GAMMA FACTORS AND CONVERSE THEOREMS FOR CLASSICAL GROUPS OVER FINITE FIELDS

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Abstract. In this paper, we prove certain multiplicity one theorems and define GL-twisted gamma factors for irreducible generic cuspidal representations of quasi-split classical groups $\mathrm{G}_r = \mathrm{Sp}_{2r}, \mathrm{U}_{2r}, \mathrm{U}_{2r+1}, \mathrm{SO}_{2r+1}$ over finite fields of odd characteristic, using Rankin-Selberg method. As applications, we prove converse theorems for these groups, namely, GL_n-twisted gamma factors, $n = 1, 2, \ldots, r$, will uniquely determine irreducible generic cuspidal representations of $\mathrm{G}_r(\mathbb{F}_q)$.

1. Introduction

Local gamma factors play essential roles in the theories of automorphic forms and representations of p-adic groups, especially in the local Langlands correspondence conjecture, Langlands functoriality conjecture, and local converse theorems. Local gamma factors can usually be defined using Langlands-Shahidi method or Rankin-Selberg method, at least for generic representations. In this paper, we prove multiplicity one theorems of certain Fourier-Jacobi models (analogs for Bessel models were proved in [GGP12b]) over finite fields and define the finite fields analogue of local gamma factors for irreducible generic cuspidal representations of quasi-split classical groups $\mathrm{G}_r = \mathrm{Sp}_{2r}, \mathrm{U}_{2r}, \mathrm{U}_{2r+1}, \mathrm{SO}_{2r+1}$, using the Rankin-Selberg method. We also obtain explicit formulas for these gamma factors in terms of corresponding Bessel functions. These gamma factors provide important invariants for generic cuspidal representations and are expected to play important roles in the representation theory of these groups over finite fields. There are interesting questions that how these invariants are related to the Deligne-Lusztig theory on virtual characters ([DL76, L84]), and to the finite fields analogue of the Gan-Gross-Prasad conjecture ([GGP12b]).

Over p-adic fields, the uniqueness of Bessel and Fourier-Jacobi models for classical groups were proved in [AGRS10] and [Su12] respectively. But over finite fields, the general statement of uniqueness of Bessel and Fourier-Jacobi models is not true for all irreducible representations of $\mathrm{G}_r(\mathbb{F}_q)$, see [GGP12b, §4,5] for example. This suggests that we cannot have a uniform proof using distribution theory as in the p-adic fields case. To conquer these difficulties, we link the finite fields case with the p-adic fields case. To do so, we need to restrict our representations of $\mathrm{G}_r(\mathbb{F}_q)$ to irreducible cuspidal representations and use the theory of depth zero representations of G_r over p-adic fields. The Bessel model case has been carried out in [GGP12b]. The main difficulty in the Fourier-Jacobi model case is to connect the Weil representations over finite fields and p-adic fields. To make this connection, we use the generalized lattice models for Weil representations.

Over finite fields, the Bessel and Fourier-Jacobi models are special cases of Generalized Gelfand-Graev models considered by Kawanaka, see [K85, K86]. There are many results on the computation of such multiplicities in more general settings, for example see [L92, Gec99, GeH08]. See also [LZ18, LZ19] for certain uniqueness results of Fourier-Jacobi models for Sp_4, U_4, and the split exceptional group of type G_2 over finite fields.

As applications of the gamma factors defined above, we prove the converse theorems for these groups, namely, GL_n-twisted gamma factors, $n = 1, 2, \ldots, r$, will uniquely determine irreducible generic cuspidal representations of $\mathrm{G}_r(\mathbb{F}_q)$. Therefore, these GL_n-twisted gamma factors form complete sets of invariants for irreducible generic cuspidal representations of $\mathrm{G}_r(\mathbb{F}_q)$.

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Over finite fields, gamma factors and converse theorems have been defined and considered for general linear groups and the split exceptional group of type G_2. Roditty [Ro10] defined gamma factors for cuspidal representations of $\mathrm{GL}_n \times \mathrm{GL}_m$ over finite fields. Nien in [Ni14] proved the converse theorem for cuspidal representations of $\mathrm{GL}_n(\mathbb{F}_q)$, using special properties of Bessel functions and the twisted gamma factors defined by Roditty. The authors defined the GL_n-twisted gamma factors for $n = 1, 2$ and proved the converse theorem for generic cuspidal representations of the split exceptional group $G_2(\mathbb{F}_q)$ in [LZ18]. Gamma factors over finite fields were defined in a more general context in [BK00].

Similar to local fields cases, it is expected that GL_n-twisted gamma factors for irreducible generic cuspidal representations of $\mathrm{G}_r(\mathbb{F}_q)$ can also be defined using Langlands-Shahidi method. In future work, the authors plan to define GL_n-twisted gamma factors using Langlands-Shahidi method and verify the consistency with those defined in this paper using Rankin-Selberg method.

In [NZ18], Nien and Zhang verified that the GL_1-twisted gamma factors will uniquely determine irreducible cuspidal representations of $\mathrm{GL}_N(\mathbb{F}_q)$, for $N \leq 5$, and irreducible generic representations of $\mathrm{GL}_N(\mathbb{F}_q)$, for $N < \frac{q-1}{2\sqrt{q}} + 1$ in the appendix by Zhiwei Yun. Note that on the dual groups side we have embeddings $\mathrm{G}_r^\vee(\mathbb{C}) \hookrightarrow \mathrm{GL}_N(\mathbb{C})$. Hence, by Langlands philosophy of functoriality, it is expected that irreducible generic cuspidal representations of $\mathrm{G}_r(\mathbb{F}_q)$ would also be uniquely determined by GL_1-twisted gamma factors when q is large. In future work, the authors plan to check this expectation directly by analyzing GL_1-twisted gamma factors for irreducible generic cuspidal representations of $\mathrm{G}_r(\mathbb{F}_q)$. The authors also plan to consider related functorial lifting and descent problems using the GL-twisted gamma factors defined in this paper.

This paper does not include the case of SO_{2n}, which is a work in progress of a student of the first named author. The converse theorem for SO_{2n} is expected to be more subtle.

Following is the structure of this paper. For $\mathrm{G}_r = \mathrm{Sp}_{2r}$, we prove a multiplicity one theorem in Section 2, define the GL-twisted gamma factors in Section 3, and prove the converse theorem in Section 4. Cases of $\mathrm{G}_r = \mathrm{U}_{2r}, \mathrm{U}_{2r+1}, \mathrm{SO}_{2r+1}$ will be considered in Sections 5, 6, 7, respectively.

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2. A multiplicity one theorem for Sp_{2r}

Let F be a p-adic field with odd residue characteristic. Let $\mathfrak{o}$ be the ring of integers of F, $\mathfrak{p}$ be the maximal ideal of $\mathfrak{o}$, and $\varpi \in \mathfrak{p}$ be a fixed generator. Let $k = \mathfrak{o}/\mathfrak{p}$ be the residue field. Let $\Pi_{\mathfrak{o}} : \mathfrak{o} \to k$ be the natural projection. Let ψ be a fixed unramified additive character of F, and let $\overline{\psi}$ be the character of k defined by

$$\overline{\psi}(\Pi_{\mathfrak{o}}(t)) = \psi(t\varpi^{-1}), t \in \mathfrak{o}.$$

2.1. **The group $\widetilde{\mathrm{Sp}}_{2r}(F)$ and the Weil representation.** Let r be a positive integer and $(W, \langle\ ,\ \rangle)$ be a symplectic space over F of rank $2r$. Let $\mathcal{H}(W)$ be the Heisenberg group of W. As a set, $\mathcal{H}(W) = W \oplus F$ and its product is given by

$$(w_1, t_1)(w_2, t_2) = (w_1 + w_2, t_1 + t_2 + \frac{1}{2}\langle w_1, w_2\rangle), w \in W, t \in F.$$

By the Stone-Von-Neumann theorem, there is a unique irreducible representation $(\rho_\psi, \mathcal{S})$ of $\mathcal{H}(W)$ with central character ψ. Let $\mathrm{Sp}(W)$ be the isometry group of $(W, \langle\ ,\ \rangle)$, i.e.,

$$\mathrm{Sp}(W) = \{g \in \mathrm{GL}(W) : \langle w_1 g, w_2 g\rangle = \langle w_1, w_2\rangle, \forall w_1, w_2 \in W\},$$

where elements in W are viewed as row vectors and $\mathrm{Sp}(W)$ acts on the right. The group $\mathrm{Sp}(W)$ acts on $\mathcal{H}(W)$ by

$$(w, t)^g = (wg, t), w \in W, t \in F, g \in \mathrm{Sp}(w).$$

For each $g \in \mathrm{Sp}(W)$, we can define a representation ρ_ψ^g of $\mathcal{H}(W)$ by

$$\rho_\psi^g(w, t) = \rho_\psi((w, t)^g).$$

Since $\mathrm{Sp}(W)$ acts trivially on the center of $\mathcal{H}(W)$, the central character of ρ_ψ^g is still ψ. Thus by the uniqueness in the Stone-Von-Neumann theorem, we have $\rho_\psi^g \cong \rho_\psi$. Fix an isomorphism

$M[g] : \rho_\psi^g \to \rho_\psi$. Then $(M[g], \mathcal{S})$ is a projective representation of $\mathrm{Sp}(W)$. It is known that it can be defined as a real representation on a double cover $\widetilde{\mathrm{Sp}}(W)$ of $\mathrm{Sp}(W)$. The corresponding representation is denoted by ω_ψ and is called the Weil representation of $\widetilde{\mathrm{Sp}}(W)$. It is well-known that, up to equivalence, there is a unique symplectic structure on W, and thus $\mathrm{Sp}(W)$ (resp. $\widetilde{\mathrm{Sp}}(W)$) is usually written as $\mathrm{Sp}_{2r}(F)$ (resp. $\widetilde{\mathrm{Sp}}_{2r}(F)$).

2.2. **Generalized lattice model of the Weil representation.** Let $(W, \langle\ ,\ \rangle)$ be a symplectic vector space over F of dimension $2r$ with symplectic form $\langle\ ,\ \rangle$. Let $e_i, i = \pm 1, \ldots, \pm r$, be a basis of W with

$$\langle e_i, e_j\rangle = 0, \langle e_{-i}, e_{-j}\rangle = 0, \langle e_i, e_{-j}\rangle = \delta_{i,j}, \forall i, j > 0.$$

Let $\mathbf{B} \subset W$ be the lattice

$$\mathbf{B} = \mathfrak{p}e_1 + \mathfrak{p}e_2 + \ldots \mathfrak{p}e_r + \mathfrak{o}e_{-r} + \mathfrak{o}e_{-(r-1)} + \ldots \mathfrak{o}e_{-1},$$

and let

$$\mathbf{B}^* = \{v \in W : \langle v, b\rangle \in \mathfrak{o}, \forall b \in \mathbf{B}\}.$$

Then one can check that

$$\mathbf{B}^* = \mathfrak{o}e_1 + \mathfrak{o}e_2 + \ldots \mathfrak{o}e_r + \varpi^{-1}\mathfrak{o}e_{-r} + \cdots + \varpi^{-1}\mathfrak{o}e_{-2} + \varpi^{-1}\mathfrak{o}e_{-1}.$$

Let $\mathbf{b}^* = \mathbf{B}^*/\mathbf{B}$, which is a lattice over k of rank $2r$. Let $\Pi_{\mathbf{B}^*} : \mathbf{B}^* \to \mathbf{b}^*$ be the natural projection. We can define a symplectic form $\langle\ ,\ \rangle_{\mathbf{b}^*} : \mathbf{b}^* \times \mathbf{b}^* \to k$ by

$$\langle \Pi_{\mathbf{B}^*}(w), \Pi_{\mathbf{B}^*}(w')\rangle_{\mathbf{b}^*} := \Pi_{\mathfrak{o}}(\langle w, w'\rangle\varpi).$$

Let $\mathrm{H}(W)$ (resp. $\mathrm{H}(\mathbf{b}^*)$) be the Heisenberg group of W (resp. $\mathbf{b}^*$) with the given symplectic structure. Let $\mathrm{H}(\mathbf{B}^*) := \mathbf{B}^* \times \mathfrak{p}^{-1}$, which is a subgroup of the Heisenberg group $\mathrm{H}(W)$. We then have a homomorphism

$$\Pi_{\mathrm{H}(\mathbf{B}^*)} : \mathrm{H}(\mathbf{B}^*) \to \mathrm{H}(\mathbf{b}^*)$$

by

$$(b, t) \mapsto (\Pi_{\mathbf{B}^*}(b), \Pi_{\mathfrak{o}}(t\varpi)).$$

Let $K_{\mathbf{B}} = \{g \in \mathrm{Sp}(W) : g\mathbf{B} = \mathbf{B}\}$ and

$$K'_{\mathbf{B}} = \{g \in K_{\mathbf{B}} : (g-1)\mathbf{B}^* \subset \mathbf{B}\}.$$

One can check that

$$K_{\mathbf{B}} = \left\{\begin{pmatrix} A & B \\ C & D\end{pmatrix} \in \mathrm{Sp}_{2r}(F), A, D \in \mathrm{Mat}_r(\mathfrak{o}), B \in \mathrm{Mat}_r(\mathfrak{p}), C \in \mathrm{Mat}_r(\mathfrak{p}^{-1})\right\}.$$

Then $K'_{\mathbf{B}}$ is a normal subgroup of $K_{\mathbf{B}}$ and $K_{\mathbf{B}}/K'_{\mathbf{B}}$ is isomorphic to $\mathrm{Sp}(\mathbf{b}^*)$.

Let $\overline{\rho}_{\overline{\psi}}$ be the representation of $\mathrm{H}(\mathbf{b}^*)$ corresponding to the character $\overline{\psi}$, and $\overline{\omega}_{\overline{\psi}}$ the Weil representation of $\mathrm{Sp}(\mathbf{b}^*)$. Let S be the space of $\overline{\rho}_{\overline{\psi}}$, and hence of $\overline{\omega}_{\overline{\psi}}$. The representation $\overline{\rho}_{\overline{\psi}}$ (resp. $\overline{\omega}_{\overline{\psi}}$) can be inflated to a representation of $\mathrm{H}(\mathbf{B}^*)$(resp. $K_{\mathbf{B}}$), which is still denoted by $\overline{\rho}_{\overline{\psi}}$ (resp. $\overline{\omega}_{\overline{\psi}}$) by abuse of notation. Let $\mathscr{S}(\mathbf{B})$ be the space of locally constant, compactly supported maps $f : W \to S$ such that

$$f(b + w) = \psi(\frac{1}{2}\langle w, b\rangle)\overline{\rho}_{\overline{\psi}}(b).(f(w)), \forall w \in W, b \in \mathbf{B}^*.$$

The Weil representation ω_ψ of $\widetilde{\mathrm{Sp}}(W)$ can be realized on $\mathscr{S}(\mathbf{B})$, which is called the generalized lattice model, see [Wp90] and a survey in [Pan02]. Note that $\widetilde{\mathrm{Sp}}(W) \to \mathrm{Sp}(W)$ splits over $K_{\mathbf{B}}$ and thus $\omega_\psi|_{K_{\mathbf{B}}}$ makes sense.

Proposition 2.1. *As a representation of $K_{\mathbf{B}}$, one has* $(\omega_\psi|_{K_{\mathbf{B}}})^{K'_{\mathbf{B}}} \cong \overline{\omega}_{\overline{\psi}}$.

This is a special case of [Pan02, Proposition 5.3].

2.3. **Genuine induced representations of $\widetilde{\mathrm{Sp}}_{2r}(F)$.** For $a \in F^\times$, let ψ_a be the character of F defined by $\psi_a(x) = \psi(ax)$. Let $\gamma(\psi)$ be the Weil index of $x \mapsto \psi(x^2)$, and let

$$\gamma_\psi(a) = \frac{\gamma(\psi_a)}{\gamma(\psi)}, a \in F^\times,$$

see [Rao93, Appendix]. It is known that $\gamma_\psi(a)\gamma_\psi(b) = \gamma_\psi(ab)(a,b)_F$, where $(a,b)_F$ is the Hilbert symbol. Moreover, under the assumption that the residue characteristic is odd, it is known that $\gamma_\psi(u) = 1$ if $u \in \mathfrak{o}^\times$.

We fix the order of basis of W by $e_1, \ldots, e_r, e_{-r}, \ldots, e_{-1}$, and then fix an embedding $\mathrm{Sp}(W) = \mathrm{Sp}_{2r}(F) \hookrightarrow \mathrm{GL}_{2r}(F)$. Define $J_r \in \mathrm{GL}_r(F)$ inductively by

$$J_r = \begin{pmatrix} & 1 \\ J_{r-1} & \end{pmatrix}, J_1 = 1.$$

Then

$$\mathrm{Sp}_{2r}(F) = \left\{ g \in \mathrm{GL}_{2r}(F) : {}^t g \begin{pmatrix} & J_r \\ -J_r & \end{pmatrix} g = \begin{pmatrix} & J_r \\ -J_r & \end{pmatrix} \right\}.$$

The upper triangular subgroup of $\mathrm{Sp}_{2r}(F)$ is a Borel subgroup. Let $P = MN$ be a fixed standard parabolic subgroup with Levi subgroup

$$M \cong \mathrm{GL}_{n_1}(F) \times \cdots \times \mathrm{GL}_{n_s}(F) \times \mathrm{Sp}_{2m}(F),$$

where $n_1 + \cdots + n_s + m = r$. Let $\widetilde{\mathrm{GL}}_{n_i}(F)$ be the double cover defined by the Hilbert symbol $(\ ,\ )_F$, i.e., $\widetilde{\mathrm{GL}}_{n_i}(F) = \mathrm{GL}_{n_i}(F) \times \{\pm 1\}$ with product

$$(a_1, \epsilon_1)(a_2, \epsilon_2) = (a_1 a_2, \epsilon_1 \epsilon_2 (\det(a_1), \det(a_2))_F),$$

for $a_1, a_2 \in \mathrm{GL}_{n_i}(F)$ and $\epsilon_1, \epsilon_2 \in \{\pm 1\}$. Let $\widetilde{M}$ be the preimage of M in $\widetilde{\mathrm{Sp}}_{2r}(F)$ under the quotient map $\widetilde{\mathrm{Sp}}_{2r}(F) \to \mathrm{Sp}_{2r}(F)$. Then the map $\widetilde{\mathrm{GL}}_{n_1}(F) \times \ldots \widetilde{\mathrm{GL}}_{n_s}(F) \times \widetilde{\mathrm{Sp}}_{2m}(F) \to M$ defined by

$$((a_1, \epsilon_1), \ldots, (a_s, \epsilon_s), (b, \epsilon)) \mapsto (\mathrm{diag}(a_1, \ldots, a_s, b, a_s^*, \ldots, a_1^*), \epsilon \prod_i \epsilon_i)$$

with $a_i \in \mathrm{GL}_{n_i}(F), \epsilon_i, \epsilon \in \{\pm 1\}, b \in \widetilde{\mathrm{Sp}}_{2m}(F)$ is a projection with kernel μ_2^s, where $\mu_2 = \{\pm 1\}$.

It is known that the double cover $\widetilde{\mathrm{Sp}}_{2r}(F) \to \mathrm{Sp}_{2r}(F)$ splits over N. Let $\widetilde{P} = \widetilde{M}N$. Let τ^i be an irreducible cuspidal representation of $\mathrm{GL}_{n_i}(F)$ and let $\widetilde{\sigma}$ be a genuine irreducible cuspidal representation of $\widetilde{\mathrm{Sp}}_{2m}(F)$. Let $\widetilde{\tau}^i = \tau^i \otimes (\gamma_\psi \circ \det)$, which is a representation of $\widetilde{\mathrm{GL}}_{n_i}(F)$. Consider the representation $\widetilde{\tau} := \widetilde{\tau}^1 \otimes \cdots \otimes \widetilde{\tau}^s \otimes \widetilde{\sigma}$, which is a representation of $\widetilde{\mathrm{GL}}_{n_1}(F) \times \ldots \widetilde{\mathrm{GL}}_{n_s}(F) \times \widetilde{\mathrm{Sp}}_{2m}(F)$ and descends to a representation $\widetilde{M}$. We then can consider the induced representation

$$\widetilde{I}(\widetilde{\tau}) := \mathrm{Ind}_{\widetilde{M}N}^{\widetilde{\mathrm{Sp}}_{2r}(F)}(\widetilde{\tau} \otimes 1_N).$$

Let τ_0^i be a cuspidal representation of $\mathrm{GL}_{n_i}(k)$ and σ_0 be a cuspidal representation of $\mathrm{Sp}_{2m}(k)$. Under the projection map $\mathrm{GL}_{n_i}(\mathfrak{o}) \cong \mathrm{GL}_{n_i}(\mathfrak{o})/(I_{n_i} + M_{n_i}(\mathfrak{p})) \cong \mathrm{GL}_{n_i}(k)$, τ_0^i can be inflated to a representation of $\mathrm{GL}_{n_i}(\mathfrak{o})$. By abuse of notation, we still denote this representation by τ_0^i. Let $\mathbf{B}_1 \subset \mathbf{B}$ be the sub-lattice of rank $2m$ which is corresponding the embedding $\mathrm{Sp}_{2m}(F) \subset \mathrm{Sp}_{2r}(F)$ and let $K_{\mathbf{B}_1}$, $K'_{\mathbf{B}_1}$ be the corresponding open compact subgroups defined by the lattice $\mathbf{B}_1$. Then one has the isomorphism $K_{\mathbf{B}_1}/K'_{\mathbf{B}_1} \cong \mathrm{Sp}_{2m}(k)$. One then inflates σ_0 to a representation of $K_{\mathbf{B}_1}$ and still denote it by σ_0.

