ERRATUM TO "ON THE NON-VANISHING OF THE CENTRAL VALUE OF THE RANKIN-SELBERG L-FUNCTIONS"

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ABSTRACT. We complete the proof of Proposition 5.3 of [GJR04].

In this note, we complete the proof of Proposition 5.3 of [GJR04] which is stated as follows.

Proposition 0.1 (Proposition 5.3, [GJR04]). *If the period, defined in (5.2) of [GJR04],*

$$\mathcal{P}_{r,r-l}(\phi_\sigma, \widetilde{\phi}_{\widetilde{\tau}}, \varphi_l),$$

does not vanish for some given $\phi_\sigma \in V_\sigma$ and $\widetilde{\phi}_{\widetilde{\tau}} \in V_{\widetilde{\tau}}$, then the integral

$$\int_{K\times \mathrm{Mat}_{r-l,2r}(\mathbb{A})\times \mathrm{PM}_{2r,l}} \phi_{\pi_\psi(\widetilde{\tau})\otimes\sigma}(mk)\mathcal{F}^\psi(\widetilde{\phi}_{\pi_\psi(\widetilde{\tau})\otimes\widetilde{\tau}})(mv^-(p_1)wk)dmdp_1dk,$$

does not vanish for some choice of data $\phi_{\pi_\psi(\widetilde{\tau})\otimes\sigma} \in \mathcal{A}_{P_{2r,l},\pi_\psi(\widetilde{\tau})\otimes\sigma}$ and $\widetilde{\phi}_{\pi_\psi(\widetilde{\tau})\otimes\widetilde{\tau}} \in \widetilde{\mathcal{A}}_{\widetilde{P}_{2r,r},\pi_\psi(\widetilde{\tau})\otimes\widetilde{\tau},0}$.

The proof of Proposition 5.3 of [GJR04] is reduced to the proof of the non-vanishing of the following integral (see (0.15) below)

$$\int_{\mathrm{Mat}_{r-l,2r}(\mathbb{A})\times \mathrm{PM}_{2r,l}} \phi_{\pi_\psi(\widetilde{\tau})\otimes\sigma}(m)\mathcal{F}^\psi(\widetilde{\phi}_{\pi_\psi(\widetilde{\tau})\otimes\widetilde{\tau}})(mv^-(p_1)w)dmdp_1$$

for a proper set of sections, which was not complete in [GJR04]. In this note, we complete this proof by proving Proposition 0.3 below. Notation in the above proposition will be explained in Section 0.1.

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0.1. **Notation and the main result in Section 4 of** [GJR04]**.** Let F be a number field. We will use the notation from [GJR04] freely. Let $\mathcal{A}_{P_{2r,l},\pi_\psi(\widetilde{\tau})\otimes\sigma}$ be the set of functions

$$\phi : M_{2r,l}(F)U_{2r,l}(\mathbb{A})\backslash \mathrm{Sp}_{4r+2l}(\mathbb{A}) \to \mathbb{C},$$

such that ϕ is right $K_{\mathrm{Sp}_{4r+2l}}(\mathbb{A})$-finite, and for each $k \in K_{\mathrm{Sp}_{4r+2l}}(\mathbb{A})$, the function $\phi_k : m \mapsto \phi(mk)$, $m \in M_{2r,l}(\mathbb{A})$, belongs to $\pi_\psi(\widetilde{\tau})\otimes\sigma$. For $\phi \in \mathcal{A}_{P_{2r,l},\pi_\psi(\widetilde{\tau})\otimes\sigma}$, let

$$\Phi(\cdot, s, \phi) = \phi(\cdot)\exp\langle s + \rho_{P_{2r,l}}, H_{P_{2r,l}}(\cdot)\rangle.$$

Then

$$\left\{\Phi(\cdot, s, \phi) : \phi \in \mathcal{A}_{P_{2r,l},\pi_\psi(\widetilde{\tau})\otimes\sigma}\right\}$$

is equivalent to $I(s, \pi_\psi(\widetilde{\tau})\otimes\sigma)$. Similarly, let $\widetilde{\mathcal{A}}_{\widetilde{P}_{2r,r},\pi_\psi(\widetilde{\tau})\otimes\widetilde{\tau}}$ be the set of functions

$$\widetilde{\phi} : \widetilde{M}_{2r,r}(F)U_{2r,r}(\mathbb{A})\backslash \widetilde{\mathrm{Sp}}_{6r}(\mathbb{A}) \to \mathbb{C},$$

such that $\widetilde{\phi}$ is right $K_{\mathrm{Sp}_{6r}}(\mathbb{A})$-finite, and for each $k \in K_{\mathrm{Sp}_{6r}}(\mathbb{A})$, the function $\widetilde{\phi}_k : \widetilde{m} \mapsto \widetilde{\phi}(\widetilde{m}k)$, $\widetilde{m} \in \widetilde{M}_{2r,r}(\mathbb{A})$, belongs to $\pi_\psi(\widetilde{\tau})\otimes\widetilde{\tau}$. For $\widetilde{\phi} \in \widetilde{\mathcal{A}}_{\widetilde{P}_{2r,r},\pi_\psi(\widetilde{\tau})\otimes\widetilde{\tau}}$, let

$$\widetilde{\Phi}(\cdot, s, \widetilde{\phi}) = \widetilde{\phi}(\cdot)\gamma_\psi(\det(\cdot))\exp\langle s + \rho_{\widetilde{P}_{2r,r}}, H_{\widetilde{P}_{2r,r}}(\cdot)\rangle.$$

Then

$$\left\{\widetilde{\Phi}(\cdot, s, \widetilde{\phi}) : \widetilde{\phi} \in \widetilde{\mathcal{A}}_{\widetilde{P}_{2r,r},\pi_\psi(\widetilde{\tau})\otimes\widetilde{\tau}}\right\}$$

is equivalent to $\widetilde{I}(s, \pi_\psi(\widetilde{\tau})\otimes\widetilde{\tau})$.

The goal of Section 4 of [GJR04] is to compute the period

(0.1)

$$\mathcal{P}_{3r,r-l}(E_{\frac{1}{2}}(\cdot,\phi), \widetilde{E}_1(\cdot,\widetilde{\phi}), \varphi_{2r+l}) = \int_{[\mathrm{Sp}_{4r+2l}]} E_{\frac{1}{2}}(g,\phi)\mathcal{F}^{\psi}_{\varphi_{2r+l}}(\widetilde{E}_1(\cdot,\widetilde{\phi}))(g)dg,$$

where, $\phi = \phi_{\pi_\psi(\widetilde{\tau})\otimes\sigma} \in \mathcal{A}_{P_{2r,l},\pi_\psi(\widetilde{\tau})\otimes\sigma}$, $E_{\frac{1}{2}}(\cdot,\phi)$ is the residue at $s = \frac{1}{2}$ of the following Eisenstein series

$$E(g, s, \phi) = \sum_{\gamma\in P_{2r,l}(F)\backslash \mathrm{Sp}_{4r+2l}(F)} \Phi(\gamma g, s, \phi),\ g \in \mathrm{Sp}_{4r+2l}(\mathbb{A});$$

$\widetilde{\phi} = \widetilde{\phi}_{\pi_\psi(\widetilde{\tau})\otimes\widetilde{\tau}} \in \widetilde{\mathcal{A}}_{\widetilde{P}_{2r,r},\pi_\psi(\widetilde{\tau})\otimes\widetilde{\tau}}$, $\widetilde{E}_1(\cdot,\widetilde{\phi})$ is the residue at $s = 1$ of the following Eisenstein series

$$\widetilde{E}(g, s, \widetilde{\phi}) = \sum_{\gamma\in P_{2r,r}(F)\backslash \mathrm{Sp}_{6r}(F)} \widetilde{\Phi}(\gamma\widetilde{g}, s, \widetilde{\phi}),\ \widetilde{g} \in \widetilde{\mathrm{Sp}}_{6r}(\mathbb{A});$$

and

$$\mathcal{F}^{\psi}_{\varphi_{2r+l}}(\widetilde{E}_1(\cdot,\widetilde{\phi}))(g) = \int_{[V_{2r+l}]} \widetilde{\theta}^{\psi^{-1}}_{\varphi_{2r+l}}(\ell_{2r+l}(v)\widetilde{g})\widetilde{E}_1(v\widetilde{g},\widetilde{\phi})\psi_{r-l}(v)dv.$$

It turns out (see (4.8) of [GJR04]) that the period (0.1) is the residue at $s=\frac{1}{2}$ of the following period

$$\mathcal{P}_{3r,r-l}(\mathcal{E}_1,\widetilde{E}_1(\cdot,\widetilde{\phi}),\varphi_{2r+l}) = \int_{[\mathrm{Sp}_{4r+2l}]} \mathcal{E}_1(g)\mathcal{F}^{\psi}_{\varphi_{2r+l}}(\widetilde{E}_1(\cdot,\widetilde{\phi}))(g)dg, \tag{0.2}$$

where,

$$\mathcal{E}_1(g) = \sum_{\gamma\in P_{2r,l}(F)\backslash \mathrm{Sp}_{4r+2l}(F)} \Phi(\gamma g,s,\phi)(1-\tau_c(H(\gamma g))).$$

Recall from (4.4) of [GJR04] that for $g = um(g)k \in \mathrm{Sp}_{4r+2l}(\mathbb{A})$ with $u\in U_{2r,l}(\mathbb{A})$, $m(g)\in M_{2r,l}(\mathbb{A})$ and $k\in K_{\mathrm{Sp}_{4r+2l}}(\mathbb{A})$, $H(g) = |\det(m(g))|$. We remark that all $\widetilde{\theta}$ occurred in Sections 4 and 5 of [GJR04], namely for the case of $r\geq l$, should be with respect to the character ψ^{-1}.

