho[-\frac{1}{2}, -\frac{1}{2}]; T^{\frac{5}{2}}_{I,1}(\pi_{sc}))$
- $L(\Delta_\rho[-\frac{1}{2}, -\frac{5}{2}], \Delta_\rho[-\frac{1}{2}, -\frac{1}{2}]; \pi_{sc})$
- $L(\Delta_\rho[-\frac{5}{2}, -\frac{5}{2}], \Delta_\rho[\frac{1}{2}, -\frac{3}{2}]; \pi_{sc})$
- $L(\Delta_\rho[-\frac{3}{2}, -\frac{5}{2}], \Delta_\rho[-\frac{1}{2}, -\frac{1}{2}], \Delta_\rho[-\frac{1}{2}, -\frac{1}{2}]; \pi_{sc})$
- $L(\Delta_\rho[-\frac{3}{2}, -\frac{3}{2}], \Delta_\rho[-\frac{1}{2}, -\frac{1}{2}], \Delta_\rho[-\frac{1}{2}, -\frac{1}{2}]; T^{\frac{5}{2}}_{I,1}(\pi_{sc}))$
- $L(\Delta_\rho[-\frac{5}{2}, -\frac{5}{2}], \Delta_\rho[-\frac{1}{2}, -\frac{3}{2}], \Delta_\rho[-\frac{1}{2}, -\frac{1}{2}]; \pi_{sc})$
- $L(\Delta_\rho[-\frac{5}{2}, -\frac{5}{2}], \Delta_\rho[-\frac{3}{2}, -\frac{3}{2}], \Delta_\rho[-\frac{1}{2}, -\frac{1}{2}], \Delta_\rho[-\frac{1}{2}, -\frac{1}{2}]; \pi_{sc})$

(b) $(\frac{1}{2}, \frac{3}{2}, \frac{3}{2}, \frac{5}{2})$*:*
- $L(\Delta_\rho[-\frac{3}{2}, -\frac{3}{2}], \Delta_\rho[-\frac{1}{2}, -\frac{1}{2}]; T^{\frac{3}{2}}_{I,1}(T^{\frac{5}{2}}_{I,1}(\pi_{sc})))$
- $L(\Delta_\rho[-\frac{3}{2}, -\frac{5}{2}], \Delta_\rho[-\frac{1}{2}, -\frac{3}{2}]; \pi_{sc})$

(c) $(\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{5}{2})$*:*

- $L(\Delta_\rho[-\frac{5}{2},-\frac{5}{2}],\Delta_\rho[-\frac{3}{2},-\frac{3}{2}],\Delta_\rho[-\frac{1}{2},-\frac{1}{2}];T_{I,1}^{\frac{5}{2}}(\pi_{sc}))$
- $L(\Delta_\rho[-\frac{5}{2},-\frac{5}{2}],\Delta_\rho[-\frac{1}{2},-\frac{3}{2}];T_{I,1}^{\frac{5}{2}}(\pi_{sc}))$

(d) $(\frac{1}{2},\frac{3}{2},\frac{5}{2},\frac{7}{2})$*:*
- $T_{I,1}^{\frac{1}{2}}(T_{I,1}^{\frac{3}{2}}(T_{I,1}^{\frac{7}{2}}(T_{I,1}^{\frac{5}{2}}(\pi_{sc}))))$
- $L(\Delta_\rho[-\frac{1}{2},-\frac{1}{2}];T_{I,1}^{\frac{3}{2}}(T_{I,1}^{\frac{7}{2}}(T_{I,1}^{\frac{5}{2}}(\pi_{sc}))))$
- $L(\Delta_\rho[-\frac{1}{2},-\frac{3}{2}];T_{I,1}^{\frac{7}{2}}(T_{I,1}^{\frac{5}{2}}(\pi_{sc})))$
- $L(\Delta_\rho[-\frac{3}{2},-\frac{3}{2}],\Delta_\rho[-\frac{1}{2},-\frac{1}{2}];T_{I,1}^{\frac{7}{2}}(T_{I,1}^{\frac{5}{2}}(\pi_{sc})))$
- $L(\Delta_\rho[-\frac{7}{2},-\frac{7}{2}],\Delta_\rho[-\frac{1}{2},-\frac{5}{2}];\pi_{sc})$
- $L(\Delta_\rho[-\frac{7}{2},-\frac{7}{2}],\Delta_\rho[-\frac{5}{2},-\frac{5}{2}],\Delta_\rho[-\frac{1}{2},-\frac{3}{2}];\pi_{sc})$
- $L(\Delta_\rho[-\frac{7}{2},-\frac{7}{2}],\Delta_\rho[-\frac{3}{2},-\frac{5}{2}],\Delta_\rho[-\frac{1}{2},-\frac{1}{2}];\pi_{sc})$
- $L(\Delta_\rho[-\frac{7}{2},-\frac{7}{2}],\Delta_\rho[-\frac{5}{2},-\frac{5}{2}],\Delta_\rho[-\frac{3}{2},-\frac{3}{2}],\Delta_\rho[-\frac{1}{2},-\frac{1}{2}];\pi_{sc})$

(e) $(\frac{3}{2},\frac{5}{2},\frac{5}{2},\frac{7}{2})$*:*
- $T_{I,1}^{\frac{5}{2}}(T_{I,1}^{\frac{3}{2}}(T_{I,1}^{\frac{7}{2}}(T_{I,1}^{\frac{5}{2}}(\pi_{sc}))))$
- $L(\Delta_\rho[-\frac{5}{2},-\frac{5}{2}],\Delta_\rho[-\frac{3}{2},-\frac{3}{2}];T_{I,1}^{\frac{7}{2}}(T_{I,1}^{\frac{5}{2}}(\pi_{sc})))$
- $L(\Delta_\rho[-\frac{5}{2},-\frac{7}{2}],\Delta_\rho[-\frac{3}{2},-\frac{5}{2}];\pi_{sc})$
- $L(\Delta_\rho[-\frac{7}{2},-\frac{7}{2}],\Delta_\rho[-\frac{5}{2},-\frac{5}{2}],\Delta_\rho[-\frac{3}{2},-\frac{3}{2}];T_{I,1}^{\frac{5}{2}}(\pi_{sc}))$

(f) $(\frac{3}{2},\frac{5}{2},\frac{7}{2},\frac{9}{2})$*:*
- $T_{I,1}^{\frac{3}{2}}(T_{I,1}^{\frac{9}{2}}(T_{I,1}^{\frac{7}{2}}(T_{I,1}^{\frac{5}{2}}(\pi_{sc}))))$
- $L(\Delta_\rho[-\frac{3}{2},-\frac{3}{2}];T_{I,1}^{\frac{9}{2}}(T_{I,1}^{\frac{7}{2}}(T_{I,1}^{\frac{5}{2}}(\pi_{sc}))))$
- $L(\Delta_\rho[-\frac{9}{2},-\frac{9}{2}],\Delta_\rho[-\frac{7}{2},-\frac{7}{2}],\Delta_\rho[-\frac{3}{2},-\frac{5}{2}];\pi_{sc})$
- $L(\Delta_\rho[-\frac{9}{2},-\frac{9}{2}],\Delta_\rho[-\frac{7}{2},-\frac{7}{2}],\Delta_\rho[-\frac{5}{2},-\frac{5}{2}],\Delta_\rho[-\frac{3}{2},-\frac{3}{2}];\pi_{sc})$

(g) $(\frac{5}{2},\frac{7}{2},\frac{9}{2},\frac{11}{2})$*:*
- $T_{I,1}^{\frac{11}{2}}(T_{I,1}^{\frac{9}{2}}(T_{I,1}^{\frac{7}{2}}(T_{I,1}^{\frac{5}{2}}(\pi_{sc}))))$
- $L(\Delta_\rho[-\frac{11}{2},-\frac{11}{2}],\Delta_\rho[-\frac{9}{2},-\frac{9}{2}],\Delta_\rho[-\frac{7}{2},-\frac{7}{2}],\Delta_\rho[-\frac{5}{2},-\frac{5}{2}];\pi_{sc})$

(7) $(\alpha=3)$*:*

(a) $(0,1,2,3)$*:*
- $T_{V,2}^{\pm}(T_{I,1}^{1}(T_{I,1}^{2}(T_{I,1}^{3}(\pi_{sc}))))$
- $L(\Delta_\rho[-1,-1];T_{IV,3}(T_{I,1}^{2}(T_{I,1}^{3}(\pi_{sc}))))$
- $L(\Delta_\rho[-1,-2];T_{IV,3}(T_{I,1}^{2}(\pi_{sc})))$
- $L(\Delta_\rho[0,-2];T_{I,1}^{3}(\pi_{sc}))$
- $L(\Delta_\rho[-1,-3];T_{IV,3}(\pi_{sc}))$
- $L(\Delta_\rho[0,-1];T_{I,1}^{2}(T_{I,1}^{3}(\pi_{sc})))$
- $L(\Delta_\rho[0,-3];\pi_{sc})$
- $L(\Delta_\rho[-2,-2],\Delta_\rho[0,-1];T_{I,1}^{3}(\pi_{sc}))$
- $L(\Delta_\rho[-2,-3],\Delta_\rho[0,-1];\pi_{sc})$
- $L(\Delta_\rho[-3,-3],\Delta_\rho[0,-2];\pi_{sc})$
- $L(\Delta_\rho[-2,-2],\Delta_\rho[-1,-1];T_{IV,3}(T_{I,1}^{3}(\pi_{sc})))$
- $L(\Delta_\rho[-2,-3],\Delta_\rho[-1,-1];T_{IV,3}(\pi_{sc}))$
- $L(\Delta_\rho[-3,-3],\Delta_\rho[-1,-2];T_{IV,3}(\pi_{sc}))$
- $L(\Delta_\rho[-3,-3],\Delta_\rho[-2,-2],\Delta_\rho[-1,-1];T_{IV,3}(\pi_{sc}))$

- $L(\Delta_\rho[-3,-3], \Delta_\rho[-2,-2], \Delta_\rho[0,-1]; \pi_{sc})$

(b) $(1,2,2,3)$*:*
- $L(\Delta_\rho[-2,-2], \Delta_\rho[-1,-1]; T^2_{I,1}(T^3_{I,1}(\pi_{sc})))$
- $L(\Delta_\rho[-2,-3], \Delta_\rho[-1,-2]; \pi_{sc})$

(c) $(1,2,3,3)$*:*
- $L(\Delta_\rho[-3,-3], \Delta_\rho[-2,-2], \Delta_\rho[-1,-1]; T^3_{I,1}(\pi_{sc}))$
- $L(\Delta_\rho[-3,-3], \Delta_\rho[-1,-2]; T^3_{I,1}(\pi_{sc}))$

(d) $(1,2,3,4)$*:*
- $T^1_{I,1}(T^2_{I,1}(T^4_{I,1}(T^3_{I,1}(\pi_{sc}))))$
- $L(\Delta_\rho[-1,-1]; T^2_{I,1}(T^4_{I,1}(T^3_{I,1}(\pi_{sc}))))$
- $L(\Delta_\rho[-1,-2]; T^4_{I,1}(T^3_{I,1}(\pi_{sc})))$
- $L(\Delta_\rho[-2,-2], \Delta_\rho[-1,-1]; T^4_{I,1}(T^3_{I,1}(\pi_{sc})))$
- $L(\Delta_\rho[-4,-4], \Delta_\rho[-1,-3]; \pi_{sc})$
- $L(\Delta_\rho[-4,-4], \Delta_\rho[-2,-3], \Delta_\rho[-1,-1]; \pi_{sc})$
- $L(\Delta_\rho[-4,-4], \Delta_\rho[-3,-3], \Delta_\rho[-1,-2]; \pi_{sc})$
- $L(\Delta_\rho[-4,-4], \Delta_\rho[-3,-3], \Delta_\rho[-2,-2], \Delta_\rho[-1,-1]; \pi_{sc})$

(e) $(2,3,3,4)$*:*
- $T^3_{I,1}(T^4_{I,1}(T^3_{I,1}(T^2_{I,1}(\pi_{sc}))))$
- $L(\Delta_\rho[-3,-3], \Delta_\rho[-2,-2]; T^4_{I,1}(T^3_{I,1}(\pi_{sc})))$
- $L(\Delta_\rho[-3,-4], \Delta_\rho[-2,-3]; \pi_{sc})$
- $L(\Delta_\rho[-4,-4], \Delta_\rho[-3,-3], \Delta_\rho[-2,-2]; T^3_{I,1}(\pi_{sc}))$

(f) $(2,3,4,5)$*:*
- $T^2_{I,1}(T^5_{I,1}(T^4_{I,1}(T^3_{I,1}(\pi_{sc}))))$
- $L(\Delta_\rho[-2,-2]; T^5_{I,1}(T^4_{I,1}(T^3_{I,1}(\pi_{sc}))))$
- $L(\Delta_\rho[-5,-5], \Delta_\rho[-4,-4], \Delta_\rho[-2,-3]; \pi_{sc})$
- $L(\Delta_\rho[-5,-5], \Delta_\rho[-4,-4], \Delta_\rho[-3,-3], \Delta_\rho[-2,-2]; \pi_{sc})$

(g) $(3,4,5,6)$*:*
- $T^6_{I,1}(T^5_{I,1}(T^4_{I,1}(T^3_{I,1}(\pi_{sc}))))$
- $L(\Delta_\rho[-6,-6], \Delta_\rho[-5,-5], \Delta_\rho[-4,-4], \Delta_\rho[-3,-3]; \pi_{sc})$

(8) $(\alpha = \frac{7}{2})$*:*

(a) $(\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2})$*:*
- $T^{\frac{1}{2}}_{I,1}(T^{\frac{3}{2}}_{I,1}(T^{\frac{5}{2}}_{I,1}(T^{\frac{7}{2}}_{I,1}(\pi_{sc}))))$
- $L(\Delta_\rho[-\frac{1}{2},-\frac{1}{2}]; T^{\frac{3}{2}}_{I,1}T^{\frac{5}{2}}_{I,1}(T^{\frac{7}{2}}_{I,1}(\pi_{sc})))$
- $L(\Delta_\rho[-\frac{1}{2},-\frac{3}{2}]; T^{\frac{5}{2}}_{I,1}(T^{\frac{7}{2}}_{I,1}(\pi_{sc})))$
- $L(\Delta_\rho[-\frac{1}{2},-\frac{5}{2}]; T^{\frac{7}{2}}_{I,1}(\pi_{sc}))$
- $L(\Delta_\rho[-\frac{1}{2},-\frac{7}{2}]; \pi_{sc})$
- $L(\Delta_\rho[-\frac{3}{2},-\frac{3}{2}], \Delta_\rho[-\frac{1}{2},-\frac{1}{2}]; T^{\frac{5}{2}}_{I,1}(T^{\frac{7}{2}}_{I,1}(\pi_{sc})))$
- $L(\Delta_\rho[-\frac{3}{2},-\frac{5}{2}], \Delta_\rho[-\frac{1}{2},-\frac{1}{2}]; T^{\frac{7}{2}}_{I,1}(\pi_{sc}))$
- $L(\Delta_\rho[-\frac{3}{2},-\frac{7}{2}], \Delta_\rho[-\frac{1}{2},-\frac{1}{2}]; \pi_{sc})$
- $L(\Delta_\rho[-\frac{5}{2},-\frac{5}{2}], \Delta_\rho[-\frac{1}{2},-\frac{3}{2}]; T^{\frac{7}{2}}_{I,1}(\pi_{sc}))$
- $L(\Delta_\rho[-\frac{5}{2},-\frac{7}{2}], \Delta_\rho[-\frac{1}{2},-\frac{3}{2}]; \pi_{sc})$
- $L(\Delta_\rho[-\frac{7}{2},-\frac{7}{2}], \Delta_\rho[-\frac{1}{2},-\frac{5}{2}]; \pi_{sc})$

- $L(\Delta_\rho[-\frac{5}{2},-\frac{5}{2}],\Delta_\rho[-\frac{3}{2},-\frac{3}{2}],\Delta_\rho[-\frac{1}{2},-\frac{1}{2}];T_{I,1}^{\frac{7}{2}}(\pi_{sc}))$
- $L(\Delta_\rho[-\frac{5}{2},-\frac{7}{2}],\Delta_\rho[-\frac{3}{2},-\frac{3}{2}],\Delta_\rho[-\frac{1}{2},-\frac{1}{2}];\pi_{sc})$
- $L(\Delta_\rho[-\frac{7}{2},-\frac{7}{2}],\Delta_\rho[-\frac{3}{2},-\frac{5}{2}],\Delta_\rho[-\frac{1}{2},-\frac{1}{2}];\pi_{sc})$
- $L(\Delta_\rho[-\frac{7}{2},-\frac{7}{2}],\Delta_\rho[-\frac{5}{2},-\frac{5}{2}],\Delta_\rho[-\frac{1}{2},-\frac{3}{2}];\pi_{sc})$
- $L(\Delta_\rho[-\frac{7}{2},-\frac{7}{2}],\Delta_\rho[-\frac{5}{2},-\frac{5}{2}],\Delta_\rho[-\frac{3}{2},-\frac{3}{2}],\Delta_\rho[-\frac{1}{2},-\frac{1}{2}];\pi_{sc})$

(b) $(\frac{3}{2},\frac{5}{2},\frac{5}{2},\frac{7}{2})$:
- $L(\Delta_\rho[-\frac{5}{2},-\frac{5}{2}],\Delta_\rho[-\frac{3}{2},-\frac{3}{2}];T_{I,1}^{\frac{5}{2}}(T_{I,1}^{\frac{7}{2}}(\pi_{sc})))$
- $L(\Delta_\rho[-\frac{5}{2},-\frac{7}{2}],\Delta_\rho[-\frac{3}{2},-\frac{5}{2}];\pi_{sc})$

(c) $(\frac{3}{2},\frac{5}{2},\frac{7}{2},\frac{7}{2})$:
- $L(\Delta_\rho[-\frac{7}{2},-\frac{7}{2}],\Delta_\rho[-\frac{3}{2},-\frac{5}{2}];T_{I,1}^{\frac{7}{2}}(\pi_{sc}))$
- $L(\Delta_\rho[-\frac{7}{2},-\frac{7}{2}],\Delta_\rho[-\frac{5}{2},-\frac{5}{2}],\Delta_\rho[-\frac{3}{2},-\frac{3}{2}];T_{I,1}^{\frac{7}{2}}(\pi_{sc}))$

(d) $(\frac{3}{2},\frac{5}{2},\frac{7}{2},\frac{9}{2})$:
- $T_{I,1}^{\frac{3}{2}}(T_{I,1}^{\frac{5}{2}}(T_{I,1}^{\frac{9}{2}}(T_{I,1}^{\frac{7}{2}}(\pi_{sc}))))$
- $L(\Delta_\rho[-\frac{3}{2},-\frac{3}{2}];T_{I,1}^{\frac{5}{2}}(T_{I,1}^{\frac{9}{2}}(T_{I,1}^{\frac{7}{2}}(\pi_{sc}))))$
- $L(\Delta_\rho[-\frac{3}{2},-\frac{5}{2}];T_{I,1}^{\frac{9}{2}}(T_{I,1}^{\frac{7}{2}}(\pi_{sc})))$
- $L(\Delta_\rho[-\frac{5}{2},-\frac{5}{2}],\Delta_\rho[-\frac{3}{2},-\frac{3}{2}];T_{I,1}^{\frac{9}{2}}(T_{I,1}^{\frac{7}{2}}(\pi_{sc})))$
- $L(\Delta_\rho[-\frac{9}{2},-\frac{9}{2}],\Delta_\rho[-\frac{3}{2},-\frac{7}{2}];\pi_{sc})$
- $L(\Delta_\rho[-\frac{9}{2},-\frac{9}{2}],\Delta_\rho[-\frac{7}{2},-\frac{7}{2}],\Delta_\rho[-\frac{3}{2},-\frac{5}{2}];\pi_{sc})$
- $L(\Delta_\rho[-\frac{9}{2},-\frac{9}{2}],\Delta_\rho[-\frac{5}{2},-\frac{7}{2}],\Delta_\rho[-\frac{3}{2},-\frac{3}{2}];\pi_{sc})$
- $L(\Delta_\rho[-\frac{9}{2},-\frac{9}{2}],\Delta_\rho[-\frac{7}{2},-\frac{7}{2}],\Delta_\rho[-\frac{5}{2},-\frac{5}{2}],\Delta_\rho[-\frac{3}{2},-\frac{3}{2}];\pi_{sc})$

(e) $(\frac{5}{2},\frac{7}{2},\frac{7}{2},\frac{9}{2})$:
- $T_{I,1}^{\frac{7}{2}}(T_{I,1}^{\frac{5}{2}}(T_{I,1}^{\frac{9}{2}}(T_{I,1}^{\frac{7}{2}}(\pi_{sc}))))$.
- $L(\Delta_\rho[-\frac{7}{2},-\frac{7}{2}],\Delta_\rho[-\frac{5}{2},-\frac{5}{2}];T_{I,1}^{\frac{9}{2}}(T_{I,1}^{\frac{7}{2}}(\pi_{sc})))$
- $L(\Delta_\rho[-\frac{7}{2},-\frac{9}{2}],\Delta_\rho[-\frac{5}{2},-\frac{7}{2}];\pi_{sc})$
- $L(\Delta_\rho[-\frac{9}{2},-\frac{9}{2}],\Delta_\rho[-\frac{7}{2},-\frac{7}{2}],\Delta_\rho[-\frac{5}{2},-\frac{5}{2}];T_{I,1}^{\frac{7}{2}}(\pi_{sc}))$

(f) $(\frac{5}{2},\frac{7}{2},\frac{9}{2},\frac{11}{2})$:
- $T_{I,1}^{\frac{5}{2}}(T_{I,1}^{\frac{11}{2}}(T_{I,1}^{\frac{9}{2}}(T_{I,1}^{\frac{7}{2}}(\pi_{sc}))))$
- $L(\Delta_\rho[-\frac{5}{2},-\frac{5}{2}];T_{I,1}^{\frac{11}{2}}(T_{I,1}^{\frac{9}{2}}(T_{I,1}^{\frac{7}{2}}(\pi_{sc}))))$
- $L(\Delta_\rho[-\frac{11}{2},-\frac{11}{2}],\Delta_\rho[-\frac{9}{2},-\frac{9}{2}],\Delta_\rho[-\frac{5}{2},-\frac{7}{2}];\pi_{sc})$
- $L(\Delta_\rho[-\frac{11}{2},-\frac{11}{2}],\Delta_\rho[-\frac{9}{2},-\frac{9}{2}],\Delta_\rho[-\frac{7}{2},-\frac{7}{2}],\Delta_\rho[-\frac{5}{2},-\frac{5}{2}];\pi_{sc})$

(g) $(\frac{7}{2},\frac{9}{2},\frac{11}{2},\frac{13}{2})$:
- $T_{I,1}^{\frac{7}{2}}(T_{I,1}^{\frac{9}{2}}(T_{I,1}^{\frac{11}{2}}(T_{I,1}^{\frac{13}{2}}(\pi_{sc}))))$
- $L(\Delta_\rho[-\frac{13}{2},-\frac{13}{2}],\Delta_\rho[-\frac{11}{2},-\frac{11}{2}],\Delta_\rho[-\frac{9}{2},-\frac{9}{2}],\Delta_\rho[-\frac{7}{2},-\frac{7}{2}];\pi_{sc})$

(9) $(\alpha\geq 4)$:

(a) $(\alpha-3,\alpha-2,\alpha-1,\alpha)$:
- $T_{I,1}^{\alpha-3}(T_{I,1}^{\alpha-2}(T_{I,1}^{\alpha-1}(T_{I,1}^{\alpha}(\pi_{sc}))))$
- $L(\Delta_\rho[-\alpha+3,-\alpha+2];T_{I,1}^{\alpha-1}(T_{I,1}^{\alpha}(\pi_{sc})))$
- $L(\Delta_\rho[-\alpha+3,-\alpha+3];T_{I,1}^{\alpha-2}(T_{I,1}^{\alpha-1}(T_{I,1}^{\alpha}(\pi_{sc}))))$

- $L(\Delta_\rho[-\alpha+3,-\alpha+1];T^{\alpha}_{I,1}(\pi_{sc}))$
- $L(\Delta_\rho[-\alpha+3,-\alpha];\pi_{sc})$
- $L(\Delta_\rho[-\alpha+2,-\alpha+2],\Delta_\rho[-\alpha+3,-\alpha+3];T^{\alpha-1}_{I,1}(T^{\alpha}_{I,1}(\pi_{sc})))$
- $L(\Delta_\rho[-\alpha,-\alpha],\Delta_\rho[-\alpha+3,-\alpha+1];\pi_{sc})+$
- $L(\Delta_\rho[-\alpha+2,-\alpha+1],\Delta_\rho[-\alpha+3,-\alpha+3];T^{\alpha}_{I,1}(\pi_{sc}))$
- $L(\Delta_\rho[-\alpha+1,-\alpha+1],\Delta_\rho[-\alpha+3,-\alpha+2];T^{\alpha}_{I,1}(\pi_{sc}))$
- $L(\Delta_\rho[-\alpha+1,-\alpha],\Delta_\rho[-\alpha+3,-\alpha+2];\pi_{sc})$
- $L(\Delta_\rho[-\alpha+2,-\alpha],\Delta_\rho[-\alpha+3,-\alpha+3];\pi_{sc})$
- $L(\Delta_\rho[-\alpha+1,-\alpha+1],\Delta_\rho[-\alpha+2,-\alpha+2],\Delta_\rho[-\alpha+3,-\alpha+3];T^{\alpha}_{I,1}(\pi_{sc}))$
- $L(\Delta_\rho[-\alpha+1,-\alpha],\Delta_\rho[-\alpha+2,-\alpha+2],\Delta_\rho[-\alpha+3,-\alpha+3];\pi_{sc})$
- $L(\Delta_\rho[-\alpha,-\alpha],\Delta_\rho[-\alpha+2,-\alpha+1],\Delta_\rho[-\alpha+3,-\alpha+3];\pi_{sc})$
- $L(\Delta_\rho[-\alpha,-\alpha],\Delta_\rho[-\alpha+1,-\alpha+1],\Delta_\rho[-\alpha+3,-\alpha+2];\pi_{sc})$
- $L(\Delta_\rho[-\alpha,-\alpha],\Delta_\rho[-\alpha+1,-\alpha+1],\Delta_\rho[-\alpha+2,-\alpha+2],\Delta_\rho[-\alpha+3,-\alpha+3];\pi_{sc})$

(b) $(\alpha-2,\alpha-1,\alpha-1,\alpha)$*:*
- $L(\Delta_\rho[-\alpha+1,-\alpha+1],\Delta_\rho[-\alpha+2,-\alpha+2];T^{\alpha-1}_{I,1}(T^{\alpha}_{I,1}(\pi_{sc})))$
- $L(\Delta_\rho[-\alpha+1,-\alpha],\Delta_\rho[-\alpha+2,-\alpha+1];\pi_{sc})$

(c) $(\alpha-2,\alpha-1,\alpha,\alpha)$*:*
- $L(\Delta_\rho[-\alpha,-\alpha];\Delta_\rho[-\alpha+2,-\alpha+1];T^{\alpha}_{I,1}(\pi_{sc}))$
- $L(\Delta_\rho[-\alpha,-\alpha],\Delta_\rho[-\alpha+1,-\alpha+1],\Delta_\rho[-\alpha+2,-\alpha+2];T^{\alpha}_{I,1}(\pi_{sc}))$

(d) $(\alpha-2,\alpha-1,\alpha,\alpha+1)$*:*
- $T^{\alpha-2}_{I,1}(T^{\alpha-1}_{I,1}(T^{\alpha+1}_{I,1}(T^{\alpha}_{I,1}(\pi_{sc}))))$
- $L(\Delta_\rho[-\alpha+2,-\alpha+2];T^{\alpha-1}_{I,1}(T^{\alpha+1}_{I,1}(T^{\alpha}_{I,1}(\pi_{sc}))))$
- $L(\Delta_\rho[-\alpha+2,-\alpha+1];T^{\alpha+1}_{I,1}(T^{\alpha}_{I,1}(\pi_{sc})))$
- $L(\Delta_\rho[-\alpha+1,-\alpha+1],\Delta_\rho[-\alpha+2,-\alpha+2];T^{\alpha+1}_{I,1}(T^{\alpha}_{I,1}(\pi_{sc})))$
- $L(\Delta_\rho[-\alpha-1,-\alpha-1],\Delta_\rho[-\alpha+2,-\alpha];\pi_{sc})$
- $L(\Delta_\rho[-\alpha-1,-\alpha-1],\Delta_\rho[-\alpha+1,-\alpha],\Delta_\rho[-\alpha+2,-\alpha+2];\pi_{sc})$
- $L(\Delta_\rho[-\alpha-1,-\alpha-1],\Delta_\rho[-\alpha,-\alpha],\Delta_\rho[-\alpha+2,-\alpha+1];\pi_{sc})$
- $L(\Delta_\rho[-\alpha-1,-\alpha-1],\Delta_\rho[-\alpha,-\alpha],\Delta_\rho[-\alpha+1,-\alpha+1],\Delta_\rho[-\alpha+2,-\alpha+2];\pi_{sc})$

(e) $(\alpha-1,\alpha,\alpha,\alpha+1)$*:*
- $T^{\alpha}_{I,1}(T^{\alpha-1}_{I,1}(T^{\alpha+1}_{I,1}(T^{\alpha}_{I,1}(\pi_{sc}))))$
- $L(\Delta_\rho[-\alpha,-\alpha],\Delta_\rho[-\alpha+1,-\alpha+1];T^{\alpha+1}_{I,1}(T^{\alpha}_{I,1}(\pi_{sc})))$
- $L(\Delta_\rho[-\alpha-1,-\alpha-1],\Delta_\rho[-\alpha+2,-\alpha];\pi_{sc})$
- $L(\Delta_\rho[-\alpha-1,-\alpha-1],\Delta_\rho[-\alpha,-\alpha],\Delta\rho[-\alpha+1,-\alpha+1];\pi_{sc})$

(f) $(\alpha-1,\alpha,\alpha+1,\alpha+2)$*:*
- $T^{\alpha-1}_{I,1}(T^{\alpha+2}_{I,1}(T^{\alpha+1}_{I,1}(T^{\alpha}_{I,1}(\pi_{sc}))))$
- $L(\Delta_\rho[-\alpha+1,-\alpha+1];T^{\alpha+2}_{I,1}(T^{\alpha+1}_{I,1}(T^{\alpha}_{I,1}(\pi_{sc}))))$
- $L(\Delta_\rho[-\alpha-2,-\alpha-2],\Delta_\rho[-\alpha-1,-\alpha-1],\Delta_\rho[-\alpha+1,-\alpha];\pi_{sc})$
- $L(\Delta_\rho[-\alpha-2,-\alpha-2],\Delta_\rho[-\alpha-1,-\alpha-1],\Delta_\rho[-\alpha,-\alpha],\Delta_\rho[-\alpha+1,-\alpha+1];\pi_{sc})$

(g) $(\alpha,\alpha+1,\alpha+2,\alpha+3)$:
- $T^{\alpha+3}_{I,1}(T^{\alpha+2}_{I,1}(T^{\alpha+1}_{I,1}(T^{\alpha}_{I,1}(\pi_{sc}))))$
- $L(\Delta_\rho[-\alpha-3,-\alpha-3],\Delta_\rho[-\alpha-2,-\alpha-2],\Delta_\rho[-\alpha-1,-\alpha-1],\Delta_\rho[-\alpha,-\alpha];\pi_{sc})$

Proof. This follows directly from Propositions 4.6 to 6.18. □

Appendix B.
List of representations of corank 4 that are of critical type but not of Arthur type, sorted by cuspidal support

In this section, we will give the opposite list as the one given in the previous section, that is, the list of all representations of corank 4 that are of critical type but not of Arthur type. This is given in Proposition B.1. Combining the two lists given in Proposition A.1 and Proposition B.1 together will allow us to classify the unitarity of all critical points, a crucial step in the construction of the unitary dual.