Consider an irreducible supercuspidal representation of $\mathrm{GL}_{n_i}(F)$

$$\tau^i := \mathrm{ind}_{F^\times \mathrm{GL}_{n_i}(\mathfrak{o})}^{\mathrm{GL}_{n_i}(F)}(\tau_0^i) \otimes |\det|^{t_i},$$

for some $t_i \in \mathbb{C}$, where $F^\times$ is identified with the center of $\mathrm{GL}_{n_i}(F)$ and τ_0^i is viewed as a representation of $F^\times \mathrm{GL}_{n_i}(\mathfrak{o})$ such that its action on $F^\times$ is trivial. Note that $\mathrm{ind}_{F^\times \mathrm{GL}_{n_i}(\mathfrak{o})}^{\mathrm{GL}_{n_i}(F)}(\tau_0^i)$ is an irreducible depth zero supercuspidal representation of $\mathrm{GL}_{n_i}(F)$. Let $\widetilde{\tau}^i = \tau^i \otimes (\gamma_\psi \circ \det)$. Similarly, let $\widetilde{\sigma}$ be an irreducible depth zero supercuspidal genuine representation of $\widetilde{\mathrm{Sp}}_{2n}(F)$ which contains σ_0 as a type. Such a representation is constructed in [HM09] and can be taken as the form $\mathrm{ind}_{\widetilde{K}_{\mathbf{B}_1}}^{\widetilde{\mathrm{Sp}}_{2m}(F)}(\sigma_0 \otimes \mathrm{sgn})$, where $\widetilde{K}_{\mathbf{B}_1} = K_{\mathbf{B}_1} \times \mu_2$ and sgn is the unique nontrivial character of μ_2. Let $\widetilde{\tau} = \widetilde{\tau}^1 \otimes \ldots \widetilde{\tau}^s \otimes \widetilde{\sigma}$ and form the induced representation $\widetilde{I}(\widetilde{\tau})$ as above. Note that, for generic choice of $\widetilde{\tau}$, the representation $\widetilde{I}(\widetilde{\tau})$ is irreducible.

On the other hand, let $P(k) = M(k)N(k)$ be the Siegel parabolic subgroup of $\mathrm{Sp}_{2r}(k)$. We can form the induced representation $I(\tau_0) = \mathrm{Ind}_{P(k)}^{\mathrm{Sp}_{2r}(k)}(\tau_0 \otimes 1_{N(k)})$, where $\tau_0 = \tau_0^1 \otimes \cdots \otimes \tau_0^s \otimes \sigma_0$. Under the projection map $K_{\mathbf{B}} \to K_{\mathbf{B}}/K'_{\mathbf{B}} \cong \mathrm{Sp}_{2r}(k)$, we can view $I(\tau_0)$ as a representation of $K_{\mathbf{B}}$. Note that in general, $I(\tau_0)$ is not irreducible.

Proposition 2.2. *As $K_{\mathbf{B}}$-modules, there is a surjective map*

$$(\widetilde{I}(\widetilde{\tau})|_{K_{\mathbf{B}}})^{K'_{\mathbf{B}}} \to I(\tau_0) \to 0.$$

Proof. Recall that $\widetilde{\tau}^i = \mathrm{ind}_{\mathrm{GL}_{n_i}(\mathfrak{o})}^{\mathrm{GL}_{n_i}(F)}(\tau_0^i) \otimes |\det|^{t_i} \otimes (\gamma_\psi \circ \det)$ and $\widetilde{\sigma} = \mathrm{ind}_{\widetilde{K}_{\mathbf{B}_1}}^{\widetilde{\mathrm{Sp}}_{2m}(F)}(\sigma_0 \otimes \mathrm{sgn})$. An element $f \in \widetilde{I}(\widetilde{\tau})$ is a function $f : \widetilde{\mathrm{Sp}}_{2r}(F) \to \widetilde{\tau}$, which can be viewed as a function

$$f : \widetilde{\mathrm{Sp}}_{2r}(F) \times \mathrm{GL}_{n_1}(F) \times \cdots \times \mathrm{GL}_{n_s}(F) \times \widetilde{\mathrm{Sp}}_{2m}(F) \to \tau_0$$

satisfying the invariance property

$$f(m(a_1, \ldots, a_s, b)ng, x_1, \ldots, x_s, y) = \prod_i (\gamma_\psi(\det(a_i))|\det(a_i)|^{t_i}) f(g, x_1a_1, \ldots, x_sa_s, yb),$$

for $a_i, x_i \in \mathrm{GL}_{n_i}(F), b, y \in \widetilde{\mathrm{Sp}}_{2m}(F), n \in N, g \in \widetilde{\mathrm{Sp}}_{2r}(F)$, where

$$m(a_1, \ldots, a_s, b) = \mathrm{diag}(a_1, \ldots, a_s, b, a_s^*, \ldots, a_1^*) \in \widetilde{M},$$

and

$$f(g, x_1a_1, \ldots, x_sa_s, yb) = \tau_0((a_1, \ldots, a_s, b))(f(g, x_1, \ldots, x_s, y)),$$

for $g \in \widetilde{\mathrm{Sp}}_{2r}(F), x_i \in \mathrm{GL}_r(F), a_i \in \mathrm{GL}_{n_i}(\mathfrak{o}), y \in \widetilde{\mathrm{Sp}}_{2m}(F), b \in K_{\mathbf{B}_1}$.

For $f \in (\widetilde{I}(\widetilde{\tau})|_{K_{\mathbf{B}}})^{K'_{\mathbf{B}}}$, define $\phi_f : K_{\mathbf{B}} \to \tau_0$ by $\phi_f(k_1) = f(k_1, 1, \ldots, 1, 1), k_1 \in K_{\mathbf{B}}$. Then ϕ_f is also $K'_{\mathbf{B}}$-invariant and thus defines a map $\mathrm{Sp}_{2r}(k) = K_{\mathbf{B}}/K'_{\mathbf{B}} \to \tau_0$, which is still denoted by ϕ_f by abuse of notation. One can check that $\phi_f \in I(\tau_0)$. The assignment $f \mapsto \phi_f$ gives a map $(\widetilde{I}(\widetilde{\tau})|_{K_{\mathbf{B}}})^{K'_{\mathbf{B}}} \to I(\tau_0)$. It is clear that the map respects the $K_{\mathbf{B}}$-action. To show that it is surjective, for $\phi \in I(\tau_0)$, we need to define a function $f \in (\widetilde{I}(\widetilde{\tau})|_{K_{\mathbf{B}}})^{K'_{\mathbf{B}}}$ such that $f(k_1, 1, \ldots, 1, 1) = \phi(k_1)$. We consider the function f which satisfies the following properties:

(1) If $f(k_1, x_1, \ldots, x_s, b) \neq 0$, then $k_1 \in K_{\mathbf{B}}$, and $x_i \in \mathrm{GL}_{n_i}(\mathfrak{o}), b \in \widetilde{K}_{\mathbf{B}_1}$;
(2) $f(k_1, x_1, \ldots, x_s, (y, \epsilon)) = \mathrm{sgn}(\epsilon)\phi(m(x_1, \ldots, x_s, y)k_1)$, for $k_1 \in K_{\mathbf{B}}, x_i \in \mathrm{GL}_{n_i}(\mathfrak{o}), (y, \epsilon) \in \widetilde{K}_{\mathbf{B}_1}$.

One can check that $f \in (\widetilde{I}(\widetilde{\tau})|_{K_{\mathbf{B}}})^{K'_{\mathbf{B}}}$ is well-defined and $\phi = \phi_f$. □

2.4. **A multiplicity one result.**

Proposition 2.3. *Let k be a finite field of odd characteristic. Let $P(k) = M(k)N(k)$ be a parabolic subgroup of $\mathrm{Sp}_{2r}(k)$. Let π_0 be an irreducible cuspidal representation of $\mathrm{Sp}_{2r}(k)$ and τ_0 be an irreducible cuspidal of $M(k)$. Then we have*

$$\dim \mathrm{Hom}_{\mathrm{Sp}_{2r}(k)}(\pi_0, I(\tau_0) \otimes \overline{\omega}_{\overline{\psi}}) \leq 1.$$

Proof. Let F be a p-adic field with residue field k. Let $P = MN$ be the corresponding Levi subgroup of $\mathrm{Sp}_{2r}(F)$. Let $\widetilde{\tau}$ be an irreducible depth zero supercuspidal representation of $\widetilde{M}$ as constructed in Section 2.3 from τ_0. Consider the induced representation $\widetilde{I}(\widetilde{\tau})$, which is irreducible for a generic choice of $\widetilde{\tau}$. Through the projection map, $K_{\mathbf{B}} \to K_{\mathbf{B}}/K'_{\mathbf{B}} \cong \mathrm{Sp}_{2r}(k)$, we can inflate π_0 to a representation of $K_{\mathbf{B}}$, which is still denoted by π_0. Let $\pi = \mathrm{ind}_{K_{\mathbf{B}}}^{\mathrm{Sp}_{2r}(F)}(\pi_0)$. It is known that π is an irreducible supercuspidal representation of $\mathrm{Sp}_{2r}(F)$. We have $\dim \mathrm{Hom}_{\mathrm{Sp}_{2r}(F)}(\pi, \widetilde{I}(\widetilde{\tau}) \otimes \omega_\psi) \leq 1$ by the main theorem of [Su12]. By Frobenius reciprocity law, we have

$$\begin{aligned}\mathrm{Hom}_{\mathrm{Sp}_{2r}(F)}(\pi, \widetilde{I}(\widetilde{\tau}) \otimes \omega_\psi) &= \mathrm{Hom}_{K_{\mathbf{B}}}(\pi_0, \widetilde{I}(\widetilde{\tau})|_{K_{\mathbf{B}}} \otimes \omega_\psi|_{K_{\mathbf{B}}}) \\ &\supset \mathrm{Hom}_{K_{\mathbf{B}}}(\pi_0, I(\tau_0) \otimes \omega_\psi|_{K_{\mathbf{B}}}) \\ &= \mathrm{Hom}_{K_{\mathbf{B}}}(\pi_0 \otimes I(\tau_0)^\vee, \omega_\psi|_{K_{\mathbf{B}}}) \\ &\supset \mathrm{Hom}_{\mathrm{Sp}_{2r}(k)}(\pi_0 \otimes I(\tau_0)^\vee, (\omega_\psi|_{K_{\mathbf{B}}})^{K'_{\mathbf{B}}}) \\ &= \mathrm{Hom}_{\mathrm{Sp}_{2r}(k)}(\pi_0 \otimes I(\tau_0)^\vee, \overline{\omega}_{\overline{\psi}}) \\ &= \mathrm{Hom}_{\mathrm{Sp}_{2r}(k)}(\pi_0, I(\tau_0) \otimes \overline{\omega}_{\overline{\psi}}),\end{aligned}$$

where the second containment is implied by Proposition 2.2. Hence, $\dim \mathrm{Hom}_{\mathrm{Sp}_{2r}(k)}(\pi_0, I(\tau_0) \otimes \overline{\omega}_{\overline{\psi}}) \leq 1$. This completes the proof of the proposition. □

2.5. **Fourier-Jacobi models.** Let n be a positive integer with $n \leq r-1$. Let $P(k) = M(k)N(k)$ be the parabolic subgroup of $\mathrm{Sp}_{2r}(k)$ with Levi subgroup

$$M(k) = \left\{\mathrm{diag}(a_1, \dots, a_{r-n}, g, a_{r-n}^{-1}, \dots, a_1^{-1}), a_i \in k^\times, g \in \mathrm{Sp}_{2n}(k)\right\} \cong \mathrm{GL}_1(k)^{r-n} \times \mathrm{Sp}_{2n}(k)$$

and unipotent subgroup $N(k)$. We view $\mathrm{Sp}_{2n}(k)$ as a subgroup of $\mathrm{Sp}_{2r}(k)$ via the embedding $\mathrm{Sp}_{2n}(k) \hookrightarrow M(k) \hookrightarrow \mathrm{Sp}_{2r}(k)$. Let

$$H(k) = \mathrm{Sp}_{2n}(k) \ltimes N(k) \subset P(k).$$

Let

$$\mathcal{H}_n = \left\{ \begin{pmatrix} 1 & w & z \\ & I_{2n} & w^* \\ & & 1 \end{pmatrix} \in \mathrm{Sp}_{2n+2}(k), w \in k^n, z \in k \right\}.$$

The group $\mathcal{H}_n$ can be viewed as the Heisenberg group of dimension $2n+1$, i.e., $\mathcal{H}_n \cong \mathcal{H}(k^{2n})$, where k^{2n} is endowed with the symplectic form defined by $\begin{pmatrix} & J_n \\ -J_n & \end{pmatrix}$. Then there is a Weil representation $\overline{\omega}_{\overline{\psi}}$ of the semi-direct product $\mathrm{Sp}_{2n}(k) \ltimes \mathcal{H}_n$. Note that $\mathrm{Sp}_{2n}(k) \ltimes \mathcal{H}_n$ can be viewed as a subgroup of $H(k)$ via the natural embedding. We now consider the representation $\nu_{\overline{\psi}}$ of $H(k)$ defined by

$$\nu_{\overline{\psi}}(uhg) = \overline{\psi}_N(u)\overline{\omega}_{\overline{\psi}}(hg),$$

where $h \in \mathcal{H}_n, g \in \mathrm{Sp}_{2n}(k)$,

$$u = \begin{pmatrix} z & v_1 & v_2 \\ & I_{2n+2} & v_1^* \\ & & z^* \end{pmatrix} \in N(k),$$

and $\overline{\psi}_N(u) = \overline{\psi}(\sum_{i=1}^{r-n-1} u_{i,i+1})$. Note that $\mathrm{Sp}_{2n}(k)$ is a quotient of $H(k)$ and thus a representation σ of $\mathrm{Sp}_{2n}(k)$ can be viewed as a representation of $H(k)$. Let σ be a representation of $\mathrm{Sp}_{2n}(k)$, then the tensor product $\sigma \otimes \nu_{\overline{\psi}}$ is also a representation of $H(k)$.

Let $P_n(k) = M_n(k)N_n(k)$ be the Siegel parabolic subgroup of $\mathrm{Sp}_{2n}(k)$ with Levi $M_n(k) \cong \mathrm{GL}_n(k)$. Let $P'(k) = M'(k)N'(k)$ be the parabolic subgroup of $\mathrm{Sp}_{2r}(k)$ with Levi subgroup

$$M'(k) = \left\{ \begin{pmatrix} a & & \\ & g & \\ & & a^* \end{pmatrix}, a \in \mathrm{GL}_{r-n}(k), g \in \mathrm{Sp}_{2n}(k) \right\}.$$

Proposition 2.4. *Let τ_0 be an irreducible representation of $\mathrm{GL}_n(k)$ and let $I(\tau_0)$ be the induced representation $\mathrm{Ind}_{M_n(k)N_n(k)}^{\mathrm{Sp}_{2n}(k)}(\tau_0)$. Let τ_0' be an irreducible cuspidal representation of $\mathrm{GL}_{r-n}(k)$ and let $I(\tau_0', \tau_0)$ be the induced representation*

$$\mathrm{Ind}_{M'(k)N'(k)}^{\mathrm{Sp}_{2r}(k)}(\tau_0' \otimes I(\tau_0)).$$

Let π_0 be an irreducible cuspidal representation of $\mathrm{Sp}_{2r}(k)$, then we have

(1) $\mathrm{Hom}_{\mathrm{Sp}_{2r}(k)}(\pi_0, I(\tau_0', \tau_0) \otimes \overline{\omega}_{\overline{\psi}}^r) = \mathrm{Hom}_{H(k)}(\pi_0, I(\tau_0) \otimes \nu_{\overline{\psi}})$, *where $\omega_{\overline{\psi}}^r$ denotes the Weil representation on $\mathrm{Sp}_{2r}(k)$;*
(2) $\dim \mathrm{Hom}_{H(k)}(\pi_0, I(\tau_0) \otimes \nu_{\overline{\psi}}) \leq 1$.

Proof. The proof of (1) is the same as that of [GGP12a, Theorem 16.1].

Note that there exists a parabolic subgroup $Q(k) = L(k)V(k)$ of $\mathrm{Sp}_{2r}(k)$ with Levi $L(k)$ and a cuspidal representation σ_0 of $L(k)$ such that $I(\tau_0', \tau_0)$ is a subrepresentation of $\mathrm{Ind}_{Q(k)}^{\mathrm{Sp}_{2r}(k)}(\sigma_0 \otimes 1_{V(k)})$. The assertion of (2) then follows from (1) and Proposition 2.3. □

For finite unitary groups, a slightly different version of Proposition 2.4 (1) was proved [LW19, Proposition 3.3], where τ_0' is not necessarily cuspidal, while π_0 is required to be unipotent. For unipotent representations of finite unitary groups, an explicit branching law was given in [LW19].

3. Gamma factors for $\mathrm{Sp}_{2r}(k) \times \mathrm{GL}_n(k)$

3.1. **Generic representations and Bessel functions.** In this subsection, we introduce the notion of Bessel functions for generic representations of $\mathrm{Sp}_{2r}(k)$. In [PS83], Bessel functions was used to study representations of $\mathrm{GL}_2(k)$ over a finite field k as an analogy of representations of GL_2 over p-adic fields. See [Co14] for a nice survey. Many constructions were extended to GL_n in [Ro10].

Let $U = U^r(k)$ be the upper triangular unipotent subgroup of $\mathrm{Sp}_{2r}(k)$. Let $\overline{\psi}_U$ be the generic character of U defined by

$$\overline{\psi}_U(u) = \overline{\psi}\left(\sum_{i=1}^{r} u_{i,i+1}\right), u = (u_{i,j}) \in U.$$

Let π be an irreducible $\overline{\psi}_U$-generic representation of $\mathrm{Sp}_{2r}(k)$. Recall that π is $\overline{\psi}_U$-generic means that $\mathrm{Hom}_U(\pi, \overline{\psi}_U) \neq 0$. A nonzero element in $\mathrm{Hom}_U(\pi, \overline{\psi}_U)$ is called a Whittaker functional of π_0. It is well-known that Whittaker functional is unique up to scalers. Let $l = l_\pi \in \mathrm{Hom}_U(\pi, \overline{\psi}_U)$ be a nonzero Whittaker functional. For $v \in \pi$, let $W_v(g) = l(\pi(g)v), g \in \mathrm{Sp}_{2r}(k)$. Let $\mathcal{W}(\pi, \overline{\psi}_U) = \{W_v, v \in \pi\}$, which is called be the $\overline{\psi}_U$-Whittaker model of π.

Let $\pi(U, \overline{\psi}_U)$ be the subspace of π generated by $\pi(u)v - \overline{\psi}_U(u)v, u \in U, v \in \pi$ and let $\pi_{U,\overline{\psi}_U} = \pi/\pi(U, \overline{\psi}_U)$ be the twisted Jacquet module. Since π is irreducible $\overline{\psi}_U$-generic, we have $\dim \pi_{U,\overline{\psi}_U} = 1$. Let $v \in \pi$ and $v \notin \pi(U, \overline{\psi}_U)$ and consider the average

$$v_0 = \frac{1}{|U|} \sum_{u \in U} \overline{\psi}_U^{-1}(u)\pi(u)v.$$

Then $v_0 \neq 0$ by Jacquet-Langlands Lemma, see [BZ76, Lemma 2.33]. Thus $l(v_0) \neq 0$. Such a vector v_0 satisfies $\pi(u)v_0 = \overline{\psi}_U(u)v_0$ for all $u \in U$, and is called a Whittaker vector of π. Let $\mathcal{B}_{\pi,\overline{\psi}}(g) = \frac{1}{l(v_0)} l(\pi(g)v_0)$. Then $\mathcal{B}_{\pi,\overline{\psi}} \in \mathcal{W}(\pi, \overline{\psi}_U)$.