Let $\Phi^c(\gamma g,s,\phi) = \Phi(\gamma g,s,\phi)(1-\tau_c(H(\gamma g)))$. By [GJR04, Proposition 4.3], the period (0.2) is equal to

(0.3)

$$\int_{M(F)U(\mathbb{A})\backslash \mathrm{Sp}_{4r+2l}(\mathbb{A})} \Phi^c(\gamma g,s,\phi)\int_{\mathrm{Mat}_{r-l,2r}(\mathbb{A})}\int_{[V_{r,l}]}\sum_{\xi_l\in F^l}\omega_{\psi^{-1}}(\ell(\overline{p}_2,\overline{z})\ell(\overline{p}_1)\widetilde{g})$$

$$\varphi_{2r+l}((0,\xi_l))\widetilde{E}_{1,\widetilde{P}_{2r,r}}(vv^-(p_1)w\widetilde{g},\widetilde{\phi})\psi_{r-l}(v)dvdp_1dg,$$

where w is the following Weyl element on p. 696 of [GJR04],

$$w = \begin{pmatrix} 0 & I_{2r} & 0 & 0 & 0 \\ I_{r-l} & 0 & 0 & 0 & 0 \\ 0 & 0 & I_{2l} & 0 & 0 \\ 0 & 0 & 0 & 0 & I_{r-l} \\ 0 & 0 & 0 & I_{2r} & 0 \end{pmatrix}, \tag{0.4}$$

$\widetilde{E}_{1,\widetilde{P}_{2r,r}}(\widetilde{g},\widetilde{\phi})$ is the constant term of the residue $\widetilde{E}_1(\widetilde{g},\widetilde{\phi})$ along the maximal parabolic subgroup $\widetilde{P}_{2r,r}$, which equals $\widetilde{\mathcal{M}}_1(\widetilde{\Phi})(\widetilde{g})$, the residue at $s=1$ of the intertwining operator $\widetilde{\mathcal{M}}(w_{2r,r},s)(\widetilde{\Phi})(\widetilde{g})$ defined in Section 3.2 of [GJR04]. $\widetilde{\mathcal{M}}(w_{2r,r},s)$ maps sections in the induced representation $\widetilde{I}(s,\pi_\psi(\widetilde{\tau})\otimes\widetilde{\tau})$ to those in the induced representation $\widetilde{I}(-s,w_{2r,r}(\pi_\psi(\widetilde{\tau})\otimes\widetilde{\tau}))$. Note that $\widetilde{\mathcal{M}}_1(\widetilde{\Phi})(\widetilde{g})$ is not identically zero, and $w_{2r,r}(\pi_\psi(\widetilde{\tau})\otimes\widetilde{\tau}) = \pi_\psi(\widetilde{\tau})\otimes\widetilde{\tau}$ since $\pi_\psi(\widetilde{\tau})$ is self-dual.

After applying the Iwasawa decomposition $\mathrm{Sp}_{4r+2l}(\mathbb{A}) = P_{2r,l}(\mathbb{A})K$, $K = K_{\mathrm{Sp}_{4r+2l}}(\mathbb{A})$, we obtain the integral (4.31) of [GJR04], in which

$\widetilde{\mathcal{M}}_1(\widetilde{\Phi})$ belongs to $\widetilde{I}(-1, \pi_\psi(\widetilde{\tau}) \otimes \widetilde{\tau})$. Since the induced representation $\widetilde{I}(-1, \pi_\psi(\widetilde{\tau}) \otimes \widetilde{\tau})$ is reducible, the image $\widetilde{\mathcal{M}}_1(\widetilde{\Phi})$ belongs to a proper subrepresentation of $\widetilde{I}(-1, \pi_\psi(\widetilde{\tau}) \otimes \widetilde{\tau})$. This is the key point that makes the original argument in the proof in [GJR04] for Proposition 5.3 insufficient. Denote the subrepresentation of $\widetilde{I}(-1, \pi_\psi(\widetilde{\tau}) \otimes \widetilde{\tau})$ consisting of the images of $\widetilde{\mathcal{M}}_1$ by $\widetilde{I}_0(-1, \pi_\psi(\widetilde{\tau}) \otimes \widetilde{\tau})$. Recall that $\widetilde{I}(-1, \pi_\psi(\widetilde{\tau}) \otimes \widetilde{\tau})$ is equivalent to

$$\left\{\widetilde{\Phi}(\cdot, -1, \widetilde{\phi}) : \widetilde{\phi} \in \widetilde{\mathcal{A}}_{\widetilde{P}_{2r,r}, \pi_\psi(\widetilde{\tau}) \otimes \widetilde{\tau}}\right\}.$$

Denote the subspace of $\widetilde{\mathcal{A}}_{\widetilde{P}_{2r,r}, \pi_\psi(\widetilde{\tau}) \otimes \widetilde{\tau}}$ corresponding to $\widetilde{I}_0(-1, \pi_\psi(\widetilde{\tau}) \otimes \widetilde{\tau})$ by $\widetilde{\mathcal{A}}_{\widetilde{P}_{2r,r}, \pi_\psi(\widetilde{\tau}) \otimes \widetilde{\tau}, 0}$.

In order to complete the proof of Proposition 5.3 in [GJR04], we find a technically more involved argument, which is not sensitive to which section to be taken in the subrepresentation $\widetilde{I}_0(-1, \pi_\psi(\widetilde{\tau}) \otimes \widetilde{\tau})$ or even in the whole induced representation $\widetilde{I}(-1, \pi_\psi(\widetilde{\tau}) \otimes \widetilde{\tau})$.

Since $\widetilde{\mathcal{M}}_1(\widetilde{\Phi}) \in \widetilde{I}_0(-1, \pi_\psi(\widetilde{\tau}) \otimes \widetilde{\tau})$, from the discussion above, there exists $\widetilde{\phi}_{\pi_\psi(\widetilde{\tau}) \otimes \widetilde{\tau}} \in \widetilde{\mathcal{A}}_{\widetilde{P}_{2r,r}, \pi_\psi(\widetilde{\tau}) \otimes \widetilde{\tau}, 0}$, such that

$$\widetilde{\mathcal{M}}_1(\widetilde{\Phi}) = \widetilde{\phi}_{\pi_\psi(\widetilde{\tau}) \otimes \widetilde{\tau}}\, \gamma_\psi(\det) \exp\langle -1 + \rho_{\widetilde{P}_{2r,r}}, H_{\widetilde{P}_{2r,r}}\rangle.$$

It follows that

$$\begin{aligned} & \widetilde{\mathcal{M}}_1(\widetilde{\Phi})(\widetilde{m}(a,b)vv^-(p_1)wk) \\ = \quad & |\det a|^{-1+2r+\frac{1}{2}}\gamma_\psi(\det a)\widetilde{\phi}_{\pi_\psi(\widetilde{\tau}) \otimes \widetilde{\tau}}(\widetilde{m}(a,b)vv^-(p_1)wk). \end{aligned}$$

After carrying out the calculations from (4.32) to (4.35) of [GJR04], one obtains in Theorem 4.4 of [GJR04] that the period (0.1) is equal to a product that a constant $\mathfrak{c}$ times the integral (4.35) of [GJR04] which is given by

(0.5)
$$\int_{K \times \mathrm{Mat}_{r-l,2r}(\mathbb{A}) \times \mathrm{PM}_{2r,l}} \phi_{\pi_\psi(\widetilde{\tau}) \otimes \sigma}(mk)\mathcal{F}^\psi(\widetilde{\phi}_{\pi_\psi(\widetilde{\tau}) \otimes \widetilde{\tau}})(mv^-(p_1)wk)dmdp_1dk,$$

where the function $\mathcal{F}^\psi(\widetilde{\phi}_{\pi_\psi(\widetilde{\tau}) \otimes \widetilde{\tau}})(mv^-(p_1)wk)$ is defined as in (4.33) of [GJR04] by