Same as before, we will sort our list of representations as subquotients of $\Pi_{x_1,x_2,x_3,x_4} := \rho|\cdot|^{x_1} \times \rho|\cdot|^{x_2} \times \rho|\cdot|^{x_3} \times \rho|\cdot|^{x_4} \rtimes \pi_{sc}$. By Theorem 2.5, this list gives exactly the set of representations that are of critical type but not unitarizable. Without loss of generality, we may assume $0 \leq x_1 \leq x_2 \leq x_3 \leq x_4$.

Proposition B.1. *Let* $\Pi_{x_1,x_2,x_3,x_4} = \rho|\cdot|^{x_1} \times \rho|\cdot|^{x_2} \times \rho|\cdot|^{x_3} \times \rho|\cdot|^{x_4} \rtimes \pi_{sc}$ *for* $0 \leq x_1 \leq x_2 \leq x_3 \leq x_4$. *Then for a given quadruple* (x_1, x_2, x_3, x_4), *the following list contains all its irreducible subquotients that are of critical type but not of Arthur type.*

(1) $(\alpha = 0)$:
- *(a)* $(0, 0, 1, 2)$:
  - $L(\Delta_\rho[-1, -2]; T^{\pm}_{V,4}(\pi_{sc}))$
  - $L(\Delta_\rho[-2, -2]; T^1_{I,1}(T^{\pm}_{V,4}(\pi_{sc}))))$
  - $L(\Delta_\rho[0, -2]; T^{\pm}_{V,2}(\pi_{sc}))$
  - $L(\Delta_\rho[-2, -2], \Delta_\rho[0, -1]; T^{\pm}_{V,2}(\pi_{sc}))$
- *(b)* $(0, 1, 1, 1)$:
  - $L(\Delta_\rho[-1, -1]; T^1_{I,2}(T^{\pm}_{V,2}(\pi_{sc})))$
  - $L(\Delta_\rho[-1, -1], \Delta_\rho[-1, -1]; T^1_{I,1}(T^{\pm}_{V,2}(\pi_{sc})))$
  - $L(\Delta_\rho[-1, -1], \Delta_\rho[-1, -1], \Delta_\rho[0, -1]; \pi_{sc})$
  - $L(\Delta_\rho[-1, -1], \Delta_\rho[-1, -1], \Delta_\rho[-1, -1]; T^{\pm}_{V,2}(\pi_{sc}))$
- *(c)* $(0, 1, 1, 2)$:
  - $L(\Delta_\rho[-1, -2]; T^{\pm}_{V,2}(T^1_{I,1}(\pi_{sc}))$
  - $L(\Delta_\rho[-2, -2]; T^1_{I,2}(T^{\pm}_{V,2}(\pi_{sc}))))$
  - $L(\Delta_\rho[-1, -1], \Delta_\rho[0, -2]; \pi_{sc})$
  - $L(\Delta_\rho[-1, -2], \Delta_\rho[-1, -1]; T^{\pm}_{V,2}(\pi_{sc}))$
  - $L(\Delta_\rho[-2, -2], \Delta_\rho[-1, -1], \Delta_\rho[0, -1]; \pi_{sc})$
- *(d)* $(0, 1, 2, 2)$:
  - $L(\Delta_\rho[-2, -2], \Delta_\rho[0, -2]; \pi_{sc})$
  - $L(\Delta_\rho[-2, -2]; T^2_{I,1}(T^1_{I,1}(T^{\pm}_{V,2}(\pi_{sc}))))$
  - $L(\Delta_\rho[-2, -2], \Delta_\rho[-1, -2]; T^{\pm}_{V,2}(\pi_{sc}))$
  - $L(\Delta_\rho[-2, -2], \Delta_\rho[-2, -2]; T_{I,1}(T^{\pm}_{V,2}(\pi_{sc})))$
  - $L(\Delta_\rho[-2, -2], \Delta_\rho[-2, -2], \Delta_\rho[-1, -1]; T^{\pm}_{V,2}(\pi_{sc}))$
  - $L(\Delta_\rho[-2, -2], \Delta_\rho[-2, -2], \Delta_\rho[0, -1]; \pi_{sc})$
- *(e)* $(0, 1, 2, 3)$:
  - $L(\Delta_\rho[0, -3]; \pi_{sc})$
  - $L(\Delta_\rho[-1, -3]; T^{\pm}_{V,2}(\pi_{sc}))$
  - $L(\Delta_\rho[-2, -3]; T^1_{I,1}(T^{\pm}_{V,2}(\pi_{sc})))$
  - $L(\Delta_\rho[-3, -3]; T^2_{I,1}(T^1_{I,1}(T^{\pm}_{V,2}(\pi_{sc}))))$

- $L(\Delta_\rho[-2,-3],\Delta_\rho[0,-1];\pi_{sc})$
- $L(\Delta_\rho[-2,-3],\Delta_\rho[-1,-1];T^{\pm}_{V,2}(\pi_{sc}))$
- $L(\Delta_\rho[-3,-3],\Delta_\rho[0,-2];\pi_{sc})$
- $L(\Delta_\rho[-3,-3],\Delta_\rho[-1,-2];T^{\pm}_{V,2}(\pi_{sc}))$
- $L(\Delta_\rho[-3,-3],\Delta_\rho[-2,-2];T^{1}_{I,1}(T^{\pm}_{V,2}(\pi_{sc})))$
- $L(\Delta_\rho[-3,-3],\Delta_\rho[-2,-2],\Delta_\rho[0,-1];\pi_{sc})$

(2) $(\alpha=\frac{1}{2})$ *:*

(a) $(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{3}{2})$*:*

- $L(\Delta_\rho[-\frac{3}{2},-\frac{3}{2}];T^{\frac{1}{2}}_{I,3}(\pi_{sc}))$
- $L(\Delta_\rho[-\frac{1}{2},-\frac{3}{2}],\Delta_\rho[-\frac{1}{2},-\frac{1}{2}],\Delta_\rho[-\frac{1}{2},-\frac{1}{2}];\pi_{sc})$

(b) $(\frac{1}{2},\frac{1}{2},\frac{3}{2},\frac{3}{2})$*:*

- $L(\Delta_\rho[-\frac{1}{2},-\frac{3}{2}];T^{\frac{3}{2}}_{I,1}(T^{\frac{1}{2}}_{I,1}(\pi_{sc})))$
- $L(\Delta_\rho[-\frac{3}{2},-\frac{3}{2}];T^{\frac{1}{2}}_{I,1}(T^{\frac{3}{2}}_{I,1}(T^{\frac{1}{2}}_{I,1}(\pi_{sc}))))$
- $L(\Delta_\rho[-\frac{1}{2},-\frac{3}{2}],\Delta_\rho[-\frac{1}{2},-\frac{3}{2}];\pi_{sc})$
- $L(\Delta_\rho[-\frac{3}{2},-\frac{3}{2}],\Delta_\rho[-\frac{1}{2},-\frac{3}{2}];T^{\frac{1}{2}}_{I,1}(\pi_{sc}))$
- $L(\Delta_\rho[-\frac{3}{2},-\frac{3}{2}],\Delta_\rho[\frac{1}{2},-\frac{3}{2}];\pi_{sc})$
- $L(\Delta_\rho[-\frac{3}{2},-\frac{3}{2}],\Delta_\rho[-\frac{3}{2},-\frac{3}{2}];T^{\frac{1}{2}}_{I,2}(\pi_{sc}))$
- $L(\Delta_\rho[-\frac{3}{2},-\frac{3}{2}],\Delta_\rho[-\frac{3}{2},-\frac{3}{2}];T^{\frac{1}{2}}_{III,2}(\pi_{sc}))$
- $L(\Delta_\rho[-\frac{3}{2},-\frac{3}{2}],\Delta_\rho[-\frac{1}{2},-\frac{3}{2}],\Delta_\rho[-\frac{1}{2},-\frac{1}{2}];\pi_{sc})$

(c) $(\frac{1}{2},\frac{1}{2},\frac{3}{2},\frac{5}{2})$*:*

- $L(\Delta_\rho[-\frac{1}{2},-\frac{5}{2}];T^{\frac{1}{2}}_{I,1}(\pi_{sc}))$
- $L(\Delta_\rho[-\frac{3}{2},-\frac{5}{2}];T^{\frac{1}{2}}_{I,2}(\pi_{sc}))$
- $L(\Delta_\rho[-\frac{5}{2},-\frac{5}{2}];T^{\frac{3}{2}}_{I,1}(T^{\frac{1}{2}}_{I,2}(\pi_{sc})))$
- $L(\Delta_\rho[-\frac{5}{2},-\frac{5}{2}];T^{\frac{3}{2}}_{I,1}(T^{\frac{1}{2}}_{III,2}(\pi_{sc}))))$
- $L(\Delta_\rho[-\frac{1}{2},-\frac{5}{2}],\Delta_\rho[-\frac{1}{2},-\frac{1}{2}];\pi_{sc})$
- $L(\Delta_\rho[\frac{1}{2},-\frac{5}{2}];\pi_{sc}))$
- $L(\Delta_\rho[-\frac{3}{2},-\frac{5}{2}],\Delta_\rho[-\frac{1}{2},-\frac{1}{2}];T^{\frac{1}{2}}_{I,1}(\pi_{sc}))$
- $L(\Delta_\rho[-\frac{5}{2},-\frac{5}{2}],\Delta_\rho[-\frac{1}{2},-\frac{1}{2}];T^{\frac{3}{2}}_{I,1}(T^{\frac{1}{2}}_{I,1}(\pi_{sc})))$
- $L(\Delta_\rho[-\frac{5}{2},-\frac{5}{2}],\Delta_\rho[-\frac{3}{2},-\frac{3}{2}];T^{\frac{1}{2}}_{I,2}(\pi_{sc}))$
- $L(\Delta_\rho[-\frac{5}{2},-\frac{5}{2}],\Delta_\rho[-\frac{3}{2},-\frac{3}{2}];T^{\frac{1}{2}}_{III,2}(\pi_{sc})))$
- $L(\Delta_\rho[-\frac{5}{2},-\frac{5}{2}],\Delta_\rho[-\frac{1}{2},-\frac{3}{2}];T^{\frac{1}{2}}_{I,1}(\pi_{sc}))$ *with* $\epsilon_{sc}(\rho\otimes S_2)=1$
- $L(\Delta_\rho[-\frac{5}{2},-\frac{5}{2}],\Delta_\rho[\frac{1}{2},-\frac{3}{2}];\pi_{sc})$
- $L(\Delta_\rho[-\frac{5}{2},-\frac{5}{2}],\Delta_\rho[-\frac{1}{2},-\frac{3}{2}],\Delta_\rho[-\frac{1}{2},-\frac{1}{2}];\pi_{sc})$
- $L(\Delta_\rho[-\frac{5}{2},-\frac{5}{2}],\Delta_\rho[-\frac{3}{2},-\frac{3}{2}],\Delta_\rho[-\frac{1}{2},-\frac{1}{2}];T^{\frac{1}{2}}_{I,1}(\pi_{sc}))$

(d) $(\frac{1}{2},\frac{3}{2},\frac{3}{2},\frac{3}{2})$*:*

- $L(\Delta_\rho[-\frac{3}{2},-\frac{3}{2}],\Delta_\rho[-\frac{3}{2},-\frac{3}{2}];T^{\frac{3}{2}}_{I,1}(T^{\frac{1}{2}}_{I,1}(\pi_{sc})))$
- $L(\Delta_\rho[-\frac{3}{2},-\frac{3}{2}],\Delta_\rho[-\frac{3}{2},-\frac{3}{2}],\Delta_\rho[-\frac{1}{2},-\frac{3}{2}];\pi_{sc}))$

- $L(\Delta_\rho[-\frac{3}{2},-\frac{3}{2}],\Delta_\rho[-\frac{3}{2},-\frac{3}{2}],\Delta_\rho[-\frac{3}{2},-\frac{3}{2}];T_{I,1}^{\frac{1}{2}}(\pi_{sc}))$
- $L(\Delta_\rho[-\frac{3}{2},-\frac{3}{2}],\Delta_\rho[-\frac{3}{2},-\frac{3}{2}],\Delta_\rho[-\frac{3}{2},-\frac{3}{2}],\Delta_\rho[-\frac{1}{2},-\frac{1}{2}];\pi_{sc})$

(e) $(\frac{1}{2},\frac{3}{2},\frac{3}{2},\frac{5}{2})$*:*

- $L(\Delta_\rho[-\frac{3}{2},-\frac{3}{2}];T_{I,1}^{\frac{5}{2}}(T_{I,1}^{\frac{3}{2}}(T_{I,1}^{\frac{1}{2}}(\pi_{sc}))))$
- $L(\Delta_\rho[-\frac{3}{2},-\frac{5}{2}];T_{I,1}^{\frac{3}{2}}(T_{I,1}^{\frac{1}{2}}(\pi_{sc})))$
- $L(\Delta_\rho[-\frac{3}{2},-\frac{3}{2}],\Delta_\rho[-\frac{1}{2},-\frac{5}{2}];\pi_{sc})$
- $L(\Delta_\rho[-\frac{3}{2},-\frac{5}{2}],\Delta_\rho[-\frac{1}{2},-\frac{3}{2}];\pi_{sc})$
- $L(\Delta_\rho[-\frac{3}{2},-\frac{5}{2}],\Delta_\rho[-\frac{3}{2},-\frac{3}{2}];T_{I,1}^{\frac{1}{2}}(\pi_{sc}))$
- $L(\Delta_\rho[-\frac{5}{2},-\frac{5}{2}],\Delta_\rho[-\frac{3}{2},-\frac{3}{2}];T_{I,1}^{\frac{3}{2}}(T_{I,1}^{\frac{1}{2}}(\pi_{sc})))$
- $L(\Delta_\rho[-\frac{3}{2},-\frac{5}{2}],\Delta_\rho[-\frac{3}{2},-\frac{3}{2}],\Delta_\rho[-\frac{1}{2},-\frac{1}{2}];\pi_{sc})$
- $L(\Delta_\rho[-\frac{5}{2},-\frac{5}{2}],\Delta_\rho[-\frac{3}{2},-\frac{3}{2}],\Delta_\rho[-\frac{1}{2},-\frac{3}{2}];\pi_{sc})$
- $L(\Delta_\rho[-\frac{5}{2},-\frac{5}{2}],\Delta_\rho[-\frac{3}{2},-\frac{3}{2}],\Delta_\rho[-\frac{3}{2},-\frac{3}{2}];T_{I,1}^{\frac{1}{2}}(\pi_{sc}))$
- $L(\Delta_\rho[-\frac{5}{2},-\frac{5}{2}],\Delta_\rho[-\frac{3}{2},-\frac{3}{2}],\Delta_\rho[-\frac{3}{2},-\frac{3}{2}],\Delta_\rho[-\frac{1}{2},-\frac{1}{2}];\pi_{sc})$

(f) $(\frac{1}{2},\frac{3}{2},\frac{5}{2},\frac{5}{2})$*:*

- $L(\Delta_\rho[-\frac{5}{2},-\frac{5}{2}];T_{I,1}^{\frac{5}{2}}(T_{I,1}^{\frac{3}{2}}(T_{I,1}^{\frac{1}{2}}(\pi_{sc}))))$
- $L(\Delta_\rho[-\frac{5}{2},-\frac{5}{2}],\Delta_\rho[-\frac{1}{2},-\frac{5}{2}];\pi_{sc})$
- $L(\Delta_\rho[-\frac{5}{2},-\frac{5}{2}],\Delta_\rho[-\frac{3}{2},-\frac{5}{2}];T_{I,1}^{\frac{1}{2}}(\pi_{sc}))$
- $L(\Delta_\rho[-\frac{5}{2},-\frac{5}{2}],\Delta_\rho[-\frac{5}{2},-\frac{5}{2}];T_{I,1}^{\frac{3}{2}}(T_{I,1}^{\frac{1}{2}}(\pi_{sc})))$
- $L(\Delta_\rho[-\frac{5}{2},-\frac{5}{2}],\Delta_\rho[-\frac{3}{2},-\frac{5}{2}],\Delta_\rho[-\frac{1}{2},-\frac{1}{2}];\pi_{sc})$
- $L(\Delta_\rho[-\frac{5}{2},-\frac{5}{2}],\Delta_\rho[-\frac{5}{2},-\frac{5}{2}],\Delta_\rho[-\frac{3}{2},-\frac{3}{2}],T_{I,1}^{\frac{1}{2}}(\pi_{sc}))$
- $L(\Delta_\rho[-\frac{5}{2},-\frac{5}{2}],\Delta_\rho[-\frac{5}{2},-\frac{5}{2}],\Delta_\rho[-\frac{1}{2},-\frac{3}{2}];\pi_{sc})$
- $L(\Delta_\rho[-\frac{5}{2},-\frac{5}{2}],\Delta_\rho[-\frac{5}{2},-\frac{5}{2}],\Delta_\rho[-\frac{3}{2},-\frac{3}{2}],\Delta_\rho[-\frac{1}{2},-\frac{1}{2}];\pi_{sc})$

(g) $(\frac{1}{2},\frac{3}{2},\frac{5}{2},\frac{7}{2})$*:*

- $L(\Delta_\rho[-\frac{1}{2},-\frac{7}{2}];\pi_{sc})$
- $L(\Delta_\rho[-\frac{3}{2},-\frac{7}{2}];T_{I,1}^{\frac{1}{2}}(\pi_{sc}))$
- $L(\Delta_\rho[-\frac{5}{2},-\frac{7}{2}];T_{I,1}^{\frac{3}{2}}(T_{I,1}^{\frac{1}{2}}(\pi_{sc})))$
- $L(\Delta_\rho[-\frac{7}{2},-\frac{7}{2}];T_{I,1}^{\frac{5}{2}}(T_{I,1}^{\frac{3}{2}}(T_{I,1}^{\frac{1}{2}}(\pi_{sc}))))$
- $L(\Delta_\rho[-\frac{3}{2},-\frac{7}{2}],\Delta_\rho[-\frac{1}{2},-\frac{1}{2}];\pi_{sc})$
- $L(\Delta_\rho[-\frac{5}{2},-\frac{7}{2}],\Delta_\rho[-\frac{1}{2},-\frac{3}{2}];\pi_{sc})$
- $L(\Delta_\rho[-\frac{5}{2},-\frac{7}{2}],\Delta_\rho[-\frac{3}{2},-\frac{3}{2}];T_{I,1}^{\frac{1}{2}}(\pi_{sc}))$
- $L(\Delta_\rho[-\frac{7}{2},-\frac{7}{2}],\Delta_\rho[-\frac{1}{2},-\frac{5}{2}];\pi_{sc})$
- $L(\Delta_\rho[-\frac{7}{2},-\frac{7}{2}],\Delta_\rho[-\frac{3}{2},-\frac{5}{2}];T_{I,1}^{\frac{1}{2}}(\pi_{sc}))$
- $L(\Delta_\rho[-\frac{7}{2},-\frac{7}{2}],\Delta_\rho[-\frac{5}{2},-\frac{5}{2}];T_{I,1}^{\frac{3}{2}}(T_{I,1}^{\frac{1}{2}}(\pi_{sc})))$
- $L(\Delta_\rho[-\frac{5}{2},-\frac{7}{2}],\Delta_\rho[-\frac{3}{2},-\frac{3}{2}],\Delta_\rho[-\frac{1}{2},-\frac{1}{2}];\pi_{sc})$
- $L(\Delta_\rho[-\frac{7}{2},-\frac{7}{2}],\Delta_\rho[-\frac{5}{2},-\frac{5}{2}],\Delta_\rho[-\frac{3}{2},-\frac{3}{2}];T_{I,1}^{\frac{1}{2}}(\pi_{sc}))$
- $L(\Delta_\rho[-\frac{7}{2},-\frac{7}{2}],\Delta_\rho[-\frac{3}{2},-\frac{5}{2}],\Delta_\rho[-\frac{1}{2},-\frac{1}{2}];\pi_{sc})$
- $L(\Delta_\rho[-\frac{7}{2},-\frac{7}{2}],\Delta_\rho[-\frac{5}{2},-\frac{5}{2}],\Delta_\rho[-\frac{1}{2},-\frac{3}{2}];\pi_{sc})$

(3) $(\alpha = 1)$*:*

(a) $(0,0,1,2)$*:*
- $L(\Delta_\rho[-2,-2]; T^{\pm}_{V,4}(T^1_{I,1}(\pi_{sc})))$
- $L(\Delta_\rho[0,-2]; T_{IV,3}(\pi_{sc}))$
- $L(\Delta_\rho[-1,-2]; T_{IV,5}(\pi_{sc}))$
- $L(\Delta_\rho[-2,-2], \Delta_\rho[-1,-1], \Delta_\rho[0,-1]; \pi_{sc})$

(b) $(0,1,1,1)$*:*
- $L(\Delta_\rho[-1,-1], \Delta_\rho[0,-1]; T^1_{I,1}(\pi_{sc}))$
- $L(\Delta_\rho[-1,-1], \Delta_\rho[-1,-1], \Delta_\rho[0,-1]; \pi_{sc})$

(c) $(0,1,1,2)$*:*
- $L(\Delta_\rho[-2,-2]; T^1_{I,2}(T_{IV,3}(\pi_{sc})))$
- $L(\Delta_\rho[-1,-1], \Delta_\rho[0,-2]; \pi_{sc})$
- $L(\Delta_\rho[-1,-2], \Delta_\rho[-1,-1], T^{\pm}_{V,2}(\pi_{sc}))$
- $L(\Delta_\rho[-2,-2], \Delta_\rho[0,-1]; T^1_{I,1}(\pi_{sc}))$

(d) $(0,1,2,2)$*:*
- $L(\Delta_\rho[-2,-2]; T^{\pm}_{V,2}(T^2_{I,1}(T^1_{I,1}(\pi_{sc}))))$
- $L(\Delta_\rho[-2,-2], \Delta_\rho[0,-2]; \pi_{sc})$
- $L(\Delta_\rho[-2,-2], \Delta_\rho[-1,-2]; T_{IV,3}(\pi_{sc}))$
- $L(\Delta_\rho[-2,-2], \Delta_\rho[-2,-2]; T^{\pm}_{V,2}(T^1_{I,1}(\pi_{sc})))$
- $L(\Delta_\rho[-2,-2], \Delta_\rho[-2,-2], \Delta_\rho[-1,-1]; T_{IV,3}(\pi_{sc}))$
- $L(\Delta_\rho[-2,-2], \Delta_\rho[-2,-2], \Delta_\rho[0,-1]; \pi_{sc})$

(e) $(0,1,2,3)$*:*
- $L(\Delta_\rho[0,-3]; \pi_{sc})$
- $L(\Delta_\rho[-2,-3]; T^{\pm}_{V,2}(T^1_{I,1}(\pi_{sc})))$
- $L(\Delta_\rho[-3,-3]; T^{\pm}_{V,2}(T^2_{I,1}(T^1_{I,1}(\pi_{sc}))))$.
- $L(\Delta_\rho[-1,-3]; T_{IV,3}(\pi_{sc}))$
- $L(\Delta_\rho[-2,-3], \Delta_\rho[0,-1]; \pi_{sc})$
- $L(\Delta_\rho[-2,-3], \Delta_\rho[-1,-1]; T_{IV,3}(\pi_{sc}))$
- $L(\Delta_\rho[-2,-3], \Delta_\rho[-1,-1]; T^{\pm}_{V,2}(\pi_{sc}))$
- $L(\Delta_\rho[-3,-3], \Delta_\rho[0,-2]; \pi_{sc})$
- $L(\Delta_\rho[-3,-3], \Delta_\rho[-1,-2]; T_{IV,3}(\pi_{sc}))$
- $L(\Delta_\rho[-3,-3], \Delta_\rho[-2,-2]; T_{IV,3}(T^1_{I,1}(\pi_{sc})))$

(f) $(1,1,1,1)$*:*
- $L(\Delta_\rho[-1,-1], \Delta_\rho[-1,-1], \Delta_\rho[-1,-1]; T^1_{I,1}(\pi_{sc}))$
- $L(\Delta_\rho[-1,-1], \Delta_\rho[-1,-1], \Delta_\rho[-1,-1], \Delta_\rho[-1,-1]; \pi_{sc})$

(g) $(1,1,1,2)$*:*
- $L(\Delta_\rho[-1,-1], \Delta_\rho[-1,-1]; T^2_{I,1}(T^1_{I,1}(\pi_{sc})))$
- $L(\Delta_\rho[-1,-2], \Delta_\rho[-1,-1]; T^1_{I,1}(\pi_{sc}))$
- $L(\Delta_\rho[-2,-2], \Delta_\rho[-1,-1], \Delta_\rho[-1,-1]; T^1_{I,1}(\pi_{sc}))$
- $L(\Delta_\rho[-1,-2], \Delta_\rho[-1,-1], \Delta_\rho[-1,-1]; \pi_{sc})$
- $L(\Delta_\rho[-2,-2], \Delta_\rho[-1,-1], \Delta_\rho[-1,-1], \Delta_\rho[-1,-1]; \pi_{sc})$

(h) $(1,1,2,2)$*:*
- $L(\Delta_\rho[-1,-2]; T^2_{I,1}(T^1_{I,1}(\pi_{sc})))$
- $L(\Delta_\rho[-2,-2], \Delta_\rho[-1,-1]; T^2_{I,1}(T^1_{I,1}(\pi_{sc})))$
- $L(\Delta_\rho[-2,-2], \Delta_\rho[-1,-2]; T^1_{I,1}(\pi_{sc}))$

- $L(\Delta_\rho[-1,-2],\Delta_\rho[-1,-2];\pi_{sc})$
- $L(\Delta_\rho[-2,-2],\Delta_\rho[-2,-2],\Delta_\rho[-1,-1];T^1_{I,1}(\pi_{sc}))$
- $L(\Delta_\rho[-2,-2],\Delta_\rho[-1,-2],\Delta_\rho[-1,-1];\pi_{sc})$
- $L(\Delta_\rho[-2,-2],\Delta_\rho[-2,-2],\Delta_\rho[-1,-1],\Delta_\rho[-1,-1];\pi_{sc})$

(i) $(1,1,2,3)$*:*
- $L(\Delta_\rho[-1,-1];T^3_{I,1}(T^2_{I,1}(T^1_{I,1}(\pi_{sc}))))$
- $L(\Delta_\rho[-1,-3];T^1_{I,1}(\pi_{sc}))$
- $L(\Delta_\rho[-1,-3],\Delta_\rho[-1,-1];\pi_{sc})$
- $L(\Delta_\rho[-2,-3],\Delta_\rho[-1,-1];T^1_{I,1}(\pi_{sc}))$
- $L(\Delta_\rho[-3,-3],\Delta_\rho[-1,-1];T^2_{I,1}(T^1_{I,1}(\pi_{sc})))$
- $L(\Delta_\rho[-3,-3],\Delta_\rho[-1,-2];T^1_{I,1}(\pi_{sc}))$
- $L(\Delta_\rho[-2,-3],\Delta_\rho[-1,-1],\Delta_\rho[-1,-1];\pi_{sc})$
- $L(\Delta_\rho[-3,-3],\Delta_\rho[-2,-2],\Delta_\rho[-1,-1];T^1_{I,1}(\pi_{sc}))$
- $L(\Delta_\rho[-3,-3],\Delta_\rho[-1,-2],\Delta_\rho[-1,-1];\pi_{sc})$
- $L(\Delta_\rho[-3,-3],\Delta_\rho[-2,-2],\Delta_\rho[-1,-1],\Delta_\rho[-1,-1];\pi_{sc})$

(j) $(1,2,2,2)$*:*
- $L(\Delta_\rho[-2,-2],\Delta_\rho[-2,-2];T^2_{I,1}(T^1_{I,1}(\pi_{sc})))$
- $L(\Delta_\rho[-2,-2],\Delta_\rho[-2,-2],\Delta_\rho[-2,-2];T^1_{I,1}(\pi_{sc}))$
- $L(\Delta_\rho[-2,-2],\Delta_\rho[-2,-2],\Delta_\rho[-1,-2];\pi_{sc})$
- $L(\Delta_\rho[-2,-2],\Delta_\rho[-2,-2],\Delta_\rho[-2,-2],\Delta_\rho[-1,-1];\pi_{sc})$

(k) $(1,2,2,3)$*:*
- $L(\Delta_\rho[-2,-2];T^3_{I,1}(T^2_{I,1}(T^1_{I,1}(\pi_{sc}))))$
- $L(\Delta_\rho[-2,-3];T^2_{I,1}(T^1_{I,1}(\pi_{sc})))$
- $L(\Delta_\rho[-2,-2],\Delta_\rho[-1,-3];\pi_{sc})$
- $L(\Delta_\rho[-2,-3],\Delta_\rho[-2,-2];T^1_{I,1}(\pi_{sc}))$
- $L(\Delta_\rho[-2,-3],\Delta_\rho[-1,-2];\pi_{sc})$
- $L(\Delta_\rho[-2,-3],\Delta_\rho[-2,-2],\Delta_\rho[-1,-1];\pi_{sc})$
- $L(\Delta_\rho[-3,-3],\Delta_\rho[-2,-2];T^2_{I,1}(T^1_{I,1}(\pi_{sc})))$
- $L(\Delta_\rho[-3,-3],\Delta_\rho[-2,-2],\Delta_\rho[-2,-2];T^1_{I,1}(\pi_{sc}))$
- $L(\Delta_\rho[-3,-3],\Delta_\rho[-2,-2],\Delta_\rho[-1,-2];\pi_{sc})$
- $L(\Delta_\rho[-3,-3],\Delta_\rho[-2,-2],\Delta_\rho[-2,-2],\Delta_\rho[-1,-1];\pi_{sc})$