Proposition 3.1. *We have*

$$\mathcal{B}_{\pi,\overline{\psi}}(1) = 1$$

and

$$\mathcal{B}_{\pi,\overline{\psi}}(u_1 g u_2) = \overline{\psi}_U(u_1 u_2)\mathcal{B}_{\pi,\overline{\psi}}(g), \forall u_1, u_2 \in U, g \in \mathrm{Sp}_{2r}(k).$$

We call $\mathcal{B}_{\pi,\overline{\psi}}$ the (normalized) Bessel function of π associated with $\overline{\psi}_U$.

3.2. **Some notation.** Let $n \leq r$ be two positive integers and we view $\mathrm{Sp}_{2n}(k)$ as a subgroup of $\mathrm{Sp}_{2r}(k)$ via the embedding

$$g \mapsto \mathrm{diag}(I_{r-n}, g, I_{r-n}), g \in \mathrm{Sp}_{2n}(k).$$

An element of $\mathrm{Sp}_{2n}(k)$ will be viewed as an element of $\mathrm{Sp}_{2r}(k)$ under the above embedding. Let $P_r(k) = M_r(k)N_r(k)$ be the Siegel parabolic subgroup of $\mathrm{Sp}_{2r}(k)$ with Levi subgroup

$$M_r(k) = \{m_r(a) := \mathrm{diag}(a, a^*), a \in \mathrm{GL}_r(k)\},$$

and unipotent subgroup $N_r(k)$. Here $a^* = J_r {}^t a^{-1} J_r$. Similarly, we have the notations $P_n(k) = M_n(k)N_n(k)$, the Siegel parabolic subgroup of $\mathrm{Sp}_{2n}(k)$, and $m_n(a)$ for $a \in \mathrm{GL}_n(k)$. Under the embedding $\mathrm{Sp}_{2n}(k) \hookrightarrow \mathrm{Sp}_{2r}(k)$, we have

$$m_n(a) = m_r\begin{pmatrix} I_{r-n} & \\ & a \end{pmatrix}, a \in \mathrm{GL}_n(k).$$

Let $Q_n^r = L_n^r V_n^r$ be the parabolic subgroup of Sp_{2r} with Levi subgroup

$$L_n^r = \{m_r(\mathrm{diag}(a, a_{n+1}, \ldots, a_r)), a \in \mathrm{GL}_n, a_i \in \mathrm{GL}_1, n+1 \leq i \leq r\}.$$

Note that $w_{r-n,n}^{-1} M_n w_{r-n,n} \subset L_n^r, w_{r-n,n}^{-1} P_n w_{r-n,n} \subset Q_n^r$. In fact, we have

$$w_{n-r,n}^{-1} m_n(a) w_{n-r,n} = m_r \begin{pmatrix} a & \\ & I_{r-n} \end{pmatrix} \in L_n^r.$$

For $a \in \mathrm{GL}_n(F)$, we write

$$t_n(a) = m_r \begin{pmatrix} a & \\ & I_{r-n} \end{pmatrix} \in L_n^r.$$

Let $B^r(k) = A^r(k)U^r(k)$ be the upper triangular Borel subgroup of $\mathrm{Sp}_{2r}(k)$ with maximal torus

$$A^r(k) = \left\{\mathrm{diag}(a_1, a_2, \ldots, a_r, a_r^{-1}, \ldots, a_1^{-1}), a_i \in k^\times, 1 \leq i \leq r\right\},$$

and maximal unipotent U^r. In the following, once r is understood, we will omit r from the notation for simplicity. Thus, we will write $B = AU$ as the upper triangular Borel subgroup of Sp_{2r} with maximal torus A and maximal unipotent U.

For an integer i with $1 \leq i \leq r$, let α_i be the simple root defined by

$$\alpha_i(\mathrm{diag}(a_1, a_2, \ldots, a_r, a_r^{-1}, \ldots, a_1^{-1})) = a_i/a_{i+1}, 1 \leq i \leq r-1,$$

and

$$\alpha_r(\mathrm{diag}(a_1, a_2, \ldots, a_r, a_r^{-1}, \ldots, a_1^{-1})) = a_r^2.$$

Let $\Delta^r = \{\alpha_i, 1 \leq i \leq r\}$ be the set of simple roots. For a root β of Sp_{2r}, let $U_\beta \subset U$ be the root space of β and $\mathbf{x}_\beta : k \to U_\beta$ be a fixed isomorphism.

Recall that $\mathcal{H}(k^{2n})$ is the Heisenberg group of k^{2n} where k^{2n} is endowed with the symplectic form defined by $\begin{pmatrix} & J_n \\ -J_r & \end{pmatrix}$. We can embed $\mathcal{H}(k^{2n})$ into Sp_{2n+2} by

$$[(x,y),z] \mapsto \begin{pmatrix} 1 & x & y & z \\ & I_n & & J_n{}^t y \\ & & I_n & -J_n{}^t x \\ & & & 1 \end{pmatrix}, (x, y \in k^n, z \in k).$$

The image of $\mathcal{H}(k^{2n})$ in $\mathrm{Sp}_{2n+2}(k)$ is denoted by $\mathcal{H}_n$. Denote $X_n = \{[(x,0),0] : x \in k^n\}$ and $Y_n = \{[(0,y),0], y \in k^n\}$. For $1 \leq n < r$, we will identify X_n, Y_n with a subgroup of $\mathrm{Sp}_{2r}(k)$ under the above identification and the embedding $\mathrm{Sp}_{2n+2} \hookrightarrow \mathrm{Sp}_{2r}$. There is a Weil representation $\overline{\omega}_{\overline{\psi}}$ of the semi-direct product $\mathrm{Sp}_{2n}(k) \ltimes \mathcal{H}_n$ on the space $\mathcal{S}(k^n)$.

3.3. **Weyl elements supporting Bessel functions.** Let $W(\mathrm{Sp}_{2r})$ be the Weyl group of Sp_{2r}. For each $\alpha_i \in \Delta(\mathrm{Sp}_{2r})$, let w_{α_i} be the simple reflection determined by the root α_i. The group $W(\mathrm{Sp}_{2r})$ is generated by $w_{\alpha_i}, 1 \leq i \leq r$. Let $\mathrm{B}(\mathrm{Sp}_{2r})$ be the set of Weyl elements $w \in W(\mathrm{Sp}_{2r})$ such that for any $\alpha_i \in \Delta(\mathrm{Sp}_{2r})$, $w\alpha_i$ is either negative or simple.

Lemma 3.2. *Let π be an irreducible generic representation of Sp_{2r} with the Bessel function $\mathcal{B}_{\pi,\overline{\psi}}$. For $w \in W(\mathrm{Sp}_{2r})$ but $w \notin \mathrm{B}(\mathrm{Sp}_{2r})$, we have*

$$\mathcal{B}_{\pi,\overline{\psi}}(g) = 0, \forall g \in BwB.$$

Proof. By Proposition 3.1, it suffices to show that $\mathcal{B}_{\pi,\overline{\psi}}(tw) = 0$ for all $t \in A$. Since $w \notin \mathrm{B}(\mathrm{Sp}_{2r})$, there exists a simple root α_i such that $w\alpha_i$ is positive but not simple. Let $x \in k$ and let $\mathbf{x}_{\alpha_i}(x)$ be an element in the root space of α_i. We have $tw\mathbf{x}_{\alpha_i}(x) = \mathbf{x}_{w\alpha_i}(x')tw$ for some $x' \in k$. Since $\overline{\psi}_U(\mathbf{x}_{\alpha_i}(x)) = \overline{\psi}(x)$ and $\overline{\psi}_U(\mathbf{x}_{w\alpha_i}(x')) = 1$. By Proposition 3.1, we have

$$\overline{\psi}(x)\mathcal{B}_{\pi,\overline{\psi}}(tw) = \mathcal{B}_{\pi,\overline{\psi}}(tw), \forall x \in k.$$

Thus we get $\mathcal{B}_{\pi,\overline{\psi}}(tw) = 0$. □

We say that Weyl elements $w \in \mathrm{B}(\mathrm{Sp}_{2r})$ support the Bessel function $\mathcal{B}_{\pi,\overline{\psi}}$.

Given $w \in \mathrm{B}(\mathrm{Sp}_{2r})$, set

$$\theta_w = \{\beta \in \Delta(\mathrm{Sp}_{2r}) : w\beta > 0\}.$$

The assignment $w \mapsto \theta_w$ defines a bijection from $\mathrm{B}(\mathrm{Sp}_{2r})$ to $\mathcal{P}(\Delta(\mathrm{Sp}_{2r}))$, the power set of $\Delta(\mathrm{Sp}_{2r})$, i.e., the set of all subsets of $\Delta(\mathrm{Sp}_{2r})$. For a subset $\theta \subset \Delta(\mathrm{Sp}_{2r})$, let w_θ be the corresponding element in $\mathrm{B}(\mathrm{Sp}_{2r})$. For example, we have $w_{\Delta(\mathrm{Sp}_{2r})} = 1$ and $w_\emptyset = w_\ell$, where w_ℓ is the longest Weyl element.

Denote

$$w_{r-n,n} = \begin{pmatrix} & I_{r-n} & & \\ I_n & & & \\ & & & I_n \\ & & I_{r-n} & \end{pmatrix} \in \mathrm{Sp}_{2r}(k), w_n = \begin{pmatrix} & I_n \\ -I_n & \end{pmatrix} \in \mathrm{Sp}_{2n}(k) \hookrightarrow \mathrm{Sp}_{2r}(k).$$

Let

$$\widetilde{w}_n = w_{r-n,n}^{-1} w_n w_{r-n,n} = \begin{pmatrix} & & I_n \\ & I_{2(r-n)} & \\ -I_n & & \end{pmatrix}. \tag{3.1}$$

Lemma 3.3. *We have $\theta_{\widetilde{w}_n} = \Delta(\mathrm{Sp}_{2r}) - \{\alpha_n\}$.*

Proof. This follows from a simple calculation. □

For $1 \le n \le r$ and $a \in \mathrm{GL}_n(k)$, recall that $t_n(a) = m_r \begin{pmatrix} a & \\ & I_{r-n} \end{pmatrix}$ as in §3.2. We then embed $W(\mathrm{GL}_n)$, the Weyl group of $\mathrm{GL}_n(k)$, into $W(\mathrm{Sp}_{2r})$ via the embedding $a \mapsto t_n(a)$.

Let $\mathrm{B}_0(\mathrm{Sp}_{2r}) = \{1\}$. For $1 \le n \le r$, let $\mathrm{B}_n(\mathrm{Sp}_{2r})$ be the subset of $\mathrm{B}(\mathrm{Sp}_{2r})$ such that each element $w \in \mathrm{B}_n(\mathrm{Sp}_{2r})$ has a representative of the form $t_n(w')\widetilde{w}_n$ where w' is a representative of some Weyl element of GL_n. Let $\mathcal{P}_n = \{\theta \subset \Delta(\mathrm{Sp}_{2r}) | w_\theta \in \mathrm{B}_n(\mathrm{Sp}_{2r})\}$. By [Zh18, Lemma 4.5], we can check that

$$\mathcal{P}_n = \{\theta \subset \Delta(\mathrm{Sp}_{2r}) : \{\alpha_{n+1}, \alpha_{n+2}, \ldots, \alpha_r\} \subseteq \theta \subseteq \Delta(\mathrm{Sp}_{2r}) - \{\alpha_n\}\}. \tag{3.2}$$

Let $\mathcal{P}_0 = \{\Delta(\mathrm{Sp}_{2r})\}$. Then we have

$$\coprod_{n=0}^{r} \mathcal{P}_n = \mathcal{P}(\Delta(\mathrm{Sp}_{2r})),$$

which implies that

$$\coprod_{n=0}^{r} \mathrm{B}_n(\mathrm{Sp}_{2r}) = \mathrm{B}(\mathrm{Sp}_{2r}). \tag{3.3}$$

Corollary 3.4. *If $w \in W(\mathrm{GL}_r) \subset W(\mathrm{Sp}_{2r})$, $w \ne 1$, we have $m_r(w) \notin \mathrm{B}(\mathrm{Sp}_{2r})$. In particular, if $a \in \mathrm{GL}_r(k)$ which is not in the upper triangular subgroup of $\mathrm{GL}_r(k)$, we have $\mathcal{B}_{\pi,\overline{\psi}}(m_r(a)) = 0$.*

Proof. In fact, for $w \in W(\mathrm{GL}_r)$ and $w \ne 1$, if $m_r(w) = t_n(w')\widetilde{w}_n$ for some n with $1 \le n \le r$ and $w' \in W(\mathrm{GL}_n)$, we would have $\widetilde{w}_n \in W(\mathrm{GL}_r)$, which is impossible. Thus we get $w \notin \coprod_{n=0}^{r} \mathrm{B}_n(\mathrm{Sp}_{2r}) = \mathrm{B}(\mathrm{Sp}_{2r})$. The second assertion follows from Lemma 3.2. □

Lemma 3.5. *For $t \in A$, the maximal torus of $\mathrm{Sp}_{2r}(k)$, if $\mathcal{B}_{\pi,\overline{\psi}}(t) \ne 0$, then t is in $\{\pm I_{2r}\}$, the center of $\mathrm{Sp}_{2r}(k)$.*

Proof. Given any $x \in k$ and any $\beta \in \Delta(\mathrm{Sp}_{2r})$, we consider the element $\mathbf{x}_\beta(x) \in U$. We have $t\mathbf{x}_\beta(x) = \mathbf{x}_\beta(\beta(t)x)t$. Thus by Proposition 3.1, we have

$$\overline{\psi}(x)\mathcal{B}_{\pi,\overline{\psi}}(t) = \overline{\psi}(\beta(t)x)\mathcal{B}_{\pi,\overline{\psi}}(t).$$

Thus if $\mathcal{B}_{\pi,\overline{\psi}}(t) \ne 0$, we have $\overline{\psi}(x) = \overline{\psi}(\beta(t)x)$, for all $x \in k$ and all $\beta \in \Delta(\mathrm{Sp}_{2r})$. Since $\overline{\psi}$ is nontrivial and x is arbitrary, we get $\beta(t) = 1$ for all $\beta \in \Delta(\mathrm{Sp}_{2r})$. Now it is easy to see that $t \in \{\pm I_{2r}\}$. □

3.4. **Induced representation on** $\mathrm{Sp}_{2n}(k)$**.** Let τ be an irreducible generic representation of $\mathrm{GL}_n(k)$ and let $I(\tau)=\mathrm{Ind}_{M_n(k)N_n(k)}^{\mathrm{Sp}_{2n}(k)}\tau$. An element $\xi\in I(\tau)$ is a function $\xi:\mathrm{Sp}_{2n}(k)\to\tau$ such that

$$\xi(m_n(a)ug)=\tau(a)(\xi(g)),\forall a\in\mathrm{GL}_n(k),u\in N_n(k),g\in\mathrm{Sp}_{2n}(k).$$

Let $Z_n(k)$ be the upper triangular unipotent subgroup of $\mathrm{GL}_n(k)$ and let $\overline{\psi}_{Z_n}^{-1}$ be the character of $Z_n(k)$ defined by

$$\overline{\psi}_{Z_n}^{-1}(z)=\overline{\psi}^{-1}\left(\sum_{i=1}^{n-1}z_{i,i+1}\right),z=(z_{i,j})\in Z_n(k).$$

Let $\Lambda=\Lambda_\tau\in\mathrm{Hom}_{Z_n}(\tau,\overline{\psi}_{Z_n}^{-1})$ be a fixed nonzero Whittaker functional of τ. For $\xi\in I(\tau)$, let $f_\xi:\mathrm{Sp}_{2n}(k)\times\mathrm{GL}_n(k)\to\mathbb{C}$ be the function defined by

$$f_\xi(g,a)=\Lambda(\tau(a)\xi(g)).$$

Let $\mathcal{I}(\tau,\overline{\psi}_{Z_n}^{-1})$ be the space of functions $f_\xi,\xi\in I(\tau)$. Note that for $f\in\mathcal{I}(\tau,\overline{\psi}_{Z_n}^{-1}),z\in Z_n(k)$, we have

$$f(zg)=\overline{\psi}_{Z_n}^{-1}(z)f(g).$$

Let τ^* denote the representation of $\mathrm{GL}_n(k)$ defined by $\tau^*(a)=\tau(a^*)$, where $a^*=J_n{}^ta^{-1}J_n\in\mathrm{GL}_n(k)$. Note that τ^* is isomorphic to the contragredient representation of τ.

There is a (standard) intertwining operator $M(\tau,\overline{\psi}^{-1}):\mathcal{I}(\tau,\overline{\psi}_{Z_n}^{-1})\to\mathcal{I}(\tau^*,\overline{\psi}_{Z_n}^{-1})$ defined by

$$M(\tau,\overline{\psi}^{-1})f(g,a)=\sum_{u\in N_n(k)}f(w_n^{-1}ug,d_na^*),$$

where $d_n=\mathrm{diag}(-1,1,\ldots,(-1)^n)\in\mathrm{GL}_n(k)$.

3.5. **Zeta "integrals".** Let π be an irreducible $\overline{\psi}_U$-generic representation of $\mathrm{Sp}_{2r}(k)$, and τ be an irreducible generic cuspidal representation of $\mathrm{GL}_n(k)$. For $W\in\mathcal{W}(\pi,\overline{\psi}_U),f\in\mathcal{I}(\tau,\overline{\psi}_{Z_n}^{-1}),\phi\in\mathcal{S}(k^n)$, we consider the following

(3.4)

$$\Psi(W,\phi,f)$$
$$=\begin{cases}\sum_{g\in U^n\backslash\mathrm{Sp}_{2n}}\sum_{u\in R^{r,n}}\sum_{x\in X_n}W(w_{r-n,n}^{-1}(uxg)w_{r-n,n})(\overline{\omega}_{\overline{\psi}^{-1}}(g)\phi)(x)f(g,I_n), & 1\le n<r,\\ \sum_{g\in U^r\backslash\mathrm{Sp}_{2r}}W(g)(\overline{\omega}_{\overline{\psi}^{-1}}(g)\phi)(e_r)f(g,I_r)dg, & n=r.\end{cases}$$

Here

$$R^{r,n}=\left\{m_r\begin{pmatrix}I_{r-n-1}&&y\\&1&\\&&I_n\end{pmatrix}\in\mathrm{Sp}_{2r}(k)\right\},$$

and $e_r=(0,\ldots,0,1)\in k^r$. Over local fields, these integrals have a rich history and were defined in [GePS87] when $n=r$, [GiRS97] when $n=1$ and [GiRS98] in the general case. See [Ka15] for a nice survey. Over finite fields, these integrals are just finite sums.

3.6. **Non-vanishing of zeta integrals.** The purpose of this subsection is to show that we can choose datum W,ϕ,f such that the zeta integral $\Psi(W,\phi,f)$ is non-vanishing.

We first construct some section of the induced representation of $\mathrm{Sp}_{2r}(k)$. Let τ be an irreducible cuspidal generic representation of $\mathrm{GL}_n(k)$. For a vector $v\in\tau$, we consider the element $\xi_v\in I(\tau)$ defined by $\mathrm{supp}(\xi_v)=M_n(k)N_n(k)$ and

$$\xi_v(m_n(a)u)=\tau(a)v,\forall a\in\mathrm{GL}_n(k),u\in N_n(k).$$

Let $f_v=f_{\xi_v}$ be the corresponding $\mathbb{C}$-valued function in $\mathcal{I}(\tau,\overline{\psi}^{-1})$, i.e., $f_v(g,a)=\Lambda(\tau(a)\xi_v(g))$. Recall that Λ is the fixed nonzero element in $\mathrm{Hom}_{Z_n}(\tau,\overline{\psi}_{Z_n}^{-1})$. Let $\widetilde{f}_v=M(\tau,\overline{\psi}^{-1})f_v$. Let $W_v(a)=\Lambda(\tau(a)v)$ and $W_v^*(a)=\Lambda(\tau(d_na^*)v)$.

Lemma 3.6. *$\widetilde{f}_v$ has the following properties.*

(1) *If $\widetilde{f}_v(g,I_n)\neq0$ for $g\in\mathrm{Sp}_{2n}(k)$, then $g\in P_n(k)w_nP_n(k)=P_n(k)w_nN_n(k)$.*

(2) *If* $x \in N_n(k)$*, we have* $\widetilde{f}_v(w_n x) = W_v^*(1)$.
(3) *For* $a \in \mathrm{GL}_n(k), x \in N_n(k)$*, we have* $\widetilde{f}_v(m_n(a)w_n x, I_n) = W_v^*(a)$.

Proof. (1) We have

$$\begin{aligned}\widetilde{f}_v(g, I_n) &= \sum_{u \in N_n(k)} f_v(w_n^{-1}ug, d_n) \\ &= \sum_{u \in N_n(k)} f_v(d_n w_n^{-1} ug, I_n).\end{aligned}$$

Note that if $f_v(d_n w_n^{-1}ug, I_n) \neq 0$ for some $u \in N_n(k)$, we must have $d_n w_n^{-1}ug \in P_n$. Thus $g \in uw_nP_n \subset P_n(k)w_nP_n(k)$.