(0.6)
$$\int_{[V_{r,l}]} \widetilde{\theta}^{\psi^{-1}}_{\varphi_{2r+l},l}(\ell(\overline{p}_2, \overline{z})\ell(\overline{p}_1)\widetilde{m}(b)k)\widetilde{\phi}_{\pi_\psi(\widetilde{\tau}) \otimes \widetilde{\tau}}(v\widetilde{m}(a,b)v^-(p_1)wk)\psi_{r-l}(v)dv,$$

the integration domain $\mathrm{PM}_{2r,l}$ is given by

(0.7) $$\mathrm{PM}_{2r,l} := (Z_{\mathrm{GL}_{2r}}(\mathbb{A})\mathrm{GL}_{2r}(F)\backslash \mathrm{GL}_{2r}(\mathbb{A})) \times (\mathrm{Sp}_{2l}(F)\backslash \mathrm{Sp}_{2l}(\mathbb{A}))$$

as in (4.34) of [GJR04], and

$$\mathfrak{c} = \frac{\mathrm{vol}(\mathbb{A}^1 \backslash F^\times)}{2rd} \tag{0.8}$$

with d being the number of the real archimedean places of the number field F. Recall from (4.29) of [GJR04] that $\widetilde{\theta}^{\psi^{-1}}_{\varphi_{2r+l},l}$ is defined as follows:

$$\widetilde{\theta}^{\psi^{-1}}_{\varphi_{2r+l},l}(\ell(\bar{p}_2,\bar{z})\ell(\bar{p}_1)\widetilde{g}) = \sum_{\xi_l \in F^l} \omega_{\psi^{-1}}(\ell(\bar{p}_2,\bar{z})\ell(\bar{p}_1)\widetilde{g})\varphi_{2r+l}((0,\xi_l)). \tag{0.9}$$

Recall that for each $k \in K_{\mathrm{Sp}_{4r+2l}}(\mathbb{A})$, $\phi \in \mathcal{A}_{P_{2r,l},\pi_\psi(\widetilde{\tau})\otimes\sigma}$, $\phi_k : m \mapsto \phi(mk)$ belongs to $\pi_\psi(\widetilde{\tau})\otimes\sigma$, and for each $k \in K_{\mathrm{Sp}_{6r}}(\mathbb{A})$, $\widetilde{\phi} \in \widetilde{\mathcal{A}}_{\widetilde{P}_{2r,r},\pi_\psi(\widetilde{\tau})\otimes\widetilde{\tau}}$, $\widetilde{\phi}_k : \widetilde{m} \mapsto \widetilde{\phi}(\widetilde{m}k)$ belongs to $\pi_\psi(\widetilde{\tau})\otimes\widetilde{\tau}$. Let $\mathcal{A}^{ss}_{P_{2r,l},\pi_\psi(\widetilde{\tau})\otimes\sigma}$ be the subset of $\mathcal{A}_{P_{2r,l},\pi_\psi(\widetilde{\tau})\otimes\sigma}$ consisting of sections $\phi_{\pi_\psi(\widetilde{\tau})\otimes\sigma}$ which are factorizable and have the property that $\phi_{\pi_\psi(\widetilde{\tau})\otimes\sigma,1}$ is decomposable in $\pi_\psi(\widetilde{\tau})\otimes\sigma$:

$$\phi_{\pi_\psi(\widetilde{\tau})\otimes\sigma,1} = \phi_{\pi_\psi(\widetilde{\tau})} \otimes \phi_\sigma \in \pi_\psi(\widetilde{\tau})\otimes\sigma, \tag{0.10}$$

where $\phi_{\pi_\psi(\widetilde{\tau})} \in \pi_\psi(\widetilde{\tau})$, $\phi_\sigma \in \sigma$. Let $\widetilde{\mathcal{A}}^{ss}_{\widetilde{P}_{2r,r},\pi_\psi(\widetilde{\tau})\otimes\widetilde{\tau},0}$ be the subset of $\widetilde{\mathcal{A}}_{\widetilde{P}_{2r,r},\pi_\psi(\widetilde{\tau})\otimes\widetilde{\tau},0}$ consisting of sections $\widetilde{\phi}_{\pi_\psi(\widetilde{\tau})\otimes\widetilde{\tau}}$ which are factorizable and have the property that $\widetilde{\phi}_{\pi_\psi(\widetilde{\tau})\otimes\widetilde{\tau},w}$ is decomposable in $\pi_\psi(\widetilde{\tau})\otimes\widetilde{\tau}$:

$$\widetilde{\phi}_{\pi_\psi(\widetilde{\tau})\otimes\widetilde{\tau},w} = \phi'_{\pi_\psi(\widetilde{\tau})} \otimes \widetilde{\phi}_{\widetilde{\tau}} \in \pi_\psi(\widetilde{\tau})\otimes\widetilde{\tau}, \tag{0.11}$$

where $\phi'_{\pi_\psi(\widetilde{\tau})} \in \pi_\psi(\widetilde{\tau})$, $\widetilde{\phi}_{\widetilde{\tau}} \in \widetilde{\tau}$, and w is the Weyl element on p. 696 of [GJR04], see (0.4).

Lemma 0.2. *$\mathcal{A}^{ss}_{P_{2r,l},\pi_\psi(\widetilde{\tau})\otimes\sigma}$ generates a dense subspace of $\mathcal{A}_{P_{2r,l},\pi_\psi(\widetilde{\tau})\otimes\sigma}$.*

Proof. Since $\pi_\psi(\widetilde{\tau})\otimes\sigma$ is generated by pure tensors $\phi_{\pi_\psi(\widetilde{\tau})}\otimes\phi_\sigma$, $\phi_{\pi_\psi(\widetilde{\tau})} \in \pi_\psi(\widetilde{\tau})$, $\phi_\sigma \in \sigma$, the set

$$\{\phi_{\pi_\psi(\widetilde{\tau})\otimes\sigma} | \phi_{\pi_\psi(\widetilde{\tau})\otimes\sigma,1} \in \pi_\psi(\widetilde{\tau})\otimes\sigma\}$$

is generated by the set

$$\{\phi_{\pi_\psi(\widetilde{\tau})\otimes\sigma} | \phi_{\pi_\psi(\widetilde{\tau})\otimes\sigma,1} = \phi_{\pi_\psi(\widetilde{\tau})}\otimes\phi_\sigma \in \pi_\psi(\widetilde{\tau})\otimes\sigma, \phi_{\pi_\psi(\widetilde{\tau})} \in \pi_\psi(\widetilde{\tau}), \phi_\sigma \in \sigma\}.$$

Hence, $\mathcal{A}^{ss}_{P_{2r,l},\pi_\psi(\widetilde{\tau})\otimes\sigma}$ generates the subset of $\mathcal{A}_{P_{2r,l},\pi_\psi(\widetilde{\tau})\otimes\sigma}$ consisting of factorizable sections. Since factorizable sections generate a dense subspace of $\mathcal{A}_{P_{2r,l},\pi_\psi(\widetilde{\tau})\otimes\sigma}$, $\mathcal{A}^{ss}_{P_{2r,l},\pi_\psi(\widetilde{\tau})\otimes\sigma}$ generates a dense subspace of $\mathcal{A}_{P_{2r,l},\pi_\psi(\widetilde{\tau})\otimes\sigma}$. ∎

0.2. **Proof of Proposition 5.3 of** [GJR04]**.** We repeat the proof for Proposition 5.3 in [GJR04] and point out the place that needs a more technical argument, which is now taken care of by Proposition 0.3 below.

Recall from (5.2), (2.11), and (2.13) of [GJR04] that the period $\mathcal{P}_{r,r-l}(\phi_\sigma, \widetilde{\phi}_{\widetilde{\tau}}, \varphi_l)$ equals

$$\int_{[\mathrm{Sp}_{2l}]} \phi_\sigma(g) \int_{[V_{r,l}]} \widetilde{\theta}^{\psi^{-1}}_{\varphi_l}(\ell_l(v)\widetilde{g})\widetilde{\phi}_{\widetilde{\tau}}(v\widetilde{g})\psi_{r-l}(v)dvdg. \tag{0.12}$$

It defines a continuous functional on the space of

$$V_\sigma \otimes \left(\widetilde{\Theta}^{\psi^{-1}}_l \otimes V_{\widetilde{\tau}}\right)^{V_{r,l},\psi_{r-l}},$$

where $\widetilde{\Theta}^{\psi^{-1}}_l$ is the space generated by the theta functions $\widetilde{\theta}^{\psi^{-1}}_{\varphi_l}$ with $\varphi_l \in \mathcal{S}(\mathbb{A}^l)$ and $(\widetilde{\Theta}^{\psi^{-1}}_l \otimes V_{\widetilde{\tau}})^{V_{r,l},\psi_{r-l}}$ is the space generated by the Fourier-Jacobi coefficients of automorphic forms in $\widetilde{\tau}$.