(l) $(1,2,3,3)$*:*
- $L(\Delta_\rho[-3,-3];T^3_{I,1}(T^2_{I,1}(T^1_{I,1}(\pi_{sc}))))$
- $L(\Delta_\rho[-3,-3],\Delta_\rho[-1,-3];\pi_{sc})$
- $L(\Delta_\rho[-3,-3],\Delta_\rho[-3,-3];T^2_{I,1}(T^1_{I,1}(\pi_{sc})))$
- $L(\Delta_\rho[-3,-3],\Delta_\rho[-2,-3];T^1_{I,1}(\pi_{sc}))$
- $L(\Delta_\rho[-3,-3],\Delta_\rho[-3,-3],\Delta_\rho[-2,-2];T^1_{I,1}(\pi_{sc}))$
- $L(\Delta_\rho[-3,-3],\Delta_\rho[-2,-3],\Delta_\rho[-1,-1];\pi_{sc})$
- $L(\Delta_\rho[-3,-3],\Delta_\rho[-3,-3],\Delta_\rho[-1,-2];\pi_{sc})$
- $L(\Delta_\rho[-3,-3],\Delta_\rho[-3,-3],\Delta_\rho[-2,-2],\Delta_\rho[-1,-1];\pi_{sc})$

(m) $(1,2,3,4)$*:*
- $L(\Delta_\rho[-1,-4];\pi_{sc})$
- $L(\Delta_\rho[-2,-4];T^1_{I,1}(\pi_{sc}))$
- $L(\Delta_\rho[-3,-4];T^2_{I,1}(T^1_{I,1}(\pi_{sc})))$
- $L(\Delta_\rho[-4,-4];T^3_{I,1}(T^2_{I,1}(T^1_{I,1}(\pi_{sc}))))$

- $L(\Delta_\rho[-2,-4], \Delta_\rho[-1,-1]; \pi_{sc})$
- $L(\Delta_\rho[-3,-4], \Delta_\rho[-1,-2]; \pi_{sc})$
- $L(\Delta_\rho[-3,-4], \Delta_\rho[-2,-2]; T^1_{I,1}(\pi_{sc}))$
- $L(\Delta_\rho[-4,-4], \Delta_\rho[-1,-3]; \pi_{sc})$
- $L(\Delta_\rho[-4,-4], \Delta_\rho[-2,-3]; T^1_{I,1}(\pi_{sc}))$
- $L(\Delta_\rho[-4,-4], \Delta_\rho[-3,-3]; T^2_{I,1}(T^1_{I,1}(\pi_{sc})))$
- $L(\Delta_\rho[-3,-4], \Delta_\rho[-2,-2], \Delta_\rho[-1,-1]; \pi_{sc})$
- $L(\Delta_\rho[-4,-4], \Delta_\rho[-3,-3], \Delta_\rho[-2,-2]; T^1_{I,1}(\pi_{sc}))$
- $L(\Delta_\rho[-4,-4], \Delta_\rho[-2,-3], \Delta_\rho[-1,-1]; \pi_{sc})$
- $L(\Delta_\rho[-4,-4], \Delta_\rho[-3,-3], \Delta_\rho[-1,-2]; \pi_{sc})$

(4) *($\alpha = \frac{3}{2}$):*

(a) *($\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{3}{2}$):*

- $L(\Delta_\rho[-\frac{3}{2},-\frac{3}{2}]; T^{\frac{1}{2}}_{I,2}(T^{\frac{3}{2}}_{I,1}(\pi_{sc})))$
- $L(\Delta_\rho[-\frac{3}{2},-\frac{3}{2}], \Delta_\rho[-\frac{1}{2},-\frac{1}{2}]; T^{\frac{1}{2}}_{I,1}(T^{\frac{3}{2}}_{I,1}(\pi_{sc})))$
- $L(\Delta_\rho[-\frac{1}{2},-\frac{3}{2}], \Delta_\rho[-\frac{1}{2},-\frac{1}{2}]; T^{\frac{3}{2}}_{I,1}(\pi_{sc}))$
- $L(\Delta_\rho[-\frac{1}{2},-\frac{3}{2}], \Delta_\rho[-\frac{1}{2},-\frac{3}{2}]; \pi_{sc})$
- $L(\Delta_\rho[-\frac{3}{2},-\frac{3}{2}], \Delta_\rho[-\frac{3}{2},-\frac{3}{2}]; T^{\frac{1}{2}}_{II,3}(\pi_{sc}))$
- $L(\Delta_\rho[-\frac{3}{2},-\frac{3}{2}], \Delta_\rho[-\frac{1}{2},-\frac{1}{2}], \Delta_\rho[-\frac{1}{2},-\frac{1}{2}]; T^{\frac{3}{2}}_{I,1}(\pi_{sc}))$
- $L(\Delta_\rho[-\frac{3}{2},-\frac{3}{2}], \Delta_\rho[-\frac{1}{2},-\frac{3}{2}], \Delta_\rho[-\frac{1}{2},-\frac{1}{2}]; \pi_{sc})$

(b) *($\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{5}{2}$):*

- $L(\Delta_\rho[-\frac{3}{2},-\frac{5}{2}]; T^{\frac{1}{2}}_{II,3}(\pi_{sc}))$
- $L(\Delta_\rho[-\frac{1}{2},-\frac{1}{2}]; T^{\frac{1}{2}}_{I,1}(T^{\frac{5}{2}}_{I,1}(T^{\frac{3}{2}}_{I,1}(\pi_{sc}))))$
- $L(\Delta_\rho[-\frac{5}{2},-\frac{5}{2}]; T^{\frac{3}{2}}_{I,1}(T^{\frac{1}{2}}_{II,3}(\pi_{sc})))$
- $L(\Delta_\rho[-\frac{5}{2},-\frac{5}{2}]; T^{\frac{1}{2}}_{I,2}(T^{\frac{3}{2}}_{I,1}(\pi_{sc})))$
- $L(\Delta_\rho[\frac{1}{2},-\frac{5}{2}]; \pi_{sc})$
- $L(\Delta_\rho[-\frac{1}{2},-\frac{5}{2}], \Delta_\rho[-\frac{1}{2},-\frac{1}{2}]; \pi_{sc})$
- $L(\Delta_\rho[-\frac{5}{2},-\frac{5}{2}], \Delta_\rho[-\frac{1}{2},-\frac{1}{2}]; T^{\frac{1}{2}}_{I,1}(T^{\frac{3}{2}}_{I,1}(\pi_{sc})))$
- $L(\Delta_\rho[-\frac{5}{2},-\frac{5}{2}], \Delta_\rho[-\frac{1}{2},-\frac{1}{2}], \Delta_\rho[-\frac{1}{2},-\frac{1}{2}]; T^{\frac{3}{2}}_{I,1}(\pi_{sc}))$
- $L(\Delta_\rho[-\frac{5}{2},-\frac{5}{2}], \Delta_\rho[-\frac{3}{2},-\frac{3}{2}], \Delta_\rho[-\frac{1}{2},-\frac{1}{2}], \Delta_\rho[-\frac{1}{2},-\frac{1}{2}]; \pi_{sc})$

(c) *($\frac{1}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}$):*

- $L(\Delta_\rho[-\frac{3}{2},-\frac{3}{2}], \Delta_\rho[-\frac{3}{2},-\frac{3}{2}]; T^{\frac{1}{2}}_{I,1}(T^{\frac{3}{2}}_{I,1}(\pi_{sc})))$
- $L(\Delta_\rho[-\frac{3}{2},-\frac{3}{2}], \Delta_\rho[-\frac{1}{2},-\frac{3}{2}]; T^{\frac{3}{2}}_{I,1}(\pi_{sc}))$
- $L(\Delta_\rho[-\frac{3}{2},-\frac{3}{2}], \Delta_\rho[-\frac{3}{2},-\frac{3}{2}], \Delta_\rho[-\frac{1}{2},-\frac{1}{2}]; T^{\frac{3}{2}}_{I,1}(\pi_{sc}))$
- $L(\Delta_\rho[-\frac{3}{2},-\frac{3}{2}], \Delta_\rho[-\frac{3}{2},-\frac{3}{2}], \Delta_\rho[-\frac{1}{2},-\frac{3}{2}]; \pi_{sc})$
- $L(\Delta_\rho[-\frac{3}{2},-\frac{3}{2}], \Delta_\rho[-\frac{3}{2},-\frac{3}{2}], \Delta_\rho[-\frac{3}{2},-\frac{3}{2}], \Delta_\rho[-\frac{1}{2},-\frac{1}{2}]; \pi_{sc})$

(d) *($\frac{1}{2}, \frac{3}{2}, \frac{3}{2}, \frac{5}{2}$):*

- $L(\Delta_\rho[-\frac{1}{2},-\frac{3}{2}]; T^{\frac{5}{2}}_{I,1}(T^{\frac{3}{2}}_{I,1}(\pi_{sc})))$
- $L(\Delta_\rho[-\frac{1}{2},-\frac{5}{2}]; T^{\frac{3}{2}}_{I,1}(\pi_{sc}))$

- $L(\Delta_\rho[-\frac{3}{2},-\frac{5}{2}];T_{I,1}^{\frac{1}{2}}(T_{I,1}^{\frac{5}{2}}(\pi_{sc})))$
- $L(\Delta_\rho[-\frac{3}{2},-\frac{3}{2}];T_{I,1}^{\frac{1}{2}}(T_{I,1}^{\frac{5}{2}}(T_{I,1}^{\frac{3}{2}}(\pi_{sc}))))$
- $L(\Delta_\rho[-\frac{5}{2},-\frac{5}{2}],\Delta_\rho[-\frac{3}{2},-\frac{3}{2}];T_{I,1}^{\frac{1}{2}}(T_{I,1}^{\frac{3}{2}}(\pi_{sc})))$
- $L(\Delta_\rho[-\frac{3}{2},-\frac{3}{2}],\Delta_\rho[-\frac{1}{2},-\frac{5}{2}];\pi_{sc})$
- $L(\Delta_\rho[-\frac{3}{2},-\frac{5}{2}],\Delta_\rho[-\frac{1}{2},-\frac{1}{2}];T_{I,1}^{\frac{3}{2}}(\pi_{sc}))$
- $L(\Delta_\rho[-\frac{3}{2},-\frac{5}{2}],\Delta_\rho[-\frac{3}{2},-\frac{3}{2}],\Delta_\rho[-\frac{1}{2},-\frac{1}{2}];\pi_{sc})$
- $L(\Delta_\rho[-\frac{5}{2},-\frac{5}{2}],\Delta_\rho[-\frac{1}{2},-\frac{3}{2}];T_{I,1}^{\frac{3}{2}}(\pi_{sc}))$
- $L(\Delta_\rho[-\frac{5}{2},-\frac{5}{2}],\Delta_\rho[-\frac{3}{2},-\frac{3}{2}],\Delta_\rho[-\frac{1}{2},-\frac{3}{2}];\pi_{sc})$
- $L(\Delta_\rho[-\frac{5}{2},-\frac{5}{2}],\Delta_\rho[-\frac{3}{2},-\frac{3}{2}],\Delta_\rho[-\frac{3}{2},-\frac{3}{2}],\Delta_\rho[-\frac{1}{2},-\frac{1}{2}];\pi_{sc})$

(e) $(\frac{1}{2},\frac{3}{2},\frac{5}{2},\frac{5}{2})$:

- $L(\Delta_\rho[-\frac{5}{2},-\frac{5}{2}];T_{I,1}^{\frac{1}{2}}(T_{I,1}^{\frac{5}{2}}(T_{I,1}^{\frac{3}{2}}(\pi_{sc}))))$
- $L(\Delta_\rho[-\frac{5}{2},-\frac{5}{2}],\Delta_\rho[-\frac{1}{2},-\frac{1}{2}];T_{I,1}^{\frac{5}{2}}(T_{I,1}^{\frac{3}{2}}(\pi_{sc})))$
- $L(\Delta_\rho[-\frac{5}{2},-\frac{5}{2}],\Delta_\rho[-\frac{1}{2},-\frac{5}{2}];\pi_{sc})$
- $L(\Delta_\rho[-\frac{5}{2},-\frac{5}{2}],\Delta_\rho[-\frac{5}{2},-\frac{5}{2}];T_{I,1}^{\frac{1}{2}}(T_{I,1}^{\frac{3}{2}}(\pi_{sc})))$
- $L(\Delta_\rho[-\frac{5}{2},-\frac{5}{2}],\Delta_\rho[-\frac{5}{2},-\frac{5}{2}],\Delta_\rho[-\frac{1}{2},-\frac{1}{2}];T_{I,1}^{\frac{3}{2}}(\pi_{sc}))$
- $L(\Delta_\rho[-\frac{5}{2},-\frac{5}{2}],\Delta_\rho[-\frac{3}{2},-\frac{5}{2}],\Delta_\rho[-\frac{1}{2},-\frac{1}{2}];\pi_{sc})$
- $L(\Delta_\rho[-\frac{5}{2},-\frac{5}{2}],\Delta_\rho[-\frac{5}{2},-\frac{5}{2}],\Delta_\rho[-\frac{1}{2},-\frac{3}{2}];\pi_{sc})$
- $L(\Delta_\rho[-\frac{5}{2},-\frac{5}{2}],\Delta_\rho[-\frac{5}{2},-\frac{5}{2}],\Delta_\rho[-\frac{3}{2},-\frac{3}{2}],\Delta_\rho[-\frac{1}{2},-\frac{1}{2}];\pi_{sc})$

(f) $(\frac{1}{2},\frac{3}{2},\frac{5}{2},\frac{7}{2})$:

- $L(\Delta_\rho[-\frac{1}{2},-\frac{7}{2}];\pi_{sc})$
- $L(\Delta_\rho[-\frac{5}{2},-\frac{7}{2}];T_{I,1}^{\frac{3}{2}}(T_{I,1}^{\frac{1}{2}}(\pi_{sc})))$
- $L(\Delta_\rho[-\frac{7}{2},-\frac{7}{2}];T_{I,1}^{\frac{1}{2}}(T_{I,1}^{\frac{5}{2}}(T_{I,1}^{\frac{3}{2}}(\pi_{sc}))))$
- $L(\Delta_\rho[-\frac{3}{2},-\frac{7}{2}],\Delta_\rho[-\frac{1}{2},-\frac{1}{2}];\pi_{sc})$
- $L(\Delta_\rho[-\frac{5}{2},-\frac{7}{2}],\Delta_\rho[-\frac{1}{2},-\frac{1}{2}];T_{I,1}^{\frac{3}{2}}(\pi_{sc}))$
- $L(\Delta_\rho[-\frac{5}{2},-\frac{7}{2}],\Delta_\rho[-\frac{1}{2},-\frac{3}{2}];\pi_{sc})$
- $L(\Delta_\rho[-\frac{7}{2},-\frac{7}{2}],\Delta_\rho[-\frac{1}{2},-\frac{1}{2}];T_{I,1}^{\frac{5}{2}}(T_{I,1}^{\frac{3}{2}}(\pi_{sc})))$
- $L(\Delta_\rho[-\frac{7}{2},-\frac{7}{2}],\Delta_\rho[-\frac{1}{2},-\frac{5}{2}];\pi_{sc})$
- $L(\Delta_\rho[-\frac{7}{2},-\frac{7}{2}],\Delta_\rho[-\frac{5}{2},-\frac{5}{2}];T_{I,1}^{\frac{1}{2}}(T_{I,1}^{\frac{3}{2}}(\pi_{sc})))$
- $L(\Delta_\rho[-\frac{5}{2},-\frac{7}{2}],\Delta_\rho[-\frac{3}{2},-\frac{3}{2}],\Delta_\rho[-\frac{1}{2},-\frac{1}{2}];\pi_{sc})$
- $L(\Delta_\rho[-\frac{7}{2},-\frac{7}{2}],\Delta_\rho[-\frac{3}{2},-\frac{5}{2}],\Delta_\rho[-\frac{1}{2},-\frac{1}{2}];\pi_{sc})$
- $L(\Delta_\rho[-\frac{7}{2},-\frac{7}{2}],\Delta_\rho[-\frac{5}{2},-\frac{5}{2}],\Delta_\rho[-\frac{1}{2},-\frac{1}{2}];T_{I,1}^{\frac{3}{2}}(\pi_{sc}))$
- $L(\Delta_\rho[-\frac{7}{2},-\frac{7}{2}],\Delta_\rho[-\frac{3}{2},-\frac{5}{2}],\Delta_\rho[-\frac{1}{2},-\frac{1}{2}];\pi_{sc})$

(g) $(\frac{3}{2},\frac{3}{2},\frac{3}{2},\frac{3}{2})$:

- $L(\Delta_\rho[-\frac{3}{2},-\frac{3}{2}],\Delta_\rho[-\frac{3}{2},-\frac{3}{2}],\Delta_\rho[-\frac{3}{2},-\frac{3}{2}];T_{I,1}^{\frac{3}{2}}(\pi_{sc}))$
- $L(\Delta_\rho[-\frac{3}{2},-\frac{3}{2}],\Delta_\rho[-\frac{3}{2},-\frac{3}{2}],\Delta_\rho[-\frac{3}{2},-\frac{3}{2}],\Delta_\rho[-\frac{3}{2},-\frac{3}{2}];\pi_{sc})$

(h) $(\frac{3}{2},\frac{3}{2},\frac{3}{2},\frac{5}{2})$:

- $L(\Delta_\rho[-\frac{3}{2},-\frac{3}{2}],\Delta_\rho[-\frac{3}{2},-\frac{3}{2}];T_{I,1}^{\frac{5}{2}}(T_{I,1}^{\frac{3}{2}}(\pi_{sc})))$

- $L(\Delta_\rho[-\frac{3}{2},-\frac{5}{2}],\Delta_\rho[-\frac{3}{2},-\frac{3}{2}];T_{I,1}^{\frac{3}{2}}(\pi_{sc}))$
- $L(\Delta_\rho[-\frac{5}{2},-\frac{5}{2}],\Delta_\rho[-\frac{3}{2},-\frac{3}{2}],\Delta_\rho[-\frac{3}{2},-\frac{3}{2}];T_{I,1}^{\frac{3}{2}}(\pi_{sc}))$
- $L(\Delta_\rho[-\frac{3}{2},-\frac{5}{2}],\Delta_\rho[-\frac{3}{2},-\frac{3}{2}],\Delta_\rho[-\frac{3}{2},-\frac{3}{2}];\pi_{sc})$
- $L(\Delta_\rho[-\frac{5}{2},-\frac{5}{2}],\Delta_\rho[-\frac{3}{2},-\frac{3}{2}],\Delta_\rho[-\frac{3}{2},-\frac{3}{2}],\Delta_\rho[-\frac{3}{2},-\frac{3}{2}];\pi_{sc})$

(i) $(\frac{3}{2},\frac{3}{2},\frac{5}{2},\frac{5}{2})$*:*

- $L(\Delta_\rho[-\frac{3}{2},-\frac{5}{2}];T_{I,1}^{\frac{5}{2}}(T_{I,1}^{\frac{3}{2}}(\pi_{sc})))$
- $L(\Delta_\rho[-\frac{5}{2},-\frac{5}{2}],\Delta_\rho[-\frac{3}{2},-\frac{3}{2}]T_{I,1}^{\frac{5}{2}}(T_{I,1}^{\frac{3}{2}}(\pi_{sc})))$
- $L(\Delta_\rho[-\frac{5}{2},-\frac{5}{2}],\Delta_\rho[-\frac{3}{2},-\frac{5}{2}];T_{I,1}^{\frac{3}{2}}(\pi_{sc}))$
- $L(\Delta_\rho[-\frac{3}{2},-\frac{5}{2}],\Delta_\rho[-\frac{3}{2},-\frac{5}{2}];\pi_{sc})$
- $L(\Delta_\rho[-\frac{5}{2},-\frac{5}{2}],\Delta_\rho[-\frac{5}{2},-\frac{5}{2}],\Delta_\rho[-\frac{3}{2},-\frac{3}{2}];T_{I,1}^{\frac{3}{2}}(\pi_{sc}))$
- $L(\Delta_\rho[-\frac{5}{2},-\frac{5}{2}],\Delta_\rho[-\frac{3}{2},-\frac{5}{2}],\Delta_\rho[-\frac{3}{2},-\frac{3}{2}];\pi_{sc})$
- $L(\Delta_\rho[-\frac{5}{2},-\frac{5}{2}],\Delta_\rho[-\frac{5}{2},-\frac{5}{2}],\Delta_\rho[-\frac{3}{2},-\frac{3}{2}],\Delta_\rho[-\frac{3}{2},-\frac{3}{2}];\pi_{sc})$

(j) $(\frac{3}{2},\frac{3}{2},\frac{5}{2},\frac{7}{2})$*:*

- $L(\Delta_\rho[-\frac{3}{2},-\frac{3}{2}];T_{I,1}^{\frac{7}{2}}(T_{I,1}^{\frac{5}{2}}(T_{I,1}^{\frac{3}{2}}(\pi_{sc}))))$
- $L(\Delta_\rho[-\frac{3}{2},-\frac{7}{2}];T_{I,1}^{\frac{3}{2}}(\pi_{sc}))$
- $L(\Delta_\rho[-\frac{3}{2},-\frac{3}{2}];T_{I,1}^{\frac{7}{2}}(T_{I,1}^{\frac{5}{2}}(T_{I,1}^{[}\frac{3}{2}(\pi_{sc}))))$
- $L(\Delta_\rho[-\frac{5}{2},-\frac{7}{2}],\Delta_\rho[-\frac{3}{2},-\frac{3}{2}];T_{I,1}^{\frac{3}{2}}(\pi_{sc}))$
- $L(\Delta_\rho[-\frac{7}{2},-\frac{7}{2}],\Delta_\rho[-\frac{3}{2},-\frac{5}{2}];T_{I,1}^{\frac{3}{2}}(\pi_{sc}))$
- $L(\Delta_\rho[-\frac{3}{2},-\frac{7}{2}],\Delta_\rho[-\frac{3}{2},-\frac{3}{2}];\pi_{sc})$
- $L(\Delta_\rho[-\frac{5}{2},-\frac{7}{2}],\Delta_\rho[-\frac{3}{2},-\frac{3}{2}];T_{I,1}^{\frac{3}{2}}(\pi_{sc}))$
- $L(\Delta_\rho[-\frac{7}{2},-\frac{7}{2}],\Delta_\rho[-\frac{5}{2},-\frac{5}{2}],\Delta_\rho[-\frac{3}{2},-\frac{3}{2}];T_{I,1}^{\frac{3}{2}}(\pi_{sc})$
- $L(\Delta_\rho[-\frac{7}{2},-\frac{7}{2}],\Delta_\rho[-\frac{3}{2},-\frac{5}{2}],\Delta_\rho[-\frac{3}{2},-\frac{3}{2}];\pi_{sc})$
- $L(\Delta_\rho[-\frac{7}{2},-\frac{7}{2}],\Delta_\rho[-\frac{5}{2},-\frac{5}{2}],\Delta_\rho[-\frac{3}{2},-\frac{3}{2}],\Delta_\rho[-\frac{3}{2},-\frac{3}{2}];\pi_{sc})$

(k) $(\frac{3}{2},\frac{5}{2},\frac{5}{2},\frac{5}{2})$*:*

- $L(\Delta_\rho[-\frac{5}{2},-\frac{5}{2}],\Delta_\rho[-\frac{5}{2},-\frac{5}{2}];T_{I,1}^{\frac{5}{2}}(T_{I,1}^{\frac{3}{2}}(\pi_{sc})))$
- $L(\Delta_\rho[-\frac{5}{2},-\frac{5}{2}],\Delta_\rho[-\frac{5}{2},-\frac{5}{2}],\Delta_\rho[-\frac{5}{2},-\frac{5}{2}];T_{I,1}^{\frac{3}{2}}(\pi_{sc}))$
- $L(\Delta_\rho[-\frac{5}{2},-\frac{5}{2}],\Delta_\rho[-\frac{5}{2},-\frac{5}{2}],\Delta_\rho[-\frac{3}{2},-\frac{5}{2}];\pi_{sc})$
- $L(\Delta_\rho[-\frac{5}{2},-\frac{5}{2}],\Delta_\rho[-\frac{5}{2},-\frac{5}{2}],\Delta_\rho[-\frac{5}{2},-\frac{5}{2}],\Delta_\rho[-\frac{3}{2},-\frac{3}{2}];\pi_{sc})$

(l) $(\frac{3}{2},\frac{5}{2},\frac{5}{2},\frac{7}{2})$*:*

- $L(\Delta_\rho[-\frac{5}{2},-\frac{5}{2}];T_{I,1}^{\frac{7}{2}}(T_{I,1}^{\frac{5}{2}}(T_{I,1}^{\frac{3}{2}}(\pi_{sc}))))$
- $L(\Delta_\rho[-\frac{5}{2},-\frac{7}{2}];T_{I,1}^{\frac{3}{2}}(\pi_{sc}))\ +$
- $L(\Delta_\rho[-\frac{7}{2},-\frac{7}{2}],\Delta_\rho[-\frac{5}{2},-\frac{5}{2}]T_{I,1}^{\frac{5}{2}}(T_{I,1}^{\frac{3}{2}}(\pi_{sc})))$
- $L(\Delta_\rho[-\frac{5}{2},-\frac{7}{2}],\Delta_\rho[-\frac{5}{2},-\frac{5}{2}];T_{I,1}^{\frac{3}{2}}(\pi_{sc}))$
- $L(\Delta_\rho[-\frac{5}{2},-\frac{7}{2}],\Delta_\rho[-\frac{3}{2},-\frac{5}{2}];\pi_{sc})$
- $L(\Delta_\rho[-\frac{5}{2},-\frac{5}{2}],\Delta_\rho[-\frac{3}{2},-\frac{7}{2}];\pi_{sc})$
- $L(\Delta_\rho[-\frac{5}{2},-\frac{7}{2}],\Delta_\rho[-\frac{5}{2},-\frac{5}{2}],\Delta_\rho[-\frac{3}{2},-\frac{3}{2}];\pi_{sc})\ +$

- $L(\Delta_\rho[-\frac{7}{2},-\frac{7}{2}],\Delta_\rho[-\frac{5}{2},-\frac{5}{2}],\Delta_\rho[-\frac{5}{2},-\frac{5}{2}];T_{I,1}^{\frac{3}{2}}(\pi_{sc}))$
- $L(\Delta_\rho[-\frac{7}{2},-\frac{7}{2}],\Delta_\rho[-\frac{5}{2},-\frac{5}{2}],\Delta_\rho[-\frac{3}{2},-\frac{5}{2}];\pi_{sc})$
- $L(\Delta_\rho[-\frac{7}{2},-\frac{7}{2}],\Delta_\rho[-\frac{5}{2},-\frac{5}{2}],\Delta_\rho[-\frac{5}{2},-\frac{5}{2}],\Delta_\rho[-\frac{3}{2},-\frac{3}{2}];\pi_{sc})$

(m) $(\frac{3}{2},\frac{5}{2},\frac{7}{2},\frac{7}{2})$*:*
- $L(\Delta_\rho[-\frac{7}{2},-\frac{7}{2}];T_{I,1}^{\frac{7}{2}}(T_{I,1}^{\frac{5}{2}}(T_{I,1}^{\frac{3}{2}}(\pi_{sc}))))$
- $L(\Delta_\rho[-\frac{7}{2},-\frac{7}{2}],\Delta_\rho[-\frac{7}{2},-\frac{7}{2}];T_{I,1}^{\frac{5}{2}}(T_{I,1}^{\frac{3}{2}}(\pi_{sc})$
- $L(\Delta_\rho[-\frac{7}{2},-\frac{7}{2}],\Delta_\rho[-\frac{5}{2},-\frac{7}{2}];T_{I,1}^{\frac{3}{2}}(\pi_{sc}))$
- $L(\Delta_\rho[-\frac{7}{2},-\frac{7}{2}],\Delta_\rho[-\frac{3}{2},-\frac{7}{2}];\pi_{sc})$
- $L(\Delta_\rho[-\frac{7}{2},-\frac{7}{2}],\Delta_\rho[-\frac{7}{2},-\frac{7}{2}],\Delta_\rho[-\frac{5}{2},-\frac{5}{2}];T_{I,1}^{\frac{3}{2}}(\pi_{sc}))$
- $L(\Delta_\rho[-\frac{7}{2},-\frac{7}{2}],\Delta_\rho[-\frac{5}{2},-\frac{7}{2}],\Delta_\rho[-\frac{3}{2},-\frac{3}{2}];\pi_{sc})$
- $L(\Delta_\rho[-\frac{7}{2},-\frac{7}{2}],\Delta_\rho[-\frac{7}{2},-\frac{7}{2}],\Delta_\rho[-\frac{3}{2},-\frac{5}{2}];\pi_{sc})$
- $L(\Delta_\rho[-\frac{7}{2},-\frac{7}{2}],\Delta_\rho[-\frac{7}{2},-\frac{7}{2}],\Delta_\rho[-\frac{5}{2},-\frac{5}{2}],\Delta_\rho[-\frac{3}{2},-\frac{3}{2}];\pi_{sc})$