(2) For $x \in N_n(k)$, we have

$$\widetilde{f}_v(w_n x, I_n) = \sum_{u \in N_n(k)} f_v(d_n w_n^{-1} u w_n x, I_n).$$

Note that $w_n^{-1}uw_nx \in P_n(k)$ if and only if $u = I_{2n}$. Thus we get

$$\widetilde{f}_v(w_n x, I_n) = f_v(d_n x, I_n) = W_v^*(1).$$

(3) This directly follows from a similar calculation as (2). □

Proposition 3.7. *Let* π *be an irreducible* $\overline{\psi}_U$*-generic cuspidal representation of* $\mathrm{Sp}_{2r}(k)$ *and* τ *be an irreducible generic representation of* $\mathrm{GL}_n(k)$ *with* $1 \leq n \leq r$*. Let* $\mathcal{B}_{\pi,\overline{\psi}} \in \mathcal{W}(\pi, \overline{\psi}_U)$ *be the Bessel function,* $\delta_0 \in \mathcal{S}(k^n)$ *be the characteristic function of* $0 = (0, \ldots, 0) \in k^n$*,* $\delta_1^r \in \mathcal{S}(k^r)$ *be the characteristic function of* $e_r = (0, \ldots, 0, 1) \in k^r$ *and* $f_v \in \mathcal{I}(\tau, \overline{\psi}^{-1})$ *be the function constructed as above for* $v \in \tau$*. We have*

$$\Psi(\mathcal{B}_{\pi,\overline{\psi}}, \delta_0, f_v) = W_v(1), 1 \leq n < r,$$

and

$$\Psi(\mathcal{B}_{\pi,\overline{\psi}}, \delta_1^r, f_v) = W_v(1), n = r.$$

In particular, there exist choices of $v \in \tau$ *such that*

$$\Psi(\mathcal{B}_{\pi,\overline{\psi}}, \delta_0, f_v) \neq 0, \Psi(\mathcal{B}_{\pi,\overline{\psi}}, \delta_1^r, f_v) \neq 0.$$

Proof. We first consider the case $1 \leq n < r$. We compute the integral $\Psi(\mathcal{B}_{\pi,\overline{\psi}}, \delta_0, f_v)$ for general v. By definition, we have

$$\Psi(\mathcal{B}_{\pi,\overline{\psi}}, \delta_0, f_v) = \sum_{g \in U^n \backslash \mathrm{Sp}_{2n}} \sum_{u \in R^{r,n}} \sum_{x \in X_n} \mathcal{B}_{\pi,\overline{\psi}}(w_{r-n,n}^{-1}(uxg)w_{r-n,n}) \overline{\omega}_{\overline{\psi}^{-1}}(g)\delta_0(x) f_v(g, I_n).$$

From the definition of f_v, we have $f_v(g, I_n) \neq 0$ if and only if $g \in M_n(k)N_n(k)$. If $g \in M_n(k)N_n(k)$, we write $g = m_n(a)u'$ for $a \in \mathrm{GL}_n(k), u' \in N_n(k)$. We have $f_v(m_n(a)u', I_n) = W_v(a)$. Thus we get

$$\Psi(\mathcal{B}_{\pi,\overline{\psi}}, \delta_0, f_v) = \sum_{a \in Z_n \backslash \mathrm{GL}_n(k)} \sum_{u \in R^{r,n}} \sum_{x \in X_n} \mathcal{B}_{\pi,\overline{\psi}}(w_{r-n,n}^{-1}(uxm_n(a))w_{r-n,n}) \overline{\omega}_{\overline{\psi}^{-1}}(m_n(a))\delta_0(x) W_v(a).$$

By the Weil representation formula, see [GH17, p.219] for example, we have

$$\omega_{\overline{\psi}^{-1}}(m_n(a))\delta_0(x) = \epsilon(\det(a))\delta_0(xa),$$

where the right side is nonzero if and only if $x = 0$. Here ϵ is the unique nontrivial quadratic character of $k^\times$. Thus we get

$$\Psi(\mathcal{B}_{\pi,\overline{\psi}}, \delta_0, f_v) = \sum_{a \in Z_n \backslash \mathrm{GL}_n(k)} \sum_{u \in R^{r,n}} \mathcal{B}_{\pi,\overline{\psi}}(w_{r-n,n}^{-1}(um_n(a))w_{r-n,n})\epsilon(\det(a)) W_v(a).$$

Write $u \in R^{r,n}$ as

$$u = m_r \begin{pmatrix} I_{r-n-1} & & y \\ & 1 & \\ & & I_n \end{pmatrix}.$$

A simple matrix calculation shows that

$$w_{r-n,n}^{-1}um_n(a)w_{r-n,n} = m_r \begin{pmatrix} a & & \\ & 1 & \\ ya & & I_{r-n-1} \end{pmatrix}.$$

By Corollary 3.4 and Lemma 3.5, if $B_{\pi,\overline{\psi}}(w_{r-n,n}^{-1}um_n(a)w_{r-n,n}) \neq 0$, then $a \in Z_n$ and $y = 0$. Thus we get

$$\Psi(\mathcal{B}_{\pi,\overline{\psi}}, \delta_0, f_v) = W_v(1). \tag{3.5}$$

We can choose $v \in \tau$ such that $W_v(1) \neq 0$. This proves the conclusion when $1 \leq n < r$.

We next consider the case $n = r$. By the definition of f_v, we have

$$\Psi(\mathcal{B}_{\pi,\overline{\psi}}, \delta_1^r, f_v) = \sum_{g \in Z_r \backslash \mathrm{GL}_r(k)} \mathcal{B}_{\pi,\overline{\psi}}(m_r(a))\overline{\omega}_{\overline{\psi}^{-1}}(m_r(a))\delta_1^r(e_r)W_v(a).$$

By Corollary 3.4, if a is not in the upper triangular subgroup of $\mathrm{GL}_r(k)$, we have $\mathcal{B}_{\pi,\overline{\psi}}(m_r(a)) = 0$. For a diagonal element t in $\mathrm{GL}_r(k)$, if $t \neq \pm I_r$, we have $\mathcal{B}_{\pi,\overline{\psi}}(m_r(t)) = 0$ by Lemma 3.5. If $t = -I_n$, we get

$$\overline{\omega}_{\overline{\psi}^{-1}}(-I_n)\delta_1^r(e_r) = 0$$

by the definition of δ_1^r. Thus we get

$$\Psi(\mathcal{B}_{\pi,\overline{\psi}}, \delta_1^r, f_v) = W_v(1). \tag{3.6}$$

It is clear that we can choose v such that $W_v(1) \neq 0$. □

3.7. **The gamma factors.**

Proposition 3.8. *Let π be an irreducible $\overline{\psi}_U$-generic representation of $\mathrm{Sp}_{2r}(k)$ and τ be an irreducible generic cuspidal representation of $\mathrm{GL}_n(k)$ with $1 \leq n \leq r$, then there exists a number $\gamma(\pi \times \tau, \overline{\psi})$ such that*

$$\Psi(W, \phi, M(\tau, \overline{\psi}^{-1})f) = \gamma(\pi \times \tau, \overline{\psi})\Psi(W, \phi, f),$$

for all $W \in \mathcal{W}(\pi, \overline{\psi}_U), f \in \mathcal{I}(\tau, \overline{\psi}_{Z_n}^{-1}), \phi \in \mathcal{S}(k^n)$.

Proof. If $1 \leq n < r$, we can check that $(W, \phi, f) \mapsto \Psi(W, \phi, f)$ and $(W, \phi, f) \mapsto \Psi(W, \phi, M(\tau, \overline{\psi}^{-1})f)$ define two elements in the space $\mathrm{Hom}_{H(k)}(\pi \otimes I(\tau) \otimes \nu_{\overline{\psi}}, \mathbb{C})$, which has dimension at most one by Proposition 2.4. Thus by the non-vanishing result of $\Psi(W, \phi, f)$, Proposition 3.7, there exists a number $\gamma(\pi \times \tau, \overline{\psi})$ such that

$$\Psi(W, \phi, M(\tau, \overline{\psi}^{-1})f) = \gamma(\pi \times \tau, \overline{\psi})\Psi(W, \phi, f),$$

for all $W \in \mathcal{W}(\pi, \overline{\psi}_U), f \in \mathcal{I}(\tau, \overline{\psi}_{Z_n}^{-1}), \phi \in \mathcal{S}(k^n)$. When $n = r$, this follows from Proposition 2.3. □

Remark 3.9. Let π be an irreducible generic representation of $\mathrm{Sp}_{2r}(k)$ and τ be an irreducible generic representation of $\mathrm{GL}_n(k)$ as usual. If $n > r$, following [GiRS98], we can still define local zeta integrals $\Psi(W, \phi, f)$ for $W \in \mathcal{W}(\pi, \overline{\psi}_U), f \in \mathcal{I}(\tau, \overline{\psi}_{Z_n}^{-1}), \phi \in \mathcal{S}(k^n)$ and hence gamma factors $\gamma(\pi \times \tau, \overline{\psi})$. However, in Theorem 4.1 next section, we will prove that the set

$$\left\{\gamma(\pi \times \tau, \overline{\psi}) : \tau \text{ irreducible generic representation of } \mathrm{GL}_n(k), 1 \leq n \leq r\right\}$$

will uniquely determine an irreducible generic cuspidal representation π. Therefore, in this paper, we do not include the gamma factors $\gamma(\pi \times \tau, \overline{\psi})$ when $n > r$.

4. A converse theorem for Sp_{2r}

In this section, we still let k be a finite field with odd characteristic. The purpose of this section is to prove the following

Theorem 4.1. *Let π and π' be two irreducible $\overline{\psi}_U$-generic cuspidal representations of $\mathrm{Sp}_{2r}(k)$ with the same central character. If*

$$\gamma(\pi\times\tau,\overline{\psi})=\gamma(\pi'\times\tau,\overline{\psi})$$

for all irreducible generic representations τ of $\mathrm{GL}_n(k)$ and for all n with $1\le n\le r$, then $\pi\cong\pi'$.

Notice that in the above theorem π and π' are assumed to be generic with respect to the same generic character. A p-adic version of Theorem 4.1 was proven in [Zh18].

4.1. An auxiliary lemma.

Lemma 4.2 ([Ni14, Lemma 3.1])**.** *Let H be a function on $\mathrm{GL}_n(k)$ such that*

$$H(ug)=\overline{\psi}_{Z_n}(u)H(g),\forall u\in Z_n, g\in \mathrm{GL}_n(k).$$

If

$$\sum_{g\in Z_n\backslash \mathrm{GL}_n(k)} H(g)W_\tau(g)=0,$$

for all $W_\tau\in\mathcal{W}(\tau,\overline{\psi}_{Z_n}^{-1})$ and all irreducible generic representations τ of $\mathrm{GL}_n(k)$, then $H(g)\equiv 0$.

4.2. **Proof of Theorem 4.1.** In the following, we fix two irreducible generic cuspidal representations π,π' with the same central character $\omega_\pi=\omega_{\pi'}$. Recall that $\mathcal{B}_{\pi,\overline{\psi}}$ and $\mathcal{B}_{\pi',\overline{\psi}}$ are the Bessel functions of π and π', respectively, and $B=AU$ is the upper triangular Borel subgroup with torus A and maximal unipotent U.

Theorem 4.3. *Let n be an integer with $1\le n\le r$. If $\gamma(\pi\times\tau,\overline{\psi})=\gamma(\pi'\times\tau,\overline{\psi})$ for all irreducible generic representations τ of $\mathrm{GL}_n(k)$, then*

$$\mathcal{B}_{\pi,\overline{\psi}}(t_n(a)\widetilde{w}_n)=\mathcal{B}_{\pi',\overline{\psi}}(t_n(a)\widetilde{w}_n),\forall a\in \mathrm{GL}_n(k).$$

Recall that $t_n(a)=m_r\begin{pmatrix}a & \\ & I_{r-n}\end{pmatrix}$ as in §3.2, and $\widetilde{w}_n$ is defined in (3.1).

Proof. We first assume that $1\le n<r$.

Recall from Proposition 3.7 that for $f_v\in\mathcal{I}(\tau,\overline{\psi}^{-1})$ as in §3.6, and $\delta_0\in\mathcal{S}(k^n)$ the characteristic function of $0=(0,\dots,0)\in k^n$, we have

$$\Psi(\mathcal{B}_{\pi,\overline{\psi}},\delta_0,f_v)=W_v(1)=\Psi(\mathcal{B}_{\pi',\overline{\psi}},\delta_0,f_v).$$

By the assumption on the gamma factors and the functional equation, we can get that

$$\Psi(\mathcal{B}_{\pi,\overline{\psi}},\delta_0,\widetilde{f}_v)=\Psi(\mathcal{B}_{\pi',\overline{\psi}},\delta_0,\widetilde{f}_v), \tag{4.1}$$

where $\widetilde{f}_v=M(\tau,\overline{\psi}^{-1})f_v$. On the other hand, we have

$$\Psi(\mathcal{B}_{\pi,\overline{\psi}},\delta_0,\widetilde{f}_v)=\sum_{g\in U^n\backslash \mathrm{Sp}_{2n}}\sum_{u\in R^{r,n}}\sum_{x\in X_n}\mathcal{B}_{\pi,\overline{\psi}}(w_{r-n,n}^{-1}(uxg)w_{r-n,n})\overline{\omega}_{\overline{\psi}^{-1}}(g)\delta_0(x)\widetilde{f}_v(g,I_n).$$

By Lemma 3.6, if $g\notin P_n(k)w_nN_n(k)$, we have $\widetilde{f}_v(g,I_n)=0$. If $g\in P_n(k)w_nN_n(k)$, we can write $g=u_2m_n(a)w_nu_1$ with $u_1,u_2\in N_n(k)$ and $a\in \mathrm{GL}_n(k)$. We have $\widetilde{f}_v(g,I_n)=W_v^*(a)$. Thus

$$\begin{aligned}\Psi(\mathcal{B}_{\pi,\overline{\psi}},\delta_0,\widetilde{f}_v)=&\sum_{a\in Z_n\backslash \mathrm{GL}_n(k)}\sum_{u_1\in N_n(k)}\sum_{u\in R^{r,n}}\sum_{x\in X_n}\mathcal{B}_{\pi,\overline{\psi}}(w_{r-n,n}^{-1}(uxm_n(a)w_nu_1)w_{r-n,n})\\&\cdot\overline{\omega}_{\overline{\psi}^{-1}}(m_n(a)w_nu_1)\delta_0(x)W_v^*(a).\end{aligned}$$

It is easy to check that $\overline{\omega}_{\overline{\psi}^{-1}}(u_1)\delta_0=\delta_0$ for $u_1\in N_n(k)$, and we have

$$w_{r-n,n}^{-1}u_1w_{r-n,n}=\begin{pmatrix}I_n & & u_1'\\ & I_{2(r-n)} & \\ & & I_n\end{pmatrix}\in N_r(k)\subset U.$$

By Proposition 3.1, we get

$$\mathcal{B}_{\pi,\overline{\psi}}(w_{r-n,n}^{-1}(uxm_n(a)w_nu_1)w_{r-n,n}) = \mathcal{B}_{\pi,\overline{\psi}}(w_{r-n,n}^{-1}(uxm_n(a)w_n)w_{r-n,n}).$$

Thus we get

$$\Psi(\mathcal{B}_{\pi,\overline{\psi}}, \delta_0, \widetilde{f}_v) = |N_n(k)| \sum_{a\in Z_n\backslash \mathrm{GL}_n(k)} \sum_{u\in R^{r,n}} \sum_{x\in X_n} \mathcal{B}_{\pi,\overline{\psi}}(w_{r-n,n}^{-1}(uxm_n(a)w_n)w_{r-n,n}) \cdot \overline{\omega}_{\overline{\psi}^{-1}}(m_n(a)w_n)\delta_0(x)\widetilde{W}_v^*(a).$$

Note that for $u \in R^{r,n}, x \in X_n$, the element ux is of the form

$$m_r\left(\begin{pmatrix} I_{r-n-1} & & y \\ & 1 & x \\ & & I_n \end{pmatrix}\right).$$

with $y \in \mathrm{Mat}_{(r-n-1)\times n}(k), x \in \mathrm{Mat}_{1\times n}(k)$. For simplicity, we write

$$u(y,x) = \left(\begin{pmatrix} I_{r-n-1} & & y \\ & 1 & x \\ & & I_n \end{pmatrix}\right).$$

By conjugation, we have

$$uxm_n(a) = m_n(a)u(ya, xa).$$

Note that $t_n(a) = w_{r-n,n}^{-1}m_n(a)w_{r-n,n}$, and $\widetilde{w}_n = w_{r-n,n}^{-1}w_nw_{r-n,n}$. We have

$$w_{r-n,n}^{-1}(uxm_n(a)w_n)w_{r-n,n} = t_n(a)w_{r-n,n}^{-1}u(ya,xa)w_nw_{r-n,n} = t_n(a)\widetilde{w}_nu'(ya,xa),$$

where $u'(ya,xa) = w_{r-n,n}^{-1}w_n^{-1}u(ya,xa)w_nw_{r-n,n}$. Plugging these calculations into the expression of $\Psi(\mathcal{B}_{\pi,\overline{\psi}}, \delta_0, \widetilde{f}_v)$, we get

$$\Psi(\mathcal{B}_{\pi,\overline{\psi}}, \delta_0, \widetilde{f}_v) = |N_n(k)| \sum_{a\in Z_n\backslash \mathrm{GL}_n(k)} \sum_{y\in \mathrm{Mat}_{(r-n-1)\times n}(k)} \sum_{x\in \mathrm{Mat}_{1\times n}(k)} \mathcal{B}_{\pi,\overline{\psi}}(t_n(a)\widetilde{w}_nu'(ya,xa)) \cdot \overline{\omega}_{\overline{\psi}^{-1}}(m_n(a)w_n)\delta_0(x)W_v^*(a).$$

Note that

$$\overline{\omega}_{\overline{\psi}^{-1}}(m_n(a)w_n)\delta_0(x) = \epsilon(\det(a))\overline{\omega}_{\overline{\psi}^{-1}}(w_n)\delta_0(xa).$$

By changing variable, we can get

$$\Psi(\mathcal{B}_{\pi,\overline{\psi}}, \delta_0, \widetilde{f}_v) = |N_n(k)| \sum_{a\in Z_n\backslash \mathrm{GL}_n(k)} \sum_{y\in \mathrm{Mat}_{(r-n-1)\times n}(k)} \sum_{x\in \mathrm{Mat}_{1\times n}(k)} \mathcal{B}_{\pi,\overline{\psi}}(t_n(a)\widetilde{w}_nu'(y,x)) \cdot \epsilon(\det(a))\overline{\omega}_{\overline{\psi}^{-1}}(w_n)\delta_0(x)W_v^*(a).$$

We can check that

$$u'(y,x) = \begin{pmatrix} I_n & & & J_n{}^tx & J_n{}^tyJ_{r-n-1} & \\ & I_{r-n-1} & & & & y \\ & & 1 & & & x \\ & & & 1 & & \\ & & & & I_{r-n-1} & \\ & & & & & I_n \end{pmatrix} \in \mathrm{Sp}_{2r}(k).$$

In particular, we have $u'(y,x) \in U$. By Proposition 3.1, we have

$$\mathcal{B}_{\pi,\overline{\psi}}(t_n(a)\widetilde{w}_nu'(y,x)) = \mathcal{B}_{\pi,\overline{\psi}}(t_n(a)\widetilde{w}_n).$$

Thus we get

$$\Psi(\mathcal{B}_{\pi,\overline{\psi}}, \delta_0, \widetilde{f}_v) = C_k \sum_{a\in Z_n\backslash \mathrm{GL}_n(k)} \mathcal{B}_{\pi,\overline{\psi}}(t_n(a)\widetilde{w}_n)\left(\sum_{x\in X_n}\overline{\omega}_{\overline{\psi}^{-1}}(w_n)\delta_0(x)\right)\epsilon(\det(a))W_v^*(a), \tag{4.2}$$

where $C_k = |N_n(k)| \cdot |\mathrm{Mat}_{(r-n-1)\times n}(k)|$ is a nonzero constant related to k (and also r, n).