It is clear that $\mathcal{S}(\mathbb{A}^{2r+l}) = \mathcal{S}(\mathbb{A}^{2r})\widehat{\otimes}\mathcal{S}(\mathbb{A}^l)$. If we take $\varphi_{2r+l} = \varphi_{2r} \otimes \varphi_l$ (separation of variables), then we have

$$\widetilde{\theta}^{\psi^{-1}}_{\varphi_{2r+l},l}(\ell(\overline{p}_2, \overline{z})\ell(\overline{p}_1)\widetilde{g}) = \varphi_{2r}(\ell(\overline{p}_1)) \cdot \widetilde{\theta}^{\psi^{-1}}_{\varphi_l}(\ell(\overline{p}_2, \overline{z})\widetilde{g})$$

for $\widetilde{g} \in \widetilde{\mathrm{Sp}}_{2l}(\mathbb{A})$ (see (0.9) for the definition of $\widetilde{\theta}^{\psi^{-1}}_{\varphi_{2r+l},l}$). For any fixed $\varphi_{2r} \in \mathcal{S}(\mathbb{A}^{2r})$, we consider all Bruhat-Schwartz functions

$$\varphi_{2r+l} = \varphi_{2r} \otimes \varphi_l \in \mathcal{S}(\mathbb{A}^{2r+l}),$$

with $\varphi_l \in \mathcal{S}(\mathbb{A}^l)$. It follows that the space generated by $\widetilde{\theta}^{\psi^{-1}}_{\varphi_{2r+l},l}(\ell(\overline{p}_2, \overline{z})\widetilde{g})$ (with a fixed $\varphi_{2r} \in \mathcal{S}(\mathbb{A}^{2r})$ and all $\varphi_l \in \mathcal{S}(\mathbb{A}^l)$) is the same as the space $\widetilde{\Theta}^{\psi^{-1}}_l$ (generated by all $\widetilde{\theta}^{\psi^{-1}}_{\varphi_l}(\ell(\overline{p}_2, \overline{z})\widetilde{g})$) as automorphic representations of the Jacobi group $\widetilde{\mathrm{Sp}}_{2l}(\mathbb{A}) \ltimes H_l(\mathbb{A})$, where H_l is the Heisenberg group generated by all $\ell(\overline{p}_2, \overline{z})$. In the following we may assume that φ_{2r} is supported in a small neighborhood of zero.

It follows that the non-vanishing of the period $\mathcal{P}_{r,r-l}(\phi_\sigma, \widetilde{\phi}_{\widetilde{\tau}}, \varphi_l)$ is equivalent to the non-vanishing of the following integral

$$\int_{[\mathrm{Sp}_{2l}]} \phi_\sigma(b) \int_{[V_{r,l}]} \widetilde{\theta}^{\psi^{-1}}_{\varphi_{2r+l},l}(\ell(\overline{p}_2, \overline{z})\widetilde{b})\widetilde{\phi}_{\widetilde{\tau}}(v\widetilde{b})\psi_{r-l}(v)dvdb.$$

On the other hand, it is clear that the integral

$$\int_{Z_{\mathrm{GL}_{2r}}(\mathbb{A})\mathrm{GL}_{2r}(F)\backslash \mathrm{GL}_{2r}(\mathbb{A})} \phi_{\pi_\psi(\widetilde{\tau})}(a)\overline{\phi}_{\pi_\psi(\widetilde{\tau})}(a)da$$

is not zero for any choice of nonzero $\phi_{\pi_\psi(\widetilde{\tau})}$, where $\overline{\phi}_{\pi_\psi(\widetilde{\tau})}$ is the complex conjugate of $\phi_{\pi_\psi(\widetilde{\tau})}$. Hence, combining the above two non-vanishing integrals, we obtain that the integral

$$(0.13) \qquad \int_{\mathrm{PM}_{2r,l}} \phi_{\pi_\psi(\widetilde{\tau})\otimes\sigma}(m)\mathcal{F}^\psi(\widetilde{\phi}_{\pi_\psi(\widetilde{\tau})\otimes\widetilde{\tau}})(mw)dm,$$

does not vanish for some choice of data $\phi_{\pi_\psi(\widetilde{\tau})\otimes\sigma} \in \mathcal{A}^{ss}_{P_{2r,l},\pi_\psi(\widetilde{\tau})\otimes\sigma}$ and $\widetilde{\phi}_{\pi_\psi(\widetilde{\tau})\otimes\widetilde{\tau}} \in \widetilde{\mathcal{A}}^{ss}_{\widetilde{P}_{2r,r},\pi_\psi(\widetilde{\tau})\otimes\widetilde{\tau},0}$ (by taking $\phi'_{\pi_\psi(\widetilde{\tau})}$ to be $\overline{\phi}_{\pi_\psi(\widetilde{\tau})}$ in (0.11)), where $\mathrm{PM}_{2r,l}$ is as in (0.7), and for $m = m(a,b)$, $\mathcal{F}^\psi(\widetilde{\phi}_{\pi_\psi(\widetilde{\tau})\otimes\widetilde{\tau}})(mw)$ is defined by

$$(0.14) \qquad \int_{[V_{r,l}]} \widetilde{\theta}^{\psi^{-1}}_{\varphi_{2r+l},l}(\ell(\overline{p}_2,\overline{z})\widetilde{m}(b))\widetilde{\phi}_{\pi_\psi(\widetilde{\tau})\otimes\widetilde{\tau}}(\widetilde{m}(a)v\widetilde{m}(b)w)\psi_{r-l}(v)dv.$$

We claim that for any choice of $\phi_{\pi_\psi(\widetilde{\tau})\otimes\sigma} \in \mathcal{A}^{ss}_{P_{2r,l},\pi_\psi(\widetilde{\tau})\otimes\sigma}$, there exists $\widetilde{\phi}_{\pi_\psi(\widetilde{\tau})\otimes\widetilde{\tau}} \in \widetilde{\mathcal{A}}^{ss}_{\widetilde{P}_{2r,r},\pi_\psi(\widetilde{\tau})\otimes\widetilde{\tau},0}$ such that the integral (0.13) does not vanish. Indeed, from the discussion above and the definitions of the sets $\mathcal{A}^{ss}_{P_{2r,l},\pi_\psi(\widetilde{\tau})\otimes\sigma}$ and $\widetilde{\mathcal{A}}^{ss}_{\widetilde{P}_{2r,r},\pi_\psi(\widetilde{\tau})\otimes\widetilde{\tau},0}$, it suffices to show that for any choice of $\phi_\sigma \in \sigma$, there exists $\widetilde{\phi}_{\widetilde{\tau}} \in \widetilde{\tau}$ such that the period $\mathcal{P}_{r,r-l}(\phi_\sigma, \widetilde{\phi}_{\widetilde{\tau}}, \varphi_l)$ is nonzero. This follows from the fact that since σ is irreducible, if the period $\mathcal{P}_{r,r-l}(\phi_\sigma, \widetilde{\phi}_{\widetilde{\tau}}, \varphi_l)$ is nonzero for some choice of $\phi_\sigma \in \sigma$ and $\widetilde{\phi}_{\widetilde{\tau}} \in \widetilde{\tau}$, then the whole σ occurs in the descent module of $\widetilde{\tau}$ (for the definition of descent modules see [GRS11, Chapter 3]).

Next we consider the following inner integration from Proposition 0.1:

$$(0.15) \quad \int_{\mathrm{Mat}_{r-l,2r}(\mathbb{A})\times \mathrm{PM}_{2r,l}} \phi_{\pi_\psi(\widetilde{\tau})\otimes\sigma}(m)\mathcal{F}^\psi(\widetilde{\phi}_{\pi_\psi(\widetilde{\tau})\otimes\widetilde{\tau}})(mv^-(p_1)w)dmdp_1.$$

Recall from p. 697 of [GJR04] that the element $v^-(p_1)$ belongs to a unipotent subgroup of Sp_{6r} consisting of elements of the form

$$v^-(p_1) = \begin{pmatrix} I_{2r} & & & & \\ p_1 & I_{r-l} & & & \\ & & I_{2l} & & \\ & & & I_{r-l} & \\ & & & p_1^* & I_{2r} \end{pmatrix}.$$

By Proposition 0.3 below and the claim above, for any choice of $\phi_{\pi_\psi(\widetilde{\tau})\otimes\sigma} \in \mathcal{A}^{ss}_{P_{2r,l},\pi_\psi(\widetilde{\tau})\otimes\sigma}$, there exists $\widetilde{\phi}_{\pi_\psi(\widetilde{\tau})\otimes\widetilde{\tau}} \in \widetilde{\mathcal{A}}_{\widetilde{P}_{2r,r},\pi_\psi(\widetilde{\tau})\otimes\widetilde{\tau},0}$ such that the integral (0.15) does not vanish. This is the place where the original argument in the proof of Proposition 5.3 of [GJR04] is not complete. Proposition 0.3 will be proved in Sections 0.3 – 0.5.

In order to prove finally the integral

$$\int_{K\times \mathrm{Mat}_{r-l,2r}(\mathbb{A})\times \mathrm{PM}_{2r,l}} \phi_{\pi_\psi(\widetilde{\tau})\otimes\sigma}(mk)\mathcal{F}^\psi(\widetilde{\phi}_{\pi_\psi(\widetilde{\tau})\otimes\widetilde{\tau}})(mv^-(p_1)wk)dmdp_1dk$$

is nonzero for some choice of data, for $k\in K$, we set

$$\Psi(k) := \int_{\mathrm{Mat}_{r-l,2r}(\mathbb{A})\times \mathrm{PM}_{2r,l}} \phi_{\pi_\psi(\widetilde{\tau})\otimes\sigma}(mk)\mathcal{F}^\psi(\widetilde{\phi}_{\pi_\psi(\widetilde{\tau})\otimes\widetilde{\tau}})(mv^-(p_1)wk)dmdp_1.$$

According to the discussion above, under the assumption of the proposition, for any choice of $\phi_{\pi_\psi(\widetilde{\tau})\otimes\sigma}\in\mathcal{A}^{ss}_{P_{2r,l},\pi_\psi(\widetilde{\tau})\otimes\sigma}$, there exists $\widetilde{\phi}_{\pi_\psi(\widetilde{\tau})\otimes\widetilde{\tau}}\in\widetilde{\mathcal{A}}_{\widetilde{P}_{2r,r},\pi_\psi(\widetilde{\tau})\otimes\widetilde{\tau},0}$ such that $\Psi(k)$ is nonzero at $k=1$, the identity. By Lemma 0.2, $\mathcal{A}^{ss}_{P_{2r,l},\pi_\psi(\widetilde{\tau})\otimes\sigma}$ generates a dense subspace of $\mathcal{A}_{P_{2r,l},\pi_\psi(\widetilde{\tau})\otimes\sigma}$, hence we have the freedom on the K-support of the factorizable section $\phi_{\pi_\psi(\widetilde{\tau})\otimes\sigma}$.