(n) $(\frac{3}{2},\frac{5}{2},\frac{7}{2},\frac{9}{2})$*:*
- $L(\Delta_\rho[-\frac{3}{2},-\frac{9}{2}];\pi_{sc})$
- $L(\Delta_\rho[-\frac{5}{2},-\frac{9}{2}];T_{I,1}^{\frac{3}{2}}(\pi_{sc}))$
- $L(\Delta_\rho[-\frac{7}{2},-\frac{9}{2}];T_{I,1}^{\frac{5}{2}}(T_{I,1}^{\frac{3}{2}}(\pi_{sc})))$
- $L(\Delta_\rho[-\frac{9}{2},-\frac{9}{2}];T_{I,1}^{\frac{7}{2}}(T_{I,1}^{\frac{5}{2}}(T_{I,1}^{[}\frac{3}{2}(\pi_{sc}))))$
- $L(\Delta_\rho[-\frac{9}{2},-\frac{9}{2}],\Delta_\rho[-\frac{7}{2},-\frac{7}{2}];T_{I,1}^{\frac{5}{2}}(T_{I,1}^{\frac{3}{2}}(\pi_{sc})))$
- $L(\Delta_\rho[-\frac{7}{2},-\frac{9}{2}],\Delta_\rho[-\frac{5}{2},-\frac{5}{2}];T_{I,1}^{\frac{3}{2}}(\pi_{sc}))$
- $L(\Delta_\rho[-\frac{9}{2},-\frac{9}{2}],\Delta_\rho[-\frac{5}{2},-\frac{7}{2}];T_{I,1}^{\frac{3}{2}}(\pi_{sc}))$
- $L(\Delta_\rho[-\frac{7}{2},-\frac{9}{2}],\Delta_\rho[-\frac{3}{2},-\frac{5}{2}];\pi_{sc})$
- $L(\Delta_\rho[-\frac{5}{2},-\frac{9}{2}],\Delta_\rho[-\frac{3}{2},-\frac{3}{2}];\pi_{sc})$
- $L(\Delta_\rho[-\frac{9}{2},-\frac{9}{2}],\Delta_\rho[-\frac{3}{2},-\frac{7}{2}];\pi_{sc})$
- $L(\Delta_\rho[-\frac{9}{2},-\frac{9}{2}],\Delta_\rho[-\frac{7}{2},-\frac{7}{2}],\Delta_\rho[-\frac{5}{2},-\frac{5}{2}];T_{I,1}^{\frac{3}{2}}(\pi_{sc}))$
- $L(\Delta_\rho[-\frac{7}{2},-\frac{9}{2}],\Delta_\rho[-\frac{5}{2},-\frac{5}{2}],\Delta_\rho[-\frac{3}{2},-\frac{3}{2}];\pi_{sc})$
- $L(\Delta_\rho[-\frac{9}{2},-\frac{9}{2}],\Delta_\rho[-\frac{5}{2},-\frac{7}{2}],\Delta_\rho[-\frac{3}{2},-\frac{3}{2}];\pi_{sc})$
- $L(\Delta_\rho[-\frac{9}{2},-\frac{9}{2}],\Delta_\rho[-\frac{7}{2},-\frac{7}{2}],\Delta_\rho[-\frac{3}{2},-\frac{5}{2}];\pi_{sc})$

(5) $(\alpha=2)$*:*

(a) $(0,1,2,2)$*:*
- $L(\Delta_\rho[0,-2];T_{I,1}^{2}(\pi_{sc}))$
- $L(\Delta_\rho[-1,-2];T_{IV,3}(T_{I,1}^{2}(\pi_{sc})))$
- $L(\Delta_\rho[-2,-2];T_{V,2}^{\pm}(T_{I,1}^{1}(T_{I,1}^{2}(\pi_{sc}))))$
- $L(\Delta_\rho[-2,-2],\Delta_\rho[0,-1];T_{I,1}^{2}(\pi_{sc}))$
- $L(\Delta_\rho[-2,-2],\Delta_\rho[0,-2];\pi_{sc})$
- $L(\Delta_\rho[-2,-2],\Delta_\rho[-1,-1];T_{IV,3}(T_{I,1}^{2}(\pi_{sc})))$
- $L(\Delta_\rho[-2,-2],\Delta_\rho[-1,-2];T_{IV,3}(\pi_{sc}))$
- $L(\Delta_\rho[-2,-2],\Delta_\rho[-2,-2],\Delta_\rho[-1,-1];T_{IV,3}(\pi_{sc}))$
- $L(\Delta_\rho[-2,-2],\Delta_\rho[-2,-2],\Delta_\rho[0,-1];\pi_{sc})$

(b) $(0,1,2,3)$*:*

- $L(\Delta_\rho[0,-1]; T^1_{I,1}(T^2_{I,1}(\pi_{sc})))$
- $L(\Delta_\rho[0,-3]; \pi_{sc})$
- $L(\Delta_\rho[-1,-3]; T_{IV,3}(\pi_{sc}))$
- $L(\Delta_\rho[-3,-3]; T^{\pm}_{V,2}(T^1_{I,1}(T^2_{I,1}(\pi_{sc}))))$
- $L(\Delta_\rho[-2,-3], \Delta_\rho[0,-1]; \pi_{sc})$
- $L(\Delta_\rho[-2,-3], \Delta_\rho[-1,-1]; T_{IV,3}(\pi_{sc}))$
- $L(\Delta_\rho[-3,-3], \Delta_\rho[0,-1]; T^2_{I,1}(\pi_{sc}))$
- $L(\Delta_\rho[-3,-3], \Delta_\rho[-1,-1]; T_{IV,3}(T^2_{I,1}(\pi_{sc})))$

(c) $(1,1,1,2)$*:*
- $L(\Delta_\rho[-1,-1], \Delta_\rho[-1,-1]; T^1_{I,1}(T^2_{I,1}(\pi_{sc})))$
- $L(\Delta_\rho[-1,-1], \Delta_\rho[-1,-1], \Delta_\rho[-1,-1]; T^2_{I,1}(\pi_{sc}))$
- $L(\Delta_\rho[-1,-2], \Delta_\rho[-1,-1], \Delta_\rho[-1,-1]; \pi_{sc})$
- $L(\Delta_\rho[-2,-2], \Delta_\rho[-1,-1], \Delta_\rho[-1,-1], \Delta_\rho[-1,-1]; \pi_{sc})$

(d) $(1,1,2,2)$*:*
- $L(\Delta_\rho[-1,-2]; T^1_{I,1}(T^2_{I,1}(\pi_{sc})))$
- $L(\Delta_\rho[-2,-2], \Delta_\rho[-1,-1]; T^1_{I,1}(T^2_{I,1}(\pi_{sc})))$
- $L(\Delta_\rho[-1,-2], \Delta_\rho[-1,-1]; T^2_{I,1}(\pi_{sc}))$
- $L(\Delta_\rho[-1,-2], \Delta_\rho[-1,-2]; \pi_{sc})$
- $L(\Delta_\rho[-2,-2], \Delta_\rho[-1,-1], \Delta_\rho[-1,-1]; T^2_{I,1}(\pi_{sc}))$
- $L(\Delta_\rho[-2,-2], \Delta_\rho[-1,-2], \Delta_\rho[-1,-1]; \pi_{sc})$
- $L(\Delta_\rho[-2,-2], \Delta_\rho[-2,-2], \Delta_\rho[-1,-1], \Delta_\rho[-1,-1]; \pi_{sc})$

(e) $(1,1,2,3)$*:*
- $L(\Delta_\rho[-1,-1]; T^1_{I,1}(T^3_{I,1}(T^2_{I,1}(\pi_{sc}))))$
- $L(\Delta_\rho[-1,-1], \Delta_\rho[-1,-1]; T^3_{I,1}(T^2_{I,1}(\pi_{sc})))$
- $L(\Delta_\rho[-3,-3], \Delta_\rho[-1,-1]; T^1_{I,1}(T^2_{I,1}(\pi_{sc}))$
- $L(\Delta_\rho[-2,-3], \Delta_\rho[-1,-1], \Delta_\rho[-1,-1]; \pi_{sc})$
- $L(\Delta_\rho[-3,-3], \Delta_\rho[-1,-1], \Delta_\rho[-1,-1]; T^2_{I,1}(\pi_{sc}))$
- $L(\Delta_\rho[-3,-3], \Delta_\rho[-1,-2], \Delta_\rho[-1,-1]; \pi_{sc})$
- $L(\Delta_\rho[-3,-3], \Delta_\rho[-2,-2], \Delta_\rho[-1,-1], \Delta_\rho[-1,-1]; \pi_{sc})$

(f) $(1,2,2,2)$*:*
- $L(\Delta_\rho[-2,-2], \Delta_\rho[-2,-2]; T^1_{I,1}(T^2_{I,1}(\pi_{sc})))$
- $L(\Delta_\rho[-2,-2], \Delta_\rho[-1,-2]; T^2_{I,1}(\pi_{sc}))$
- $L(\Delta_\rho[-2,-2], \Delta_\rho[-2,-2], \Delta_\rho[-1,-1]; T^2_{I,1}(\pi_{sc}))$
- $L(\Delta_\rho[-2,-2], \Delta_\rho[-2,-2], \Delta_\rho[-1,-2]; \pi_{sc})$
- $L(\Delta_\rho[-2,-2], \Delta_\rho[-2,-2], \Delta_\rho[-2,-2], \Delta_\rho[-1,-1]; \pi_{sc})$

(g) $(1,2,2,3)$*:*
- $L(\Delta_\rho[-1,-2]; T^3_{I,1}(T^2_{I,1}(\pi_{sc})))$
- $L(\Delta_\rho[-2,-3]; T^2_{I,1}(T^1_{I,1}(\pi_{sc})))$
- $L(\Delta_\rho[-2,-2]; T^1_{I,1}(T^3_{I,1}(T^2_{I,1}(\pi_{sc}))))$
- $L(\Delta_\rho[-2,-3]; T^1_{I,1}(T^2_{I,1}(\pi_{sc})))$
- $L(\Delta_\rho[-1,-3]; T^2_{I,1}(\pi_{sc}))$
- $L(\Delta_\rho[-3,-3], \Delta_\rho[-2,-2]; T^1_{I,1}(T^2_{I,1}(\pi_{sc})))$
- $L(\Delta_\rho[-3,-3], \Delta_\rho[-1,-2]; T^2_{I,1}(\pi_{sc}))$
- $L(\Delta_\rho[-2,-3], \Delta_\rho[-1,-1]; T^2_{I,1}(\pi_{sc}))$
- $L(\Delta_\rho[-2,-2], \Delta_\rho[-1,-3]; \pi_{sc})$

- $L(\Delta_\rho[-2,-3],\Delta_\rho[-2,-2],\Delta_\rho[-1,-1];\pi_{sc})$
- $L(\Delta_\rho[-3,-3],\Delta_\rho[-2,-2],\Delta_\rho[-1,-2];\pi_{sc})$
- $L(\Delta_\rho[-3,-3],\Delta_\rho[-2,-2],\Delta_\rho[-2,-2],\Delta_\rho[-1,-1];\pi_{sc})$

(h) $(1,2,3,3)$*:*
- $L(\Delta_\rho[-3,-3];T_{I,1}^1(T_{I,1}^3(T_{I,1}^2(\pi_{sc}))))$
- $L(\Delta_\rho[-3,-3],\Delta_\rho[-3,-3];T_{I,1}^1(T_{I,1}^2(\pi_{sc})))$
- $L(\Delta_\rho[-3,-3],\Delta_\rho[-1,-1];T_{I,1}^3(T_{I,1}^2(\pi_{sc})))$
- $L(\Delta_\rho[-3,-3],\Delta_\rho[-1,-3];\pi_{sc})$
- $L(\Delta_\rho[-3,-3],\Delta_\rho[-3,-3],\Delta_\rho[-1,-1];T_{I,1}^2(\pi_{sc}))$
- $L(\Delta_\rho[-3,-3],\Delta_\rho[-2,-3],\Delta_\rho[-1,-1];\pi_{sc})$
- $L(\Delta_\rho[-3,-3],\Delta_\rho[-3,-3],\Delta_\rho[-1,-2];\pi_{sc})$
- $L(\Delta_\rho[-3,-3],\Delta_\rho[-2,-2],\Delta_\rho[-1,-1],\Delta_\rho[-1,-1];\pi_{sc})$

(i) $(1,2,3,4)$*:*
- $L(\Delta_\rho[-1,-4];\pi_{sc})$
- $L(\Delta_\rho[-4,-4];T_{I,1}^1(T_{I,1}^3(T_{I,1}^2(\pi_{sc}))))$
- $L(\Delta_\rho[-3,-4];T_{I,1}^1(T_{I,1}^2(\pi_{sc})))$
- $L(\Delta_\rho[-4,-4],\Delta_\rho[-3,-3];T_{I,1}^1(T_{I,1}^2(\pi_{sc})))$
- $L(\Delta_\rho[-4,-4],\Delta_\rho[-1,-1];T_{I,1}^3(T_{I,1}^2(\pi_{sc})))$
- $L(\Delta_\rho[-4,-4],\Delta_\rho[-1,-3];\pi_{sc})$
- $L(\Delta_\rho[-3,-4],\Delta_\rho[-1,-2];\pi_{sc})$
- $L(\Delta_\rho[-2,-4],\Delta_\rho[-1,-1];\pi_{sc})$
- $L(\Delta_\rho[-3,-4],\Delta_\rho[-2,-2],\Delta_\rho[-1,-1];\pi_{sc})$
- $L(\Delta_\rho[-4,-4],\Delta_\rho[-3,-3],\Delta\rho[-1,-1];T_{I,1}^2(\pi_{sc}))$
- $L(\Delta_\rho[-4,-4],\Delta_\rho[-2,-3],\Delta_\rho[-1,-1];\pi_{sc})$
- $L(\Delta_\rho[-4,-4],\Delta_\rho[-3,-3],\Delta_\rho[-1,-2];\pi_{sc})$

(j) $(2,2,2,2)$*:*
- $L(\Delta_\rho[-2,-2],\Delta_\rho[-2,-2],\Delta_\rho[-2,-2];T_{I,1}^2(\pi_{sc}))$
- $L(\Delta_\rho[-2,-2],\Delta_\rho[-2,-2],\Delta_\rho[-2,-2],\Delta_\rho[-2,-2];\pi_{sc})$

(k) $(2,2,2,3)$*:*
- $L(\Delta_\rho[-2,-2],\Delta_\rho[-2,-2];T_{I,1}^3(T_{I,1}^2(\pi_{sc})))$
- $L(\Delta_\rho[-2,-3],\Delta_\rho[-2,-2];T_{I,1}^2(\pi_{sc}))$
- $L(\Delta_\rho[-3,-3],\Delta_\rho[-2,-2],\Delta_\rho[-2,-2];T_{I,1}^2(\pi_{sc}))$
- $L(\Delta_\rho[-2,-3],\Delta_\rho[-2,-2],\Delta_\rho[-2,-2];\pi_{sc})$
- $L(\Delta_\rho[-3,-3],\Delta_\rho[-2,-2],\Delta_\rho[-2,-2],\Delta_\rho[-2,-2];\pi_{sc})$

(l) $(2,2,3,3)$*:*
- $L(\Delta_\rho[-2,-3];T_{I,1}^3(T_{I,1}^2(\pi_{sc})))$
- $L(\Delta_\rho[-3,-3],\Delta_\rho[-2,-2];T_{I,1}^3(T_{I,1}^2(\pi_{sc})))$
- $L(\Delta_\rho[-3,-3],\Delta_\rho[-2,-3];T_{I,1}^2(\pi_{sc}))$
- $L(\Delta_\rho[-2,-3],\Delta_\rho[-2,-3];\pi_{sc})$
- $L(\Delta_\rho[-3,-3],\Delta_\rho[-3,-3],\Delta_\rho[-2,-2];T_{I,1}^2(\pi_{sc}))$
- $L(\Delta_\rho[-3,-3],\Delta_\rho[-2,-3],\Delta_\rho[-2,-2];\pi_{sc})$
- $L(\Delta_\rho[-3,-3],\Delta_\rho[-3,-3],\Delta_\rho[-2,-2],\Delta_\rho[-2,-2];\pi_{sc})$

(m) $(2,2,3,4)$*:*
- $L(\Delta_\rho[-2,-2];T_{I,1}^4(T_{I,1}^3(T_{I,1}^2(\pi_{sc}))))$
- $L(\Delta_\rho[-2,-4];T_{I,1}^2(\pi_{sc}))$

- $L(\Delta_\rho[-3,-4], \Delta_\rho[-2,-2]; T_{I,1}^2(\pi_{sc}))$
- $L(\Delta_\rho[-4,-4], \Delta_\rho[-2,-3]; T_{I,1}^2(\pi_{sc}))$
- $L(\Delta_\rho[-4,-4], \Delta_\rho[-2,-2]; T_{I,1}^3(T_{I,1}^2(\pi_{sc})))$
- $L(\Delta_\rho[-2,-4], \Delta_\rho[-2,-2]; \pi_{sc})$
- $L(\Delta_\rho[-4,-4], \Delta_\rho[-3,-3], \Delta_\rho[-2,-2]; T_{I,1}^2(\pi_{sc}))$
- $L(\Delta_\rho[-3,-4], \Delta_\rho[-2,-2], \Delta_\rho[-2,-2]; \pi_{sc})$
- $L(\Delta_\rho[-4,-4], \Delta_\rho[-2,-3], \Delta_\rho[-2,-2]; \pi_{sc})$
- $L(\Delta_\rho[-4,-4], \Delta_\rho[-3,-3], \Delta_\rho[-2,-2], \Delta_\rho[-2,-2]; \pi_{sc})$

(n) $(2,3,3,3)$*:*
- $L(\Delta_\rho[-3,-3], \Delta_\rho[-3,-3]; T_{I,1}^3(T_{I,1}^2(\pi_{sc})))$
- $L(\Delta_\rho[-3,-3], \Delta_\rho[-3,-3], \Delta_\rho[-2,-3]; \pi_{sc})$
- $L(\Delta_\rho[-3,-3], \Delta_\rho[-3,-3], \Delta_\rho[-3,-3]; T_{I,1}^2(\pi_{sc}))$
- $L(\Delta_\rho[-3,-3], \Delta_\rho[-3,-3], \Delta_\rho[-3,-3], \Delta_\rho[-2,-2]; \pi_{sc})$

(o) $(2,3,3,4)$*:*
- $L(\Delta_\rho[-3,-3]; T_{I,1}^4(T_{I,1}^3(T_{I,1}^2(\pi_{sc}))))$
- $L(\Delta_\rho[-4,-4], \Delta_\rho[-3,-3]; T_{I,1}^3(T_{I,1}^2(\pi_{sc})))$
- $L(\Delta_\rho[-3,-4], \Delta_\rho[-3,-3]; T_{I,1}^2(\pi_{sc}))$
- $L(\Delta_\rho[-3,-4], \Delta_\rho[-2,-3]; \pi_{sc})$
- $L(\Delta_\rho[-2,-4], \Delta_\rho[-3,-3]; \pi_{sc})$
- $L(\Delta_\rho[-3,-3], \Delta_\rho[-2,-4]; \pi_{sc})$
- $L(\Delta_\rho[-4,-4], \Delta_\rho[-3,-3], \Delta_\rho[-3,-3]; T_{I,1}^2(\pi_{sc}))$
- $L(\Delta_\rho[-3,-4], \Delta_\rho[-3,-3], \Delta_\rho[-2,-2]; \pi_{sc})$
- $L(\Delta_\rho[-4,-4], \Delta_\rho[-3,-3], \Delta_\rho[-2,-3]; \pi_{sc})$
- $L(\Delta_\rho[-4,-4], \Delta_\rho[-3,-3], \Delta_\rho[-3,-3], \Delta_\rho[-2,-2]; \pi_{sc})$

(p) $(2,3,4,4)$*:*
- $L(\Delta_\rho[-4,-4]; T_{I,1}^4(T_{I,1}^3(T_{I,1}^2(\pi_{sc}))))$
- $L(\Delta_\rho[-4,-4], \Delta_\rho[-4,-4]; T_{I,1}^3(T_{I,1}^2(\pi_{sc})))$
- $L(\Delta_\rho[-4,-4], \Delta_\rho[-3,-4]; T_{I,1}^2(\pi_{sc}))$
- $L(\Delta_\rho[-4,-4], \Delta_\rho[-2,-4]; \pi_{sc})$
- $L(\Delta_\rho[-4,-4], \Delta_\rho[-4,-4], \Delta_\rho[-2,-3]; \pi_{sc})$
- $L(\Delta_\rho[-4,-4], \Delta_\rho[-4,-4], \Delta_\rho[-3,-3]; T_{I,1}^2(\pi_{sc}))$
- $L(\Delta_\rho[-4,-4], \Delta_\rho[-3,-4], \Delta_\rho[-2,-2]; \pi_{sc})$
- $L(\Delta_\rho[-4,-4], \Delta_\rho[-4,-4], \Delta_\rho[-3,-3], \Delta_\rho[-2,-2]; \pi_{sc})$

(q) $(2,3,4,5)$*:*
- $L(\Delta_\rho[-5,-5]; T_{I,1}^4(T_{I,1}^3(T_{I,1}^2(\pi_{sc}))))$
- $L(\Delta_\rho[-4,-5]; T_{I,1}^3(T_{I,1}^2(\pi_{sc}))$
- $L(\Delta_\rho[-3,-5]; T_{I,1}^2(\pi_{sc}))$
- $L(\Delta_\rho[-2,-5]; \pi_{sc})$
- $L(\Delta_\rho[-5,-5], \Delta_\rho[-4,-4]; T_{I,1}^3(T_{I,1}^2(\pi_{sc})))$
- $L(\Delta_\rho[-4,-5], \Delta_\rho[-3,-3]; T_{I,1}^2(\pi_{sc}))$
- $L(\Delta_\rho[-5,-5], \Delta_\rho[-3,-4]; T_{I,1}^2(\pi_{sc}))$
- $L(\Delta_\rho[-4,-5], \Delta_\rho[-2,-3]; \pi_{sc})$
- $L(\Delta_\rho[-3,-5], \Delta_\rho[-2,-2]; \pi_{sc})$
- $L(\Delta_\rho[-5,-5], \Delta_\rho[-2,-4]; \pi_{sc})$
- $L(\Delta_\rho[-4,-5], \Delta_\rho[-3,-3], \Delta_\rho[-2,-2]; \pi_{sc})$

- $L(\Delta_\rho[-5,-5], \Delta_\rho[-4,-4], \Delta_\rho[-2,-3]'\pi_{sc})$
- $L(\Delta_\rho[-5,-5], \Delta_\rho[-4,-4], \Delta_\rho[-3,-3]; T_{I,1}^{2}(\pi_{sc}))$
- $L(\Delta_\rho[-5,-5], \Delta_\rho[-3,-4], \Delta_\rho[-2,-2]; \pi_{sc})$

(6) $(\alpha = \frac{5}{2})$:

(a) $(\frac{1}{2}, \frac{3}{2}, \frac{3}{2}, \frac{5}{2})$:
- $L(\Delta_\rho[-\frac{3}{2},-\frac{3}{2}]; T_{I,1}^{\frac{1}{2}}(T_{I,1}^{\frac{3}{2}}(T_{I,1}^{\frac{5}{2}}(\pi_{sc}))))$
- $L(\Delta_\rho[-\frac{1}{2},-\frac{3}{2}]; T_{I,1}^{\frac{3}{2}}(T_{I,1}^{\frac{5}{2}}(\pi_{sc})))$
- $L(\Delta_\rho[-\frac{3}{2},-\frac{3}{2}], \Delta_\rho[-\frac{1}{2},-\frac{3}{2}]; T_{I,1}^{\frac{5}{2}}(\pi_{sc}))$
- $L(\Delta_\rho[-\frac{3}{2},-\frac{3}{2}], \Delta_\rho[-\frac{1}{2},-\frac{5}{2}]; \pi_{sc})$
- $L(\Delta_\rho[-\frac{3}{2},-\frac{3}{2}], \Delta_\rho[-\frac{3}{2},-\frac{3}{2}], \Delta_\rho[-\frac{1}{2},-\frac{1}{2}]; T_{I,1}^{\frac{5}{2}}(\pi_{sc}))$
- $L(\Delta_\rho[-\frac{5}{2},-\frac{5}{2}], \Delta_\rho[-\frac{3}{2},-\frac{3}{2}], \Delta_\rho[-\frac{1}{2},-\frac{3}{2}]; \pi_{sc})$
- $L(\Delta_\rho[-\frac{5}{2},-\frac{5}{2}], \Delta_\rho[-\frac{3}{2},-\frac{3}{2}], \Delta_\rho[-\frac{3}{2},-\frac{3}{2}], \Delta_\rho[-\frac{1}{2},-\frac{1}{2}]; \pi_{sc})$

(b) $(\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{5}{2})$:
- $L(\Delta_\rho[-\frac{1}{2},-\frac{5}{2}]; T_{I,1}^{\frac{5}{2}}(\pi_{sc}))$
- $L(\Delta_\rho[-\frac{5}{2},-\frac{5}{2}]; T_{I,1}^{\frac{1}{2}}(T_{I,1}^{\frac{3}{2}}(T_{I,1}^{\frac{5}{2}}(\pi_{sc}))))$
- $L(\Delta_\rho[-\frac{3}{2},-\frac{5}{2}], \Delta_\rho[-\frac{1}{2},-\frac{1}{2}]; T_{I,1}^{\frac{5}{2}}(\pi_{sc}))$
- $L(\Delta_\rho[-\frac{5}{2},-\frac{5}{2}], \Delta_\rho[-\frac{1}{2},-\frac{1}{2}]; T_{I,1}^{\frac{3}{2}}(T_{I,1}^{\frac{5}{2}}(\pi_{sc})))$
- $L(\Delta_\rho[-\frac{5}{2},-\frac{5}{2}], \Delta_\rho[-\frac{1}{2},-\frac{5}{2}]; \pi_{sc})$
- $L(\Delta_\rho[-\frac{5}{2},-\frac{5}{2}], \Delta_\rho[-\frac{5}{2},-\frac{5}{2}], \Delta_\rho[-\frac{1}{2},-\frac{3}{2}]; \pi_{sc})$
- $L(\Delta_\rho[-\frac{5}{2},-\frac{5}{2}], \Delta_\rho[-\frac{3}{2},-\frac{5}{2}], \Delta_\rho[-\frac{1}{2},-\frac{1}{2}]; \pi_{sc})$
- $L(\Delta_\rho[-\frac{1}{2},-\frac{1}{2}], \Delta_\rho[-\frac{3}{2},-\frac{3}{2}], \Delta_\rho[-\frac{5}{2},-\frac{5}{2}], \Delta_\rho[-\frac{5}{2},-\frac{5}{2}]; \pi_{sc})$

(c) $(\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2})$:
- $L(\Delta_\rho[-\frac{7}{2},-\frac{7}{2}]; T_{I,1}^{\frac{1}{2}}(T_{I,1}^{\frac{3}{2}}(T_{I,1}^{\frac{5}{2}}(\pi_{sc}))))$
- $L(\Delta_\rho[-\frac{1}{2},-\frac{7}{2}]; \pi_{sc})$
- $L(\Delta_\rho[-\frac{5}{2},-\frac{7}{2}], \Delta_\rho[-\frac{1}{2},-\frac{3}{2}]; \pi_{sc})$
- $L(\Delta_\rho[-\frac{7}{2},-\frac{7}{2}], \Delta_\rho[-\frac{1}{2},-\frac{3}{2}]; T_{I,1}^{\frac{5}{2}}(\pi_{sc}))$
- $L(\Delta_\rho[-\frac{7}{2},-\frac{7}{2}], \Delta_\rho[-\frac{1}{2},-\frac{1}{2}]; T_{I,1}^{\frac{3}{2}}(T_{I,1}^{\frac{5}{2}}(\pi_{sc})))$
- $L(\Delta_\rho[-\frac{3}{2},\frac{7}{2}], \Delta_\rho[-\frac{1}{2},-\frac{1}{2}]; \pi_{sc})$
- $L(\Delta_\rho[-\frac{7}{2},-\frac{7}{2}], \Delta_\rho[-\frac{3}{2},-\frac{3}{2}], \Delta_\rho[-\frac{1}{2},-\frac{1}{2}]; T_{I,1}^{\frac{5}{2}}(\pi_{sc}))$
- $L(\Delta_\rho[-\frac{5}{2},-\frac{7}{2}], \Delta_\rho[-\frac{3}{2},-\frac{3}{2}], \Delta_\rho[-\frac{1}{2},-\frac{1}{2}]; \pi_{sc})$

(d) $(\frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{5}{2})$:
- $L(\Delta_\rho[-\frac{3}{2},-\frac{3}{2}], \Delta_\rho[-\frac{3}{2},-\frac{3}{2}]; T_{I,1}^{\frac{3}{2}}(T_{I,1}^{\frac{5}{2}}(\pi_{sc}))$
- $L(\Delta_\rho[-\frac{3}{2},-\frac{3}{2}], \Delta_\rho[-\frac{3}{2},-\frac{3}{2}], \Delta_\rho[-\frac{3}{2},-\frac{3}{2}]; T_{I,1}^{\frac{5}{2}}(\pi_{sc}))$
- $L(\Delta_\rho[-\frac{3}{2},-\frac{5}{2}], \Delta_\rho[-\frac{3}{2},-\frac{3}{2}], \Delta_\rho[-\frac{3}{2},-\frac{3}{2}]; \pi_{sc})$
- $L(\Delta_\rho[-\frac{5}{2},-\frac{5}{2}], \Delta_\rho[-\frac{3}{2},-\frac{3}{2}], \Delta_\rho[-\frac{3}{2},-\frac{3}{2}], \Delta_\rho[-\frac{3}{2},-\frac{3}{2}]; \pi_{sc})$

(e) $(\frac{3}{2}, \frac{3}{2}, \frac{5}{2}, \frac{5}{2})$:
- $L(\Delta_\rho[-\frac{3}{2},-\frac{5}{2}]; T_{I,1}^{\frac{3}{2}}(T_{I,1}^{\frac{5}{2}}(\pi_{sc})))$
- $L(\Delta_\rho[-\frac{3}{2},-\frac{5}{2}], \Delta_\rho[-\frac{3}{2},-\frac{3}{2}]; T_{I,1}^{\frac{5}{2}}(\pi_{sc}))$

- $L(\Delta_\rho[-\frac{3}{2},-\frac{5}{2}],\Delta_\rho[-\frac{3}{2},-\frac{5}{2}];\pi_{sc})$
- $L(\Delta_\rho[-\frac{5}{2},-\frac{5}{2}],\Delta_\rho[-\frac{3}{2},-\frac{3}{2}];T_{I,1}^{\frac{5}{2}}(T_{I,1}^{\frac{3}{2}}(\pi_{sc})))$
- $L(\Delta_\rho[-\frac{5}{2},-\frac{5}{2}],\Delta_\rho[-\frac{3}{2},-\frac{3}{2}],\Delta_\rho[-\frac{3}{2},-\frac{3}{2}];T_{I,1}^{\frac{5}{2}}(\pi_{sc}))$
- $L(\Delta_\rho[-\frac{5}{2},-\frac{5}{2}],\Delta_\rho[-\frac{3}{2},-\frac{5}{2}],\Delta_\rho[-\frac{3}{2},-\frac{3}{2}];\pi_{sc})$
- $L(\Delta_\rho[-\frac{5}{2},-\frac{5}{2}],\Delta_\rho[-\frac{5}{2},-\frac{5}{2}],\Delta_\rho[-\frac{3}{2},-\frac{3}{2}],\Delta_\rho[-\frac{3}{2},-\frac{3}{2}];\pi_{sc})$