By the Weil representation formula, see [GH17, p.220] or [LZ19, p.76], we have

$$\overline{\omega}_{\overline{\psi}^{-1}}(w_n)\delta_0(x) = \frac{1}{\gamma(I_n, \overline{\psi}^{-1})},$$

where $\gamma(I_n, \overline{\psi}^{-1}) = \sum_{x\in k^n} \overline{\psi}(xJ_n {}^t x)$, which is nonzero. Thus we get

$$\sum_{x\in X_n} \overline{\omega}_{\overline{\psi}^{-1}}\delta_0(x) = \frac{|k^n|}{\gamma(I_n, \overline{\psi}^{-1})}.$$

Let

$$C_k' = C_k \sum_{x\in X_n} \overline{\omega}_{\overline{\psi}^{-1}}(w_n)\delta_0(x) \neq 0.$$

Thus (4.2) becomes

$$\Psi(\mathcal{B}_{\pi,\overline{\psi}}, \delta_0, \widetilde{f}_v) = C_k' \sum_{a\in Z_n\backslash \mathrm{GL}_n(k)} \mathcal{B}_{\pi,\overline{\psi}}(t_n(a)\widetilde{w}_n)\epsilon(\det(a))W_v^*(a). \tag{4.3}$$

The above equation is also valid if we replace π by π'. By (4.1), we get

$$\sum_{a\in Z_n\backslash \mathrm{GL}_n(k)} \left(\mathcal{B}_{\pi,\overline{\psi}}(t_n(a)\widetilde{w}_n) - \mathcal{B}_{\pi',\overline{\psi}}(t_n(a)\widetilde{w}_n)\right)\epsilon(\det(a))W_v^*(a) = 0.$$

Note that this is true for all $v\in\tau$ and all irreducible generic representations τ of $\mathrm{GL}_n(k)$. By Lemma 4.2, we have

$$\mathcal{B}_{\pi,\overline{\psi}}(t_n(a)\widetilde{w}_n) = \mathcal{B}_{\pi',\overline{\psi}}(t_n(a)\widetilde{w}_n), \forall a\in \mathrm{GL}_n(k).$$

This concludes the proof when $1\leq n<r$.

We next consider the case when $n=r$. Recall that $\delta_1^r\in\mathcal{S}(k^r)$ is the characteristic function of $e_r=(0,\dots,0,1)\in k^r$. Let $v\in\tau$ and $f_v\in\mathcal{I}(\tau,\overline{\psi}^{-1})$ still be the function considered in §3.6. By (3.6), we have

$$\Psi(\mathcal{B}_{\pi,\overline{\psi}}, \delta_1^r, f_v) = W_v(1).$$

In particular, we have

$$\Psi(\mathcal{B}_{\pi,\overline{\psi}}, \delta_1^r, f_v) = \Psi(\mathcal{B}_{\pi',\overline{\psi}}, \delta_1^r, f_v).$$

By the functional equation and the assumption on gamma factors, we then get

$$\Psi(\mathcal{B}_{\pi,\overline{\psi}}, \delta_1^r, \widetilde{f}_v) = \Psi(\mathcal{B}_{\pi',\overline{\psi}}, \delta_1^r, \widetilde{f}_v),$$

where $\widetilde{f}_v = M(\tau,\overline{\psi}^{-1})f_v$. From the definition, we get

$$\Psi(\mathcal{B}_{\pi,\overline{\psi}}, \delta_1^r, \widetilde{f}_v) = \sum_{g\in U\backslash \mathrm{Sp}_{2r}(k)} \mathcal{B}_{\pi,\overline{\psi}}(g)\overline{\omega}_{\overline{\psi}^{-1}}(g)\delta_1^r(e_r)\widetilde{f}_v(g, I_r).$$

By Lemma 3.6, if $g\notin P_r(k)w_rN_r(k)$, we have $\widetilde{f}_v(g,I_r)=0$. For $g\in P_r(k)w_rN_r(k)$, we write $g=u_1m_r(a)w_ru_2$. We have $\widetilde{f}_v(g,I_r)=W_v^*(a)$. Thus we have

$$\Psi(\mathcal{B}_{\pi,\overline{\psi}}, \delta_1^r, \widetilde{f}_v) = \sum_{a\in Z_r\backslash \mathrm{GL}_r(k)} \mathcal{B}_{\pi,\overline{\psi}}(m_r(a)w_ru_2)\overline{\omega}_{\overline{\psi}^{-1}}(m_r(a)w_ru_2)\delta_1^r(e_r)W_v^*(a).$$

By the Weil representation formula (see [GH17, p.220]),

$$\overline{\omega}_{\overline{\psi}^{-1}}(u_2)\delta_1^r = \overline{\psi}_U^{-1}(u_2)\delta_1^r.$$

And by Proposition 3.1,

$$\mathcal{B}_{\pi,\overline{\psi}}(m_r(a)w_ru_2) = \overline{\psi}_U(u_2)\mathcal{B}_{\pi,\overline{\psi}}(m_r(a)w_r).$$

We thus get

$$\Psi(\mathcal{B}_{\pi,\overline{\psi}}, \delta_1^r, \widetilde{f}_v) = |N_r(k)| \sum_{a\in Z_r\backslash \mathrm{GL}_r(k)} \mathcal{B}_{\pi,\overline{\psi}}(m_r(a)w_r)\overline{\omega}_{\overline{\psi}^{-1}}(m_r(a)w_r)\delta_1^r(e_r)W_v^*(a).$$

Write $a=(a_{ij})\in \mathrm{GL}_r(k)$. By the Weil representation formula (see [GH17, p.220] or [LZ19, p.76]), we have

$$\begin{aligned}\overline{\omega}_{\overline{\psi}^{-1}}(m_r(a)w_r)\delta_1^r(e_r)&=\epsilon(\det(a))\overline{\omega}_{\overline{\psi}^{-1}}(w_r)\delta_1^r(e_r a)\\&=\frac{1}{\gamma(I_r,\overline{\psi}^{-1})}\epsilon(\det(a))\sum_{x\in k^r}\overline{\psi}^{-1}(2(e_r a)J_r{}^t x)\delta_1^r(x)\\&=\frac{1}{\gamma(I_r,\overline{\psi}^{-1})}\epsilon(\det(a))\overline{\psi}^{-1}(2a_{r1}).\end{aligned}$$

Hence,

$$\Psi(\mathcal{B}_{\pi,\overline{\psi}},\delta_1^r,\widetilde{f}_v)=\frac{|N_r(k)|}{\gamma(I_r,\overline{\psi}^{-1})}\sum_{a\in Z_r\backslash \mathrm{GL}_r(k)}\mathcal{B}_{\pi,\overline{\psi}}(m_r(a)w_r)\overline{\psi}^{-1}(2a_{r1})\epsilon(\det(a))W_v^*(a). \tag{4.4}$$

There is a similar equation for $\Psi(\mathcal{B}_{\pi',\overline{\psi}},\delta_1^r,\widetilde{f}_v)$. The condition

$$\Psi(\mathcal{B}_{\pi,\psi},\delta_1^r,\widetilde{f}_v)=\Psi(\mathcal{B}_{\pi',\psi},\delta_1^r,\widetilde{f}_v)$$

thus implies that

$$\sum_{a\in Z_r\backslash \mathrm{GL}_r(k)}(\mathcal{B}_{\pi,\overline{\psi}}(m_r(a)w_r)-\mathcal{B}_{\pi',\overline{\psi}}(m_r(a)w_r))\overline{\psi}^{-1}(2a_{r1})\epsilon(\det(a))W_v^*(a)=0.$$

Note that this is true for all $v\in\tau$ and all irreducible generic representations τ of $\mathrm{GL}_r(k)$. Therefore, by Lemma 4.2,

$$(\mathcal{B}_{\pi,\overline{\psi}}(m_r(a)w_r)-\mathcal{B}_{\pi',\overline{\psi}}(m_r(a)w_r))\overline{\psi}^{-1}(2a_{r1})=0,\forall a\in \mathrm{GL}_r(k).$$

Since $\overline{\psi}^{-1}(2a_{r1})\neq 0$, we get

$$\mathcal{B}_{\pi,\overline{\psi}}(m_r(a)w_r)=\mathcal{B}_{\pi',\overline{\psi}}(m_r(a)w_r).$$

This concludes the proof. □

Proof of Theorem 4.1. Denote $\mathcal{B}(g)=\mathcal{B}_{\pi,\overline{\psi}}(g)-\mathcal{B}_{\pi',\overline{\psi}}(g)$. By assumption and Theorem 4.3, we have that

$$\mathcal{B}(t_n(a)\widetilde{w}_n)=0,\forall a\in \mathrm{GL}_n(k),\forall 1\leq n\leq r. \tag{4.5}$$

For $w\in \mathrm{B}_n(\mathrm{Sp}_{2r})$, we show that $\mathcal{B}(tw)=0$ for all $t\in A$. Let $i\geq n+1$, we consider the unipotent element $\mathbf{x}_{\alpha_i}(x)$ in the root space of α_i. By (3.2), we can check that

$$w\mathbf{x}_{\alpha_i}(x)=\mathbf{x}_{\alpha_i}(x)w.$$

Hence,

$$tw\mathbf{x}_{\alpha_i}(x)=\mathbf{x}_{\alpha_i}(\alpha_i(t)x)tw.$$

Thus by Proposition 3.1, we have

$$\overline{\psi}(x)\mathcal{B}(tw)=\overline{\psi}(\alpha_i(t)x)\mathcal{B}(tw).$$

If $\alpha_i(t)\neq 1$ for some $i>n$, we get $\mathcal{B}(tw)=0$ since x is arbitrary in the above equation and $\overline{\psi}$ is nontrivial. We next assume that $\alpha_i(t)=1$ for all $i>n$. Then t has the form

$$t=zm_r(\mathrm{diag}(a_1,\dots,a_n,1,\dots,1))=zt_n(\mathrm{diag}(a_1,\dots,a_n)),$$

for some $z=\pm I_{2r}$ in the center of $\mathrm{Sp}_{2r}(k)$. By (4.5), we get

$$\mathcal{B}(tw)=\omega(z)\mathcal{B}(t_n(\mathrm{diag}(a_1,\dots,a_n))t_n(w')\widetilde{w}_n)=0.$$

Here $\omega=\omega_\pi=\omega_{\pi'}$ is the central character of π and π'. Therefore, we have proved that $\mathcal{B}(tw)=0$ for all $t\in A$ and all $w\in \mathrm{B}_n(\mathrm{Sp}_{2r})$ with $1\leq n\leq r$.

If $w=1$, we have $\mathcal{B}(t)=0$ for all $t\in A$ by Lemma 3.5 and the assumption $\omega_\pi=\omega_{\pi'}$. Then we get $\mathcal{B}(tw)=0$ for all $t\in A$ and all $w\in \mathrm{B}(\mathrm{Sp}_{2r})$ by (3.3). By Lemma 3.2, we get $\mathcal{B}(g)=0$ for all $g\in \mathrm{Sp}_{2r}(k)$, i.e., $\mathcal{B}_{\pi,\overline{\psi}}(g)=\mathcal{B}_{\pi',\overline{\psi}}(g)$ for all $g\in\mathrm{Sp}_{2r}(k)$. By the uniqueness of Whittaker model, we get $\pi\cong\pi'$. This completes the proof of Theorem 4.1. □

4.3. **Computation of the gamma factors.** As a corollary of the proof of Theorem 4.3, we can obtain an expression of the gamma factors $\gamma(\pi\times\tau,\psi)$. Let τ be an irreducible generic representation of $\mathrm{GL}_n(k)$. Let $\mathcal{B}_{\tau,\overline{\psi}^{-1}}$ be the (normalized) Bessel function of τ, which is the unique function in $\mathcal{W}(\tau,\overline{\psi}_{Z_n}^{-1})$ such that

$$\mathcal{B}_{\tau,\overline{\psi}^{-1}}(z_1gz_2)=\overline{\psi}_{Z_n}^{-1}(z_1z_2)\mathcal{B}_{\tau,\overline{\psi}^{-1}}(g),\forall z_1,z_2\in Z_n,$$

and

$$\mathcal{B}_{\tau,\overline{\psi}^{-1}}(1)=1.$$

The existence of $\mathcal{B}_{\tau,\overline{\psi}^{-1}}$ can be proved similarly as in the Sp_{2r} case, see §3.1.

Proposition 4.4. *Let π be an irreducible $\overline{\psi}_U$-generic cuspidal representation of $\mathrm{Sp}_{2r}(k)$ and τ be an irreducible generic representation of $\mathrm{GL}_n(k)$.*

(1) *If $1\le n<r$, we have*

$$\gamma(\pi\times\tau,\overline{\psi})=\frac{q^{(2rn-n^2+n)/2}}{\gamma(I_n,\overline{\psi}^{-1})}\sum_{a\in Z_n\backslash\mathrm{GL}_n(k)}\mathcal{B}_{\pi,\overline{\psi}}(t_n(a)\widetilde{w}_n)\epsilon(\det(a))\mathcal{B}_{\tau,\overline{\psi}^{-1}}(d_na^*).$$

(2) *If $n=r$, we have*

$$\gamma(\pi\times\tau,\overline{\psi})=\frac{q^{r(r+1)/2}}{\gamma(I_r,\overline{\psi}^{-1})}\sum_{a\in Z_r\backslash\mathrm{GL}_r(k)}\mathcal{B}_{\pi,\overline{\psi}}(m_r(a)w_r)\overline{\psi}^{-1}(2a_{r1})\epsilon(\det(a))\mathcal{B}_{\tau,\overline{\psi}^{-1}}(d_ra^*).$$

Proof. (1) For $v\in\tau$, let $f_v\in\mathcal{I}(\tau,\overline{\psi}^{-1})$ be the section defined in §3.6. From the calculation in the proof of Proposition 3.7 or (3.5), we get

$$\Psi(\mathcal{B}_{\pi,\overline{\psi}},\delta_0,f_v)=W_v(1).$$

From the proof of Theorem 4.3, in particular from (4.3), we get

$$\Psi(\mathcal{B}_{\pi,\overline{\psi}},\delta_0,\widetilde{f}_v)=C_k'\sum_{g\in Z_n\backslash\mathrm{GL}_n(k)}\mathcal{B}_{\pi,\overline{\psi}}(t_n(a)\widetilde{w}_n)\epsilon(\det(a))W_v^*(a),$$

where

$$C_k'=C_k\sum_{x\in X_n}\overline{\omega}_{\overline{\psi}^{-1}}(w_n)\delta_0(x)=|N_n(k)|\cdot|\mathrm{Mat}_{(r-n-1)\times n}(k)|\frac{|k^n|}{\gamma(I_n,\overline{\psi}^{-1})}.$$

Now we fix a Whittaker vector $v\in\tau$, i.e., a vector v such that $\tau(z)v=\overline{\psi}_{Z_n}^{-1}(z)v$ for all $z\in Z_n$. We also assume that the Whittaker functional $\Lambda\in\mathrm{Hom}_{Z_n}(\tau,\overline{\psi}_{Z_n}^{-1})$ is chosen such that $W_v(1)=\Lambda(v)=1$. The Bessel function $\mathcal{B}_{\tau,\overline{\psi}^{-1}}$ is the Whittaker function associated with v, i.e., $\mathcal{B}_{\tau,\overline{\psi}^{-1}}(a)=\Lambda(\tau(a)v)$.

From the functional equation, we get

$$\gamma(\pi\times\tau,\overline{\psi})=\frac{q^{(2rn-n^2+n)/2}}{\gamma(I_n,\overline{\psi}^{-1})}\sum_{g\in Z_n\backslash\mathrm{GL}_n(k)}\mathcal{B}_{\pi,\overline{\psi}}(t_n(a)\widetilde{w}_n)\epsilon(\det(a))W_v^*(a),$$

where $W_v^*(a)=\Lambda(\tau(d_na^*)v)=\mathcal{B}_{\tau,\overline{\psi}^{-1}}(d_na^*)$. The assertion follows.

(2) The proof is similar as the above case. For $v\in\tau$ and $f_v\in\mathcal{I}(\tau,\overline{\psi}^{-1})$ be as the above. Let $\delta_1^r\in\mathcal{S}(k^r)$ be the characteristic function of e_r. By (3.6), We have

$$\Psi(\mathcal{B}_{\pi,\overline{\psi}},\delta_1^r,f_v)=W_v(1).$$

By (4.4), we have

$$\Psi(\mathcal{B}_{\pi,\overline{\psi}},\delta_1^r,\widetilde{f}_v)=\frac{|N_r(k)|}{\gamma(I_r,\overline{\psi}^{-1})}\sum_{a\in Z_r\backslash\mathrm{GL}_r(k)}\mathcal{B}_{\pi,\overline{\psi}}(m_r(a)w_r)\overline{\psi}^{-1}(2a_{r1})\epsilon(\det(a))W_v^*(a).$$

Now we pick $v\in\tau$ to be the Whittaker vector and $\Lambda\in\mathrm{Hom}_{Z_r}(\tau,\overline{\psi}_{Z_n}^{-1})$ also be as above. Then we have $\mathcal{B}_{\tau,\overline{\psi}^{-1}}(a)=W_v(a)=\Lambda(\tau(a)v)$. The assertion then follows from the definition of $\gamma(\pi\times\tau,\psi)$ directly. □

Remark 4.5. The formulas in Proposition 4.4 show that the gamma factor $\gamma(\pi\times\tau,\overline{\psi})$ is certain Mellin transform of Bessel functions. Similar formulas have been proved for $\mathrm{GL}_2(k)$ in [PS83], see also [Co14, §2], $\mathrm{GL}_n(k)$ in [Ro10], and $\mathrm{G}_2(k)$ in [LZ18]. It is desirable to express gamma factors as certain Mellin transform of Bessel functions for more general groups, see [Co14, §6].

Remark 4.6. If $r>1$, $n=1$ and χ a character of $\mathrm{GL}_1(k)=k^\times$, then the formula in the above proposition reads

$$\gamma(\pi\times\chi,\overline{\psi})=\frac{q^r}{\gamma(1,\overline{\psi}^{-1})}\sum_{a\in k^\times}\mathcal{B}_{\pi,\overline{\psi}}(t_1(a)\widetilde{w}_1)\epsilon(a)\chi^{-1}(a). \tag{4.6}$$

It was proved in [NZ18] that when q is large, an irreducible generic cuspidal representation π of $\mathrm{GL}_n(k)=\mathrm{GL}_n(\mathbb{F}_q)$ can be uniquely determined by the gamma factors of its various GL_1-twists. It is an interesting question that if an irreducible generic cuspidal representation of $\mathrm{Sp}_{2r}(\mathbb{F}_q)$ can be determined by the above $\mathrm{GL}_1(k)$-twists when q is large.

5. Gamma factors and a converse theorem for U_{2r}

The technique used in the previous sections can also be used to define gamma factors for $\mathrm{U}_{2r}(k)\times\mathrm{GL}_n(k_2)$ for $1\le n\le r$ and then give a proof of the local converse theorem for $\mathrm{U}_{2r}(k)$, where k is a finite field of odd characteristic, k_2 is the quadratic extension of k, and $\mathrm{U}_{2r}(k)$ is the quasi-split unitary group of size $2r$ associated with the extension k_2/k. Since the proof is quite similar, we just give a sketch in this section and highlight the differences in the proof.

Let F be a p-adic field with odd residue characteristic, and let E be a fixed unramified quadratic extension of F. Denote by $x\mapsto x^\iota$ the unique nontrivial element in $\mathrm{Gal}(E/F)$. Let $\mathfrak{o}_F$ (resp. $\mathfrak{o}_E$) be the ring of intergers of F (resp. E), $\mathfrak{p}_F$ (resp. $\mathfrak{p}_E$) be the maximal ideal of $\mathfrak{o}_F$ (resp. $\mathfrak{o}_E$). Let $\omega_{E/F}$ be the class field character of $F^\times$ associated with the extension E/F. We assume E and F are chosen such that $k=\mathfrak{o}_F/\mathfrak{o}_F$ and $k_2=\mathfrak{o}_E/\mathfrak{p}_E$. We also denote by $x\mapsto x^\iota$ the nontrivial Galois element in $\mathrm{Gal}(k_2/k)$. For $x\in k_2$, let $\mathrm{Tr}_{k_2/k}(x)=x+x^\iota$. Let $\Pi_{\mathfrak{o}_F}:\mathfrak{o}_F\to k$ be the natural projection. Let ψ be a fixed unramified additive character of F and let $\overline{\psi}$ be the character of k defined by $\overline{\psi}(\Pi_{\mathfrak{o}_F}(t))=\psi(t\varpi_F^{-1})$, where ϖ_F is a fixed generator of $\mathfrak{p}_F$.

5.1. **Weil representations.** Let $(V,\langle\ ,\ \rangle)$ be a skew-Hermitian space of dimension $2r$ over E. We fix an isomorphism $V\cong E^{2r}$ and assume the skew-Hermitian form is given by

$$\langle v_1,v_2\rangle=v_1\begin{pmatrix} & J_r\\ -J_r & \end{pmatrix}{}^t(v_2^\iota).$$

Let $\mathrm{U}_{2r}(F)$ be the isometry group of $(V,\langle\ ,\ \rangle)$. Similarly, we can define the group $\mathrm{U}_{2r}(k)$ for the extension k_2/k.