Therefore, at non-archimedean ramified local places v, we can choose a small support $\Omega_v\subset K_v$ of $\phi_{\pi_\psi(\widetilde{\tau})\otimes\sigma}$ near the identity, such that

$$\Psi(k_\infty\cdot k_v)=\Psi(k_\infty).$$

At the archimedean local places v, by using the continuity at $k=1$ of $\Psi(k)$, there is a small support $\Omega_\infty\subset K_\infty$ for $\phi_{\pi_\psi(\widetilde{\tau})\otimes\sigma}$ such that the integral

$$\int_K \Psi(k)dk = c_f\cdot\int_{\Omega_\infty}\Psi(k_\infty)dk_\infty\neq 0,$$

with a constant c_f depending on the ramified finite local places.

This completes the proof of Proposition 5.3 of [GJR04], up to proving Proposition 0.3 below. □

Proposition 0.3. *For any choice of data* $\phi_{\pi_\psi(\widetilde{\tau})\otimes\sigma}\in\mathcal{A}^{ss}_{P_{2r,l},\pi_\psi(\widetilde{\tau})\otimes\sigma}$, *there exists* $\widetilde{\phi}_{\pi_\psi(\widetilde{\tau})\otimes\widetilde{\tau}}\in\widetilde{\mathcal{A}}_{\widetilde{P}_{2r,r},\pi_\psi(\widetilde{\tau})\otimes\widetilde{\tau},0}$ *such that the integral* (0.15), *which equals*

$$\int_{\mathrm{Mat}_{r-l,2r}(\mathbb{A})\times \mathrm{PM}_{2r,l}} \phi_{\pi_\psi(\widetilde{\tau})\otimes\sigma}(m)\mathcal{F}^\psi(\widetilde{\phi}_{\pi_\psi(\widetilde{\tau})\otimes\widetilde{\tau}})(mv^-(p_1)w)dmdp_1,$$

does not vanish.

The proof of this proposition will be given in following sections.

0.3. **The idea for proving Proposition 0.3.** In this section, we briefly introduce the idea for proving Proposition 0.3. First, we recall a lemma from [GRS11], which plays the same role as [GJR04, Lemma 4.2].

Let H be any F-quasisplit classical group, including the general linear group. Let C be an F-subgroup of a maximal unipotent subgroup of H, and let ψ_C be a non-trivial character of $[C] = C(F)\backslash C(\mathbb{A})$. X, Y are two unipotent F-subgroups, satisfying the following conditions:

(1) X and Y normalize C;
(2) $X \cap C$ and $Y \cap C$ are normal in X and Y, respectively, $(X\cap C)\backslash X$ and $(Y \cap C)\backslash Y$ are abelian;
(3) $X(\mathbb{A})$ and $Y(\mathbb{A})$ preserve ψ_C;
(4) ψ_C is trivial on $(X \cap C)(\mathbb{A})$ and $(Y \cap C)(\mathbb{A})$;
(5) $[X, Y] \subset C$;
(6) there is a non-degenerate pairing $(X\cap C)(\mathbb{A})\times(Y\cap C)(\mathbb{A}) \to \mathbb{C}^*$, given by $(x, y) \mapsto \psi_C([x, y])$, which is multiplicative in each coordinate, and identifies $(Y \cap C)(F)\backslash Y(F)$ with the dual of $X(F)(X \cap C)(\mathbb{A})\backslash X(\mathbb{A})$, and $(X \cap C)(F)\backslash X(F)$ with the dual of $Y(F)(Y \cap C)(\mathbb{A})\backslash Y(\mathbb{A})$.

Let $B = CY$ and $D = CX$, and extend ψ_C trivially to characters of $[B] = B(F)\backslash B(\mathbb{A})$ and $[D] = D(F)\backslash D(\mathbb{A})$, which will be denoted by ψ_B and ψ_D respectively. When there is no confusion, we may denote ψ_B and ψ_D all by ψ_C.

Lemma 0.4 (Lemma 7.1 of [GRS11]). *Assume that the quadruple (C, ψ_C, X, Y) satisfies all the above conditions. Let f be an automorphic form on $H(\mathbb{A})$. Then for any $g \in H(\mathbb{A})$,*

$$\int_{[B]} f(vg)\psi_B^{-1}(v)dv = \int_{(Y\cap C)(\mathbb{A})\backslash Y(\mathbb{A})} \int_{[D]} f(uyg)\psi_D^{-1}(u)dudy.$$

The right hand side of the the above equality is convergent in the sense

$$\int_{(Y\cap C)(\mathbb{A})\backslash Y(\mathbb{A})} |\int_{[D]} f(uyg)\psi_D^{-1}(u)du|dy < \infty,$$

and this convergence is uniform as g varies in compact subsets of $H(\mathbb{A})$.

The idea for proving Proposition 0.3.

First, based on the discussion in Section 0.2, for any choice of $\phi_{\pi_\psi(\widetilde{\tau})\otimes\sigma} \in \mathcal{A}^{ss}_{P_{2r,l},\pi_\psi(\widetilde{\tau})\otimes\sigma}$, there exists $\widetilde{\phi}_{\pi_\psi(\widetilde{\tau})\otimes\widetilde{\tau}} \in \widetilde{\mathcal{A}}^{ss}_{\widetilde{P}_{2r,r},\pi_\psi(\widetilde{\tau})\otimes\widetilde{\tau},0}$ such that the integral (0.13):

$$\int_{\mathrm{PM}_{2r,l}} \phi_{\pi_\psi(\widetilde{\tau})\otimes\sigma}(m)\mathcal{F}^\psi(\widetilde{\phi}_{\pi_\psi(\widetilde{\tau})\otimes\widetilde{\tau}})(mw)dm, \tag{0.16}$$

does not vanish, where $\mathrm{PM}_{2r,l}$ is as in (0.7), and for $m = m(a, b)$, $\mathcal{F}^\psi(\widetilde{\phi}_{\pi_\psi(\widetilde{\tau})\otimes\widetilde{\tau}})(mw)$ is defined in (0.14).

The proof of Proposition 0.3 briefly consists of the following 4 steps.

(1) Reversing the calculations from (4.29) – (4.35) and reversing the step of taking Fourier expansion of $\widetilde{E}_1$ along $[\mathcal{C}_1]$ as in [GJR04,

Section 4], we can transform the integral (0.16) to a nonzero constant times the residue at $s = \frac{1}{2}$ of a multiple integral over $[M]$ and $[V^{(1)}]$ (see (0.18), (0.31) below).

(2) Note that $V^{(1)} = \prod_{n=2}^{r-l} \mathcal{C}_n V^{(0)}$, we consider the integral over $[V^{(1)}]$. Applying Lemma 0.4 repeatedly to exchange roots from $\prod_{i=2}^{r-l} \mathcal{C}_i$ to $\prod_{j=1}^{r-l-1} \mathcal{R}_j$, we obtain a multiple integral over $\prod_{i=2}^{r-l} \mathcal{C}_i(\mathbb{A})$, $[\prod_{j=1}^{r-l-1} \mathcal{R}_j]$ and $[V^{(0)}]$ (see (0.20), (0.33) below). Combining with the outer integral over $[M]$, after changing of variables, we obtain a non-vanishing multiple integral over $\prod_{i=2}^{r-l} \mathcal{C}_i(\mathbb{A})$, $[M]$, $[\prod_{j=1}^{r-l-1} \mathcal{R}_j]$ and $[V^{(0)}]$ (see (0.22), (0.34) below). Then we drop the outer integral over $\prod_{i=2}^{r-l} \mathcal{C}_i(\mathbb{A})$. Clearly the inner multiple integral over $[M]$, $[\prod_{j=1}^{r-l-1} \mathcal{R}_j]$ and $[V^{(0)}]$ is non-vanishing.