(f) $(\frac{3}{2},\frac{3}{2},\frac{5}{2},\frac{7}{2})$:

- $L(\Delta_\rho[-\frac{3}{2},-\frac{3}{2}];T_{I,1}^{\frac{3}{2}}(T_{I,1}^{\frac{7}{2}}(T_{I,1}^{\frac{5}{2}}(\pi_{sc}))))$
- $L(\Delta_\rho[-\frac{3}{2},-\frac{3}{2}];T_{I,1}^{\frac{3}{2}}(T_{I,1}^{\frac{7}{2}}(T_{I,1}^{\frac{5}{2}}(\pi_{sc}))))$
- $L(\Delta_\rho[-\frac{7}{2},-\frac{7}{2}],\Delta_\rho[-\frac{3}{2},-\frac{3}{2}];T_{I,1}^{\frac{3}{2}}(T_{I,1}^{\frac{5}{2}}(\pi_{sc})))$
- $L(\Delta_\rho[-\frac{3}{2},-\frac{3}{2}],\Delta_\rho[-\frac{3}{2},-\frac{3}{2}];T_{I,1}^{\frac{7}{2}}(T_{I,1}^{\frac{5}{2}}(\pi_{sc})))$
- $L(\Delta_\rho[-\frac{3}{2},\frac{7}{2}],\Delta_\rho[-\frac{3}{2},-\frac{3}{2}];\pi_{sc})$
- $L(\Delta_\rho[-\frac{7}{2},-\frac{7}{2}],\Delta_\rho[-\frac{3}{2},-\frac{3}{2}],\Delta_\rho[-\frac{3}{2},-\frac{3}{2}];T_{I,1}^{\frac{5}{2}}(\pi_{sc}))$
- $L(\Delta_\rho[-\frac{5}{2},-\frac{7}{2}],\Delta_\rho[-\frac{3}{2},-\frac{3}{2}],\Delta_\rho[-\frac{3}{2},-\frac{3}{2}];\pi_{sc})$
- $L(\Delta_\rho[-\frac{7}{2},-\frac{7}{2}],\Delta_\rho[-\frac{5}{2},-\frac{3}{2}],\Delta_\rho[-\frac{3}{2},-\frac{3}{2}];\pi_{sc})$
- $L(\Delta_\rho[-\frac{7}{2},-\frac{7}{2}],\Delta_\rho[-\frac{5}{2},-\frac{5}{2}],\Delta_\rho[-\frac{3}{2},-\frac{3}{2}],\Delta_\rho[-\frac{3}{2},-\frac{3}{2}];\pi_{sc})$

(g) $(\frac{3}{2},\frac{5}{2},\frac{5}{2},\frac{5}{2})$:

- $L(\Delta_\rho[-\frac{5}{2},-\frac{5}{2}],\Delta_\rho[-\frac{3}{2},-\frac{5}{2}];T_{I,1}^{\frac{5}{2}}(\pi_{sc}))$
- $L(\Delta_\rho[-\frac{5}{2},-\frac{5}{2}],\Delta_\rho[-\frac{5}{2},-\frac{5}{2}];T_{I,1}^{\frac{3}{2}}(T_{I,1}^{\frac{5}{2}}(\pi_{sc})))$
- $L(\Delta_\rho[-\frac{5}{2},-\frac{5}{2}],\Delta_\rho[-\frac{5}{2},-\frac{5}{2}],\Delta_\rho[-\frac{3}{2},-\frac{3}{2}];T_{I,1}^{\frac{5}{2}}(\pi_{sc}))$
- $L(\Delta_\rho[-\frac{5}{2},-\frac{5}{2}],\Delta_\rho[-\frac{5}{2},-\frac{5}{2}],\Delta_\rho[-\frac{3}{2},-\frac{5}{2}];\pi_{sc})$
- $L(\Delta_\rho[-\frac{5}{2},-\frac{5}{2}],\Delta_\rho[-\frac{5}{2},-\frac{5}{2}],\Delta_\rho[-\frac{5}{2},-\frac{5}{2}],\Delta_\rho[-\frac{3}{2},-\frac{3}{2}];\pi_{sc})$

(h) $(\frac{3}{2},\frac{5}{2},\frac{5}{2},\frac{7}{2})$:

- $L(\Delta_\rho[-\frac{5}{2},-\frac{5}{2}];T_{I,1}^{\frac{3}{2}}(T_{I,1}^{\frac{7}{2}}(T_{I,1}^{\frac{5}{2}}(\pi_{sc}))))$
- $L(\Delta_\rho[-\frac{5}{2},-\frac{5}{2}];T_{I,1}^{\frac{3}{2}}(T_{I,1}^{\frac{7}{2}}(T_{I,1}^{\frac{5}{2}}(\pi_{sc}))))$
- $L(\Delta_\rho[-\frac{3}{2},-\frac{7}{2}];T_{I,1}^{\frac{5}{2}}(\pi_{sc}))$
- $L(\Delta_\rho[-\frac{5}{2},-\frac{5}{2}],\Delta_\rho[-\frac{3}{2},-\frac{3}{2}];T_{I,1}^{\frac{5}{2}}(T_{I,1}^{\frac{7}{2}}(\pi_{sc}))$
- $L(\Delta_\rho[-\frac{5}{2},-\frac{7}{2}],\Delta_\rho[-\frac{3}{2},-\frac{3}{2}];T_{I,1}^{\frac{5}{2}}(\pi_{sc}))$
- $L(\Delta_\rho[-\frac{7}{2},-\frac{7}{2}],\Delta_\rho[-\frac{3}{2},-\frac{5}{2}];T_{I,1}^{\frac{5}{2}}(\pi_{sc}))$
- $L(\Delta_\rho[-\frac{3}{2},\frac{7}{2}],\Delta_\rho[-\frac{5}{2},-\frac{5}{2}];\pi_{sc})$
- $L(\Delta_\rho[-\frac{5}{2},-\frac{5}{2}],\Delta_\rho[-\frac{3}{2},-\frac{7}{2}];\pi_{sc})$
- $L(\Delta_\rho[-\frac{5}{2},-\frac{7}{2}],\Delta_\rho[-\frac{5}{2},-\frac{5}{2}],\Delta_\rho[-\frac{3}{2},-\frac{3}{2}];T_{I,1}^{\frac{5}{2}}(\pi_{sc}))$
- $L(\Delta_\rho[-\frac{7}{2},-\frac{7}{2}],\Delta_\rho[-\frac{5}{2},-\frac{5}{2}],\Delta_\rho[-\frac{3}{2},-\frac{5}{2}];\pi_{sc})$
- $L(\Delta_\rho[-\frac{7}{2},-\frac{7}{2}],\Delta_\rho[-\frac{5}{2},-\frac{5}{2}],\Delta_\rho[-\frac{5}{2},-\frac{5}{2}],\Delta_\rho[-\frac{3}{2},-\frac{3}{2}];\pi_{sc})$

(i) $(\frac{3}{2},\frac{5}{2},\frac{7}{2},\frac{7}{2})$:

- $L(\Delta_\rho[-\frac{7}{2},-\frac{7}{2}];T_{I,1}^{\frac{3}{2}}(T_{I,1}^{\frac{7}{2}}(T_{I,1}^{\frac{5}{2}}(\pi_{sc}))))$
- $L(\Delta_\rho[-\frac{7}{2},-\frac{7}{2}];T_{I,1}^{\frac{3}{2}}(T_{I,1}^{\frac{7}{2}}(T_{I,1}^{\frac{5}{2}}(\pi_{sc}))))$

- $L(\Delta_\rho[-\frac{7}{2},-\frac{7}{2}],\Delta_\rho[-\frac{7}{2},-\frac{7}{2}];T_{I,1}^{\frac{3}{2}}(T_{I,1}^{\frac{5}{2}}(\pi_{sc})))$
- $L(\Delta_\rho[-\frac{7}{2},\frac{7}{2}],\Delta_\rho[-\frac{3}{2},-\frac{7}{2}];\pi_{sc})$
- $L(\Delta_\rho[-\frac{7}{2},-\frac{7}{2}],\Delta_\rho[-\frac{7}{2},-\frac{7}{2}],\Delta_\rho[-\frac{3}{2},-\frac{3}{2}];T_{I,1}^{\frac{5}{2}}(\pi_{sc}))$
- $L(\Delta_\rho[-\frac{7}{2},-\frac{7}{2}],\Delta_\rho[-\frac{5}{2},-\frac{7}{2}],\Delta_\rho[-\frac{3}{2},-\frac{3}{2}];\pi_{sc})$
- $L(\Delta_\rho[-\frac{7}{2},-\frac{7}{2}],\Delta_\rho[-\frac{7}{2},-\frac{7}{2}],\Delta_\rho[-\frac{3}{2},-\frac{5}{2}];\pi_{sc})$
- $L(\Delta_\rho[-\frac{7}{2},-\frac{7}{2}],\Delta_\rho[-\frac{7}{2},-\frac{7}{2}],\Delta_\rho[-\frac{5}{2},-\frac{5}{2}],\Delta_\rho[-\frac{3}{2},-\frac{3}{2}];\pi_{sc})$

(j) $(\frac{3}{2},\frac{5}{2},\frac{7}{2},\frac{9}{2})$:

- $L(\Delta_\rho[-\frac{9}{2},-\frac{9}{2}];T_{I,1}^{\frac{3}{2}}(T_{I,1}^{\frac{7}{2}}(T_{I,1}^{\frac{5}{2}}(\pi_{sc}))))$
- $L(\Delta_\rho[-\frac{7}{2},-\frac{9}{2}];T_{I,1}^{\frac{3}{2}}(T_{I,1}^{\frac{5}{2}}(\pi_{sc})))$
- $L(\Delta_\rho[-\frac{3}{2},-\frac{9}{2}];\pi_{sc})$
- $L(\Delta_\rho[-\frac{7}{2},-\frac{9}{2}],\Delta_\rho[-\frac{3}{2},-\frac{5}{2}];\pi_{sc})$
- $L(\Delta_\rho[-\frac{5}{2},\frac{9}{2}],\Delta_\rho[-\frac{3}{2},-\frac{3}{2}];\pi_{sc})$
- $L(\Delta_\rho[-\frac{9}{2},\frac{9}{2}],\Delta_\rho[-\frac{3}{2},-\frac{7}{2}];\pi_{sc})$
- $L(\Delta_\rho[-\frac{9}{2},-\frac{9}{2}],\Delta_\rho[-\frac{3}{2},-\frac{3}{2}];T_{I,1}^{\frac{7}{2}}(T_{I,1}^{\frac{5}{2}}(\pi_{sc})))$
- $L(\Delta_\rho[-\frac{7}{2},-\frac{9}{2}],\Delta_\rho[-\frac{3}{2},-\frac{3}{2}];T_{I,1}^{\frac{5}{2}}(\pi_{sc}))$
- $L(\Delta_\rho[-\frac{9}{2},-\frac{9}{2}],\Delta_\rho[-\frac{7}{2},-\frac{7}{2}];T_{I,1}^{\frac{3}{2}}(T_{I,1}^{\frac{5}{2}}(\pi_{sc})))$
- $L(\Delta_\rho[-\frac{9}{2},-\frac{9}{2}],\Delta_\rho[-\frac{7}{2},-\frac{7}{2}],\Delta_\rho[-\frac{3}{2},-\frac{3}{2}];T_{I,1}^{\frac{5}{2}}(\pi_{sc}))$
- $L(\Delta_\rho[-\frac{7}{2},-\frac{9}{2}],\Delta_\rho[-\frac{5}{2},-\frac{5}{2}],\Delta_\rho[-\frac{3}{2},-\frac{3}{2}];\pi_{sc})$
- $L(\Delta_\rho[-\frac{9}{2},-\frac{9}{2}],\Delta_\rho[-\frac{5}{2},-\frac{7}{2}],\Delta_\rho[-\frac{3}{2},-\frac{3}{2}];\pi_{sc})$

(k) $(\frac{5}{2},\frac{5}{2},\frac{5}{2},\frac{5}{2})$:

- $L(\Delta_\rho[-\frac{5}{2},-\frac{5}{2}],\Delta_\rho[-\frac{5}{2},-\frac{5}{2}],\Delta_\rho[-\frac{5}{2},-\frac{5}{2}];T_{I,1}^{\frac{5}{2}}(\pi_{sc}))$
- $L(\Delta_\rho[-\frac{5}{2},-\frac{5}{2}],\Delta_\rho[-\frac{5}{2},-\frac{5}{2}],\Delta_\rho[-\frac{5}{2},-\frac{5}{2}],\Delta_\rho[-\frac{5}{2},-\frac{5}{2}];\pi_{sc})$

(l) $(\frac{5}{2},\frac{5}{2},\frac{5}{2},\frac{7}{2})$:

- $L(\Delta_\rho[-\frac{5}{2},-\frac{7}{2}],\Delta_\rho[-\frac{5}{2},-\frac{5}{2}];T_{I,1}^{\frac{5}{2}}(\pi_{sc}))$
- $L(\Delta_\rho[-\frac{5}{2},-\frac{5}{2}],\Delta_\rho[-\frac{5}{2},-\frac{5}{2}];T_{I,1}^{\frac{7}{2}}(T_{I,1}^{\frac{5}{2}}(\pi_{sc})))$
- $L(\Delta_\rho[-\frac{7}{2},-\frac{7}{2}],\Delta_\rho[-\frac{5}{2},-\frac{5}{2}],\Delta_\rho[-\frac{5}{2},-\frac{5}{2}];T_{I,1}^{\frac{5}{2}}(\pi_{sc}))$
- $L(\Delta_\rho[-\frac{5}{2},-\frac{7}{2}],\Delta_\rho[-\frac{5}{2},-\frac{5}{2}],\Delta_\rho[-\frac{5}{2},-\frac{5}{2}];\pi_{sc})$
- $L(\Delta_\rho[-\frac{7}{2},-\frac{7}{2}],\Delta_\rho[-\frac{5}{2},-\frac{5}{2}],\Delta_\rho[-\frac{5}{2},-\frac{5}{2}],\Delta_\rho[-\frac{5}{2},-\frac{5}{2}];\pi_{sc})$

(m) $(\frac{5}{2},\frac{5}{2},\frac{7}{2},\frac{7}{2})$:

- $L(\Delta_\rho[-\frac{7}{2},-\frac{7}{2}],\Delta_\rho[-\frac{5}{2},-\frac{5}{2}];T_{I,1}^{\frac{7}{2}}(T_{I,1}^{\frac{5}{2}}(\pi_{sc})))$
- $L(\Delta_\rho[-\frac{7}{2},-\frac{7}{2}],\Delta_\rho[-\frac{5}{2},-\frac{5}{2}];T_{I,1}^{\frac{5}{2}}(T_{I,1}^{\frac{7}{2}}(\pi_{sc}))$
- $L(\Delta_\rho[-\frac{7}{2},-\frac{7}{2}],\Delta_\rho[-\frac{5}{2},-\frac{7}{2}];T_{I,1}^{\frac{5}{2}}(\pi_{sc}))$
- $L(\Delta_\rho[-\frac{5}{2},-\frac{7}{2}],\Delta_\rho[-\frac{5}{2},-\frac{7}{2}];\pi_{sc})$
- $L(\Delta_\rho[-\frac{7}{2},-\frac{7}{2}],\Delta_\rho[-\frac{7}{2},-\frac{7}{2}],\Delta_\rho[-\frac{5}{2},-\frac{5}{2}];T_{I,1}^{\frac{5}{2}}(\pi_{sc}))$
- $L(\Delta_\rho[-\frac{7}{2},-\frac{7}{2}],\Delta_\rho[-\frac{7}{2},-\frac{7}{2}],\Delta_\rho[-\frac{5}{2},-\frac{5}{2}],\Delta_\rho[-\frac{5}{2},-\frac{5}{2}];\pi_{sc})$
- $L(\Delta_\rho[-\frac{7}{2},-\frac{7}{2}],\Delta_\rho[-\frac{7}{2},-\frac{7}{2}],\Delta_\rho[-\frac{5}{2},-\frac{5}{2}],\Delta_\rho[-\frac{5}{2},-\frac{5}{2}];\pi_{sc})$

(n) $(\frac{5}{2},\frac{5}{2},\frac{7}{2},\frac{9}{2})$:

- $L(\Delta_\rho[-\frac{5}{2},-\frac{5}{2}];T_{I,1}^{\frac{9}{2}}(T_{I,1}^{\frac{7}{2}}(T_{I,1}^{\frac{5}{2}}(\pi_{sc}))))$
- $L(\Delta_\rho[-\frac{5}{2},-\frac{9}{2}];T_{I,1}^{\frac{5}{2}}(\pi_{sc}))$
- $L(\Delta_\rho[-\frac{7}{2},-\frac{9}{2}],\Delta_\rho[-\frac{5}{2},-\frac{5}{2}];T_{I,1}^{\frac{5}{2}}(\pi_{sc}))$
- $L(\Delta_\rho[-\frac{9}{2},-\frac{9}{2}],\Delta_\rho[-\frac{5}{2},-\frac{7}{2}];T_{I,1}^{\frac{5}{2}}(\pi_{sc}))$
- $L(\Delta_\rho[-\frac{5}{2},\frac{9}{2}],\Delta_\rho[-\frac{5}{2},-\frac{5}{2}];\pi_{sc})$
- $L(\Delta_\rho[-\frac{9}{2},-\frac{9}{2}],\Delta_\rho[-\frac{5}{2},-\frac{5}{2}];T_{I,1}^{\frac{7}{2}}(T_{I,1}^{\frac{5}{2}}(\pi_{sc})))$
- $L(\Delta_\rho[-\frac{9}{2},-\frac{9}{2}],\Delta_\rho[-\frac{7}{2},-\frac{7}{2}],\Delta_\rho[-\frac{5}{2},-\frac{5}{2}];T_{I,1}^{\frac{5}{2}}(\pi_{sc}))$
- $L(\Delta_\rho[-\frac{7}{2},-\frac{9}{2}],\Delta_\rho[-\frac{5}{2},-\frac{5}{2}],\Delta_\rho[-\frac{5}{2},-\frac{5}{2}];\pi_{sc})$
- $L(\Delta_\rho[-\frac{9}{2},-\frac{9}{2}],\Delta_\rho[-\frac{5}{2},-\frac{7}{2}],\Delta_\rho[-\frac{5}{2},-\frac{5}{2}];\pi_{sc})$
- $L(\Delta_\rho[-\frac{9}{2},-\frac{9}{2}],\Delta_\rho[-\frac{7}{2},-\frac{7}{2}],\Delta_\rho[-\frac{5}{2},-\frac{5}{2}],\Delta_\rho[-\frac{5}{2},-\frac{5}{2}];\pi_{sc})$

(o) $(\frac{5}{2},\frac{7}{2},\frac{7}{2},\frac{7}{2})$

- $L(\Delta_\rho[-\frac{7}{2},-\frac{7}{2}],\Delta_\rho[-\frac{7}{2},-\frac{7}{2}];T_{I,1}^{\frac{7}{2}}(T_{I,1}^{\frac{5}{2}}(\pi_{sc})))$
- $L(\Delta_\rho[-\frac{3=7}{2},-\frac{7}{2}],\Delta_\rho[-\frac{7}{2},-\frac{7}{2}],\Delta_\rho[-\frac{7}{2},-\frac{7}{2}];T_{I,1}^{\frac{5}{2}}(\pi_{sc}))$
- $L(\Delta_\rho[-\frac{7}{2},-\frac{7}{2}],\Delta_\rho[-\frac{7}{2},-\frac{7}{2}],\Delta_\rho[-\frac{5}{2},-\frac{7}{2}];\pi_{sc})$
- $L(\Delta_\rho[-\frac{7}{2},-\frac{7}{2}],\Delta_\rho[-\frac{7}{2},-\frac{7}{2}],\Delta_\rho[-\frac{7}{2},-\frac{7}{2}],\Delta_\rho[-\frac{5}{2},-\frac{5}{2}];\pi_{sc})$

(p) $(\frac{5}{2},\frac{7}{2},\frac{7}{2},\frac{9}{2})$*:*

- $L(\Delta_\rho[-\frac{7}{2},-\frac{7}{2}];T_{I,1}^{\frac{9}{2}}(T_{I,1}^{\frac{7}{2}}(T_{I,1}^{\frac{5}{2}}(\pi_{sc}))))$
- $L(\Delta_\rho[-\frac{9}{2},-\frac{9}{2}],\Delta_\rho[-\frac{7}{2},-\frac{7}{2}];T_{I,1}^{\frac{7}{2}}(T_{I,1}^{\frac{5}{2}}(\pi_{sc})))$
- $L(\Delta_\rho[-\frac{9}{2},-\frac{9}{2}],\Delta_\rho[-\frac{7}{2},-\frac{7}{2}];T_{I,1}^{\frac{5}{2}}(T_{I,1}^{\frac{7}{2}}(\pi_{sc})))$
- $L(\Delta_\rho[-\frac{7}{2},-\frac{9}{2}],\Delta_\rho[-\frac{7}{2},-\frac{7}{2}];T_{I,1}^{\frac{5}{2}}(\pi_{sc}))$
- $L(\Delta_\rho[-\frac{7}{2},-\frac{9}{2}],\Delta_\rho[-\frac{5}{2},-\frac{7}{2}];\pi_{sc})$
- $L(\Delta_\rho[-\frac{5}{2},\frac{9}{2}],\Delta_\rho[-\frac{7}{2},-\frac{7}{2}];\pi_{sc})$
- $L(\Delta_\rho[-\frac{7}{2},\frac{7}{2}],\Delta_\rho[-\frac{5}{2},-\frac{9}{2}];\pi_{sc})$
- $L(\Delta_\rho[-\frac{9}{2},-\frac{9}{2}],\Delta_\rho[-\frac{7}{2},-\frac{7}{2}],\Delta_\rho[-\frac{7}{2},-\frac{7}{2}];T_{I,1}^{\frac{5}{2}}(\pi_{sc}))$
- $L(\Delta_\rho[-\frac{9}{2},-\frac{9}{2}],\Delta_\rho[-\frac{7}{2},-\frac{7}{2}],\Delta_\rho[-\frac{5}{2},-\frac{7}{2}];\pi_{sc})$
- $L(\Delta_\rho[-\frac{9}{2},-\frac{9}{2}],\Delta_\rho[-\frac{7}{2},-\frac{7}{2}],\Delta_\rho[-\frac{7}{2},-\frac{7}{2}],\Delta_\rho[-\frac{5}{2},-\frac{5}{2}];\pi_{sc})$

(q) $(\frac{5}{2},\frac{7}{2},\frac{9}{2},\frac{9}{2})$*:*

- $L(\Delta_\rho[-\frac{9}{2},-\frac{9}{2}];T_{I,1}^{\frac{9}{2}}(T_{I,1}^{\frac{7}{2}}(T_{I,1}^{\frac{5}{2}}(\pi_{sc}))))$
- $L(\Delta_\rho[-\frac{9}{2},-\frac{9}{2}],\Delta_\rho[-\frac{9}{2},-\frac{9}{2}];T_{I,1}^{\frac{7}{2}}(T_{I,1}^{\frac{5}{2}}(\pi_{sc})))$
- $L(\Delta_\rho[-\frac{9}{2},-\frac{9}{2}],\Delta_\rho[-\frac{7}{2},-\frac{9}{2}];T_{I,1}^{\frac{5}{2}}(\pi_{sc}))$
- $L(\Delta_\rho[-\frac{9}{2},\frac{9}{2}],\Delta_\rho[-\frac{5}{2},-\frac{9}{2}];\pi_{sc})$
- $L(\Delta_\rho[-\frac{9}{2},-\frac{9}{2}],\Delta_\rho[-\frac{9}{2},-\frac{9}{2}],\Delta_\rho[-\frac{7}{2},-\frac{7}{2}];T_{I,1}^{\frac{5}{2}}(\pi_{sc}))$
- $L(\Delta_\rho[-\frac{9}{2},-\frac{9}{2}],\Delta_\rho[-\frac{7}{2},-\frac{9}{2}],\Delta_\rho[-\frac{5}{2},-\frac{5}{2}];\pi_{sc})$
- $L(\Delta_\rho[-\frac{9}{2},-\frac{9}{2}],\Delta_\rho[-\frac{9}{2},-\frac{9}{2}],\Delta_\rho[-\frac{5}{2},-\frac{7}{2}];\pi_{sc})$
- $L(\Delta_\rho[-\frac{9}{2},-\frac{9}{2}],\Delta_\rho[-\frac{9}{2},-\frac{9}{2}],\Delta_\rho[-\frac{7}{2},-\frac{7}{2}],\Delta_\rho[-\frac{5}{2},-\frac{5}{2}];\pi_{sc})$

(r) $(\frac{5}{2},\frac{7}{2},\frac{9}{2},\frac{11}{2})$*:*

- $L(\Delta_\rho[-\frac{11}{2},-\frac{11}{2}];T_{I,1}^{\frac{9}{2}}(T_{I,1}^{\frac{7}{2}}(T_{I,1}^{\frac{5}{2}}(\pi_{sc}))))$

- $L(\Delta_\rho[-\frac{9}{2},-\frac{11}{2}];T_{I,1}^{\frac{7}{2}}(T_{I,1}^{\frac{5}{2}}(\pi_{sc})))$
- $L(\Delta_\rho[-\frac{7}{2},-\frac{11}{2}];T_{I,1}^{\frac{5}{2}}(\pi_{sc}))$
- $L(\Delta_\rho[-\frac{5}{2},-\frac{11}{2}];\pi_{sc})$
- $L(\Delta_\rho[-\frac{11}{2},-\frac{11}{2}],\Delta_\rho[-\frac{9}{2},-\frac{9}{2}];T_{I,1}^{\frac{7}{2}}(T_{I,1}^{\frac{5}{2}}(\pi_{sc})))$
- $L(\Delta_\rho[-\frac{9}{2},-\frac{11}{2}],\Delta_\rho[-\frac{7}{2},-\frac{7}{2}];T_{I,1}^{\frac{5}{2}}(\pi_{sc}))$
- $L(\Delta_\rho[-\frac{11}{2},-\frac{11}{2}],\Delta_\rho[-\frac{7}{2},-\frac{9}{2}];T_{I,1}^{\frac{5}{2}}(\pi_{sc}))$
- $L(\Delta_\rho[-\frac{9}{2},-\frac{11}{2}],\Delta_\rho[-\frac{5}{2},-\frac{7}{2}];\pi_{sc})$
- $L(\Delta_\rho[-\frac{7}{2},\frac{11}{2}],\Delta_\rho[-\frac{5}{2},-\frac{5}{2}];\pi_{sc})$
- $L(\Delta_\rho[-\frac{11}{2},\frac{11}{2}],\Delta_\rho[-\frac{5}{2},-\frac{9}{2}];\pi_{sc})$
- $L(\Delta_\rho[-\frac{11}{2},-\frac{11}{2}],\Delta_\rho[-\frac{9}{2},-\frac{9}{2}],\Delta_\rho[-\frac{7}{2},-\frac{7}{2}];T_{I,1}^{\frac{5}{2}}(\pi_{sc}))$
- $L(\Delta_\rho[-\frac{9}{2},-\frac{11}{2}],\Delta_\rho[-\frac{7}{2},-\frac{7}{2}],\Delta_\rho[-\frac{5}{2},-\frac{5}{2}];\pi_{sc})$
- $L(\Delta_\rho[-\frac{11}{2},-\frac{11}{2}],\Delta_\rho[-\frac{7}{2},-\frac{9}{2}],\Delta_\rho[-\frac{5}{2},-\frac{5}{2}];\pi_{sc})$
- $L(\Delta_\rho[-\frac{11}{2},-\frac{11}{2}],\Delta_\rho[-\frac{9}{2},-\frac{9}{2}],\Delta_\rho[-\frac{5}{2},-\frac{7}{2}];\pi_{sc})$

(7) $(\alpha=3)$*:*

(a) $(1,1,2,3)$*:*
- $L(\Delta_\rho[-1,-1];T_{I,1}^{1}(T_{I,1}^{2}(T_{I,1}^{3}(\pi_{sc}))))$
- $L(\Delta_\rho[-1,-1],\Delta_\rho[-1,-1];T_{I,1}^{2}(T_{I,1}^{3}(\pi_{sc})))$
- $L(\Delta_\rho[-1,-2],\Delta_\rho[-1,-1];T_{I,1}^{3}(\pi_{sc}))$
- $L(\Delta_\rho[-1,-3],\Delta_\rho[-1,-1];\pi_{sc})$
- $L(\Delta_\rho[-2,-2],\Delta_\rho[-1,-1],\Delta_\rho[-1,-1];T_{I,1}^{3}(\pi_{sc}))$
- $L(\Delta_\rho[-2,-3],\Delta_\rho[-1,-1],\Delta_\rho[-1,-1];\pi_{sc})$
- $L(\Delta_\rho[-3,-3],\Delta_\rho[-1,-2],\Delta_\rho[-1,-1];\pi_{sc})$
- $L(\Delta_\rho[-3,-3],\Delta_\rho[-2,-2],\Delta_\rho[-1,-1],\Delta_\rho[-1,-1];\pi_{sc})$

(b) $(1,2,2,3)$*:*
- $L(\Delta_\rho[-1,-2];T_{I,1}^{2}(T_{I,1}^{3}(\pi_{sc})))$
- $L(\Delta_\rho[-2,-2];T_{I,1}^{1}(T_{I,1}^{2}(T_{I,1}^{3}(\pi_{sc}))))$
- $L(\Delta_\rho[-2,-2],\Delta_\rho[-1,-2];T_{I,1}^{3}(\pi_{sc}))$
- $L(\Delta_\rho[-2,-2],\Delta_\rho[-1,-3];\pi_{sc})$
- $L(\Delta_\rho[-2,-2],\Delta_\rho[-2,-2],\Delta_\rho[-1,-1];T_{I,1}^{3}(\pi_{sc}))$
- $L(\Delta_\rho[-2,-3],\Delta_\rho[-2,-2],\Delta_\rho[-1,-1];\pi_{sc})$
- $L(\Delta_\rho[-3,-3],\Delta_\rho[-2,-2],\Delta_\rho[-1,-2];\pi_{sc})$
- $L(\Delta_\rho[-3,-3],\Delta_\rho[-2,-2],\Delta_\rho[-2,-2],\Delta_\rho[-1,-1];\pi_{sc})$