Let $V'\cong E$ be the one dimensional Hermitian space with Hermitian structure defined by $\langle v_1,v_2\rangle=v_1v_2^\iota, v_1,v_2\in V'$. Let $\mathrm{U}_1(F)=\mathrm{U}(V')$ be the isometry group of V'. Then the tensor product $W=V\otimes_E V'$ is a skew-Hermitian space and its underlying space over F is a symplectic space with the symplectic structure

$$\langle v_1\otimes v_1',v_2\otimes v_2'\rangle'=\mathrm{Tr}_{E/F}(\langle v_1,v_2\rangle v_1'(v_2')^\iota).$$

In this way, $\mathrm{U}_{2r}(F)\times\mathrm{U}_1(F)$ is a reductive dual pair in the group $\mathrm{Sp}(W)\cong\mathrm{Sp}_{4r}(F)$. Thus we have an embedding $\mathrm{U}_{2r}(F)\times\mathrm{U}(V')\hookrightarrow\mathrm{Sp}_{4r}(F)$. Let ω_ψ be the Weil representation of $\widetilde{\mathrm{Sp}}_{4r}(F)$. Let μ be a character of $E^\times$ such that its restriction to $F^\times$ is $\omega_{E/F}$. Then it is known that the projection $\widetilde{\mathrm{Sp}}_{4r}(F)\to\mathrm{Sp}_{2r}(F)$ splits over $\mathrm{U}_{2r}(F)\times\mathrm{U}(V')$. More precisely, there is a group homomorphism $s_\mu:\mathrm{U}_{2r}(F)\times\mathrm{U}(V')\to\widetilde{\mathrm{Sp}}_{4r}(F)$ which depends on μ, such that the composition $\mathrm{U}_{2r}(F)\to\widetilde{\mathrm{Sp}}_{4r}(F)\to\mathrm{Sp}_{4r}(F)$ is the embedding, see [HKS96, §1]. Thus we get a Weil representation $\omega_{\psi,\mu}$ of $\mathrm{U}_{2r}(F)\times\mathrm{U}(V')$ via the embedding s_μ. Via the embedding $\mathrm{U}_{2r}(F)\hookrightarrow\mathrm{U}_{2r}(F)\times\mathrm{U}(V')$, we also view $\omega_{\psi,\mu}$ as a representation of $\mathrm{U}_{2r}(F)$.

Over the finite field k, the group $\mathrm{U}_{2r}(k)$ can be embedded into $\mathrm{Sp}_{4r}(k)$ naturally. Recall that we have a Weil representation $\overline{\omega}_\psi$ of $\mathrm{Sp}_{4r}(k)$. Let ϵ be the unique non-trivial quadratic character of $k_2^1=\{x\in k_2: xx^\iota=1\}$. We view ϵ as a character of $\mathrm{U}_{2r}(k)$ via the determinant map. For the finite unitary group $\mathrm{U}_{2r}(k)$, the Weil representation $\overline{\omega}'_{\overline{\psi}}$ associated with the character $\overline{\psi}$ is defined to be

$\overline{\omega}'_{\overline{\psi}} = \epsilon \otimes \overline{\omega}_{\overline{\psi}}$, where $\overline{\omega}_{\overline{\psi}}$ is the pull-back of the Weil representation of $\mathrm{Sp}_{4r}(k)$ to $\mathrm{U}_{2r}(k)$, see [Ge77, Theorem 3.3]. Note that the space of $\overline{\omega}_{\overline{\psi}}$ can be identified with $\mathcal{S}(k_2^r)$.

5.2. **General lattice model of the Weil representation.** Let $e_i, i = \pm 1, \dots, \pm r$ be a basis of V with

$$\langle e_i, e_j \rangle = 0 = \langle e_{-i}, e_{-j} \rangle, \langle e_i, e_{-j} \rangle = \delta_{i,j}, \forall i, j > 0.$$

Let $L \subset V$ be the lattice

$$L = \mathfrak{p}_E e_1 + \mathfrak{p}_E e_2 + \cdots + \mathfrak{p}_E e_r + \mathfrak{o}_E e_{-r} + \mathfrak{o}_E e_{-(r-1)} + \cdots + \mathfrak{o}_E e_{-1}.$$

Let $L^* = \{v \in V : \langle v, b \rangle \in \mathfrak{o}_E, \forall b \in L\}$. Then we can check that

$$L^* = \mathfrak{o}_E e_1 + \cdots + \mathfrak{o}_E e_r + \varpi_E^{-1} \mathfrak{o}_E e_{-r} + \cdots + \varpi_E^{-1} \mathfrak{o}_E e_{-1},$$

i.e., $\varpi_E L^* = L$. Set

$$G_L = \{g \in \mathrm{U}_{2r}(F) : g.L = L\},$$
$$G_{L,0+} = \{g \in G_L : (g-1).L^* \subset L\}.$$

Then we can check that $G_L/G_{L,0+} \cong \mathrm{U}_{2r}(k)$.

Recall that $V' \cong E$ is the one dimensional Hermitian space defined in the above section. Let $L' = \mathfrak{o}_E$, viewed as a lattice in V'. Then $L'^* = \{v \in V' : \langle v, b \rangle \in \mathfrak{o}_E, \forall v \in L'\} = L'$. Similarly as G_L and $G_{L,0+}$, we can define $G_{L'}$ and $G_{L',0+}$. Then we have $G_{L'}/G_{L',0+} \cong \mathrm{U}_1(k)$.

Let

$$\mathbf{B} = L^* \otimes L' \cap L \otimes L'^* = L \otimes L',$$

which is a lattice in W, the underlying F space of $V \otimes_E V'$. As in Section 2.2, we can define the group $K_{\mathbf{B}}$ and $K'_{\mathbf{B}}$. We have that $K_{\mathbf{B}}/K'_{\mathbf{B}} \cong \mathrm{Sp}_{4r}(k)$. Moreover, the Weil representation of ω_ψ of $\widetilde{\mathrm{Sp}}_{4r}(F)$ and the Weil representation $\overline{\omega}_{\overline{\psi}}$ of $\mathrm{Sp}_{4r}(k)$ have the relation

$$(\omega_\psi|_{K_{\mathbf{B}}})^{K'_{\mathbf{B}}} \cong \overline{\omega}_{\overline{\psi}},$$

see [Pan02, Proposition 5.3] or Proposition 2.1.

Lemma 5.1. *There exists a character μ such that the natural homomorphism*

$$\omega_{\psi,\mu}|_{G_L} \to \overline{\omega}'_{\overline{\psi}}$$

is surjective between representations of $\mathrm{U}_{2r}(k)$.

Proof. Consider the dual pair $\mathrm{U}(V) \times \mathrm{U}(V')$ as above. Let μ be any character of $E^\times$ such that $\mu|_{F^\times}$ is the class field theory character associated with the extension E/F. Such a μ determines a map $\beta_\mu : \mathrm{U}(V) \to \mathbb{C}^\times$ such that

$$s_\mu : \mathrm{U}(V) \to \widetilde{\mathrm{U}}(V)$$
$$g \mapsto (g, \beta_\mu(g))$$

is a group homomorphism, where $\widetilde{\mathrm{U}}(V)$ is the inverse image of $\mathrm{U}(V)$ in $\widetilde{\mathrm{Sp}}_{4r}(F)$ via the inclusion $\mathrm{U}(V) \to \mathrm{Sp}_{4r}(F)$, see [HKS96, (1.14), (1.15), p.952]. Moreover, the Weil representation $\omega_{\psi,\mu}$ of $\mathrm{U}(V)$ is exactly $(\omega_\psi)|_{\widetilde{\mathrm{U}}(V)} \circ s_\mu$. By [Pan01, Theorem A], there exists a choice of μ such that $\beta_\mu|_{G_L}$ is a nontrivial quadratic character, say ϵ'. We fix such a character μ. As a representation of G_L, we have $\omega_{\psi,\mu}|_{G_L} = (\omega_\psi|_{K_B})|_{G_L} \otimes \epsilon' = \epsilon' \otimes \omega_\psi|_{G_L}$. Note that for $g \in G_{L,0+}$, we have $\det(g) = 1$. Thus $\epsilon'|_{G_{L,0}} = 1$. Thus ϵ' defines a character of $\mathrm{U}_{2r}(k) \cong G_L/G_{L,0+}$, which must be ϵ by the uniqueness of quadratic characters on $\mathrm{U}_{2r}(k)$. By [Pan02, Proposition 5.3] or Proposition 5.1, there is a surjective map $\omega_\psi|_{K_B} \to \overline{\omega}_{\overline{\psi}}$ of $\mathrm{Sp}_{4r}(k)$ representations. By restriction, we get a surjective map

$$\omega_{\psi,\mu}|_{G_L} \cong \epsilon' \otimes \omega_\psi|_{G_L} \to \overline{\omega}'_{\overline{\psi}} \cong \epsilon \otimes \overline{\omega}_{\overline{\psi}}$$

of $\mathrm{U}_{2r}(k)$-representations. □

5.3. Uniqueness of Fourier-Jacobi model.

Proposition 5.2. *Let k be a finite field of odd characteristic and k_2 be the unique quadratic extension of k (in a fixed algebraic closure). Let $P = MN$ be a parabolic subgroup of $\mathrm{U}_{2r}(k)$. Let π_0 be an irreducible cuspidal representation of $\mathrm{U}_{2r}(k)$ and τ_0 be an irreducible cuspidal representation of M. Then we have*

$$\dim \mathrm{Hom}_{\mathrm{U}_{2r}(k)}(\pi_0, I(\tau_0) \otimes \overline{\omega}'_{\overline{\psi}}) \leq 1,$$

where $I(\tau_0) = \mathrm{Ind}_{MN}^{\mathrm{U}_{2r}(k)}(\tau_0)$.

Proof. Applying Lemma 5.1, the proof is similar to that of Proposition 2.3 and can be reduced to the uniqueness of Fourier-Jacobi models over p-adic fields for unitary groups proved by Sun [Su12]. Note that, we also have an analogue of Proposition 2.2. □

Similarly, as in the Sp_{2r} case, the above result can also be extended to more general settings. See §2.5 for the Sp_{2r} case.

Assume that $n \leq r-1$. Let $P = MN$ be the parabolic subgroup of $\mathrm{U}_{2r}(k)$ with Levi subgroup

$$M = \left\{\mathrm{diag}(a_1, \ldots, a_{r-n}, g, a_{r-n}^{\iota}, \ldots, a_1^{\iota}), a_i \in k_2^{\times}, g \in \mathrm{U}_{2n}(k)\right\} \cong (k_2^{\times})^{r-n} \times \mathrm{U}_{2n}(k)$$

and unipotent subgroup N. Let

$$H = \mathrm{U}_{2n}(k) \ltimes N \subset P.$$

Let

$$\mathcal{H}_n = \left\{ \begin{pmatrix} 1 & x & z \\ & I_{2n} & x^* \\ & & 1 \end{pmatrix}, x \in k_2^n, z \in k \right\} \subset \mathrm{U}_{2n+2}(k).$$

Note that $\mathcal{H}_n$ can be embedded into N. Note that, the group $\mathcal{H}_n$ is indeed the $4n+1$ dimensional Heisenberg group. There is a Weil representation $\overline{\omega}'_{\overline{\psi}}$ of $\mathrm{U}_{2n}(k) \ltimes \mathcal{H}_n$. Note that $\mathrm{U}_{2n}(k) \ltimes \mathcal{H}_n$ is a subgroup of H, and any element $h \in H$ is of the form $h = uh_0g$ with $h_0 \in \mathcal{H}_n, g \in \mathrm{U}_{2r}(k)$ and

$$u = \begin{pmatrix} z & v_1 & v_2 \\ & I_{2n+2} & v_1^* \\ & & z \end{pmatrix}$$

where z is in the standard upper triangular subgroup of $\mathrm{GL}_{r-n-1}(k_2)$. Let $\nu_{\overline{\psi}}$ be the representation of H defined by

$$\nu_{\overline{\psi}}(uh_0g) = \overline{\psi}(u)\overline{\omega}'_{\overline{\psi}}(h_0g),$$

where $h_0 \in \mathcal{H}_n, g \in \mathrm{U}_{2r}(k)$ and

$$u = \begin{pmatrix} z & v_1 & v_2 \\ & I_{2n+2} & v_1^* \\ & & z \end{pmatrix}$$

as above, and

$$\overline{\psi}(u) := \overline{\psi}\left(\mathrm{Tr}_{k_2/k}\left(\sum_{i=1}^{r-n-1} z_{i,i+1}\right)\right).$$

The representation $\nu_{\overline{\psi}}$ is well-defined, see [GGP12a, §12]. Let σ be a representation of $\mathrm{U}_{2n}(k)$. We can view σ as a representation of H via the quotient map $H \to \mathrm{U}_{2n}(k)$. Thus we can form the representation $\sigma \otimes \nu_{\overline{\psi}}$ of H.

Proposition 5.3. *Let τ_0 be an irreducible representation of $\mathrm{GL}_n(k_2)$ and π_0 be an irreducible cuspidal representation of $\mathrm{U}_{2r}(k)$. Then we have*

$$\dim \mathrm{Hom}_H(\pi_0, I(\tau_0) \otimes \nu_{\overline{\psi}}) \leq 1.$$

The proof is similar to that of Proposition 2.4 Part (2) and we omit the details.

5.4. **The gamma factors.** Over a p-adic field extension E/F, the local and global zeta integrals for generic representations of $\mathrm{U}_{2r}(F)$ twisted by generic representations of $\mathrm{GL}_n(E)$ for $n \le r$ were studied in [BAS09]. In this subsection, we consider their finite fields analogues (in fact, integrals over finite fields are finite sums). These integrals are quite similar to those defined for $\mathrm{Sp}_{2r}(k)$, see §3.5.

Let τ be an irreducible generic representation of $\mathrm{GL}_n(k_2)$ and let $I(\tau) := \mathrm{Ind}_{P_n}^{\mathrm{U}_{2n}(k)}(\tau)$. Recall that P_n is the Siegel parabolic subgroup of $\mathrm{U}_{2n}(k)$. Let Z_n be the upper triangular unipotent subgroup of $\mathrm{GL}_n(k_2)$ and let $\overline{\psi}_{Z_n}^{-1}$ be the character of Z_n defined by

$$\overline{\psi}_{Z_n}^{-1}(z) = \overline{\psi}^{-1}\left(\mathrm{Tr}_{k_2/k}\left(\sum_{i=1}^{n-1} z_{i,i+1}\right)\right), z = (z_{ij}) \in Z_n.$$

Fix a nontrivial Whittaker functional $\Lambda = \Lambda_\tau \in \mathrm{Hom}_{Z_n}(\tau, \overline{\psi}_{Z_n}^{-1})$. Let $W_v(a) = \Lambda(\tau(a)v)$ and $W_v^*(a) = \Lambda(\tau(d_n a^*)v)$ for $a \in \mathrm{GL}_n(k_2)$. Here $d_n = \mathrm{diag}(-1, 1, \ldots, (-1)^n) \in \mathrm{GL}_n$. For $\xi \in I(\tau)$, let $f_\xi(g, a) = \Lambda(\tau(a)\xi(g)), a \in \mathrm{GL}_n(k_2), g \in \mathrm{U}_{2n}(k)$. Let $\mathcal{I}(\tau, \overline{\psi}_{Z_n}^{-1})$ be the space of all f_ξ as $\xi \in I(\tau)$.

Let π be an irreducible cuspidal representation of $\mathrm{U}_{2r}(k)$. Let $\overline{\omega}'_{\overline{\psi}}$ be the Weil representation of $\mathrm{U}_{2n}(k)$. Let U^n be the upper triangular unipotent subgroup of $\mathrm{U}_{2n}(k)$ and $\overline{\psi}_{U^r}$ be any fixed generic character of U^r. For $W \in \mathcal{W}(\pi, \overline{\psi}_{U^r})$, $f \in \mathcal{I}(\tau, \overline{\psi}_{Z_n}^{-1})$, $\phi \in \mathcal{S}(k_2^n)$, we can define

(5.1)

$$\Psi(W, \phi, f) = \begin{cases} \sum_{g \in U^n \backslash \mathrm{U}_{2n}(k)} \sum_{v \in R^{r,n}} \sum_{x \in X_n} W(w_{r-n,n}^{-1}(vxg)w_{r-n,n}) \overline{\omega}'_{\overline{\psi}^{-1}}(g)\phi(x) f(g, I_n), & 1 \le n < r, \\ \sum_{g \in U^r \backslash \mathrm{U}_{2r}(k)} W(g) \overline{\omega}'_{\overline{\psi}^{-1}}(g)\phi(e_r) f(g, I_r) dg, & n = r. \end{cases}$$

Here $X_n, R^{r,n}$ and $w_{r-n,n}$ can be defined similarly as in the Sp_{2r} case, see §3.5.

Similarly, as in Proposition 3.7, we can pick W, ϕ, f such that $\Psi(W, \phi, f) \neq 0$. Following Proposition 5.3, analogous to Proposition 3.8, we have the following

Proposition 5.4. *Let k be a finite field with odd characteristic and k_2/k be a fixed quadratic extension. Let π be an irreducible generic cuspidal representation of $\mathrm{U}_{2r}(k)$ and τ be an irreducible generic representation of $\mathrm{GL}_n(k_2)$ with $n \le r$. Then there exists a complex number $\gamma(\pi \times \tau, \overline{\psi})$ such that*

$$\Psi(W, \phi, M(\tau, \overline{\psi}^{-1})f) = \gamma(\pi \times \tau, \overline{\psi})\Psi(W, \Phi, f),$$

for all $W \in \mathcal{W}(\pi, \overline{\psi}_{U^r}), \phi \in \mathcal{S}(k_2^n)$ and $f \in \mathcal{I}(\tau, \overline{\psi}_{Z_n}^{-1})$. Here $M(\tau, \overline{\psi}^{-1})$ is the intertwining operator defined similarly as in the Sp_{2r} case.

5.5. **A converse theorem.** Similarly, as in the Sp_{2r} case, the gamma factors defined in Proposition 5.4 are enough to determine the representation π itself, i.e., we have the following

Theorem 5.5. *Let π, π' be two irreducible generic cuspidal representations of $\mathrm{U}_{2r}(k)$. Assume that π, π' are generic with respect to the same additive character $\overline{\psi}_{U^r}$ and*

$$\gamma(\pi \times \tau, \overline{\psi}) = \gamma(\pi' \times \tau, \overline{\psi})$$

for all irreducible generic representations τ of $\mathrm{GL}_n(k_2)$ and for all n with $1 \le n \le r$. Then $\pi \cong \pi'$.

Since the proof of the above theorem is similar to that of Theorem 4.1, we omit the details.

5.6. **Computation of the gamma factors.** Although we omitted the details of the proof of Theorem 5.5, we would like to include the computation of gamma factors for $\mathrm{U}_{2r}(k)$, which will be useful for applications.

Let π be an irreducible $\overline{\psi}_{U^r}$-generic cuspidal representation of $\mathrm{U}_{2r}(k)$ and τ be an irreducible generic representation of $\mathrm{GL}_n(k_2)$. Let $\mathcal{B}_{\pi,\overline{\psi}}$ (resp. $\mathcal{B}_{\tau,\overline{\psi}^{-1}}$) be the Bessel function of π (resp. τ) as usual. Let $q = |k|$. Thus $|k_2| = q^2$. Recall that we have the Weil representation formula

$$\overline{\omega}'_{\overline{\psi}^{-1}}(w_n)\phi(x) = \epsilon_0(\overline{\psi})q^{-n} \sum_{y \in k_2^n} \overline{\psi}(\mathrm{Tr}_{k_2/k}(xJ_n {}^t y^\iota))\phi(y),$$

where $\epsilon_0(\overline{\psi})$ is certain Weil index associated with $\overline{\psi}$.

Proposition 5.6. (1) *If* $1 \leq n < r$, *we have*

$$\gamma(\pi \times \tau, \overline{\psi}) = \epsilon_0(\overline{\psi}) q^{2nr-n^2-n} \sum_{a \in Z_n \backslash \mathrm{GL}_n(k_2)} \mathcal{B}_{\pi,\overline{\psi}}(t_n(a)\widetilde{w}_n)\mathcal{B}_{\tau,\overline{\psi}^{-1}}(d_n a^*),$$

where $t_n(a) = \mathrm{diag}(a, I_{2r-2n}, a^*)$, *and* $a^* = J_n {}^t a^\iota J_n$.

(2) *If* $n = r$, *we have*

$$\gamma(\pi \times \tau, \overline{\psi}) = \epsilon_0(\overline{\psi}) q^{r^2-r} \sum_{a \in Z_r \backslash \mathrm{GL}_r(k_2)} \mathcal{B}_{\pi,\overline{\psi}}(m_r(a)w_r)\overline{\psi}(\mathrm{Tr}_{k_2/k}(a_{r1}))\mathcal{B}_{\tau,\overline{\psi}^{-1}}(d_n a^*),$$

where $a = (a_{ij}) \in \mathrm{GL}_r(k_2), 1 \leq i, j \leq r$.