(3) Then we consider the non-vanishing inner multiple integral over $[\prod_{j=1}^{r-l-1} \mathcal{R}_j]$ and $[V^{(0)}]$. Applying Lemma 0.4 repeatedly to exchange roots from $\prod_{j=1}^{r-l-1} \mathcal{R}_j$ to $\prod_{i=2}^{r-l} \mathcal{C}_i$, we obtain a multiple integral over $\prod_{j=1}^{r-l-1} \mathcal{R}_j(\mathbb{A})$, $[\prod_{i=2}^{r-l} \mathcal{C}_i]$ and $[V^{(0)}]$ (see (0.25), (0.37) below). Combining with the outer integral over $[M]$, after changing of variables, we obtain a non-vanishing multiple integral over $\prod_{j=1}^{r-l-1} \mathcal{R}_j(\mathbb{A})$, $[M]$, $[\prod_{i=2}^{r-l} \mathcal{C}_i]$ and $[V^{(0)}]$ (see (0.27), (0.39) below). Note that $\prod_{i=2}^{r-l} \mathcal{C}_i V^{(0)} = V^{(1)}$.

(4) By choosing appropriate $\varphi_{2r} \in \mathcal{S}(\mathbb{A}^{2r})$, we obtain a non-vanishing multiple integral over $\prod_{j=1}^{r-l} \mathcal{R}_j(\mathbb{A})$, $[M]$ and $[V^{(1)}]$ (see (0.28), (0.40) below). Note that $\prod_{j=1}^{r-l} \mathcal{R}_j(\mathbb{A})$ is exactly the group $\mathrm{Mat}_{r-l,2r}(\mathbb{A})$. After taking Fourier expansion of $\widetilde{E}_1$ along $[\mathcal{C}_1]$ and the calculations from (4.29) – (4.35) as in [GJR04, Section 4], we obtain a non-vanishing integral which is exactly a product of a nonzero constant with the integral in (0.15), for $\phi_{\pi_\psi(\widetilde{\tau})\otimes\sigma}$ and some right translation of $\widetilde{\phi}_{\pi_\psi(\widetilde{\tau})\otimes\widetilde{\tau}}$.

0.4. **Proof of Proposition 0.3: special case $l = 1, r = 3$.** In this section, we prove Proposition 0.3 for the first non-trivial case: $l = 1, r = 3$.

We start from the non-vanishing integral (0.16). Reversing the calculations from (4.29) – (4.35) as in [GJR04, Section 4], the integral (0.16) is equal to $\frac{1}{\mathfrak{c}}$ times the residue at $s = \frac{1}{2}$ of

$$\int_{[M]} \Phi^c(m, s, \phi)|\det a|^{-\frac{21}{2}} \int_{[V_{3,1}]} \sum_{\xi_1 \in F^1} \omega_{\psi^{-1}}(\ell(\bar{p}_2, \bar{z})\widetilde{m}) \tag{0.17}$$

$$\varphi_7((0,\xi_1))\widetilde{E}_{1,\widetilde{P}_{6,3}}(v\widetilde{m}w,\widetilde{\phi})\psi_2(v)dvdm,$$

where $\mathfrak{c}$ is as in (0.8).

Reversing the step of taking Fourier expansion of $\widetilde{E}_1$ along $[\mathcal{C}_1]$ as in [GJR04, Section 4], the integral (0.17) is equal to

$$\int_{[M]}\Phi^c(m,s,\phi)|\det a|^{-\frac{21}{2}}\int_{[V^{(1)}]}\sum_{\xi_1\in F^1}\omega_{\psi^{-1}}(\ell(\bar{p}_2,\bar{z})\widetilde{m}) \tag{0.18}$$

$$\varphi_7((0,\xi_1))\widetilde{E}_1(v^{(1)}\widetilde{m}w,\widetilde{\phi})\psi_2(v^{(1)})dv^{(1)}dm.$$

Recall that $V^{(1)}$ consists of elements of the type

$$v^{(1)}=\begin{pmatrix} I_6 & q & y & p_3^* & z' \\ & n & p_2 & z & p_3 \\ & & I_2 & p_2^* & y^* \\ & & & n^* & q^* \\ & & & & I_6\end{pmatrix},$$

where $q\in \mathrm{Mat}_{6,2}$ with the first column being zero.

Recall that $V^{(0)}$ consists of elements of the type

$$v^{(0)}=\begin{pmatrix} I_6 & 0 & y & p_3^* & z' \\ & n & p_2 & z & p_3 \\ & & I_2 & p_2^* & y^* \\ & & & n^* & 0 \\ & & & & I_6\end{pmatrix}.$$

And for $1\le t\le 2$,

$$\mathcal{C}_t=\left\{\begin{pmatrix} I_6 & q & 0 & 0 & 0 \\ & I_2 & 0 & 0 & 0 \\ & & I_2 & 0 & 0 \\ & & & I_2 & q^* \\ & & & & I_6\end{pmatrix} : q\in \mathrm{Mat}_{6,2}, q_{i,j}=0, j\neq t\right\},$$

$$\mathcal{R}_t=\left\{\begin{pmatrix} I_6 & 0 & 0 & 0 & 0 \\ p & I_2 & 0 & 0 & 0 \\ & & I_2 & 0 & 0 \\ & & & I_2 & 0 \\ & & & p^* & I_6\end{pmatrix} : p\in \mathrm{Mat}_{2,6}, q_{i,j}=0, i\neq t\right\}.$$

Note that $V^{(1)}=\mathcal{C}_2V^{(0)}$. Next, we consider the integral over $[V^{(1)}]$ and apply Lemma 0.4 to exchange the roots from $\mathcal{C}_2$ to $\mathcal{R}_1$. It is easy to see that the quadruple

$$(V^{(0)},\psi_2,\mathcal{R}_1,\mathcal{C}_2)$$

satisfies all the conditions in Lemma 0.4. Hence, applying Lemma 0.4 to the quadruple $(V^{(0)}, \psi_2, \mathcal{R}_1, \mathcal{C}_2)$, the integral

$$\int_{[V^{(1)}]} \sum_{\xi_1 \in F^1} \omega_{\psi^{-1}}(\ell(\bar{p}_2, \bar{z})\widetilde{m})\varphi_7((0, \xi_1)) \widetilde{E}_1(v^{(1)}\widetilde{m}w, \widetilde{\phi})\psi_2(v^{(1)})dv^{(1)} \tag{0.19}$$

is equal to

$$\int_{\mathcal{C}_2(\mathbb{A})} \int_{[\mathcal{R}_1]} \int_{[V^{(0)}]} \sum_{\xi_1 \in F^1} \omega_{\psi^{-1}}(\ell(\bar{p}_2, \bar{z})\ell(\bar{p}_1)\widetilde{m})\varphi_7((0, \xi_1)) \widetilde{E}_1(v^{(0)}v^-(p_1)v\widetilde{m}w, \widetilde{\phi})\psi_2(v^{(0)})dv^{(0)}dp_1 dv. \tag{0.20}$$

Hence, the integral (0.18) is equal to

$$\int_{[M]} \Phi^c(m, s, \phi)|\det a|^{-\frac{21}{2}} \int_{\mathcal{C}_2(\mathbb{A})} \int_{[\mathcal{R}_1]} \int_{[V^{(0)}]} \sum_{\xi_1 \in F^1} \omega_{\psi^{-1}}(\ell(\bar{p}_2, \bar{z})\ell(\bar{p}_1)\widetilde{m}) \varphi_7((0, \xi_1))\widetilde{E}_1(v^{(0)}v^-(p_1)v\widetilde{m}w, \widetilde{\phi})\psi_2(v^{(0)})dv^{(0)}dp_1 dv dm. \tag{0.21}$$

Since $[M]$ normalizes the group $\mathcal{C}_2(\mathbb{A})$, after changing of variables, we obtain the following non-vanishing integral

$$\int_{\mathcal{C}_2(\mathbb{A})} \int_{[M]} \Phi^c(m, s, \phi)|\det a|^{-\frac{23}{2}} \int_{[\mathcal{R}_1]} \int_{[V^{(0)}]} \sum_{\xi_1 \in F^1} \omega_{\psi^{-1}}(\ell(\bar{p}_2, \bar{z})\ell(\bar{p}_1)\widetilde{m}) \varphi_7((0, \xi_1))\widetilde{E}_1(v^{(0)}v^-(p_1)\widetilde{m}w(w^{-1}vw), \widetilde{\phi})\psi_2(v^{(0)})dv^{(0)}dp_1 dm dv. \tag{0.22}$$

Therefore, as an inner integral, the following integral is non-vanishing

$$\int_{[M]} \Phi^c(m, s, \phi)|\det a|^{-\frac{23}{2}} \int_{[\mathcal{R}_1]} \int_{[V^{(0)}]} \sum_{\xi_1 \in F^1} \omega_{\psi^{-1}}(\ell(\bar{p}_2, \bar{z})\ell(\bar{p}_1)\widetilde{m}) \varphi_7((0, \xi_1))\widetilde{E}_1(v^{(0)}v^-(p_1)\widetilde{m}wg, \widetilde{\phi})\psi_2(v^{(0)})dv^{(0)}dp_1 dm, \tag{0.23}$$

where $g = w^{-1}vw$, for some $v \in \mathcal{C}_2(\mathbb{A})$.