(c) $(1,2,3,3)$*:*
- $L(\Delta_\rho[-3,-3];T_{I,1}^{1}(T_{I,1}^{2}(T_{I,1}^{3}(\pi_{sc}))))$
- $L(\Delta_\rho[-1,-3];T_{I,1}^{3}(\pi_{sc}))$
- $L(\Delta_\rho[-3,-3],\Delta_\rho[-1,-1];T_{I,1}^{2}(T_{I,1}^{3}(\pi_{sc})))$
- $L(\Delta_\rho[-2,-3],\Delta_\rho[-1,-1];T_{I,1}^{3}(\pi_{sc}))$
- $L(\Delta_\rho[-3,-3],\Delta_\rho[-1,-3];\pi_{sc})$
- $L(\Delta_\rho[-3,-3],\Delta_\rho[-2,-3],\Delta_\rho[-1,-1];\pi_{sc})$
- $L(\Delta_\rho[-3,-3],\Delta_\rho[-3,-3],\Delta_\rho[-1,-2];\pi_{sc})$
- $L(\Delta_\rho[-3,-3],\Delta_\rho[-3,-3],\Delta_\rho[-2,-2],\Delta_\rho[-1,-1];\pi_{sc})$

(d) $(1,2,3,4)$*:*

- $L(\Delta_\rho[-4,-4]; T^1_{I,1}(T^2_{I,1}(T^3_{I,1}(\pi_{sc}))))$
- $L(\Delta_\rho[-1,-4]; \pi_{sc})$
- $L(\Delta_\rho[-3,-4], \Delta_\rho[-1,-2]; \pi_{sc})$
- $L(\Delta_\rho[-2,-4], \Delta_\rho[-1,-1]; \pi_{sc})$
- $L(\Delta_\rho[-4,-4], \Delta_\rho[-1,-2]; T^3_{I,1}(\pi_{sc}))$
- $L(\Delta_\rho[-4,-4], \Delta_\rho[-1,-1]; T^2_{I,1}(T^3_{I,1}(\pi_{sc})))$
- $L(\Delta_\rho[-4,-4], \Delta_\rho[-2,-2], \Delta_\rho[-1,-1]; T^3_{I,1}(\pi_{sc}))$
- $L(\Delta_\rho[-3,-4], \Delta_\rho[-2,-2], \Delta_\rho[-1,-1]; \pi_{sc})$

(e) $(2,2,2,3)$*:*
- $L(\Delta_\rho[-2,-2], \Delta_\rho[-2,-2]; T^2_{I,1}(T^3_{I,1}(\pi_{sc})))$
- $L(\Delta_\rho[-2,-2], \Delta_\rho[-2,-2], \Delta_\rho[-2,-2]; T^3_{I,1}(\pi_{sc}))$
- $L(\Delta_\rho[-2,-3], \Delta_\rho[-2,-2], \Delta_\rho[-2,-2]; \pi_{sc})$
- $L(\Delta_\rho[-3,-3], \Delta_\rho[-2,-2], \Delta_\rho[-2,-2], \Delta_\rho[-2,-2]; \pi_{sc})$

(f) $(2,2,3,3)$*:*
- $L(\Delta_\rho[-2,-3]; T^2_{I,1}(T^3_{I,1}(\pi_{sc})))$
- $L(\Delta_\rho[-3,-3], \Delta_\rho[-2,-2]; T^2_{I,1}(T^3_{I,1}(\pi_{sc})))$
- $L(\Delta_\rho[-2,-3], \Delta_\rho[-2,-2]; T^3_{I,1}(\pi_{sc}))$
- $L(\Delta_\rho[-2,-3], \Delta_\rho[-2,-3]; \pi_{sc})$
- $L(\Delta_\rho[-3,-3], \Delta_\rho[-2,-2], \Delta_\rho[-2,-2]; T^3_{I,1}(\pi_{sc}))$
- $L(\Delta_\rho[-3,-3], \Delta_\rho[-2,-3], \Delta_\rho[-2,-2]; \pi_{sc})$
- $L(\Delta_\rho[-3,-3], \Delta_\rho[-3,-3], \Delta_\rho[-2,-2], \Delta_\rho[-2,-2]; \pi_{sc})$

(g) $(2,2,3,4)$*:*
- $L(\Delta_\rho[-2,-2]; T^2_{I,1}(T^4_{I,1}(T^3_{I,1}(\pi_{sc}))))$
- $L(\Delta_\rho[-2,-4], \Delta_\rho[-2,-2]; \pi_{sc})$
- $L(\Delta_\rho[-2,-2], \Delta_\rho[-2,-2]; T^4_{I,1}(T^3_{I,1}(\pi_{sc})))$
- $L(\Delta_\rho[-4,-4], \Delta_\rho[-2,-2]; T^2_{I,1}(T^3_{I,1}(\pi_{sc})))$
- $L(\Delta_\rho[-4,-4], \Delta_\rho[-2,-2], \Delta_\rho[-2,-2]; T^3_{I,1}(\pi_{sc}))$
- $L(\Delta_\rho[-3,-4], \Delta_\rho[-2,-2], \Delta_\rho[-2,-2]; \pi_{sc})$
- $L(\Delta_\rho[-4,-4], \Delta_\rho[-2,-3], \Delta_\rho[-2,-2]; \pi_{sc})$
- $L(\Delta_\rho[-4,-4], \Delta_\rho[-3,-3], \Delta_\rho[-2,-2], \Delta_\rho[-2,-2]; \pi_{sc})$

(h) $(2,3,3,3)$*:*
- $L(\Delta_\rho[-3,-3], \Delta_\rho[-2,-3]; T^3_{I,1}(\pi_{sc}))\ +$
- $L(\Delta_\rho[-3,-3], \Delta_\rho[-3,-3]; T^2_{I,1}(T^3_{I,1}(\pi_{sc})))$
- $L(\Delta_\rho[-3,-3], \Delta_\rho[-3,-3], \Delta_\rho[-2,-2]; T^3_{I,1}(\pi_{sc}))$
- $L(\Delta_\rho[-3,-3], \Delta_\rho[-3,-3], \Delta_\rho[-2,-3]; \pi_{sc})$
- $L(\Delta_\rho[-3,-3], \Delta_\rho[-3,-3], \Delta_\rho[-3,-3], \Delta_\rho[-2,-2]; \pi_{sc})$

(i) $(2,3,3,4)$*:*
- $L(\Delta_\rho[-3,-3]; T^2_{I,1}(T^4_{I,1}(T^3_{I,1}(\pi_{sc}))))$
- $L(\Delta_\rho[-2,-3]; T^4_{I,1}(T^3_{I,1}(\pi_{sc})))$
- $L(\Delta_\rho[-2,-4]; T^3_{I,1}(\pi_{sc}))$
- $L(\Delta_\rho[-3,-4]; T^2_{I,1}(T^3_{I,1}(\pi_{sc})))$
- $L(\Delta_\rho[-4,-4], \Delta_\rho[-3,-3]; T^2_{I,1}(T^3_{I,1}(\pi_{sc})))$
- $L(\Delta_\rho[-3,-2], \Delta_\rho[-2,-2]; T^3_{I,1}(\pi_{sc}))$
- $L(\Delta_\rho[-3,-4], \Delta_\rho[-2,-2]; T^3_{I,1}(\pi_{sc}))$
- $L(\Delta_\rho[-4,-4], \Delta_\rho[-2,-3]; T^3_{I,1}(\pi_{sc}))$

- $L(\Delta_\rho[-2,-4],\Delta_\rho[-3,-3];\pi_{sc})$
- $L(\Delta_\rho[-3,-3],\Delta_\rho[-2,-4];\pi_{sc})$
- $L(\Delta_\rho[-3,-4],\Delta_\rho[-3,-3],\Delta_\rho[-2,-2];\pi_{sc})$
- $L(\Delta_\rho[-4,-4],\Delta_\rho[-3,-3],\Delta_\rho[-2,-3];\pi_{sc})$
- $L(\Delta_\rho[-4,-4],\Delta_\rho[-3,-3],\Delta_\rho[-3,-3],\Delta_\rho[-2,-2];\pi_{sc})$

(j) $(2,3,4,4)$*:*
- $L(\Delta_\rho[-4,-4];T_{I,1}^2(T_{I,1}^4(T_{I,1}^3(\pi_{sc}))))$
- $L(\Delta_\rho[-4,-4],\Delta_\rho[-4,-4];T_{I,1}^2(T_{I,1}^3(\pi_{sc})))$
- $L(\Delta_\rho[-4,-4],\Delta_\rho[-2,-2];T_{I,1}^4(T_{I,1}^3(\pi_{sc})))$
- $L(\Delta_\rho[-4,-4],\Delta_\rho[-2,-4];\pi_{sc})$
- $L(\Delta_\rho[-4,-4],\Delta_\rho[-4,-4],\Delta_\rho[-2,-2];T_{I,1}^3(\pi_{sc}))$
- $L(\Delta_\rho[-4,-4],\Delta_\rho[-3,-4],\Delta_\rho[-2,-2];\pi_{sc})$
- $L(\Delta_\rho[-4,-4],\Delta_\rho[-4,-4],\Delta_\rho[-2,-3];\pi_{sc})$
- $L(\Delta_\rho[-4,-4],\Delta_\rho[-4,-4],\Delta_\rho[-3,-3],\Delta_\rho[-2,-2];\pi_{sc})$

(k) $(2,3,4,5)$*:*
- $L(\Delta_\rho[-4,-5];T_{I,1}^2(T_{I,1}^3(\pi_{sc})))+$
- $L(\Delta_\rho[-2,-5];\pi_{sc})$
- $L(\Delta_\rho[-5,-5];T_{I,1}^2(T_{I,1}^4(T_{I,1}^3(\pi_{sc}))))$
- $L(\Delta_\rho[-5,-5],\Delta_\rho[-4,-4];T_{I,1}^2(T_{I,1}^3(\pi_{sc})))$
- $L(\Delta_\rho[-5,-5],\Delta_\rho[-2,-2];T_{I,1}^4(T_{I,1}^3(\pi_{sc})))$
- $L(\Delta_\rho[-4,-5],\Delta_\rho[-2,-2];T_{I,1}^3(\pi_{sc}))$
- $L(\Delta_\rho[-4,-5],\Delta_\rho[-2,-3];\pi_{sc})$
- $L(\Delta_\rho[-3,-5],\Delta_\rho[-2,-2];\pi_{sc})$
- $L(\Delta_\rho[-5,-5],\Delta_\rho[-2,-4];\pi_{sc})$
- $L(\Delta_\rho[-4,-5],\Delta_\rho[-3,-3],\Delta_\rho[-2,-2];\pi_{sc})$
- $L(\Delta_\rho[-5,-5],\Delta_\rho[-4,-4],\Delta_\rho[-2,-2];T_{I,1}^3(\pi_{sc}))$
- $L(\Delta_\rho[-5,-5],\Delta_\rho[-3,-4],\Delta_\rho[-2,-2];\pi_{sc})$

(l) $(3,3,3,3)$*:*
- $L(\Delta_\rho[-3,-3],\Delta_\rho[-3,-3],\Delta_\rho[-3,-3];T_{I,1}^3(\pi_{sc}))$
- $L(\Delta_\rho[-3,-3],\Delta_\rho[-3,-3],\Delta_\rho[-3,-3],\Delta_\rho[-3,-3];\pi_{sc})$

(m) $(3,3,3,4)$*:*
- $L(\Delta_\rho[-3,-3],\Delta_\rho[-3,-3];T_{I,1}^4(T_{I,1}^3(\pi_{sc})))$
- $L(\Delta_\rho[-3,-4],\Delta_\rho[-3,-3];T_{I,1}^3(\pi_{sc}))$
- $L(\Delta_\rho[-4,-4],\Delta_\rho[-3,-3],\Delta_\rho[-3,-3];T_{I,1}^3(\pi_{sc}))$
- $L(\Delta_\rho[-3,-4],\Delta_\rho[-3,-3],\Delta_\rho[-3,-3];\pi_{sc})$
- $L(\Delta_\rho[-4,-4],\Delta_\rho[-3,-3],\Delta_\rho[-3,-3],\Delta_\rho[-3,-3];\pi_{sc})$

(n) $(3,3,4,4)$*:*
- $L(\Delta_\rho[-3,-4];T_{I,1}^4(T_{I,1}^3(\pi_{sc})))$
- $L(\Delta_\rho[-4,-4],\Delta_\rho[-3,-4];T_{I,1}^3(\pi_{sc}))$
- $L(\Delta_\rho[-4,-4],\Delta_\rho[-3,-3];T_{I,1}^4(T_{I,1}^3(\pi_{sc})))$
- $L(\Delta_\rho[-3,-4],\Delta_\rho[-3,-4];\pi_{sc})$
- $L(\Delta_\rho[-4,-4],\Delta_\rho[-4,-4],\Delta_\rho[-3,-3];T_{I,1}^3(\pi_{sc}))$
- $L(\Delta_\rho[-4,-4],\Delta_\rho[-3,-4],\Delta_\rho[-3,-3];\pi_{sc})$
- $L(\Delta_\rho[-4,-4],\Delta_\rho[-4,-4],\Delta_\rho[-3,-3],\Delta_\rho[-3,-3];\pi_{sc})$

(o) $(3,3,4,5)$:

- $L(\Delta_\rho[-3,-3]; T^5_{I,1}(T^4_{I,1}(T^3_{I,1}(\pi_{sc}))))$
- $L(\Delta_\rho[-3,-5]; T^3_{I,1}(\pi_{sc}))$
- $L(\Delta_\rho[-5,-5], \Delta_\rho[-3,-3]; T^4_{I,1}(T^3_{I,1}(\pi_{sc})))$
- $L(\Delta_\rho[-4,-5], \Delta_\rho[-3,-3]; T^3_{I,1}(\pi_{sc}))$
- $L(\Delta_\rho[-3,-5], \Delta_\rho[-3,-3]; \pi_{sc})$
- $L(\Delta_\rho[-5,-5], \Delta_\rho[-3,-4]; T^3_{I,1}(\pi_{sc}))$
- $L(\Delta_\rho[-5,-5], \Delta_\rho[-4,-4], \Delta_\rho[-3,-3]; T^3_{I,1}(\pi_{sc}))$
- $L(\Delta_\rho[-4,-5], \Delta_\rho[-3,-3], \Delta_\rho[-3,-3]; \pi_{sc})$
- $L(\Delta_\rho[-5,-5], \Delta_\rho[-3,-4], \Delta_\rho[-3,-3]; \pi_{sc})$
- $L(\Delta_\rho[-5,-5], \Delta_\rho[-4,-4], \Delta_\rho[-3,-3], \Delta_\rho[-3,-3]; \pi_{sc})$

(p) $(3,4,4,4)$*:*
- $L(\Delta_\rho[-4,-4], \Delta_\rho[-4,-4]; T^4_{I,1}(T^3_{I,1}(\pi_{sc})))$
- $L(\Delta_\rho[-4,-4], \Delta_\rho[-4,-4], \Delta_\rho[-4,-4]; T^3_{I,1}(\pi_{sc}))$
- $L(\Delta_\rho[-4,-4], \Delta_\rho[-4,-4], \Delta_\rho[-3,-4]; \pi_{sc})$
- $L(\Delta_\rho[-4,-4], \Delta_\rho[-4,-4], \Delta_\rho[-4,-4], \Delta_\rho[-3,-3]; \pi_{sc})$

(q) $(3,4,4,5)$*:*
- $L(\Delta_\rho[-4,-4]; T^5_{I,1}(T^3_{I,1}(T^3_{I,1}(\pi_{sc}))))$
- $L(\Delta_\rho[-4,-5]; T^4_{I,1}(\pi_{sc}))$
- $L(\Delta_\rho[-5,-5], \Delta_\rho[-4,-4]; T^4_{I,1}(T^3_{I,1}(\pi_{sc})))$
- $L(\Delta_\rho[-4,-5], \Delta_\rho[-4,-4]; T^3_{I,1}(\pi_{sc}))$
- $L(\Delta_\rho[-4,-5], \Delta_\rho[-3,-4]; \pi_{sc})$
- $L(\Delta_\rho[-3,-5], \Delta_\rho[-4,-4]; \pi_{sc})$
- $L(\Delta_\rho[-4,-4], \Delta_\rho[-3,-5]; \pi_{sc})$
- $L(\Delta_\rho[-4,-5], \Delta_\rho[-4,-4], \Delta_\rho[-3,-3]; \pi_{sc})$
- $L(\Delta_\rho[-5,-5], \Delta_\rho[-4,-4], \Delta_\rho[-4,-4]; T^3_{I,1}(\pi_{sc}))$
- $L(\Delta_\rho[-5,-5], \Delta_\rho[-4,-4], \Delta_\rho[-3,-4]; \pi_{sc})$
- $L(\Delta_\rho[-5,-5], \Delta_\rho[-4,-4], \Delta_\rho[-4,-4], \Delta_\rho[-3,-3]; \pi_{sc})$

(r) $(3,4,5,5)$*:*
- $L(\Delta_\rho[-5,-5]; T^5_{I,1}(T^4_{I,1}(T^3_{I,1}(\pi_{sc}))))$
- $L(\Delta_\rho[-5,-5], \Delta_\rho[-3,-5]; \pi_{sc})$
- $L(\Delta_\rho[-5,-5], \Delta_\rho[-4,-5]; T^3_{I,1}(\pi_{sc}))$
- $L(\Delta_\rho[-5,-5], \Delta_\rho[-5,-5]; T^4_{I,1}(T^3_{I,1}(\pi_{sc})))$
- $L(\Delta_\rho[-5,-5], \Delta_\rho[-5,-5], \Delta_\rho[-4,-4]; T^3_{I,1}(\pi_{sc}))$
- $L(\Delta_\rho[-5,-5], \Delta_\rho[-4,-5], \Delta_\rho[-3,-3]; \pi_{sc})$
- $L(\Delta_\rho[-5,-5], \Delta_\rho[-5,-5], \Delta_\rho[-3,-4]; \pi_{sc})$
- $L(\Delta_\rho[-5,-5], \Delta_\rho[-5,-5], \Delta_\rho[-4,-4], \Delta_\rho[-3,-3]; \pi_{sc})$

(s) $(3,4,5,6)$*:*
- $L(\Delta_\rho[-3,-6]; \pi_{sc})$
- $L(\Delta_\rho[-6,-6]; T^5_{I,1}(T^4_{I,1}(T^3_{I,1}(\pi_{sc}))))$
- $L(\Delta_\rho[-4,-6]; T^3_{I,1}(\pi_{sc}))$
- $L(\Delta_\rho[-5,-6]; T^4_{I,1}(T^3_{I,1}(\pi_{sc})))$
- $L(\Delta_\rho[-6,-6], \Delta_\rho[5,-5]; T^4_{I,1}(T^3_{I,1}(\pi_{sc})))$
- $L(\Delta_\rho[-5,-6], \Delta_\rho[-4,-4]; T^3_{I,1}(\pi_{sc}))$
- $L(\Delta_\rho[-5,-6], \Delta_\rho[-3,-4]; \pi_{sc})$
- $L(\Delta_\rho[-4,-6], \Delta_\rho[-3,-3]; \pi_{sc})$

- $L(\Delta_\rho[-6,-6], \Delta_\rho[-3,-5]; \pi_{sc})$
- $L(\Delta_\rho[-6,-6], \Delta_\rho[-4,-5]; T_{I,1}^{3}(\pi_{sc}))$
- $L(\Delta_\rho[-6,-6], \Delta_\rho[-5,-5], \Delta_\rho[-4,-4]; T_{I,1}^{3}(\pi_{sc}))$
- $L(\Delta_\rho[-5,-6], \Delta_\rho[-4,-4], \Delta_\rho[-3,-3]; \pi_{sc})$
- $L(\Delta_\rho[-6,-6], \Delta_\rho[-4,-5], \Delta_\rho[-3,-3]; \pi_{sc})$
- $L(\Delta_\rho[-6,-6], \Delta_\rho[-5,-5], \Delta_\rho[-3,-4]; \pi_{sc})$

(8) $(\alpha = \frac{7}{2})$*:*

(a) $(\frac{3}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}):$
- $L(\Delta_\rho[-\frac{3}{2},-\frac{3}{2}]; T_{I,1}^{\frac{3}{2}}(T_{I,1}^{\frac{5}{2}}(T_{I,1}^{\frac{7}{2}}(\pi_{sc}))))$
- $L(\Delta_\rho[-\frac{3}{2},-\frac{3}{2}], \Delta_\rho[-\frac{3}{2},-\frac{3}{2}]; T_{I,1}^{\frac{5}{2}}(T_{I,1}^{\frac{7}{2}}(\pi_{sc})))$
- $L(\Delta_\rho[-\frac{3}{2},-\frac{5}{2}], \Delta_\rho[-\frac{3}{2},-\frac{3}{2}]; T_{I,1}^{\frac{7}{2}}(\pi_{sc}))$
- $L(\Delta_\rho[-\frac{3}{2},-\frac{7}{2}], \Delta_\rho[-\frac{3}{2},-\frac{3}{2}]; \pi_{sc})$
- $L(\Delta_\rho[-\frac{5}{2},-\frac{5}{2}], \Delta_\rho[-\frac{3}{2},-\frac{3}{2}], \Delta_\rho[-\frac{3}{2},-\frac{3}{2}]; T_{I,1}^{\frac{7}{2}}(\pi_{sc}))$
- $L(\Delta_\rho[-\frac{5}{2},-\frac{7}{2}], \Delta_\rho[-\frac{3}{2},-\frac{3}{2}], \Delta_\rho[-\frac{3}{2},-\frac{3}{2}]; \pi_{sc})$
- $L(\Delta_\rho[-\frac{7}{2},-\frac{7}{2}], \Delta_\rho[-\frac{3}{2},-\frac{5}{2}], \Delta_\rho[-\frac{3}{2},-\frac{3}{2}]; \pi_{sc})$
- $L(\Delta_\rho[-\frac{7}{2},-\frac{7}{2}], \Delta_\rho[-\frac{5}{2},-\frac{5}{2}], \Delta_\rho[-\frac{3}{2},-\frac{3}{2}], \Delta_\rho[-\frac{3}{2},-\frac{3}{2}]; \pi_{sc})$

(b) $(\frac{3}{2}, \frac{5}{2}, \frac{5}{2}, \frac{7}{2}):$
- $L(\Delta_\rho[-\frac{5}{2},-\frac{5}{2}]; T_{I,1}^{\frac{3}{2}}(T_{I,1}^{\frac{5}{2}}(T_{I,1}^{\frac{7}{2}}(\pi_{sc}))))$
- $L(\Delta_\rho[-\frac{3}{2},-\frac{5}{2}]; T_{I,1}^{\frac{5}{2}}(T_{I,1}^{\frac{7}{2}}(\pi_{sc})))$
- $L(\Delta_\rho[-\frac{5}{2},-\frac{5}{2}], \Delta_\rho[-\frac{3}{2},-\frac{7}{2}]; \pi_{sc})$
- $L(\Delta_\rho[-\frac{5}{2},-\frac{5}{2}], \Delta_\rho[-\frac{3}{2},-\frac{5}{2}]; T_{I,1}^{\frac{7}{2}}(\pi_{sc}))$
- $L(\Delta_\rho[-\frac{5}{2},-\frac{7}{2}], \Delta_\rho[-\frac{5}{2},-\frac{5}{2}], \Delta_\rho[-\frac{3}{2},-\frac{3}{2}]; \pi_{sc})$
- $L(\Delta_\rho[-\frac{5}{2},-\frac{5}{2}], \Delta_\rho[-\frac{5}{2},-\frac{5}{2}], \Delta_\rho[-\frac{3}{2},-\frac{3}{2}]; T_{I,1}^{\frac{7}{2}}(\pi_{sc}))$
- $L(\Delta_\rho[-\frac{7}{2},-\frac{7}{2}], \Delta_\rho[-\frac{5}{2},-\frac{5}{2}], \Delta_\rho[-\frac{3}{2},-\frac{5}{2}]; \pi_{sc})$
- $L(\Delta_\rho[-\frac{7}{2},-\frac{7}{2}], \Delta_\rho[-\frac{5}{2},-\frac{5}{2}], \Delta_\rho[-\frac{5}{2},-\frac{5}{2}], \Delta_\rho[-\frac{3}{2},-\frac{3}{2}]; \pi_{sc})$

(c) $(\frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \frac{7}{2}):$
- $L(\Delta_\rho[-\frac{7}{2},-\frac{7}{2}]; T_{I,1}^{\frac{3}{2}}(T_{I,1}^{\frac{5}{2}}(T_{I,1}^{\frac{7}{2}}(\pi_{sc}))))$
- $L(\Delta_\rho[-\frac{3}{2},-\frac{7}{2}]; T_{I,1}^{\frac{7}{2}}(\pi_{sc}))$
- $L(\Delta_\rho[-\frac{7}{2},-\frac{7}{2}], \Delta_\rho[-\frac{3}{2},-\frac{3}{2}]; T_{I,1}^{\frac{5}{2}}(T_{I,1}^{\frac{7}{2}}(\pi_{sc})))$
- $L(\Delta_\rho[-\frac{5}{2},-\frac{7}{2}], \Delta_\rho[-\frac{3}{2},-\frac{3}{2}]; T_{I,1}^{\frac{7}{2}}(\pi_{sc}))$
- $L(\Delta_\rho[-\frac{7}{2},-\frac{7}{2}], \Delta_\rho[-\frac{3}{2},-\frac{7}{2}]; \pi_{sc})$
- $L(\Delta_\rho[-\frac{7}{2},-\frac{7}{2}], \Delta_\rho[-\frac{5}{2},-\frac{7}{2}], \Delta_\rho[-\frac{3}{2},-\frac{3}{2}]; \pi_{sc})$
- $L(\Delta_\rho[-\frac{7}{2},-\frac{7}{2}], \Delta_\rho[-\frac{7}{2},-\frac{7}{2}], \Delta_\rho[-\frac{3}{2},-\frac{5}{2}]; \pi_{sc})$
- $L(\Delta_\rho[-\frac{7}{2},-\frac{7}{2}], \Delta_\rho[-\frac{7}{2},-\frac{7}{2}], \Delta_\rho[-\frac{5}{2},-\frac{5}{2}], \Delta_\rho[-\frac{3}{2},-\frac{3}{2}]; \pi_{sc})$

(d) $(\frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \frac{9}{2}):$
- $L(\Delta_\rho[-\frac{9}{2},-\frac{9}{2}]; T_{I,1}^{\frac{3}{2}}(T_{I,1}^{\frac{5}{2}}(T_{I,1}^{\frac{7}{2}}(\pi_{sc}))))$
- $L(\Delta_\rho[-\frac{3}{2},-\frac{9}{2}]; \pi_{sc})$
- $L(\Delta_\rho[-\frac{7}{2},-\frac{9}{2}], \Delta_\rho[-\frac{3}{2},-\frac{5}{2}]; \pi_{sc})$
- $L(\Delta_\rho[-\frac{5}{2},-\frac{9}{2}], \Delta_\rho[-\frac{3}{2},-\frac{3}{2}]; \pi_{sc})$

- $L(\Delta_\rho[-\frac{9}{2},-\frac{9}{2}],\Delta_\rho[-\frac{5}{2},-\frac{5}{2}],\Delta_\rho[-\frac{3}{2},-\frac{3}{2}];T_{I,1}^{\frac{7}{2}}(\pi_{sc}))$
- $L(\Delta_\rho[-\frac{9}{2},-\frac{9}{2}],\Delta_\rho[-\frac{3}{2},-\frac{3}{2}];T_{I,1}^{\frac{5}{2}}(T_{I,1}^{\frac{7}{2}}(\pi_{sc})))+$
- $L(\Delta_\rho[-\frac{9}{2},-\frac{9}{2}],\Delta_\rho[-\frac{3}{2},-\frac{5}{2}];T_{I,1}^{\frac{7}{2}}(\pi_{sc}))$
- $L(\Delta_\rho[-\frac{7}{2},-\frac{9}{2}],\Delta_\rho[-\frac{5}{2},-\frac{5}{2}],\Delta_\rho[-\frac{3}{2},-\frac{3}{2}];\pi_{sc})$

(e) $(\frac{5}{2},\frac{5}{2},\frac{5}{2},\frac{7}{2})$:
- $L(\Delta_\rho[-\frac{5}{2},-\frac{5}{2}],\Delta_\rho[-\frac{5}{2},-\frac{5}{2}];T_{I,1}^{\frac{5}{2}}(T_{I,1}^{\frac{7}{2}}(\pi_{sc})))$
- $L(\Delta_\rho[-\frac{5}{2},-\frac{5}{2}],\Delta_\rho[-\frac{5}{2},-\frac{5}{2}],\Delta_\rho[-\frac{5}{2},-\frac{5}{2}],T_{I,1}^{\frac{7}{2}}(\pi_{sc}))$
- $L(\Delta_\rho[-\frac{5}{2},-\frac{7}{2}],\Delta_\rho[-\frac{5}{2},-\frac{5}{2}],\Delta_\rho[-\frac{5}{2},-\frac{5}{2}];\pi_{sc})$
- $L(\Delta_\rho[-\frac{7}{2},-\frac{7}{2}],\Delta_\rho[-\frac{5}{2},-\frac{5}{2}],\Delta_\rho[-\frac{5}{2},-\frac{5}{2}],\Delta_\rho[-\frac{5}{2},-\frac{5}{2}];\pi_{sc})$