Proof. The proof is similar to that of Proposition 4.4 and we only sketch some details for (1). Let $v \in \tau$ and $\xi_v \in I(\tau)$ be the section such that $\mathrm{supp}(\xi_v) = P_n = M_n N_n$, and $\xi_v(m_n(a)u) = \tau(a)v$, where $a \in \mathrm{GL}_n(k_2), u \in N_n$ and $m_n(a) = \mathrm{diag}(a, a^*) \in \mathrm{U}_{2n}(k)$. Let $f_v = f_{\xi_v}$ be the corresponding function in $\mathcal{I}(\tau, \overline{\psi}^{-1}_{Z_n})$ and $\widetilde{f}_v = M(\tau, \overline{\psi}^{-1}) f_v$. Let $\delta_0 \in \mathcal{S}(k_2^n)$ be the characteristic function of $0 \in k_2^n$. Similarly as in Proposition 3.7, we can compute that

$$\Psi(\mathcal{B}_{\pi,\overline{\psi}}, \delta_0, f_v) = W_v(1).$$

Similar to the calculation in the proof of Theorem 4.3, we have

$$\Psi(\mathcal{B}_{\pi,\overline{\psi}}, \delta_0, \widetilde{f}_v) = C'_k \sum_{a \in Z_n \backslash \mathrm{GL}_n(k_2)} \mathcal{B}_{\pi,\overline{\psi}}(t_n(a)\widetilde{w}_n) W_v^*(a),$$

where the notations are similar to the Sp_{2r}-case, and

$$C'_k = |N_n(k)| \cdot |\mathrm{Mat}_{(r-n-1)\times n}(k_2)| \sum_{x \in X_n} \overline{\omega}'_{\overline{\psi}^{-1}}(w_n)\delta_0(x).$$

Note that in our case

$$N_n(k) = \left\{ \begin{pmatrix} 1 & X \\ & 1 \end{pmatrix}, X \in \mathrm{Mat}_{n\times n}(k_2), {}^t X^\iota = J_n X J_n \right\}.$$

Thus we get $|N_n(k)| = q^{n^2}$. On the other hand, we have

$$\overline{\omega}'_{\overline{\psi}^{-1}}(w_n)\delta_0(x) = \epsilon_0(\overline{\psi}) q^{-n} \sum_{y \in k_2^n} \psi(\mathrm{Tr}_{k_2/k}(x J_n {}^t y^\iota))\delta_0(y) = \epsilon_0(\overline{\psi}) q^{-n}.$$

Thus

$$\sum_{x \in X_n} \overline{\omega}'_{\overline{\psi}^{-1}}(w_n)\delta_0(x) = \epsilon_0(\overline{\psi}) q^n.$$

We then get

$$C'_k = \epsilon_0(\overline{\psi}) q^{n^2+n+2n(r-n-1)}.$$

Thus if we take $v \in \tau$ to be the Whittaker vector such that $W_v(1) = 1$, we can get

$$\begin{aligned}\gamma(\pi \times \tau, \overline{\psi}^{-1}) &= \epsilon_0(\overline{\psi}) q^{2nr-n^2-n} \sum_{a \in Z_n \backslash \mathrm{GL}_n(k_2)} \mathcal{B}_{\pi,\overline{\psi}}(t_n(a)\widetilde{w}_n) W_v^*(a) \\ &= \epsilon_0(\overline{\psi}) q^{2nr-n^2-n} \sum_{a \in Z_n \backslash \mathrm{GL}_n(k_2)} \mathcal{B}_{\pi,\overline{\psi}}(t_n(a)\widetilde{w}_n)\mathcal{B}_{\tau,\overline{\psi}^{-1}}(d_n a^*).\end{aligned}$$

This concludes the proof. □

6. Gamma factors and a converse theorem for $\mathrm{U}_{2r+1}(k)$

Let k be a finite field of odd characteristic and k_2 be the quadratic extension of k. Let $x \mapsto x^\iota$ be the nontrivial Galois element in $\mathrm{Gal}(k_2/k)$. Let $k_2^1 = \{x \in k_2 : xx^\iota = 1\}$. As in previous sections, for a positive integer m, set

$$J_m = \begin{pmatrix} & 1 \\ J_{m-1} & \end{pmatrix} \in \mathrm{GL}_m(k), J_1 = (1).$$

Let

$$\mathrm{U}_m(k) = \left\{ g \in \mathrm{GL}_m(k_2) : gJ_m {}^t g^\iota = g \right\}.$$

Note that, if $m = 2r$ is even, then the definition of U_{2r} is a little bit different from that defined in §5. In fact, the unitary group U_{2r} in §5 was defined by a skew-Hermitian form and the group $\mathrm{U}_{2r}(k)$ considered in this section is defined by a Hermitian form. One can check that they are isomorphic. Here by abuse of notation, we use the same notation to denote the "Hermitian" version of the unitary group.

6.1. **Notations.** The notations used here follows from that of [BAS09, Zh18] closely. For $m = 2n$ even, we denote by $P_n = M_n \ltimes N_n$ the Siegel type parabolic subgroup of U_{2n}, where

$$M_n = \left\{ m_n(a) := \begin{pmatrix} a & \\ & a^* \end{pmatrix} \in \mathrm{U}_{2n}(k), a \in \mathrm{GL}_n(k_2) \right\}, \quad N_n = \left\{ u(x) := \begin{pmatrix} I_n & x \\ & I_n \end{pmatrix} \in \mathrm{U}_{2n}(k) \right\}.$$

Here a^* is determined by a such that $m(a) \in \mathrm{U}_{2n}(k)$. In fact, we have $a^* = J_n {}^t(a^{-1})^\iota J_n$. Denote $w_n = \begin{pmatrix} & I_n \\ I_n & \end{pmatrix} \in \mathrm{U}_{2n}$. Let $B_m = A_m U_m$ be the upper triangular Borel subgroup with maximal torus A_m and maximal unipotent U_m.

The center of $\mathrm{U}_{2r+1}(k)$ is consisting of elements of the form $\mathrm{diag}(z, z, \dots, z, z), z \in k_2^1$. We will identify k_2^1 with the center of $\mathrm{U}_{2r+1}(k)$ via the map $z \mapsto \mathrm{diag}(z, z, \dots, z, z)$. A typical element $t \in A_{2r+1}$ has the form

$$t = z\mathrm{diag}(a_1, \dots, a_r, 1, (a_r^{-1})^\iota, \dots, (a_1^{-1})^\iota), z \in k_2^1, a_1, \dots, a_r \in k_2^\times.$$

Define characters $\alpha_i, 1 \le i \le r$ on A_{2r+1} by

$$\alpha_i(t) = a_i/a_{i+1}, 1 \le i \le r-1; \alpha_r(t) = a_r,$$

where $t = z\mathrm{diag}(a_1, \dots, a_r, 1, (a_r^{-1})^\iota, \dots, (a_1^{-1})^\iota)$. Denote $\Delta(\mathrm{U}_{2r+1}) = \{\alpha_i, 1 \le i \le r\}$, which is the set of simple roots of $\mathrm{U}_{2r+1}(k)$.

For $n \le r$, we consider the embedding $\mathrm{U}_{2n}(k) \to \mathrm{U}_{2r+1}(k)$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} I_{r-n} & & & & \\ & a & & b & \\ & & 1 & & \\ & c & & d & \\ & & & & I_{r-n} \end{pmatrix}.$$

Through this embedding, we will identify $\mathrm{U}_{2n}(k)$ as a subgroup of $\mathrm{U}_{2r+1}(k)$ without further notice. Thus the element $w_n \in \mathrm{U}_{2n}$ can be viewed as an element of U_{2r+1} for $n \le r$.

For $a \in \mathrm{GL}_r(k_2)$, following [BAS09], we denote

$$a^\wedge = \begin{pmatrix} a & & \\ & 1 & \\ & & a^* \end{pmatrix}.$$

For $n \le r$, denote $w_{n,r-n} = \begin{pmatrix} & I_n \\ I_{r-n} & \end{pmatrix}^\wedge$. Note that $w_{n,r-n}^{-1} = w_{r-n,n}$. Denote

$$\widetilde{w}_n = w_{n,r-n} w_n w_{n,r-n}^{-1} = \begin{pmatrix} & & I_n \\ & I_{2(r-n)+1} & \\ I_n & & \end{pmatrix}.$$

6.2. **Uniqueness of Bessel models.** We review a special case of Bessel models considered in [GGP12a].

Fix a pair of integers n, r with $n \leq r$. We denote $P = MN$ the parabolic subgroup of $\mathrm{U}_{2r+1}(k)$ with Levi subgroup

$$M = \left\{\mathrm{diag}(a_1, \ldots, a_{r-n}, g, (a_{r-n}^{-1})^{\iota}, \ldots, (a_1^{-1})^{\iota}), a_i \in k_2^{\times}, g \in \mathrm{U}_{2n+1}(k)\right\},$$

and unipotent subgroup N. Consider the character $\overline{\psi}_N$ defined by

$$\overline{\psi}_N(u_{ij}) = \overline{\psi}\left(\sum_{i=1}^{r-n-1} \mathrm{Tr}_{k_2/k}(u_{i,i+1}) + \mathrm{Tr}_{k_2/k}(u_{r-n,r+1})\right), u = (u_{ij})_{1\leq i,j\leq 2r+1} \in N.$$

Denote by H the following subgroup of P:

$$H = \mathrm{U}_{2n}(k) \ltimes N.$$

Recall that we always identify U_{2n} with a subgroup of U_{2r+1} and with this identification, U_{2n} is a subgroup of M. In matrix form, we have

$$H = \left\{\begin{pmatrix} u & * & * & * & * \\ & a & & b & * \\ & & 1 & & * \\ & c & & d & * \\ & & & & u^* \end{pmatrix}, u \in Z_{r-n}, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{U}_{2n}, a, b, c, d \in \mathrm{Mat}_{n\times n}(k_2)\right\}.$$

Here Z_{r-n} is the upper triangular unipotent subgroup of $\mathrm{GL}_{r-n}(k_2)$. There exists a representation ν of H such that $\nu|_{\mathrm{U}_{2n}(k)} = 1$, and $\nu|_N = \overline{\psi}_N$. See [GGP12a, §12]. Then, we have the following

Proposition 6.1. *Let $P' = M'N'$ be a parabolic subgroup of $\mathrm{U}_{2n}(k)$ and σ be an irreducible representation of the Levi M'. Let π be an irreducible cuspidal representation of $\mathrm{U}_{2r+1}(k)$ and $\tau = \mathrm{Ind}_{M'N'}^{\mathrm{U}_{2n}(k)}(\sigma \otimes 1_{N'})$. Then*

(1) *If $n = r$, we have*

$$\dim \mathrm{Hom}_{\mathrm{U}_{2r}(k)}(\pi, \tau) \leq 1.$$

(2) *For $1 \leq n < r$. We have*

$$\dim \mathrm{Hom}_H(\pi, \tau \otimes \nu) \leq 1.$$

Proof. If σ is cuspidal, the assertion is [GGP12b, Propositions 5.1 and 5.3]. If σ is not cuspidal, then there exists a parabolic subgroup $P_0 = M_0N_0$ of M' and an irreducible cuspidal representation σ_0 of M_0 such that $\sigma \subset \mathrm{Ind}_{M_0N_0}^{M'}(\sigma_0 \otimes 1_{N_0})$. Since $\mathrm{Ind}_{M'N'}^{\mathrm{U}_{2n}(k)}(\mathrm{Ind}_{M_0N_0}^{M'}(\sigma_0 \otimes 1_{N_0}) \otimes 1_{N'}) = \mathrm{Ind}_{M_0N_0N'}^{\mathrm{U}_{2n}(k)}(\sigma_0 \otimes 1_{N_0N'})$, the assertion follows from the case that σ is cuspidal. □

6.3. **Gamma factors.** As an analogue of the p-adic fields case, we define gamma factors for generic representations of $\mathrm{U}_{2r+1}(k) \times \mathrm{GL}_n(k_2)$.

Let τ be an irreducible generic representation of $\mathrm{GL}_n(k_2)$. We view τ as a representation of $M_n \cong \mathrm{GL}_n(k_2)$ and consider the induced representation $I(\tau) = \mathrm{Ind}_{P_n}^{\mathrm{U}_{2n}(k)}(\tau \otimes 1_{N_n})$. Let Z_n be the upper triangular unipotent subgroup of $\mathrm{GL}_n(k_2)$ and let $\overline{\psi}_{Z_n}^{-1}$ be the character of Z_n defined similarly as in §3.4, and let $\Lambda = \Lambda_\tau \in \mathrm{Hom}_{Z_n}(\tau, \overline{\psi}_{Z_n}^{-1})$ be a fixed nonzero Whittaker functional. Let $W_v(a) = \Lambda(\tau(a)v)$ and $W_v^*(a) = \Lambda(\tau(d_na^*)v)$, where $d_n = \mathrm{diag}(-1, 1, \ldots, (-1)^n)$. For $\xi \in I(\tau)$, let $f_\xi : \mathrm{U}_{2n}(k) \times \mathrm{GL}_n(k) \to \mathbb{C}$ be the function defined by

$$f_\xi(g, a) = \Lambda(\tau(a)\xi(g)).$$

Let $\mathcal{I}(\tau, \overline{\psi}_{Z_n}^{-1})$ be the space of functions $f_\xi, \xi \in I(\tau)$. Similarly, as in §3.4, we have a standard intertwining operator $M(\tau, \overline{\psi}^{-1}) : \mathcal{I}(\tau, \overline{\psi}_{Z_n}^{-1}) \to \mathcal{I}(\tau^*, \overline{\psi}_{Z_n}^{-1})$.

Recall that U_m is the upper triangular subgroup of $\mathrm{U}_m(k)$. For simplicity, write $U^n = U_{2n}$ for $1 \leq n \leq r$ and write $U = U_{2r+1}$. Let $\overline{\psi}_U$ be the generic character of U defined by

$$\overline{\psi}_U(u) = \overline{\psi}\left(\sum_{i=1}^{r} u_{i,i+1}\right), u = (u_{i,j}) \in U.$$

Let π be an irreducible $\overline{\psi}_U$-generic cuspidal representation of $\mathrm{U}_{2r+1}(k)$. Let $\mathcal{W}(\pi,\overline{\psi}_U)$ be the Whittaker model of π.

For $W\in\mathcal{W}(\pi,\overline{\psi}_U)$, $f\in\mathcal{I}(\tau,\overline{\psi}_{Z_n}^{-1})$, we consider the "integral"

$$\Psi(W,f)=\sum_{h\in U^n\backslash \mathrm{U}_{2n}(k)}\sum_{x\in \mathrm{Mat}_{(r-n)\times n}(k_2)} W\left(w_{n,r-n}\begin{pmatrix} I_{r-n} & x\\ & I_n\end{pmatrix}^{\wedge} h w_{n,r-n}^{-1}\right) f(h,I_n).$$

The above "integral" is the finite fields analogue of the corresponding integral over p-adic fields, see [BAS09].

We first show that the above integral is non-vanishing for some choices of W,f. Let π be any irreducible $\overline{\psi}_U$-generc cuspidal representation of $\mathrm{U}_{2r+1}(k)$. Let $\mathcal{B}_{\pi,\overline{\psi}}$ be the (normalized) Bessel function with the following properties

$$\mathcal{B}_{\pi,\overline{\psi}}(1)=1,$$

and

$$\mathcal{B}_{\pi,\overline{\psi}}(u_1gu_2)=\overline{\psi}_U(u_1u_2)\mathcal{B}_{\pi,\overline{\psi}}(g),\forall g\in \mathrm{U}_{2r+1}(k),u_1,u_2\in U.$$

The existence of such function can be proved similarly as in the Sp_{2r} case, see §3.1. Given $v\in\tau$, a fixed irreducible generic representation of $\mathrm{GL}_n(k_2)$, we consider the section $\xi_v\in I(\tau)$ such that $\mathrm{supp}(\xi_v)=P_n=M_nN_n$ and

$$\xi_v(m_n(a)u)=\tau(a)v, a\in \mathrm{GL}_n(k_2), u\in N_n.$$

Let $f_v=f_{\xi_v}\in\mathcal{I}(\tau,\overline{\psi}_{Z_n}^{-1})$ and $\widetilde{f}_v=M(\tau,\overline{\psi}^{-1})f_v$.

Lemma 6.2. *There exists a choice of $v\in\tau$ such that*

$$\Psi(\mathcal{B}_{\pi,\overline{\psi}},f_v)\neq 0.$$

Proof. The proof is similar to that of Proposition 3.7 and we give a sketch here. By the definition of f_v, we have

$$\begin{aligned}\Psi(\mathcal{B}_{\pi,\overline{\psi}},f_v)&=\sum_{a\in Z_n\backslash \mathrm{GL}_n(k_2)}\sum_{x\in \mathrm{Mat}_{(r-n)\times n}(k_2)}\mathcal{B}_{\pi,\overline{\psi}}\left(w_{n,r-n}\begin{pmatrix} I_{r-n} & x\\ & I_n\end{pmatrix}^{\wedge} m_n(a)w_{n,r-n}^{-1}\right)W_v(a)\\ &=\sum_{a\in Z_n\backslash \mathrm{GL}_n(k_2)}\sum_{x\in \mathrm{Mat}_{(r-n)\times n}(k_2)}\mathcal{B}_{\pi,\overline{\psi}}\left(\begin{pmatrix} a & \\ x & I_{r-n}\end{pmatrix}^{\wedge}\right)W_v(a).\end{aligned}$$

By similar results as in Corollary 3.4 and Lemma 3.5, we get

$$\mathcal{B}_{\pi,\overline{\psi}}\left(\begin{pmatrix} a & \\ x & I_{r-n}\end{pmatrix}^{\wedge}\right)=0,\ \text{unless } x=0,\ \text{and } a\in Z_n.$$

Thus we get

$$\Psi(\mathcal{B}_{\pi,\overline{\psi}},f_v)=W_v(1).$$

We can pick $v\in\tau$ such that $W_v(1)\neq 0$ and the assertion follows. □

Proposition 6.3. *There is a constant $\gamma(\pi\times\tau,\overline{\psi})\in\mathbb{C}$ such that*

$$\Psi(W,M(\tau,\overline{\psi}^{-1})f)=\gamma(\pi\times\tau,\overline{\psi})\Psi(W,f),$$

for all $W\in\mathcal{W}(\pi,\overline{\psi}_U)$, $f\in\mathcal{I}(\tau,\overline{\psi}_{Z_n}^{-1})$.