Now, we consider the multiple integral over $[\mathcal{R}_1]$ and $[V^{(0)}]$, and apply Lemma 0.4 to exchange the roots from $\mathcal{R}_1$ to $\mathcal{C}_2$. Precisely, applying Lemma 0.4 to the quadruple $(V^{(0)}, \psi_2, \mathcal{C}_2, \mathcal{R}_1)$ (which also satisfies all the conditions in Lemma 0.4), the integral

$$\int_{[\mathcal{R}_1]} \int_{[V^{(0)}]} \sum_{\xi_1 \in F^1} \omega_{\psi^{-1}}(\ell(\bar{p}_2, \bar{z})\ell(\bar{p}_1)\widetilde{m})\varphi_7((0, \xi_1)) \widetilde{E}_1(v^{(0)}v^-(p_1)\widetilde{m}wg, \widetilde{\phi})\psi_2(v^{(0)})dv^{(0)}dp_1 \tag{0.24}$$

is equal to

$$\int_{\mathcal{R}_1(\mathbb{A})}\int_{[V^{(1)}]}\sum_{\xi_1\in F^1}\omega_{\psi^{-1}}(\ell(\bar{p}_2,\bar{z})\ell(\bar{p}_1)\widetilde{m})\varphi_7((0,\xi_1))$$
$$\widetilde{E}_1(v^{(1)}v^-(p_1)\widetilde{m}wg,\widetilde{\phi})\psi_2(v^{(1)})dv^{(1)}dp_1. \tag{0.25}$$

Hence, the integral (0.23) is equal to

$$\int_{[M]}\Phi^c(m,s,\phi)|\det a|^{-\frac{23}{2}}\int_{\mathcal{R}_1(\mathbb{A})}\int_{[V^{(1)}]}\sum_{\xi_1\in F^1}\omega_{\psi^{-1}}(\ell(\bar{p}_2,\bar{z})\ell(\bar{p}_1)\widetilde{m})\varphi_7((0,\xi_1))$$
$$\widetilde{E}_1(v^{(1)}v^-(p_1)\widetilde{m}wg,\widetilde{\phi})\psi_2(v^{(1)})dv^{(1)}dp_1dm. \tag{0.26}$$

Since $[M]$ normalizes $\mathcal{R}_1(\mathbb{A})$, after changing of variables, we obtain the following non-vanishing integral

$$\int_{\mathcal{R}_1(\mathbb{A})}\int_{[M]}\Phi^c(m,s,\phi)|\det a|^{-\frac{21}{2}}\int_{[V^{(1)}]}\sum_{\xi_1\in F^1}\omega_{\psi^{-1}}(\ell(\bar{p}_2,\bar{z})\widetilde{m}\ell(\bar{p}_1))$$
$$\varphi_7((0,\xi_1))\widetilde{E}_1(v^{(1)}\widetilde{m}v^-(p_1)wg,\widetilde{\phi})\psi_2(v^{(1)})dv^{(1)}dmdp_1. \tag{0.27}$$

By choosing appropriate $\varphi_2\in\mathcal{S}(\mathbb{A}^6)$, the integral (0.27) is non-vanishing if and only if the following integral is non-vanishing

$$\int_{\mathcal{R}_1\mathcal{R}_2(\mathbb{A})}\int_{[M]}\Phi^c(m,s,\phi)|\det a|^{-\frac{21}{2}}\int_{[V^{(1)}]}\sum_{\xi_1\in F^1}\omega_{\psi^{-1}}(\ell(\bar{p}_2,\bar{z})\widetilde{m}\ell(\bar{p}_1))$$
$$\varphi_7((0,\xi_1))\widetilde{E}_1(v^{(1)}\widetilde{m}v^-(p_1)wg,\widetilde{\phi})\psi_2(v^{(1)})dv^{(1)}dmdp_1. \tag{0.28}$$

After taking Fourier expansion of $\widetilde{E}_1$ along $[\mathcal{C}_1]$, arguing as in [GJR04, Section 4], the integral (0.28) is equal to

$$\int_{\mathcal{R}_1\mathcal{R}_2(\mathbb{A})}\int_{[M]}\Phi^c(m,s,\phi)|\det a|^{-\frac{21}{2}}\int_{[V_{3,1}]}\sum_{\xi_1\in F^1}\omega_{\psi^{-1}}(\ell(\bar{p}_2,\bar{z})\widetilde{m}\ell(\bar{p}_1))$$
$$\varphi_7((0,\xi_1))\widetilde{E}_{1,P_{6,3}}(v\widetilde{m}v^-(p_1)wg,\widetilde{\phi})\psi_2(v)dvdmdp_1. \tag{0.29}$$

Then, following the calculations from (4.29) – (4.35) in [GJR04], we obtain that the following integral

$$\mathfrak{c}\int_{\mathrm{Mat}_{2,6}(\mathbb{A})\times\mathrm{PM}_{6,1}}\phi_{\pi_\psi(\widetilde{\tau})\otimes\sigma}(m)\mathcal{F}^\psi(R(g)\widetilde{\phi}_{\pi_\psi(\widetilde{\tau})\otimes\widetilde{\tau}})(mv^-(p_1)w)dmdp_1$$

does not vanish, where $R(g)$ is the right translation operator. Hence, for $\phi_{\pi_\psi(\widetilde{\tau})\otimes\sigma}\in\mathcal{A}^{ss}_{P_{2r,l},\pi_\psi(\widetilde{\tau})\otimes\sigma}$, $R(g)\widetilde{\phi}_{\pi_\psi(\widetilde{\tau})\otimes\widetilde{\tau}}\in\widetilde{\mathcal{A}}_{\widetilde{P}_{2r,r},\pi_\psi(\widetilde{\tau})\otimes\widetilde{\tau},0}$ would be suffice, in order for the integral (0.15) to be non-vanishing.

This completes the proof of Proposition 0.3 for the special case $l = 1$, $r = 3$. $\square$

0.5. **Proof of Proposition 0.3: the general case.** In this section, we prove Proposition 0.3 for the general case.

Again, we start from the non-vanishing integral (0.16). Reversing the calculations from (4.29) – (4.35) as in [GJR04, Section 4], the integral (0.16) is equal to $\frac{1}{\mathfrak{c}}$ times the residue at $s = \frac{1}{2}$ of

(0.30)
$$\int_{[M]} \Phi^c(m, s, \phi) |\det a|^{-3r-l-\frac{1}{2}} \int_{[V_{r,l}]} \sum_{\xi_l \in F^l} \omega_{\psi^{-1}}(\ell(\bar{p}_2, \bar{z})\widetilde{m})\varphi_{2r+l}((0, \xi_l))$$
$$\widetilde{E}_{1,\widetilde{P}_{2r,r}}(v\widetilde{m}w, \widetilde{\phi})\psi_{r-l}(v) dv dm,$$

where again $\mathfrak{c}$ is as in (0.8).

Reversing the step of taking Fourier expansion of $\widetilde{E}_1$ along $[\mathcal{C}_1]$, the integral (0.30) is equal to

(0.31)
$$\int_{[M]} \Phi^c(m, s, \phi) |\det a|^{-3r-l-\frac{1}{2}} \int_{[V^{(1)}]} \sum_{\xi_l \in F^l} \omega_{\psi^{-1}}(\ell(\bar{p}_2, \bar{z})\widetilde{m})\varphi_{2r+l}((0, \xi_l))$$
$$\widetilde{E}_1(v^{(1)}\widetilde{m}w, \widetilde{\phi})\psi_{r-l}(v^{(1)}) dv^{(1)} dm.$$

Recall that $V^{(1)}$ consists of elements of the type

$$v^{(1)} = \begin{pmatrix} I_{2r} & q & y & p_3^* & z' \\ & n & p_2 & z & p_3 \\ & & I_{2l} & p_2^* & y^* \\ & & & n^* & q^* \\ & & & & I_{2r} \end{pmatrix},$$

where $q \in \mathrm{Mat}_{2r,r-l}$ with the first column being zero.

Recall that $V^{(0)}$ consists of elements of the type

$$v^{(0)} = \begin{pmatrix} I_{2r} & 0 & y & p_3^* & z' \\ & n & p_2 & z & p_3 \\ & & I_{2l} & p_2^* & y^* \\ & & & n^* & 0 \\ & & & & I_{2r} \end{pmatrix}.$$

And for $1 \le t \le r-l$,

$$\mathcal{C}_t = \left\{ \begin{pmatrix} I_{2r} & q & 0 & 0 & 0 \\ & I_{r-l} & 0 & 0 & 0 \\ & & I_{2l} & 0 & 0 \\ & & & I_{r-l} & q^* \\ & & & & I_{2r} \end{pmatrix} : q \in \mathrm{Mat}_{2r,r-l}, q_{i,j} = 0, j \neq t \right\},$$

$$\mathcal{R}_t = \left\{ \begin{pmatrix} I_{2r} & 0 & 0 & 0 & 0 \\ p & I_{r-l} & 0 & 0 & 0 \\ & & I_{2l} & 0 & 0 \\ & & & I_{r-l} & 0 \\ & & & p^* & I_{2r} \end{pmatrix} : p \in \mathrm{Mat}_{r-l,2r}, q_{i,j} = 0, i \neq t \right\}.$$