(f) $(\frac{5}{2},\frac{5}{2},\frac{7}{2},\frac{7}{2})$:
- $L(\Delta_\rho[-\frac{7}{2},-\frac{7}{2}],\Delta_\rho[-\frac{5}{2},-\frac{5}{2}];T_{I,1}^{\frac{5}{2}}(T_{I,1}^{\frac{7}{2}}(\pi_{sc})))$
- $L(\Delta_\rho[-\frac{5}{2},-\frac{7}{2}],\Delta_\rho[-\frac{5}{2},-\frac{5}{2}];T_{I,1}^{\frac{7}{2}}(\pi_{sc}))$
- $L(\Delta_\rho[-\frac{5}{2},-\frac{7}{2}],\Delta_\rho[-\frac{5}{2},-\frac{7}{2}];\pi_{sc})$
- $L(\Delta_\rho[-\frac{7}{2},-\frac{7}{2}],\Delta_\rho[-\frac{5}{2},-\frac{5}{2}],\Delta_\rho[-\frac{5}{2},-\frac{5}{2}];T_{I,1}^{\frac{7}{2}}(\pi_{sc}))$
- $L(\Delta_\rho[-\frac{7}{2},-\frac{7}{2}],\Delta_\rho[-\frac{5}{2},-\frac{7}{2}],\Delta_\rho[-\frac{5}{2},-\frac{5}{2}];\pi_{sc})$
- $L(\Delta_\rho[-\frac{7}{2},-\frac{7}{2}],\Delta_\rho[-\frac{7}{2},-\frac{7}{2}],\Delta_\rho[-\frac{5}{2},-\frac{5}{2}],\Delta_\rho[-\frac{5}{2},-\frac{5}{2}];\pi_{sc})$
- $L(\Delta_\rho[-\frac{7}{2},-\frac{7}{2}],\Delta_\rho[-\frac{7}{2},-\frac{7}{2}],\Delta_\rho[-\frac{5}{2},-\frac{5}{2}],\Delta_\rho[-\frac{5}{2},-\frac{5}{2}];\pi_{sc})$

(g) $(\frac{5}{2},\frac{5}{2},\frac{7}{2},\frac{9}{2})$:
- $L(\Delta_\rho[-\frac{5}{2},-\frac{5}{2}];T_{I,1}^{\frac{5}{2}}(T_{I,1}^{\frac{9}{2}}(T_{I,1}^{\frac{7}{2}}(\pi_{sc}))))$
- $L(\Delta_\rho[-\frac{5}{2},-\frac{5}{2}],\Delta_\rho[-\frac{5}{2},-\frac{5}{2}];T_{I,1}^{\frac{9}{2}}(T_{I,1}^{\frac{7}{2}}(\pi_{sc})))$
- $L(\Delta_\rho[-\frac{9}{2},-\frac{9}{2}],\Delta_\rho[-\frac{5}{2},-\frac{5}{2}];T_{I,1}^{\frac{5}{2}}(T_{I,1}^{\frac{7}{2}}(\pi_{sc})))$
- $L(\Delta_\rho[-\frac{5}{2},-\frac{9}{2}],\Delta_\rho[-\frac{5}{2},-\frac{5}{2}];\pi_{sc})$
- $L(\Delta_\rho[-\frac{9}{2},-\frac{9}{2}],\Delta_\rho[-\frac{5}{2},-\frac{5}{2}],\Delta_\rho[-\frac{5}{2},-\frac{5}{2}];T_{I,1}^{\frac{7}{2}}(\pi_{sc}))$
- $L(\Delta_\rho[-\frac{7}{2},-\frac{9}{2}],\Delta_\rho[-\frac{5}{2},-\frac{5}{2}],\Delta_\rho[-\frac{5}{2},-\frac{5}{2}];\pi_{sc})$
- $L(\Delta_\rho[-\frac{9}{2},-\frac{9}{2}],\Delta_\rho[-\frac{5}{2},-\frac{7}{2}],\Delta_\rho[-\frac{5}{2},-\frac{5}{2}];\pi_{sc})$
- $L(\Delta_\rho[-\frac{9}{2},-\frac{9}{2}],\Delta_\rho[-\frac{7}{2},-\frac{7}{2}],\Delta_\rho[-\frac{5}{2},-\frac{5}{2}],\Delta_\rho[-\frac{5}{2},-\frac{5}{2}];\pi_{sc})$

(h) $(\frac{5}{2},\frac{7}{2},\frac{7}{2},\frac{7}{2})$:
- $L(\Delta_\rho[-\frac{7}{2},-\frac{7}{2}],\Delta_\rho[-\frac{7}{2},-\frac{7}{2}];T_{I,1}^{\frac{5}{2}}(T_{I,1}^{\frac{7}{2}}(\pi_{sc})))$
- $L(\Delta_\rho[-\frac{7}{2},-\frac{7}{2}],\Delta_\rho[-\frac{7}{2},-\frac{7}{2}];T_{I,1}^{\frac{7}{2}}(T_{I,1}^{\frac{5}{2}}(\pi_{sc})))$
- $L(\Delta_\rho[-\frac{7}{2},-\frac{7}{2}],\Delta_\rho[-\frac{7}{2},-\frac{7}{2}],\Delta_\rho[-\frac{5}{2},-\frac{5}{2}];T_{I,1}^{\frac{7}{2}}(\pi_{sc}))$
- $L(\Delta_\rho[-\frac{7}{2},-\frac{7}{2}],\Delta_\rho[-\frac{7}{2},-\frac{7}{2}],\Delta_\rho[-\frac{5}{2},-\frac{7}{2}];\pi_{sc})$
- $L(\Delta_\rho[-\frac{7}{2},-\frac{7}{2}],\Delta_\rho[-\frac{7}{2},-\frac{7}{2}],\Delta_\rho[-\frac{7}{2},-\frac{7}{2}],\Delta_\rho[-\frac{5}{2},-\frac{5}{2}];\pi_{sc})$

(i) $(\frac{5}{2},\frac{7}{2},\frac{7}{2},\frac{9}{2})$:
- $L(\Delta_\rho[-\frac{7}{2},-\frac{7}{2}];T_{I,1}^{\frac{5}{2}}(T_{I,1}^{\frac{9}{2}}(T_{I,1}^{\frac{7}{2}}(\pi_{sc}))))$
- $L(\Delta_\rho[-\frac{5}{2},-\frac{9}{2}];T_{I,1}^{\frac{7}{2}}(\pi_{sc}))$
- $L(\Delta_\rho[-\frac{7}{2},-\frac{9}{2}];T_{I,1}^{\frac{5}{2}}(T_{I,1}^{\frac{7}{2}}(\pi_{sc})))$
- $L(\Delta_\rho[-\frac{5}{2},-\frac{7}{2}];T_{I,1}^{\frac{9}{2}}(T_{I,1}^{\frac{7}{2}}(\pi_{sc})))$

- $L(\Delta_\rho[-\frac{9}{2},-\frac{9}{2}],\Delta_\rho[-\frac{7}{2},-\frac{7}{2}];T_{I,1}^{\frac{5}{2}}(T_{I,1}^{\frac{7}{2}}(\pi_{sc})))$
- $L(\Delta_\rho[-\frac{9}{2},-\frac{9}{2}],\Delta_\rho[-\frac{7}{2},-\frac{7}{2}];T_{I,1}^{\frac{7}{2}}(T_{I,1}^{\frac{5}{2}}(\pi_{sc})))$
- $L(\Delta_\rho[-\frac{7}{2},-\frac{9}{2}],\Delta_\rho[-\frac{5}{2},-\frac{5}{2}];T_{I,1}^{\frac{7}{2}}(\pi_{sc}))$
- $L(\Delta_\rho[-\frac{5}{2},-\frac{9}{2}],\Delta_\rho[-\frac{7}{2},-\frac{7}{2}];\pi_{sc})$
- $L(\Delta_\rho[-\frac{7}{2},-\frac{7}{2}],\Delta_\rho[-\frac{5}{2},-\frac{9}{2}];\pi_{sc})$
- $L(\Delta_\rho[-\frac{7}{2},-\frac{9}{2}],\Delta_\rho[-\frac{7}{2},-\frac{7}{2}],\Delta_\rho[-\frac{5}{2},-\frac{5}{2}];\pi_{sc})$
- $L(\Delta_\rho[-\frac{9}{2},-\frac{9}{2}],\Delta_\rho[-\frac{7}{2},-\frac{7}{2}],\Delta_\rho[-\frac{5}{2},-\frac{7}{2}];\pi_{sc})$
- $L(\Delta_\rho[-\frac{9}{2},-\frac{9}{2}],\Delta_\rho[-\frac{7}{2},-\frac{7}{2}],\Delta_\rho[-\frac{7}{2},-\frac{7}{2}],\Delta_\rho[-\frac{5}{2},-\frac{5}{2}];\pi_{sc})$

(j) $(\frac{5}{2},\frac{7}{2},\frac{9}{2},\frac{9}{2})$*:*

- $L(\Delta_\rho[-\frac{9}{2},-\frac{9}{2}];T_{I,1}^{\frac{5}{2}}(T_{I,1}^{\frac{9}{2}}(T_{I,1}^{\frac{7}{2}}(\pi_{sc}))))$
- $L(\Delta_\rho[-\frac{9}{2},-\frac{9}{2}],\Delta_\rho[-\frac{9}{2},-\frac{9}{2}];T_{I,1}^{\frac{5}{2}}(T_{I,1}^{\frac{7}{2}}(\pi_{sc})))$
- $L(\Delta_\rho[-\frac{9}{2},-\frac{9}{2}],\Delta_\rho[-\frac{5}{2},-\frac{5}{2}];T_{I,1}^{\frac{9}{2}}(T_{I,1}^{\frac{7}{2}}(\pi_{sc})))$
- $L(\Delta_\rho[-\frac{9}{2},-\frac{9}{2}],\Delta_\rho[-\frac{5}{2},-\frac{9}{2}];\pi_{sc})$
- $L(\Delta_\rho[-\frac{9}{2},-\frac{9}{2}],\Delta_\rho[-\frac{9}{2},-\frac{9}{2}],\Delta_\rho[-\frac{5}{2},-\frac{5}{2}];T_{I,1}^{\frac{7}{2}}(\pi_{sc}))$
- $L(\Delta_\rho[-\frac{9}{2},-\frac{9}{2}],\Delta_\rho[-\frac{7}{2},-\frac{9}{2}],\Delta_\rho[-\frac{5}{2},-\frac{5}{2}];\pi_{sc})$
- $L(\Delta_\rho[-\frac{9}{2},-\frac{9}{2}],\Delta_\rho[-\frac{9}{2},-\frac{9}{2}],\Delta_\rho[-\frac{5}{2},-\frac{7}{2}];\pi_{sc})$
- $L(\Delta_\rho[-\frac{9}{2},-\frac{9}{2}],\Delta_\rho[-\frac{9}{2},-\frac{9}{2}],\Delta_\rho[-\frac{7}{2},-\frac{7}{2}],\Delta_\rho[-\frac{5}{2},-\frac{5}{2}];\pi_{sc})$

(k) $(\frac{5}{2},\frac{7}{2},\frac{9}{2},\frac{11}{2})$*:*

- $L(\Delta_\rho[-\frac{11}{2},-\frac{11}{2}];T_{I,1}^{\frac{5}{2}}(T_{I,1}^{\frac{9}{2}}(T_{I,1}^{\frac{7}{2}}(\pi_{sc}))))$
- $L(\Delta_\rho[-\frac{9}{2},-\frac{11}{2}];T_{I,1}^{\frac{5}{2}}(T_{I,1}^{\frac{7}{2}}(\pi_{sc})))\ +$
- $L(\Delta_\rho[-\frac{5}{2},-\frac{11}{2}];\pi_{sc})$
- $L(\Delta_\rho[-\frac{11}{2},-\frac{11}{2}],\Delta_\rho[-\frac{9}{2},-\frac{9}{2}];T_{I,1}^{\frac{5}{2}}(T_{I,1}^{\frac{7}{2}}(\pi_{sc})))$
- $L(\Delta_\rho[-\frac{11}{2},-\frac{11}{2}],\Delta_\rho[-\frac{5}{2},-\frac{5}{2}];T_{I,1}^{\frac{9}{2}}(T_{I,1}^{\frac{7}{2}}(\pi_{sc}))$
- $L(\Delta_\rho[-\frac{9}{2},-\frac{11}{2}],\Delta_\rho[-\frac{5}{2},-\frac{5}{2}];T_{I,1}^{\frac{7}{2}}(\pi_{sc}))$
- $L(\Delta_\rho[-\frac{9}{2},-\frac{11}{2}],\Delta_\rho[-\frac{5}{2},-\frac{7}{2}];\pi_{sc})$
- $L(\Delta_\rho[-\frac{7}{2},-\frac{11}{2}],\Delta_\rho[-\frac{5}{2},-\frac{5}{2}];\pi_{sc})$
- $L(\Delta_\rho[-\frac{11}{2},-\frac{11}{2}],\Delta_\rho[-\frac{5}{2},-\frac{9}{2}];\pi_{sc})$
- $L(\Delta_\rho[-\frac{11}{2},-\frac{11}{2}],\Delta_\rho[-\frac{9}{2},-\frac{9}{2}],\Delta_\rho[-\frac{5}{2},-\frac{5}{2}];T_{I,1}^{\frac{7}{2}}(\pi_{sc}))$
- $L(\Delta_\rho[-\frac{9}{2},-\frac{11}{2}],\Delta_\rho[-\frac{7}{2},-\frac{7}{2}],\Delta_\rho[-\frac{5}{2},-\frac{5}{2}];\pi_{sc})$
- $L(\Delta_\rho[-\frac{11}{2},-\frac{11}{2}],\Delta_\rho[-\frac{7}{2},-\frac{9}{2}],\Delta_\rho[-\frac{5}{2},-\frac{5}{2}];\pi_{sc})$

(l) $(\frac{7}{2},\frac{7}{2},\frac{7}{2},\frac{7}{2})$*:*

- $L(\Delta_\rho[-\frac{7}{2},-\frac{7}{2}],\Delta_\rho[-\frac{7}{2},-\frac{7}{2}],\Delta_\rho[-\frac{7}{2},-\frac{7}{2}],T_{I,1}^{\frac{7}{2}}(\pi_{sc}))$
- $L(\Delta_\rho[-\frac{7}{2},-\frac{7}{2}],\Delta_\rho[-\frac{7}{2},-\frac{7}{2}],\Delta_\rho[-\frac{7}{2},-\frac{7}{2}],\Delta_\rho[-\frac{7}{2},-\frac{7}{2}];\pi_{sc})$

(m) $(\frac{7}{2},\frac{7}{2},\frac{7}{2},\frac{9}{2})$*:*

- $L(\Delta_\rho[-\frac{7}{2},-\frac{7}{2}],\Delta_\rho[-\frac{7}{2},-\frac{7}{2}];T_{I,1}^{\frac{9}{2}}(T_{I,1}^{\frac{7}{2}}(\pi_{sc})))$
- $L(\Delta_\rho[-\frac{7}{2},-\frac{9}{2}],\Delta_\rho[-\frac{7}{2},-\frac{7}{2}];T_{I,1}^{\frac{7}{2}}(\pi_{sc}))$
- $L(\Delta_\rho[-\frac{9}{2},-\frac{9}{2}],\Delta_\rho[-\frac{7}{2},-\frac{7}{2}],\Delta_\rho[-\frac{7}{2},-\frac{7}{2}];T_{I,1}^{\frac{7}{2}}(\pi_{sc}))$

- $L(\Delta_\rho[-\frac{7}{2},-\frac{9}{2}],\Delta_\rho[-\frac{7}{2},-\frac{7}{2}],\Delta_\rho[-\frac{7}{2},-\frac{7}{2}];\pi_{sc})$
- $L(\Delta_\rho[-\frac{9}{2},-\frac{9}{2}],\Delta_\rho[-\frac{7}{2},-\frac{7}{2}],\Delta_\rho[-\frac{7}{2},-\frac{7}{2}],\Delta_\rho[-\frac{7}{2},-\frac{7}{2}];\pi_{sc})$

(n) $(\frac{7}{2},\frac{7}{2},\frac{9}{2},\frac{9}{2}):$
- $L(\Delta_\rho[-\frac{7}{2},-\frac{9}{2}],\Delta_\rho[-\frac{7}{2},-\frac{9}{2}];\pi_{sc})$
- $L(\Delta_\rho[-\frac{9}{2},-\frac{9}{2}],\Delta_\rho[-\frac{7}{2},-\frac{7}{2}];T_{I,1}^{\frac{9}{2}}(T_{I,1}^{\frac{7}{2}}(\pi_{sc})))$
- $L(\Delta_\rho[-\frac{7}{2},-\frac{9}{2}],\Delta_\rho[-\frac{7}{2},-\frac{9}{2}];\pi_{sc})$
- $L(\Delta_\rho[-\frac{9}{2},-\frac{9}{2}],\Delta_\rho[-\frac{7}{2},-\frac{9}{2}];T_{I,1}^{\frac{7}{2}}(\pi_{sc}))$
- $L(\Delta_\rho[-\frac{9}{2},-\frac{9}{2}],\Delta_\rho[-\frac{9}{2},-\frac{9}{2}],\Delta_\rho[-\frac{7}{2},-\frac{7}{2}];T_{I,1}^{\frac{7}{2}}(\pi_{sc}))$
- $L(\Delta_\rho[-\frac{9}{2},-\frac{9}{2}],\Delta_\rho[-\frac{7}{2},-\frac{9}{2}],\Delta_\rho[-\frac{7}{2},-\frac{7}{2}];\pi_{sc})$
- $L(\Delta_\rho[-\frac{9}{2},-\frac{9}{2}],\Delta_\rho[-\frac{9}{2},-\frac{9}{2}],\Delta_\rho[-\frac{7}{2},-\frac{7}{2}],\Delta_\rho[-\frac{7}{2},-\frac{7}{2}];\pi_{sc})$

(o) $(\frac{7}{2},\frac{7}{2},\frac{9}{2},\frac{11}{2}):$
- $L(\Delta_\rho[-\frac{7}{2},-\frac{7}{2}];T_{I,1}^{\frac{11}{2}}(T_{I,1}^{\frac{9}{2}}(T_{I,1}^{\frac{7}{2}}(\pi_{sc}))))$
- $L(\Delta_\rho[-\frac{7}{2},-\frac{11}{2}];T_{I,1}^{\frac{7}{2}}(\pi_{sc}))$
- $L(\Delta_\rho[-\frac{9}{2},-\frac{11}{2}],\Delta_\rho[-\frac{7}{2},-\frac{7}{2}];T_{I,1}^{\frac{7}{2}}(\pi_{sc}))$
- $L(\Delta_\rho[-\frac{7}{2},-\frac{11}{2}],\Delta_\rho[-\frac{7}{2},-\frac{7}{2}];\pi_{sc})$
- $L(\Delta_\rho[-\frac{11}{2},-\frac{11}{2}],\Delta_\rho[-\frac{7}{2},-\frac{7}{2}];T_{I,1}^{\frac{9}{2}}(T_{I,1}^{\frac{7}{2}}(\pi_{sc})))$
- $L(\Delta_\rho[-\frac{11}{2},-\frac{11}{2}],\Delta_\rho[-\frac{7}{2},-\frac{9}{2}],T_{I,1}^{\frac{7}{2}}(\pi_{sc}))\ +$
- $L(\Delta_\rho[-\frac{11}{2},-\frac{11}{2}],\Delta_\rho[-\frac{9}{2},-\frac{9}{2}],\Delta_\rho[-\frac{7}{2},-\frac{7}{2}];T_{I,1}^{\frac{7}{2}}(\pi_{sc}))$
- $L(\Delta_\rho[-\frac{9}{2},-\frac{11}{2}],\Delta_\rho[-\frac{7}{2},-\frac{7}{2}],\Delta_\rho[-\frac{7}{2},-\frac{7}{2}];\pi_{sc})$
- $L(\Delta_\rho[-\frac{11}{2},-\frac{11}{2}],\Delta_\rho[-\frac{7}{2},-\frac{9}{2}],\Delta_\rho[-\frac{7}{2},-\frac{7}{2}];\pi_{sc})$
- $L(\Delta_\rho[-\frac{11}{2},-\frac{11}{2}],\Delta_\rho[-\frac{9}{2},-\frac{9}{2}],\Delta_\rho[-\frac{7}{2},-\frac{7}{2}],\Delta_\rho[-\frac{7}{2},-\frac{7}{2}];\pi_{sc})$

(p) $(\frac{7}{2},\frac{9}{2},\frac{9}{2},\frac{9}{2})$:
- $L(\Delta_\rho[-\frac{9}{2},-\frac{9}{2}],\Delta_\rho[-\frac{9}{2},-\frac{9}{2}];T_{I,1}^{\frac{9}{2}}(T_{I,1}^{\frac{7}{2}}(\pi_{sc})))$
- $L(\Delta_\rho[-\frac{9}{2},-\frac{9}{2}],\Delta_\rho[-\frac{9}{2},-\frac{9}{2}],\Delta_\rho[-\frac{9}{2},-\frac{9}{2}],T_{I,1}^{\frac{7}{2}}(\pi_{sc}))$
- $L(\Delta_\rho[-\frac{9}{2},-\frac{9}{2}],\Delta_\rho[-\frac{9}{2},-\frac{9}{2}],\Delta_\rho[-\frac{7}{2},-\frac{9}{2}];\pi_{sc})$
- $L(\Delta_\rho[-\frac{9}{2},-\frac{9}{2}],\Delta_\rho[-\frac{9}{2},-\frac{9}{2}],\Delta_\rho[-\frac{9}{2},-\frac{9}{2}],\Delta_\rho[-\frac{7}{2},-\frac{7}{2}];\pi_{sc})$

(q) $(\frac{7}{2},\frac{9}{2},\frac{9}{2},\frac{11}{2}):$
- $L(\Delta_\rho[-\frac{9}{2},-\frac{9}{2}];T_{I,1}^{\frac{11}{2}}(T_{I,1}^{\frac{9}{2}}(T_{I,1}^{\frac{7}{2}}(\pi_{sc}))))$
- $L(\Delta_\rho[-\frac{9}{2},-\frac{11}{2}];T_{I,1}^{\frac{9}{2}}(T_{I,1}^{\frac{7}{2}}(\pi_{sc})))$
- $L(\Delta_\rho[-\frac{11}{2},-\frac{11}{2}],\Delta_\rho[-\frac{9}{2},-\frac{9}{2}];T_{I,1}^{\frac{9}{2}}(T_{I,1}^{\frac{7}{2}}(\pi_{sc})))$
- $L(\Delta_\rho[-\frac{9}{2},-\frac{11}{2}],\Delta_\rho[-\frac{9}{2},-\frac{9}{2}];T_{I,1}^{\frac{7}{2}}(\pi_{sc}))$
- $L(\Delta_\rho[-\frac{9}{2},-\frac{11}{2}],\Delta_\rho[-\frac{7}{2},-\frac{9}{2}];\pi_{sc})$
- $L(\Delta_\rho[-\frac{7}{2},-\frac{11}{2}],\Delta_\rho[-\frac{9}{2},-\frac{9}{2}];\pi_{sc})$
- $L(\Delta_\rho[-\frac{9}{2},-\frac{9}{2}],\Delta_\rho[-\frac{7}{2},-\frac{11}{2}];\pi_{sc})$
- $L(\Delta_\rho[-\frac{9}{2},-\frac{11}{2}],\Delta_\rho[-\frac{9}{2},-\frac{9}{2}],\Delta_\rho[-\frac{7}{2},-\frac{7}{2}];\pi_{sc})$
- $L(\Delta_\rho[-\frac{11}{2},-\frac{11}{2}],\Delta_\rho[-\frac{9}{2},-\frac{9}{2}],\Delta_\rho[-\frac{9}{2},-\frac{9}{2}];T_{I,1}^{\frac{7}{2}}(\pi_{sc}))$
- $L(\Delta_\rho[-\frac{11}{2},-\frac{11}{2}],\Delta_\rho[-\frac{9}{2},-\frac{9}{2}],\Delta_\rho[-\frac{7}{2},-\frac{9}{2}];\pi_{sc})$
- $L(\Delta_\rho[-\frac{11}{2},-\frac{11}{2}],\Delta_\rho[-\frac{9}{2},-\frac{9}{2}],\Delta_\rho[-\frac{9}{2},-\frac{9}{2}],\Delta_\rho[-\frac{7}{2},-\frac{7}{2}];\pi_{sc})$

(r) $(\frac{7}{2}, \frac{9}{2}, \frac{11}{2}, \frac{11}{2})$:
- $L(\Delta_\rho[-\frac{11}{2}, -\frac{11}{2}]; T_{I,1}^{\frac{11}{2}}(T_{I,1}^{\frac{9}{2}}(T_{I,1}^{\frac{7}{2}}(\pi_{sc}))))$
- $L(\Delta_\rho[-\frac{11}{2}, -\frac{11}{2}], \Delta_\rho[-\frac{11}{2}, -\frac{11}{2}]; T_{I,1}^{\frac{9}{2}}(T_{I,1}^{\frac{7}{2}}(\pi_{sc})))$
- $L(\Delta_\rho[-\frac{11}{2}, -\frac{11}{2}], \Delta_\rho[-\frac{7}{2}, -\frac{11}{2}]; \pi_{sc})$
- $L(\Delta_\rho[-\frac{11}{2}, -\frac{11}{2}], \Delta_\rho[-\frac{9}{2}, -\frac{11}{2}]; T_{I,1}^{\frac{7}{2}}(\pi_{sc}))$
- $L(\Delta_\rho[-\frac{11}{2}, -\frac{11}{2}], \Delta_\rho[-\frac{11}{2}, -\frac{11}{2}], \Delta_\rho[-\frac{9}{2}, -\frac{9}{2}]; T_{I,1}^{\frac{7}{2}}(\pi_{sc}))$
- $L(\Delta_\rho[-\frac{11}{2}, -\frac{11}{2}], \Delta_\rho[-\frac{9}{2}, -\frac{11}{2}], \Delta_\rho[-\frac{7}{2}, -\frac{7}{2}]; \pi_{sc})$
- $L(\Delta_\rho[-\frac{11}{2}, -\frac{11}{2}], \Delta_\rho[-\frac{11}{2}, -\frac{11}{2}], \Delta_\rho[-\frac{7}{2}, -\frac{9}{2}]; \pi_{sc})$
- $L(\Delta_\rho[-\frac{11}{2}, -\frac{11}{2}], \Delta_\rho[-\frac{11}{2}, -\frac{11}{2}], \Delta_\rho[-\frac{9}{2}, -\frac{9}{2}], \Delta_\rho[-\frac{7}{2}, -\frac{7}{2}]; \pi_{sc})$

(s) $(\frac{7}{2}, \frac{9}{2}, \frac{11}{2}, \frac{13}{2})$:
- $L(\Delta_\rho[-\frac{13}{2}, -\frac{13}{2}]; T_{I,1}^{\frac{11}{2}}(T_{I,1}^{\frac{9}{2}}(T_{I,1}^{\frac{7}{2}}(\pi_{sc}))))$
- $L(\Delta_\rho[-\frac{9}{2}, -\frac{13}{2}]; T_{I,1}^{\frac{7}{2}}(\pi_{sc}))$
- $L(\Delta_\rho[-\frac{7}{2}, -\frac{13}{2}]; \pi_{sc})$
- $L(\Delta_\rho[-\frac{11}{2}, -\frac{13}{2}]; T_{I,1}^{\frac{9}{2}}(T_{I,1}^{\frac{7}{2}}(\pi_{sc})))$
- $L(\Delta_\rho[-\frac{13}{2}, -\frac{13}{2}], \Delta_\rho[-\frac{11}{2}, -\frac{11}{2}]; T_{I,1}^{\frac{9}{2}}(T_{I,1}^{\frac{7}{2}}(\pi_{sc})))$
- $L(\Delta_\rho[-\frac{11}{2}, -\frac{13}{2}], \Delta_\rho[-\frac{9}{2}, -\frac{9}{2}]; T_{I,1}^{\frac{7}{2}}(\pi_{sc}))$
- $L(\Delta_\rho[-\frac{13}{2}, -\frac{13}{2}], \Delta_\rho[-\frac{9}{2}, -\frac{11}{2}], T_{I,1}^{\frac{7}{2}}(\pi_{sc}))$
- $L(\Delta_\rho[-\frac{11}{2}, -\frac{13}{2}], \Delta_\rho[-\frac{7}{2}, -\frac{9}{2}]; \pi_{sc})$
- $L(\Delta_\rho[-\frac{9}{2}, -\frac{13}{2}], \Delta_\rho[-\frac{7}{2}, -\frac{7}{2}]; \pi_{sc})$
- $L(\Delta_\rho[-\frac{13}{2}, -\frac{13}{2}], \Delta_\rho[-\frac{7}{2}, -\frac{11}{2}]; \pi_{sc})$
- $L(\Delta_\rho[-\frac{11}{2}, -\frac{13}{2}], \Delta_\rho[-\frac{9}{2}, -\frac{9}{2}], \Delta_\rho[-\frac{7}{2}, -\frac{7}{2}]; \pi_{sc})$
- $L(\Delta_\rho[-\frac{13}{2}, -\frac{13}{2}], \Delta_\rho[-\frac{11}{2}, -\frac{11}{2}], \Delta_\rho[-\frac{9}{2}, -\frac{9}{2}]; T_{I,1}^{\frac{7}{2}}(\pi_{sc}))$
- $L(\Delta_\rho[-\frac{13}{2}, -\frac{13}{2}], \Delta_\rho[-\frac{11}{2}, -\frac{11}{2}], \Delta_\rho[-\frac{7}{2}, -\frac{9}{2}]; \pi_{sc})$
- $L(\Delta_\rho[-\frac{13}{2}, -\frac{13}{2}], \Delta_\rho[-\frac{9}{2}, -\frac{11}{2}], \Delta_\rho[-\frac{7}{2}, -\frac{7}{2}]; \pi_{sc})$

(9) $(\alpha \geq 4)$:

(a) $(\alpha-2, \alpha-2, \alpha-1, \alpha)$:
- $L(\Delta_\rho[-\alpha+2, -\alpha+2]; T_{I,1}^{\alpha-2}(T_{I,1}^{\alpha-1}(T_{I,1}^{\alpha}(\pi_{sc}))))$
- $L(\Delta_\rho[-\alpha+2, -\alpha+2], \Delta_\rho[-\alpha+2, -\alpha+2]; T_{I,1}^{\alpha-1}(T_{I,1}^{\alpha}(\pi_{sc})))$
- $L(\Delta_\rho[-\alpha+2, -\alpha+1], \Delta_\rho[-\alpha+2, -\alpha+2]; T_{I,1}^{\alpha}(\pi_{sc}))$
- $L(\Delta_\rho[-\alpha+2, -\alpha], \Delta_\rho[-\alpha+2, -\alpha+2]; \pi_{sc})$
- $L(\Delta_\rho[-\alpha+1, -\alpha+1], \Delta_\rho[-\alpha+2, -\alpha+2], \Delta_\rho[-\alpha+2, -\alpha+2]; T_{I,1}^{\alpha}(\pi_{sc}))$
- $L(\Delta_\rho[-\alpha+1, -\alpha], \Delta_\rho[-\alpha+2, -\alpha+2], \Delta_\rho[-\alpha+2, -\alpha+2]; \pi_{sc})$
- $L(\Delta_\rho[-\alpha, -\alpha], \Delta_\rho[-\alpha+2, -\alpha+1], \Delta_\rho[-\alpha+2, -\alpha+2]; \pi_{sc})$
- $L(\Delta_\rho[-\alpha, -\alpha], \Delta_\rho[-\alpha+1, -\alpha+1], \Delta_\rho[-\alpha+2, \alpha+2], \Delta_\rho[-\alpha+2, -\alpha+2]; \pi_{sc})$