Proof. One can check that both $(W,f)\mapsto\Psi(W,f)$ and $(W,f)\mapsto\Psi(W,M(\tau,\overline{\psi}^{-1})f)$ define linear functionals in $\mathrm{Hom}_H(\pi\otimes I(\tau),\nu)$. Thus the result follows from Proposition 6.1 and Lemma 6.2 directly. □

Similarly, as in the Sp_{2r} and U_{2r} cases, the gamma factor $\gamma(\pi\times\tau,\overline{\psi})$ can be expressed in terms of Bessel functions $\mathcal{B}_{\pi,\overline{\psi}}$ and $\mathcal{B}_{\tau,\overline{\psi}^{-1}}$ of π and τ respectively. For $1\le n\le r$ and $a\in\mathrm{GL}_n(k_2)$, denote

$$t_n(a)=\mathrm{diag}(a,I_{2(r-n)+1},a^*)\in \mathrm{U}_{2r+1}(k).$$

Proposition 6.4. *We have*

$$\gamma(\pi \times \tau, \overline{\psi}) = q^{2rn-n^2} \sum_{a \in Z_n \backslash \mathrm{GL}_n(k_2)} \mathcal{B}_{\pi,\overline{\psi}}(t_n(a)\widetilde{w}_n)\, \mathcal{B}_{\tau,\overline{\psi}^{-1}}(d_n a^*).$$

Proof. For $v \in \tau$, let $f_v \in \mathcal{I}(\tau, \overline{\psi}_{Z_n}^{-1})$ be the section as in the proof of Lemma 6.2. We have

$$\Psi(\mathcal{B}_{\pi,\overline{\psi}}, f_v) = W_v(1).$$

We now compute $\Psi(\mathcal{B}_{\pi,\overline{\psi}}, \widetilde{f}_v)$ where $\widetilde{f}_v = M(\tau, \overline{\psi}^{-1}) f_v$. By an analogue of Lemma 3.6, we have $\mathrm{supp}(\widetilde{f}_v) = M_n w_n N_n$, and for $a \in \mathrm{GL}_n(k_2), u \in N_n$, we have

$$\widetilde{f}_v(m_n(a) w_n u, I_n) = W_v^*(a).$$

Thus we get

$$\begin{aligned}
&\Psi(\mathcal{B}_{\pi,\overline{\psi}}, \widetilde{f}_v)\\
&= \sum_{h \in U^n \backslash \mathrm{U}_{2n}(k)} \sum_{x \in \mathrm{Mat}_{(r-n)\times n}(k_2)} \mathcal{B}_{\pi,\overline{\psi}}\left(w_{n,r-n} \begin{pmatrix} I_{r-n} & x \\ & I_n \end{pmatrix}^{\wedge} h w_{n,r-n}^{-1} \right) \widetilde{f}_v(h, I_n)\\
&= |N_n| \sum_{a \in Z_n \backslash \mathrm{GL}_n(k_2)} \sum_{x \in \mathrm{Mat}_{(r-n)\times n}(k_2)} \mathcal{B}_{\pi,\overline{\psi}}\left(w_{n,r-n} \begin{pmatrix} I_{r-n} & x \\ & I_n \end{pmatrix}^{\wedge} m_n(a) w_n w_{n,r-n}^{-1} \right) W_v^*(a)\\
&= |N_n| \sum_{a \in Z_n \backslash \mathrm{GL}_n(k_2)} \sum_{x \in \mathrm{Mat}_{(r-n)\times n}(k_2)} \mathcal{B}_{\pi,\overline{\psi}}(t_n(a)\widetilde{w}_n u_1(x))\, W_v^*(a)\\
&= |N_n| |\mathrm{Mat}_{(r-n)\times n}(k_2)| \sum_{a \in Z_n \backslash \mathrm{GL}_n(k_2)} \mathcal{B}_{\pi,\overline{\psi}}(t_n(a)\widetilde{w}_n)\, W_v^*(a),
\end{aligned}$$

where

$$u_1(x) = \begin{pmatrix} I_n & & & x & \\ & I_{r-n} & & & x' \\ & & 1 & & \\ & & & I_{r-n} & \\ & & & & I_n \end{pmatrix} \in \mathrm{U}_{2r+1}.$$

Note that $|N_n| = q^{n^2}$ and $|\mathrm{Mat}_{(r-n)\times n}(k_2)| = q^{2rn-2n^2}$. Thus we get

$$\Psi(\mathcal{B}_{\pi,\overline{\psi}}, \widetilde{f}_v) = q^{2rn-n^2} \sum_{a \in Z_n \backslash \mathrm{GL}_n(k_2)} \mathcal{B}_{\pi,\overline{\psi}}(t_n(a)\widetilde{w}_n)\, W_v^*(a). \tag{6.1}$$

By taking $v \in \tau$ the Whittaker vector and applying the functional equation, we get the result. □

6.4. **A converse theorem.** Similarly, as in the Sp_{2r} and U_{2r} cases, we also have the following

Theorem 6.5. *Let π, π' be two irreducible $\overline{\psi}_U$-generic cuspidal representations of $\mathrm{U}_{2r+1}(k)$ with the same central character. If*

$$\gamma(\pi \times \tau, \overline{\psi}) = \gamma(\pi' \times \tau, \overline{\psi})$$

for all irreducible generic representations τ of $\mathrm{GL}_n(k_2)$, for all n with $1 \le n \le r$, then $\pi \cong \pi'$.

Note that in the U_{2r+1} case, there is only one class of generic characters of U. Thus if π is $\overline{\psi}_U$-generic, then it is generic with respect to any generic character $\overline{\psi}'_U$ of U. A p-adic version of Theorem 6.5 was proven in [Zh19].

The proof of Theorem 6.5 is quite similar to that of Sp_{2r} case and is sketched below.

Write $B = B_{2r+1}$ and $A = A_{2r+1}$. Then $B = AU$ is the upper-triangular Borel subgroup of $\mathrm{U}_{2r+1}(k)$ with torus A and maximal unipotent U. Recall that π, π' are the two irreducible $\overline{\psi}_U$-generic cuspidal representations in the assumption of Theorem 6.5, let $\mathcal{B}_{\pi,\overline{\psi}}$ and $\mathcal{B}_{\pi',\overline{\psi}}$ be the corresponding (normalized) Bessel representations. The idea is to prove that

$$\mathcal{B}_{\pi,\overline{\psi}}(g) = \mathcal{B}_{\pi',\overline{\psi}}(g), \forall g \in BwB, \forall w \in W(\mathrm{U}_{2r+1}), \tag{6.2}$$

where $W(\mathrm{U}_{2r+1})$ is the Weyl group of $\mathrm{U}_{2r+1}(k)$. Since $\mathrm{U}_{2r+1}(k) = \coprod_{w \in W(\mathrm{U}_{2r+1})} BwB$, the above equality implies that $\pi \cong \pi'$ by the uniqueness of Whittaker models of π, π'.

Let

$$\mathrm{B}(\mathrm{U}_{2r+1}) = \{w \in W(\mathrm{U}_{2r+1}) : w\alpha_i \text{ is either simple or negative}, \forall \alpha_i \in \Delta(\mathrm{U}_{2r+1})\}.$$

The same proof as in Lemma 3.2 will show that, if $w \notin \mathrm{B}(\mathrm{U}_{2r+1})$, then

$$\mathcal{B}_{\pi,\overline{\psi}}(g) = \mathcal{B}_{\pi',\overline{\psi}}(g) = 0, \forall g \in BwB.$$

Thus it suffices to show that

$$\mathcal{B}_{\pi,\overline{\psi}}(g) = \mathcal{B}_{\pi',\overline{\psi}}(g), \quad \forall g \in BwB, \forall w \in \mathrm{B}(\mathrm{U}_{2r+1}). \tag{6.3}$$

Given $w \in \mathrm{B}(\mathrm{U}_{2r+1})$, set $\theta_w = \{\beta \in \Delta(\mathrm{U}_{2r+1}), w(\beta) > 0\}$. As in the Sp_{2r} case, the assignment $w \mapsto \theta_w$ defines a bijection from $\mathrm{B}(\mathrm{U}_{2r+1})$ to $\mathcal{P}(\Delta(\mathrm{U}_{2r+1}))$, the set of all subsets of $\Delta(\mathrm{U}_{2r+1})$. For simplicity, we will write $\Delta(\mathrm{U}_{2r+1})$ as Δ in the rest of this section. Given a subset $\theta \subset \Delta$, let $w_\theta \in \mathrm{B}(\mathrm{U}_{2r+1})$ be the corresponding element such that $\theta_{w_\theta} = \theta$. As Lemma 3.3, We can check that

$$\widetilde{w}_n \text{ is a representative of } w_{\Delta-\{\alpha_n\}}, 1 \le n \le r. \tag{6.4}$$

In the following, we don't distinguish a Weyl element and its representative and just write that $\widetilde{w}_n = w_{\Delta-\{\alpha_n\}}$.

The main step towards a proof of (6.3) is the following analogue of Theorem 4.3.

Proposition 6.6. *The condition $\gamma(\pi \times \tau, \overline{\psi}) = \gamma(\pi' \times \tau, \overline{\psi})$ for all irreducible generic representations τ of $\mathrm{GL}_n(k)$ implies that*

$$\mathcal{B}_{\pi,\overline{\psi}}(t_n(a)\widetilde{w}_n) = \mathcal{B}_{\pi',\overline{\psi}}(t_n(a)\widetilde{w}_n), \forall a \in \mathrm{GL}_n(k_2).$$

Proof. For an irreducible generic representation τ and $v \in \tau$, let $f_v \in \mathcal{I}(\tau, \overline{\psi}_{Z_n}^{-1})$ be the section used in Proposition 6.4. We have

$$\Psi(\mathcal{B}_{\pi,\overline{\psi}}, f_v) = \Psi(\mathcal{B}_{\pi',\overline{\psi}}, f_v) = W_v(1).$$

By the condition $\gamma(\pi \times \tau, \overline{\psi}) = \gamma(\pi' \times \tau, \overline{\psi})$, we get that

$$\Psi(\mathcal{B}_{\pi,\overline{\psi}}, \widetilde{f}_v) = \Psi(\mathcal{B}_{\pi',\overline{\psi}}, \widetilde{f}_v).$$

By the calculation of $\Psi(\mathcal{B}_{\pi,\overline{\psi}}, \widetilde{f}_v)$ in (6.1) and its analogue of $\Psi(\mathcal{B}_{\pi',\overline{\psi}}, \widetilde{f}_v)$, we get that

$$\sum_{a \in Z_n \backslash \mathrm{GL}_n(k_2)} (\mathcal{B}_{\pi,\overline{\psi}}(t_n(a)\widetilde{w}_n) - \mathcal{B}_{\pi',\overline{\psi}}(t_n(a)\widetilde{w}_n)) W_v^*(a) = 0.$$

Since the last equation is true for all irreducible generic τ and all $v \in \tau$, by Lemma 4.2, we get

$$\mathcal{B}_{\pi,\overline{\psi}}(t_n(a)\widetilde{w}_n) - \mathcal{B}_{\pi',\overline{\psi}}(t_n(a)\widetilde{w}_n) = 0, \forall a \in \mathrm{GL}_n(k_2).$$

This concludes the proof. □

Following Proposition 6.6, the rest of the proof of Theorem 6.5 is similar to that of Theorem 4.1 and we omit the details.

7. A converse theorem for $\mathrm{SO}_{2r+1}(k)$

In this section, let k be a finite field without any restriction on the characteristic.

7.1. **Notations.** The notation we adopt here follows from the analogous notation in the U_{2r+1} case as in §6, see also [BAS09, Zh18], which is slightly different from the original context [So93] and [Ka15].

For a positive integer m, we still denote
$$J_m = \begin{pmatrix} & 1 \\ J_{m-1} & \end{pmatrix}, J_1 = (1).$$
Let
$$\mathrm{SO}_m(k) = \left\{ g \in \mathrm{GL}_m(k) : {}^t g J_m g = J_m \right\}.$$
For $m = 2n$ even, we denote by $P_n = M_n \ltimes N_n$ the Siegel type parabolic subgroup of SO_m, where
$$M_n = \left\{ m_n(a) := \begin{pmatrix} a & \\ & a^* \end{pmatrix} \in \mathrm{SO}_{2n}(k), a \in \mathrm{GL}_n(k) \right\}, \quad N_n = \left\{ u_n(x) := \begin{pmatrix} I_n & x \\ & I_n \end{pmatrix} \in \mathrm{SO}_{2n}(k) \right\}.$$
Here a^* is determined by a such that $m_n(a) \in \mathrm{SO}_{2n}$. One can check that $a^* = J_n {}^t a^{-1} J_n$. Denote $w_n = \begin{pmatrix} & I_n \\ I_n & \end{pmatrix} \in \mathrm{SO}_{2n}(k)$.

Let $B_m = A_m U_m$ be the upper triangular Borel subgroup of $\mathrm{SO}_m(k)$ with maximal torus A_m and maximal unipotent U_m. In this section, we will mainly consider the case that m is an odd positive integer, i.e., $m = 2r+1$. A typical element $t \in A_{2r+1}$ has the form
$$t = \mathrm{diag}(a_1, \ldots, a_r, 1, a_r^{-1}, \ldots, a_1^{-1}), a_1, \ldots, a_r \in k^\times.$$
Define characters $\alpha_i, 1 \le i \le r$, on A_{2r+1} by
$$\alpha_i(t) = a_i/a_{i+1}, 1 \le i \le r-1; \alpha_r(t) = a_r,$$
where $t = \mathrm{diag}(a_1, \ldots, a_r, 1, a_r^{-1}, \ldots, a_1^{-1})$. Denote $\Delta(\mathrm{SO}_{2r+1}) = \{\alpha_i, 1 \le i \le r\}$, which is the set of simple roots of SO_{2r+1}.

For $n \le r$, we fix an embedding $\mathrm{SO}_{2n}(k) \to \mathrm{SO}_{2r+1}(k)$
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} I_{r-n} & & & & \\ & a & & b & \\ & & 1 & & \\ & c & & d & \\ & & & & I_{r-n} \end{pmatrix}.$$
An element of $\mathrm{SO}_{2n}(k)$ will be viewed as an element of SO_{2r+1} through this embedding without further notice.

For $a \in \mathrm{GL}_r(k)$, we denote
$$a^\wedge = \begin{pmatrix} a & & \\ & 1 & \\ & & a^* \end{pmatrix} \in \mathrm{SO}_{2r+1}(k).$$
For $n \le r$, denote $w_{n,r-n} = \begin{pmatrix} & I_n \\ I_{r-n} & \end{pmatrix}^\wedge$. Note that $w_{n,r-n}^{-1} = w_{r-n,n}$. Denote
$$\widetilde{w}_n = w_{n,r-n} w_n w_{n,r-n}^{-1} = \begin{pmatrix} & & I_n \\ & I_{2(r-n)+1} & \\ I_n & & \end{pmatrix}.$$

7.2. **A multiplicity one result.** Bessel models for SO_{2r+1} are similar to those in the U_{2r+1} case considered in §6. We state the definition here for completeness.

Let $P = MN$ be the parabolic subgroup of SO_{2r+1} with Levi subgroup
$$M = \left\{ \mathrm{diag}(a_1, \ldots, a_{r-n}, g, a_{r-n}^{-1}, \ldots, a_1^{-1}), a_i \in k^\times, g \in \mathrm{SO}_{2n+1}(k) \right\},$$
and unipotent subgroup N. Denote by Z_{r-n} the upper triangular unipotent subgroup of $\mathrm{GL}_{r-n}(k)$, which is embedded into N in the natural way. Denote by $\overline{\psi}_N$ the character on N defined by
$$\overline{\psi}_N(u_{ij}) = \overline{\psi}\left(\sum_{i=1}^{r-n-1} u_{i,i+1} + u_{r-n,r+1} \right), u = (u_{ij})_{1 \le i,j \le 2r+1} \in N.$$

For example, if $n=1, r=2$, we have

$$\overline{\psi}_N\left(\begin{pmatrix}1 & x_1 & x_2 & x_3 & x_4\\ & 1 & & & x_3'\\ & & 1 & & x_2'\\ & & & 1 & x_1'\\ & & & & 1\end{pmatrix}\right)=\overline{\psi}(x_2).$$

Denote by H the following subgroup of SO_{2r+1}:

$$H=\mathrm{SO}_{2n}\ltimes N.$$

In matrix form, we have

$$H=\left\{\begin{pmatrix}u & * & * & * & *\\ & a & & b & *\\ & & 1 & & *\\ & c & & d & *\\ & & & & u^*\end{pmatrix}, u\in Z_{r-n}, \begin{pmatrix}a & b\\ c & d\end{pmatrix}\in \mathrm{SO}_{2n}, a,b,c,d\in \mathrm{Mat}_{n\times n}(k)\right\}.$$

There is a representation ν of H such that $\nu|_N=\overline{\psi}_N$, and $\nu|_{\mathrm{SO}_{2n}(k)}=1$, see [GGP12a, §12]. The pair (H,ν) is called a Bessel data of $\mathrm{SO}_{2r+1}(k)$.

Proposition 7.1. *Let $P'=M'N'$ be a parabolic subgroup of $\mathrm{SO}_{2n}(k)$ and σ be an irreducible representation of the Levi M'. Let π be an irreducible cuspidal representation of $\mathrm{SO}_{2r+1}(k)$ and $\tau=\mathrm{Ind}_{P'}^{\mathrm{SO}_{2n}(k)}(\sigma)$. Then*

(1) *If $n=r$, we have*

$$\dim \mathrm{Hom}_{\mathrm{SO}_{2r}(k)}(\pi,\tau)\le 1.$$

(2) *For $1\le n<r$. We have*

$$\dim \mathrm{Hom}_H(\pi,\tau\otimes\nu)\le 1.$$

Proof. The proof is similar to that of Proposition 6.1. We omit the details. □

7.3. **The gamma factors.** Twisted local gamma factors for $\mathrm{SO}_{2r+1}\times \mathrm{GL}_n$ over a p-adic field have been defined by Soudry in [So93]. We consider their analogues over finite fields.

Let τ be an irreducible generic representation of $\mathrm{GL}_n(k)$. We view τ as a representation of $M_n\cong \mathrm{GL}_n(k)$ and consider the induced representation $I(\tau)=\mathrm{Ind}_{P_n}^{\mathrm{SO}_{2n}(k)}(\tau)$. Let $\overline{\psi}_{Z_n}^{-1}$ be the character of $Z_n(k)$ defined as in §3.4, and let $\Lambda_\tau\in \mathrm{Hom}_{Z_n}(\tau,\overline{\psi}_{Z_n}^{-1})$ be a fixed nonzero Whittaker functional. For $\xi\in I(\tau)$, let $f_\xi:\mathrm{SO}_{2n}(k)\times \mathrm{GL}_n(k)\to\mathbb{C}$ be the function defined by

$$f_\xi(g,a)=\Lambda_\tau(\tau(a)\xi(g)).$$

Let $\mathcal{I}(\tau,\overline{\psi}_{Z_n}^{-1})$ be the space of functions $f_\xi, \xi\in I(\tau)$. Similarly, as in §3.4, we have a standard intertwining operator $M(\tau,\overline{\psi}^{-1}):\mathcal{I}(\tau,\overline{\psi}_{Z_n}^{-1})\to\mathcal{I}(\tau^*,\overline{\psi}_{Z_n}^{-1})$. Let $W_v(a)=\Lambda(\tau(a)v)$ and $W_v^*(a)=\Lambda(\tau(d_na^*)v)$ for $a\in \mathrm{GL}_n(k)$, where $d_n=\mathrm{diag}(-1,1,\dots,(-1)^n)\in \mathrm{GL}_n$ as usual.

Recall that U_m is the upper triangular subgroup of $\mathrm{SO}_m(k)$. For simplicity, write $U^n=U_{2n}$ for $1\le n\le r$ and $U=U_{2r+1}$. Let $\overline{\psi}_U$ be the generic character of U defined by

$$\overline{\psi}_U(u)=\overline{\psi}\left(\sum_{i=1}^r u_{i,i+1}\right), u=(u_{i,j})\in U.$$

Let π be an irreducible $\overline{\psi}_U$-generic cuspidal representation of $\mathrm{SO}_{2r+1}(k)$. Let $\mathcal{W}(\pi,\overline{\psi}_U)$ be the Whittaker model of π.

For $W\in\mathcal{W}(\pi,\overline{\psi}_U), f\in\mathcal{I}(\tau,\overline{\psi}_{Z_n}^{-1})$, we consider the "integral"

$$\Psi(W,f)=\sum_{h\in U^n\backslash \mathrm{SO}_{2n}(k)}\ \sum_{x\in \mathrm{Mat}_{(r-n)\times n}(k)} W\left(w_{n,r-n}\begin{pmatrix}I_{r-n} & x\\ & I_n\end{pmatrix}^{\wedge} h w_{n,r-n}^{-1}\right)f(h,I_n).$$

Similarly, as in the U_{2r+1} case, analogous to Lemma 6.2, we can show that there exist choices W,f such that $\Psi(W,f)\neq 0$.

Proposition 7.2. *There is a constant* $\gamma(\pi \times \tau, \overline{\psi}) \in \mathbb{C}$ *such that*

$$\Psi(W, M(\tau, \overline{\psi}^{-1})f) = \gamma(\pi \times \tau, \psi)\Psi(W, f),$$

for all $W \in \mathcal{W}(\pi, \overline{\psi}_U)$, $f \in \mathcal{I}(\tau, \overline{\psi}_{Z_n}^{-1})$.

Proof. This follows from Proposition 7.1 directly. □

Similarly, as in Proposition 6.4, we can express $\gamma(\pi \times \tau, \overline{\psi})$ in terms of Bessel functions. Let π be an irreducible generic cuspidal representation of $\mathrm{SO}_{2r+1}(k)$ and τ be an irreducible generic representation representation of $\mathrm{GL}_n(F)$ with $1 \le n \le r$. Let $\mathcal{B}_{\pi,\overline{\psi}}$ and $\mathcal{B}_{\tau,\overline{\psi}^{-1}}$ be the corresponding (normalized) Bessel functions of π and τ respectively.

Proposition 7.3. *We have*

$$\gamma(\pi \times \tau, \overline{\psi}) = q^{rn-n(n+1)/2} \sum_{a \in Z_n \backslash \mathrm{GL}_n(k)} \mathcal{B}_{\pi,\overline{\psi}}(t_n(a)\widetilde{w}_n)\, \mathcal{B}_{\tau,\overline{\psi}^{-1}}(d_n a^*)$$

The proof is similar to that of Proposition 6.4 and we thus omit the details.

7.4. **A converse theorem.** Similarly, as in the Sp_{2r} and U_{2r} cases, we also have the following

Theorem 7.4. *Let* π, π' *be two irreducible* $\overline{\psi}_U$*-generic cuspidal representations of* $\mathrm{SO}_{2r+1}(k)$. *If*

$$\gamma(\pi \times \tau, \overline{\psi}) = \gamma(\pi' \times \tau, \overline{\psi})$$

for all irreducible generic representations τ *of* $\mathrm{GL}_n(k)$, *for all* n *with* $1 \le n \le r$, *then* $\pi \cong \pi'$.

Note that in the SO_{2r+1} case, there is only one class of generic characters of U. Thus if π is $\overline{\psi}_U$-generic, then it is generic with respect to any generic character $\overline{\psi}'_U$ of U.

The proof of Theorem 7.4 is similar to that of the U_{2r+1} case and is thus omitted.

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