(b) $(\alpha-2, \alpha-1, \alpha-1, \alpha)$:
- $L(\Delta_\rho[-\alpha+2, -\alpha+1]; T_{I,1}^{\alpha-1}(T_{I,1}^{\alpha}(\pi_{sc})))$
- $L(\Delta_\rho[-\alpha+1, -\alpha+1]; T_{I,1}^{\alpha-2}(T_{I,1}^{\alpha-1}(T_{I,1}^{\alpha}(\pi_{sc}))))$
- $L(\Delta_\rho[-\alpha+1, -\alpha+1], \Delta_\rho[-\alpha+2, -\alpha+1]; T_{I,1}^{\alpha}(\pi_{sc}))$
- $L(\Delta_\rho[-\alpha+1, -\alpha+1], \Delta_\rho[-\alpha+2, -\alpha]; \pi_{sc})$
- $L(\Delta_\rho[-\alpha+1, -\alpha+1], \Delta_\rho[-\alpha+1, -\alpha+1], \Delta_\rho[-\alpha+2, -\alpha+2]; T_{I,1}^{\alpha}(\pi_{sc}))$

- $L(\Delta_\rho[-\alpha+1,-\alpha],\Delta_\rho[-\alpha+1,-\alpha+1],\Delta_\rho[-\alpha+2,-\alpha+2];\pi_{sc})$
- $L(\Delta_\rho[-\alpha,-\alpha],\Delta_\rho[-\alpha+1,-\alpha+1],\Delta_\rho[-\alpha+2,-\alpha+1];\pi_{sc})$
- $L(\Delta_\rho[-\alpha,-\alpha],\Delta_\rho[-\alpha+1,-\alpha+1],\Delta_\rho[-\alpha+1,\alpha+1],\Delta_\rho[-\alpha+2,-\alpha+2];\pi_{sc})$

(c) $(\alpha-2,\alpha-1,\alpha,\alpha)$*:*
- $L(\Delta_\rho[-\alpha,-\alpha];T_{I,1}^{\alpha-2}(T_{I,1}^{\alpha-1}(T_{I,1}^{\alpha}(\pi_{sc}))))$
- $L(\Delta_\rho[-\alpha+2,-\alpha];T_{I,1}^{\alpha}(\pi_{sc}))$
- $L(\Delta_\rho[-\alpha+1,-\alpha],\Delta_\rho[-\alpha+2,-\alpha+2];T_{I,1}^{\alpha}(\pi_{sc}))$
- $L(\Delta_\rho[-\alpha,-\alpha],\Delta_\rho[-\alpha+2,-\alpha+2];T_{I,1}^{\alpha-1}(T_{I,1}^{\alpha}(\pi_{sc})))$
- $L(\Delta_\rho[-\alpha,-\alpha],\Delta_\rho[-\alpha+2,-\alpha];\pi_{sc})$
- $L(\Delta_\rho[-\alpha,-\alpha],\Delta_\rho[-\alpha+1,-\alpha],\Delta_\rho[-\alpha+2,-\alpha+2];\pi_{sc})$
- $L(\Delta_\rho[-\alpha,-\alpha],\Delta_\rho[-\alpha,-\alpha],\Delta_\rho[-\alpha+2,-\alpha+1];\pi_{sc})$
- $L(\Delta_\rho[-\alpha,-\alpha],\Delta_\rho[-\alpha,-\alpha],\Delta_\rho[-\alpha+1,\alpha+1],\Delta_\rho[-\alpha+2,-\alpha+2];\pi_{sc})$

(d) $(\alpha-2,\alpha-1,\alpha,\alpha+1)$*:*
- $L(\Delta_\rho[-\alpha+2,-\alpha-1];\pi_{sc})$
- $L(\Delta_\rho[-\alpha-1,-\alpha-1];T_{I,1}^{\alpha-2}(T_{I,1}^{\alpha-1}(T_{I,1}^{\alpha}(\pi_{sc}))))$
- $L(\Delta_\rho[-\alpha+1,-\alpha-1],\Delta_\rho[-\alpha+2,-\alpha+2];\pi_{sc})$
- $L(\Delta_\rho[-\alpha,-\alpha-1],\Delta_\rho[-\alpha+2,-\alpha+1];\pi_{sc})$
- $L(\Delta_\rho[-\alpha-1,-\alpha-1],\Delta_\rho[-\alpha+2,-\alpha+2];T_{I,1}^{\alpha-1}(T_{I,1}^{\alpha}(\pi_{sc})))$
- $L(\Delta_\rho[-\alpha-1,-\alpha-1],\Delta_\rho[-\alpha+2,-\alpha+1];T_{I,1}^{\alpha}(\pi_{sc}))$
- $L(\Delta_\rho[-\alpha,-\alpha-1],\Delta_\rho[-\alpha+1,-\alpha+1],\Delta_\rho[-\alpha+2,-\alpha+2];\pi_{sc})$
- $L(\Delta_\rho[-\alpha-1,-\alpha-1],\Delta_\rho[-\alpha+1,-\alpha+1],\Delta_\rho[-\alpha+2,-\alpha+2];T_{I,1}^{\alpha}(\pi_{sc}))$

(e) $(\alpha-1,\alpha-1,\alpha-1,\alpha)$*:*
- $L(\Delta_\rho[-\alpha+1,-\alpha+1],\Delta_\rho[-\alpha+1,-\alpha+1];T_{I,1}^{\alpha-1}(T_{I,1}^{\alpha}(\pi_{sc})))$
- $L(\Delta_\rho[-\alpha+1,-\alpha+1],\Delta_\rho[-\alpha+1,-\alpha+1],\Delta_\rho[-\alpha+1,-\alpha+1],T_{I,1}^{\alpha}(\pi_{sc}))$
- $L(\Delta_\rho[-\alpha+1,-\alpha],\Delta_\rho[-\alpha+1,-\alpha+1],\Delta_\rho[-\alpha+1,-\alpha+1];\pi_{sc})$
- $L(\Delta_\rho[-\alpha,-\alpha],\Delta_\rho[-\alpha+1,-\alpha+1],\Delta_\rho[-\alpha+1,-\alpha+1],\Delta_\rho[-\alpha+1,-\alpha+1];\pi_{sc})$

(f) $(\alpha-1,\alpha-1,\alpha,\alpha)$*:*
- $L(\Delta_\rho[-\alpha+1,-\alpha];T_{I,1}^{\alpha-1}(T_{I,1}^{\alpha}(\pi_{sc})))$
- $L(\Delta_\rho[-\alpha,-\alpha],\Delta_\rho[-\alpha+1,-\alpha+1];T_{I,1}^{\alpha-1}(T_{I,1}^{\alpha}(\pi_{sc})))$
- $L(\Delta_\rho[-\alpha+1,-\alpha],\Delta_\rho[-\alpha+1,-\alpha+1];T_{I,1}^{\alpha}(\pi_{sc}))$
- $L(\Delta_\rho[-\alpha+1,-\alpha],\Delta_\rho[-\alpha+1,-\alpha];\pi_{sc})$
- $L(\Delta_\rho[-\alpha,-\alpha],\Delta_\rho[-\alpha+1,-\alpha+1],\Delta_\rho[-\alpha+1,-\alpha+1];T_{I,1}^{\alpha}(\pi_{sc}))$
- $L(\Delta_\rho[-\alpha,-\alpha],\Delta_\rho[-\alpha+1,-\alpha],\Delta_\rho[-\alpha+1,-\alpha+1];\pi_{sc})$
- $L(\Delta_\rho[-\alpha,-\alpha],\Delta_\rho[-\alpha,-\alpha],\Delta_\rho[-\alpha+1,-\alpha+1],\Delta_\rho[-\alpha+1,-\alpha+1];\pi_{sc})$

(g) $(\alpha-1,\alpha-1,\alpha,\alpha+1)$*:*
- $L(\Delta_\rho[-\alpha+1,-\alpha+1];T_{I,1}^{\alpha-1}(T_{I,1}^{\alpha+1}(T_{I,1}^{\alpha}(\pi_{sc}))))$
- $L(\Delta_\rho[-\alpha-1,-\alpha-1],\Delta_\rho[-\alpha+1,-\alpha+1];T_{I,1}^{\alpha-1}(T_{I,1}^{\alpha}(\pi_{sc})))$
- $L(\Delta_\rho[-\alpha+1,-\alpha+1],\Delta_\rho[-\alpha+1,-\alpha+1];T_{I,1}^{\alpha+1}(T_{I,1}^{\alpha}(\pi_{sc})))$
- $L(\Delta_\rho[-\alpha+1,-\alpha-1],\Delta_\rho[-\alpha+1,-\alpha+1];\pi_{sc})$
- $L(\Delta_\rho[-\alpha-1,-\alpha-1],\Delta_\rho[-\alpha+1,-\alpha+1],\Delta_\rho[-\alpha+1,-\alpha+1];T_{I,1}^{\alpha}(\pi_{sc}))$
- $L(\Delta_\rho[-\alpha-1,-\alpha-1],\Delta_\rho[-\alpha+1,-\alpha],\Delta_\rho[-\alpha+1,-\alpha+1];\pi_{sc})$
- $L(\Delta_\rho[-\alpha,-\alpha-1],\Delta_\rho[-\alpha+1,-\alpha+1],\Delta_\rho[-\alpha+1,-\alpha+1];\pi_{sc})$
- $L(\Delta_\rho[-\alpha-1,-\alpha-1],\Delta_\rho[-\alpha,-\alpha],\Delta_\rho[-\alpha+1,\alpha+1],\Delta_\rho[-\alpha+1,-\alpha+1];\pi_{sc})$

(h) $(\alpha-1,\alpha,\alpha,\alpha)$*:*

- $L(\Delta_\rho[-\alpha,-\alpha],\Delta_\rho[-\alpha+1,-\alpha];T_{I,1}^{\alpha}(\pi_{sc}))$
- $L(\Delta_\rho[-\alpha,-\alpha],\Delta_\rho[-\alpha,-\alpha];T_{I,1}^{\alpha-1}(T_{I,1}^{\alpha}(\pi_{sc})))$
- $L(\Delta_\rho[-\alpha,-\alpha],\Delta_\rho[-\alpha,-\alpha],\Delta_\rho[-\alpha+1,-\alpha+1];T_{I,1}^{\alpha}(\pi_{sc}))$
- $L(\Delta_\rho[-\alpha,-\alpha],\Delta_\rho[-\alpha,-\alpha],\Delta_\rho[-\alpha+1,-\alpha];\pi_{sc})$
- $L(\Delta_\rho[-\alpha,-\alpha],\Delta_\rho[-\alpha,-\alpha],\Delta_\rho[-\alpha,-\alpha],\Delta_\rho[-\alpha+1,-\alpha+1];\pi_{sc})$

(i) $(\alpha-1,\alpha,\alpha,\alpha+1)$*:*
- $L(\Delta_\rho[-\alpha+1,-\alpha];T_{I,1}^{\alpha+1}(T_{I,1}^{\alpha}(\pi_{sc})))$
- $L(\Delta_\rho[-\alpha+1,-\alpha-1];T_{I,1}^{\alpha}(\pi_{sc}))$
- $L(\Delta_\rho[-\alpha,-\alpha];T_{I,1}^{\alpha-1}(T_{I,1}^{\alpha+1}(T_{I,1}^{\alpha}(\pi_{sc}))))$
- $L(\Delta_\rho[-\alpha,-\alpha-1];T_{I,1}^{\alpha-1}(T_{I,1}^{\alpha}(\pi_{sc})))$
- $L(\Delta_\rho[-\alpha-1,-\alpha-1],\Delta_\rho[-\alpha,-\alpha];T_{I,1}^{\alpha-1}(T_{I,1}^{\alpha}(\pi_{sc})))$
- $L(\Delta_\rho[-\alpha-1,-\alpha-1],\Delta_\rho[-\alpha+1,-\alpha];T_{I,1}^{\alpha}(\pi_{sc}))$
- $L(\Delta_\rho[-\alpha,-\alpha-1],\Delta_\rho[-\alpha+1,-\alpha+1];T_{I,1}^{\alpha}(\pi_{sc}))$
- $L(\Delta_\rho[-\alpha,-\alpha-1],\Delta_\rho[-\alpha+1,-\alpha];\pi_{sc})$
- $L(\Delta_\rho[-\alpha,-\alpha],\Delta_\rho[-\alpha+1,-\alpha-1];\pi_{sc})$
- $L(\Delta_\rho[-\alpha,-\alpha-1],\Delta_\rho[-\alpha,-\alpha],\Delta_\rho[-\alpha+1,-\alpha+1];\pi_{sc})$
- $L(\Delta_\rho[-\alpha-1,-\alpha-1],\Delta_\rho[-\alpha,-\alpha],\Delta_\rho[-\alpha+1,-\alpha];\pi_{sc})$

(j) $(\alpha-1,\alpha,\alpha+1,\alpha+1)$*:*
- $L(\Delta_\rho[-\alpha-1,-\alpha-1];T_{I,1}^{\alpha-1}(T_{I,1}^{\alpha+1}(T_{I,1}^{\alpha}(\pi_{sc}))))$
- $L(\Delta_\rho[-\alpha-1,-\alpha-1],\Delta_\rho[-\alpha-1,-\alpha-1];T_{I,1}^{\alpha-1}(T_{I,1}^{\alpha}(\pi_{sc})))$
- $L(\Delta_\rho[-\alpha-1,-\alpha-1],\Delta_\rho[-\alpha+1,-\alpha-1];\pi_{sc})$
- $L(\Delta_\rho[-\alpha-1,-\alpha-1],\Delta_\rho[-\alpha+1,-\alpha+1],T_{I,1}^{\alpha+1}(T_{I,1}^{\alpha}(\pi_{sc})))$
- $L(\Delta_\rho[-\alpha-1,-\alpha-1],\Delta_\rho[-\alpha-1,-\alpha-1],\Delta_\rho[-\alpha+1,-\alpha+1];T_{I,1}^{\alpha}(\pi_{sc}))$
- $L(\Delta_\rho[-\alpha-1,-\alpha-1],\Delta_\rho[-\alpha,-\alpha-1],\Delta_\rho[-\alpha+1,-\alpha+1];\pi_{sc})$
- $L(\Delta_\rho[-\alpha-1,-\alpha-1],\Delta_\rho[-\alpha-1,-\alpha-1],\Delta_\rho[-\alpha+1,-\alpha];\pi_{sc})$
- $L(\Delta_\rho[-\alpha-1,-\alpha-1],\Delta_\rho[-\alpha-1,-\alpha-1],\Delta_\rho[-\alpha,\alpha],\Delta_\rho[-\alpha+1,-\alpha+1];\pi_{sc})$

(k) $(\alpha-1,\alpha,\alpha+1,\alpha+2)$*:*
- $L(\Delta_\rho[-\alpha-2,-\alpha-2];T_{I,1}^{\alpha-1}(T_{I,1}^{\alpha+1}(T_{I,1}^{\alpha}(\pi_{sc}))))$
- $L(\Delta_\rho[-\alpha-1,-\alpha-2];T_{I,1}^{\alpha}(T_{I,1}^{\alpha-1}(\pi_{sc})))$
- $L(\Delta_\rho[-\alpha+1,-\alpha-2];\pi_{sc})$
- $L(\Delta_\rho[-\alpha-2,-\alpha-2],\Delta_\rho[-\alpha-1,-\alpha-1];T_{I,1}^{\alpha-1}(T_{I,1}^{\alpha}(\pi_{sc})))$
- $L(\Delta_\rho[-\alpha-2,-\alpha-2],\Delta_\rho[-\alpha+1,-\alpha+1];T_{I,1}^{\alpha+1}(T_{I,1}^{\alpha}(\pi_{sc})))$
- $L(\Delta_\rho[-\alpha-1,-\alpha-2],\Delta_\rho[-\alpha+1,-\alpha+1];T_{I,1}^{\alpha}(\pi_{sc}))$
- $L(\Delta_\rho[-\alpha-1,-\alpha-2],\Delta_\rho[-\alpha+1,-\alpha];\pi_{sc})$
- $L(\Delta_\rho[-\alpha,-\alpha-2],\Delta_\rho[-\alpha+1,-\alpha+1];\pi_{sc})$
- $L(\Delta_\rho[-\alpha-2,-\alpha-2],\Delta_\rho[-\alpha+1,-\alpha-1];\pi_{sc})$
- $L(\Delta_\rho[-\alpha-1,-\alpha-2],\Delta_\rho[-\alpha,-\alpha],\Delta_\rho[-\alpha+1,-\alpha+1];\pi_{sc})$
- $L(\Delta_\rho[-\alpha-2,-\alpha-2],\Delta_\rho[-\alpha-1,-\alpha-1],\Delta_\rho[-\alpha+1,-\alpha+1];T_{I,1}^{\alpha}(\pi_{sc}))$
- $L(\Delta_\rho[-\alpha-2,-\alpha-2],\Delta_\rho[-\alpha,-\alpha-1],\Delta_\rho[-\alpha+1,-\alpha+1];\pi_{sc})$

(l) $(\alpha,\alpha,\alpha,\alpha)$*:*
- $L(\Delta_\rho[-\alpha,-\alpha],\Delta_\rho[-\alpha,-\alpha],\Delta_\rho[-\alpha,-\alpha],T_{I,1}^{\alpha}(\pi_{sc}))$
- $L(\Delta_\rho[-\alpha,-\alpha],\Delta_\rho[-\alpha,-\alpha],\Delta_\rho[-\alpha,-\alpha],\Delta_\rho[-\alpha,-\alpha];\pi_{sc})$

(m) $(\alpha,\alpha,\alpha,\alpha+1)$*:*
- $L(\Delta_\rho[-\alpha,-\alpha],\Delta_\rho[-\alpha,-\alpha];T_{I,1}^{\alpha+1}(T_{I,1}^{\alpha}(\pi_{sc})))$

- $L(\Delta_\rho[-\alpha,-\alpha-1],\Delta_\rho[-\alpha,-\alpha];T_{I,1}^{\alpha}(\pi_{sc}))$
- $L(\Delta_\rho[-\alpha-1,-\alpha-1],\Delta_\rho[-\alpha,-\alpha],\Delta_\rho[-\alpha,-\alpha];T_{I,1}^{\alpha}(\pi_{sc}))$
- $L(\Delta_\rho[-\alpha,-\alpha-1],\Delta_\rho[-\alpha,-\alpha],\Delta_\rho[-\alpha,-\alpha];\pi_{sc})$
- $L(\Delta_\rho[-\alpha-1,-\alpha-1],\Delta_\rho[-\alpha,-\alpha],\Delta_\rho[-\alpha,-\alpha],\Delta_\rho[-\alpha,-\alpha];\pi_{sc})$

(n) $(\alpha,\alpha,\alpha+1,\alpha+1)$*:*
- $L(\Delta_\rho[-\alpha,-\alpha-1];T_{I,1}^{\alpha+1}(T_{I,1}^{\alpha}(\pi_{sc})))$
- $L(\Delta_\rho[-\alpha-1,-\alpha-1],\Delta_\rho[-\alpha,-\alpha];T_{I,1}^{\alpha+1}(T_{I,1}^{\alpha}(\pi_{sc})))$
- $L(\Delta_\rho[-\alpha-1,-\alpha-1],\Delta_\rho[-\alpha,-\alpha-1];T_{I,1}^{\alpha}(\pi_{sc}))$
- $L(\Delta_\rho[-\alpha,-\alpha-1],\Delta_\rho[-\alpha,-\alpha-1];\pi_{sc})$
- $L(\Delta_\rho[-\alpha-1,-\alpha-1],\Delta_\rho[-\alpha-1,-\alpha-1],\Delta_\rho[-\alpha,-\alpha];T_{I,1}^{\alpha}(\pi_{sc}))$
- $L(\Delta_\rho[-\alpha-1,-\alpha-1],\Delta_\rho[-\alpha,-\alpha-1],\Delta_\rho[-\alpha,-\alpha];\pi_{sc})$
- $L(\Delta_\rho[-\alpha-1,-\alpha-1],\Delta_\rho[-\alpha-1,-\alpha-1],\Delta_\rho[-\alpha,-\alpha],\Delta_\rho[-\alpha,-\alpha];\pi_{sc})$

(o) $(\alpha,\alpha,\alpha+1,\alpha+2)$*:*
- $L(\Delta_\rho[-\alpha,-\alpha];T_{I,1}^{\alpha+2}(T_{I,1}^{\alpha+1}(T_{I,1}^{\alpha}(\pi_{sc}))))$
- $L(\Delta_\rho[-\alpha,-\alpha-2];T_{I,1}^{\alpha}(\pi_{sc}))$
- $L(\Delta_\rho[-\alpha-1,-\alpha-2],\Delta_\rho[-\alpha,-\alpha];T_{I,1}^{\alpha}(\pi_{sc}))$
- $L(\Delta_\rho[-\alpha,-\alpha-2],\Delta_\rho[-\alpha,-\alpha];\pi_{sc})$
- $L(\Delta_\rho[-\alpha-2,-\alpha-2],\Delta_\rho[-\alpha,-\alpha];T_{I,1}^{\alpha+1}(T_{I,1}^{\alpha}(\pi_{sc})))$
- $L(\Delta_\rho[-\alpha-2,-\alpha-2],\Delta_\rho[-\alpha,-\alpha-1];T_{I,1}^{\alpha}(\pi_{sc}))$
- $L(\Delta_\rho[-\alpha-2,-\alpha-2],\Delta_\rho[-\alpha-1,-\alpha-1],\Delta_\rho[-\alpha,-\alpha];T_{I,1}^{\alpha}(\pi_{sc}))$
- $L(\Delta_\rho[-\alpha-1,-\alpha-2],\Delta_\rho[-\alpha,-\alpha],\Delta_\rho[-\alpha,-\alpha],\pi_{sc})$
- $L(\Delta_\rho[-\alpha-2,-\alpha-2],\Delta_\rho[-\alpha,-\alpha-1],\Delta_\rho[-\alpha,-\alpha];\pi_{sc})$
- $L(\Delta_\rho[-\alpha-2,-\alpha-2],\Delta_\rho[-\alpha-1,-\alpha-1],\Delta_\rho[-\alpha,-\alpha],\Delta_\rho[-\alpha,-\alpha];\pi_{sc})$

(p) $(\alpha,\alpha+1,\alpha+1,\alpha+1)$*:*
- $L(\Delta_\rho[-\alpha-1,-\alpha-1],\Delta_\rho[-\alpha-1,-\alpha-1];T_{I,1}^{\alpha+1}(T_{I,1}^{\alpha}(\pi_{sc})))$
- $L(\Delta_\rho[-\alpha-1,-\alpha-1],\Delta_\rho[-\alpha-1,-\alpha-1],\Delta_\rho[-\alpha-1,-\alpha-1];T_{I,1}^{\alpha}(\pi_{sc}))$
- $L(\Delta_\rho[-\alpha-1,-\alpha-1],\Delta_\rho[-\alpha-1,-\alpha-1],\Delta_\rho[-\alpha,-\alpha-1];\pi_{sc})$
- $L(\Delta_\rho[-\alpha-1,-\alpha-1],\Delta_\rho[-\alpha-1,-\alpha-1],\Delta_\rho[-\alpha-1,-\alpha-1],\Delta_\rho[-\alpha,-\alpha];\pi_{sc})$

(q) $(\alpha,\alpha+1,\alpha+1,\alpha+2)$*:*
- $L(\Delta_\rho[-\alpha-1,-\alpha-1];T_{I,1}^{\alpha+2}(T_{I,1}^{\alpha+1}(T_{I,1}^{\alpha}(\pi_{sc}))))$
- $L(\Delta_\rho[-\alpha-1,-\alpha-2];T_{I,1}^{\alpha+1}(T_{I,1}^{\alpha}(\pi_{sc})))$
- $L(\Delta_\rho[-\alpha-1,-\alpha-2],\Delta_\rho[-\alpha-1,-\alpha-1];T_{I,1}^{\alpha}(\pi_{sc}))$
- $L(\Delta_\rho[-\alpha-1,-\alpha-2],\Delta_\rho[-\alpha,-\alpha-1];\pi_{sc})$
- $L(\Delta_\rho[-\alpha-1,-\alpha-1],\Delta_\rho[-\alpha,-\alpha-2];\pi_{sc})$
- $L(\Delta_\rho[-\alpha-2,-\alpha-2],\Delta_\rho[-\alpha-1,-\alpha-1];T_{I,1}^{\alpha+1}(T_{I,1}^{\alpha}(\pi_{sc})))$
- $L(\Delta_\rho[-\alpha-1,-\alpha-2],\Delta_\rho[-\alpha-1,-\alpha-1],\Delta_\rho[-\alpha,-\alpha];\pi_{sc})$
- $L(\Delta_\rho[-\alpha-2,-\alpha-2],\Delta_\rho[-\alpha-1,-\alpha-1],\Delta_\rho[-\alpha,-\alpha-1];\pi_{sc})$
- $L(\Delta_\rho[-\alpha-2,-\alpha-2],\Delta_\rho[-\alpha-1,-\alpha-1],\Delta_\rho[-\alpha-1,-\alpha-1];T_{I,1}^{\alpha}(\pi_{sc}))$
- $L(\Delta_\rho[-\alpha-2,-\alpha-2],\Delta_\rho[-\alpha-1,-\alpha-1],\Delta_\rho[-\alpha-1,\alpha-1],\Delta_\rho[-\alpha,-\alpha];\pi_{sc})$

(r) $(\alpha,\alpha+1,\alpha+2,\alpha+2)$*:*
- $L(\Delta_\rho[-\alpha,-2-\alpha-2];T_{I,1}^{\alpha+2}(T_{I,1}^{\alpha+1}(T_{I,1}^{\alpha}(\pi_{sc}))))$
- $L(\Delta_\rho[-\alpha-2,-\alpha-2],\Delta_\rho[-\alpha-2,-\alpha-2];T_{I,1}^{\alpha+1}(T_{I,1}^{\alpha}(\pi_{sc})))$
- $L(\Delta_\rho[-\alpha-2,-\alpha-2],\Delta_\rho[-\alpha-1,-\alpha-2];T_{I,1}^{\alpha}(\pi_{sc}))$
- $L(\Delta_\rho[-\alpha-2,-\alpha-2],\Delta_\rho[-\alpha,-\alpha-2];\pi_{sc})$
- $L(\Delta_\rho[-\alpha-2,-\alpha-2],\Delta_\rho[-\alpha-2,-\alpha-2],\Delta_\rho[-\alpha-1,-\alpha-1];T_{I,1}^{\alpha}(\pi_{sc}))$

- $L(\Delta_\rho[-\alpha-2,-\alpha-2],\Delta_\rho[-\alpha-1,-\alpha-2],\Delta_\rho[-\alpha,-\alpha];\pi_{sc})$
- $L(\Delta_\rho[-\alpha-2,-\alpha-2],\Delta_\rho[-\alpha-2,-\alpha-2],\Delta_\rho[-\alpha,-\alpha-1];\pi_{sc})$
- $L(\Delta_\rho[-\alpha-2,-\alpha-2],\Delta_\rho[-\alpha-2,-\alpha-2],\Delta_\rho[-\alpha-1,\alpha-1],\Delta_\rho[-\alpha,-\alpha];\pi_{sc})$

(s) $(\alpha,\alpha+1,\alpha+2,\alpha+3)$*:*
- $L(\Delta_\rho[-\alpha-3,-\alpha-3];T_{I,1}^{\alpha+2}(T_{I,1}^{\alpha+1}(T_{I,1}^{\alpha}(\pi_{sc}))))$
- $L(\Delta_\rho[-\alpha-2,-\alpha-3];T_{I,1}^{\alpha+1}(T_{I,1}^{\alpha}(\pi_{sc})))$
- $L(\Delta_\rho[-\alpha-1,-\alpha-3];T_{I,1}^{\alpha}(\pi_{sc}))$
- $L(\Delta_\rho[-\alpha-3,-\alpha-3],\Delta_\rho[-\alpha-2,-\alpha-2];T_{I,1}^{\alpha+1}(T_{I,1}^{\alpha}(\pi_{sc})))$
- $L(\Delta_\rho[-\alpha,-\alpha-3];\pi_{sc})$
- $L(\Delta_\rho[-\alpha-3,-\alpha-3],\Delta_\rho[-\alpha-1,-\alpha-2];T_{I,1}^{\alpha}(\pi_{sc}))$
- $L(\Delta_\rho[-\alpha-2,-\alpha-3],\Delta_\rho[-\alpha-1,-\alpha-1];T_{I,1}^{\alpha}(\pi_{sc}))$
- $L(\Delta_\rho[-\alpha-2,-\alpha-3],\Delta_\rho[-\alpha,-\alpha-1];\pi_{sc})$
- $L(\Delta_\rho[-\alpha-1,-\alpha-3],\Delta_\rho[-\alpha,-\alpha];\pi_{sc})$
- $L(\Delta_\rho[-\alpha-3,-\alpha-3],\Delta_\rho[-\alpha,-\alpha-2];\pi_{sc})$
- $L(\Delta_\rho[-\alpha-3,-\alpha-3],\Delta_\rho[-\alpha-2,-\alpha-2],\Delta_\rho[-\alpha-1,-\alpha-1];T_{I,1}^{\alpha}(\pi_{sc}))$
- $L(\Delta_\rho[-\alpha-2,-\alpha-3],\Delta_\rho[-\alpha-1,-\alpha-1],\Delta_\rho[-\alpha,-\alpha];\pi_{sc})$
- $L(\Delta_\rho[-\alpha-3,-\alpha-3],\Delta_\rho[-\alpha-1,-\alpha-2],\Delta_\rho[-\alpha,-\alpha];\pi_{sc})$
- $L(\Delta_\rho[-\alpha-3,-\alpha-3],\Delta_\rho[-\alpha-2,-\alpha-2],\Delta_\rho[-\alpha,-\alpha-1];\pi_{sc})$

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