Note that $V^{(1)} = \prod_{i=2}^{r-l} \mathcal{C}_i V^{(0)}$. Next, we consider the integral over $[V^{(1)}]$ and apply Lemma 0.4 repeatedly to exchange roots from $\prod_{i=2}^{r-l} \mathcal{C}_i$ to $\prod_{j=1}^{r-l-1} \mathcal{R}_j$. First, one can see that the quadruples

$$\left(\prod_{j=1}^{t-1} \mathcal{R}_j \prod_{i=t+2}^{r-l} \mathcal{C}_i V^{(0)}, \psi_{r-l}, \mathcal{R}_t, \mathcal{C}_{t+1} \right), 1 \le t \le r-l-1,$$

satisfy all the conditions in Lemma 0.4. Hence, applying Lemma 0.4 to the following ordered sequence of quadruples

$$\left(\prod_{i=3}^{r-l} \mathcal{C}_i V^{(0)}, \psi_{r-l}, \mathcal{R}_1, \mathcal{C}_2 \right),$$

$$\left(\mathcal{R}_1 \prod_{i=4}^{r-l} \mathcal{C}_i V^{(0)}, \psi_{r-l}, \mathcal{R}_2, \mathcal{C}_3 \right),$$

$$\cdots$$

$$\left(\prod_{j=1}^{t-1} \mathcal{R}_j \prod_{i=t+2}^{r-l} \mathcal{C}_i V^{(0)}, \psi_{r-l}, \mathcal{R}_t, \mathcal{C}_{t+1} \right),$$

$$\cdots$$

$$\left(\prod_{j=1}^{r-l-2} \mathcal{R}_j V^{(0)}, \psi_{r-l}, \mathcal{R}_{r-l-1}, \mathcal{C}_{r-l} \right),$$

the integral

$$\int_{[V^{(1)}]} \sum_{\xi_l \in F^l} \omega_{\psi^{-1}}(\ell(\bar{p}_2, \bar{z})\widetilde{m})\varphi_{2r+l}((0, \xi_l)) \tag{0.32}$$

$$\widetilde{E}_1(v^{(1)}\widetilde{m}w, \widetilde{\phi})\psi_{r-l}(v^{(1)})dv^{(1)}$$

is equal to
(0.33)
$$\int_{\prod_{i=2}^{r-l}\mathcal{C}_i(\mathbb{A})}\int_{[\prod_{j=1}^{r-l-1}\mathcal{R}_j]}\int_{[V^{(0)}]}\sum_{\xi_l\in F^l}\omega_{\psi^{-1}}(\ell(\bar{p}_2,\bar{z})\ell(\bar{p}_1)\widetilde{m})\varphi_{2r+l}((0,\xi_l))$$
$$\widetilde{E}_1(v^{(0)}v^-(p_1)v\widetilde{m}w,\widetilde{\phi})\psi_{r-l}(v^{(0)})dv^{(0)}dp_1dv.$$

Since M normalizes the group $\prod_{i=2}^{r-l}\mathcal{C}_i$, after changing of variables, we obtain the following non-vanishing integral
(0.34)
$$\int_{\prod_{i=2}^{r-l}\mathcal{C}_i(\mathbb{A})}\int_{[M]}\Phi^c(m,s,\phi)|\det a|^{-4r+\frac{1}{2}}\int_{[\prod_{j=1}^{r-l-1}\mathcal{R}_j]}\int_{[V^{(0)}]}\sum_{\xi_l\in F^l}\omega_{\psi^{-1}}(\ell(\bar{p}_2,\bar{z})\ell(\bar{p}_1)\widetilde{m})$$
$$\varphi_{2r+l}((0,\xi_l))\widetilde{E}_1(v^{(0)}v^-(p_1)\widetilde{m}w(w^{-1}vw),\widetilde{\phi})\psi_{r-l}(v^{(0)})dv^{(0)}dp_1dmdv.$$
Therefore, as an inner integral, the following integral is non-vanishing
(0.35)
$$\int_{[M]}\Phi^c(m,s,\phi)|\det a|^{-4r+\frac{1}{2}}\int_{[\prod_{j=1}^{r-l-1}\mathcal{R}_j]}\int_{[V^{(0)}]}\sum_{\xi_l\in F^l}\omega_{\psi^{-1}}(\ell(\bar{p}_2,\bar{z})\ell(\bar{p}_1)\widetilde{m})$$
$$\varphi_{2r+l}((0,\xi_l))\widetilde{E}_1(v^{(0)}v^-(p_1)\widetilde{m}wg,\widetilde{\phi})\psi_{r-l}(v^{(0)})dv^{(0)}dp_1dm,$$
where $g=w^{-1}vw$, for some $v\in\prod_{i=2}^{r-l}\mathcal{C}_i(\mathbb{A})$.

Now, we consider the multiple integral over $[\prod_{j=1}^{r-l-1}\mathcal{R}_j]$ and $[V^{(0)}]$, and apply Lemma 0.4 repeatedly to exchange roots from $\prod_{j=1}^{r-l-1}\mathcal{R}_j$ to $\prod_{i=2}^{r-l}\mathcal{C}_i$. One can see that the quadruples
$$\left(\prod_{i=2}^{t}\mathcal{C}_m\prod_{j=t+1}^{r-l-1}\mathcal{R}_jV^{(0)},\psi_{r-l},\mathcal{C}_{t+1},\mathcal{R}_t\right),1\leq t\leq r-l-1,$$
also satisfy all the conditions in Lemma 0.4. Hence, applying Lemma 0.4 to the following ordered sequence of quadruples
$$\left(\prod_{j=2}^{r-l-1}\mathcal{R}_jV^{(0)},\psi_{r-l},\mathcal{C}_2,\mathcal{R}_1\right),$$
$$\left(\mathcal{C}_2\prod_{j=3}^{r-l-1}\mathcal{R}_jV^{(0)},\psi_{r-l},\mathcal{C}_3,\mathcal{R}_2\right),$$
$$\cdots$$
$$\left(\prod_{i=2}^{t}\mathcal{C}_i\prod_{j=t+1}^{r-l-1}\mathcal{R}_nV^{(0)},\psi_{r-l},\mathcal{C}_{t+1},\mathcal{R}_t\right),$$
$$\cdots$$

$$\left(\prod_{i=2}^{r-l-1} \mathcal{C}_i V^{(0)}, \psi_{r-l}, \mathcal{C}_{r-l}, \mathcal{R}_{r-l-1}\right),$$

the integral

(0.36)
$$\int_{[\prod_{j=1}^{r-l-1} \mathcal{R}_j]} \int_{[V^{(0)}]} \sum_{\xi_l \in F^l} \omega_{\psi^{-1}}(\ell(\bar{p}_2, \bar{z})\ell(\bar{p}_1)\widetilde{m})\varphi_{2r+l}((0, \xi_l))$$
$$\widetilde{E}_1(v^{(0)} v^-(p_1)\widetilde{m}wg, \widetilde{\phi})\psi_{r-l}(v^{(0)})dv^{(0)}dp_1$$

is equal to

(0.37)
$$\int_{\prod_{j=1}^{r-l-1} \mathcal{R}_j(\mathbb{A})} \int_{[\prod_{i=2}^{r-l} \mathcal{C}_i]} \int_{[V^{(0)}]} \sum_{\xi_l \in F^l} \omega_{\psi^{-1}}(\ell(\bar{p}_2, \bar{z})\ell(\bar{p}_1)\widetilde{m})\varphi_{2r+l}((0, \xi_l))$$
$$\widetilde{E}_1(v^{(0)} v v^-(p_1)\widetilde{m}wg, \widetilde{\phi})\psi_{r-l}(v^{(0)})dv^{(0)}dvdp_1.$$

Hence, the integral (0.35) becomes

(0.38)
$$\int_{[M]} \Phi^c(m, s, \phi)|\det a|^{-4r+\frac{1}{2}} \int_{\prod_{j=1}^{r-l-1} \mathcal{R}_j(\mathbb{A})} \int_{[V^{(1)}]} \sum_{\xi_l \in F^l} \omega_{\psi^{-1}}(\ell(\bar{p}_2, \bar{z})\ell(\bar{p}_1)\widetilde{m})$$
$$\varphi_{2r+l}((0, \xi_l))\widetilde{E}_1(v^{(1)} v^-(p_1)\widetilde{m}wg, \widetilde{\phi})\psi_{r-l}(v^{(1)})dv^{(1)}dp_1dm.$$

Since M normalizes the group $\prod_{j=1}^{r-l-1} \mathcal{R}_j$, after changing of variables, we obtain the following non-vanishing integral

(0.39)
$$\int_{\prod_{j=1}^{r-l-1} \mathcal{R}_j(\mathbb{A})} \int_{[M]} \Phi^c(m, s, \phi)|\det a|^{-3r-l-\frac{1}{2}} \int_{[V^{(1)}]} \sum_{\xi_l \in F^l} \omega_{\psi^{-1}}(\ell(\bar{p}_2, \bar{z})\widetilde{m}\ell(\bar{p}_1))$$
$$\varphi_{2r+l}((0, \xi_l))\widetilde{E}_1(v^{(1)}\widetilde{m}v^-(p_1)wg, \widetilde{\phi})\psi_{r-l}(v^{(1)})dv^{(1)}dmdp_1.$$

By choosing appropriate $\varphi_2 \in \mathcal{S}(\mathbb{A}^{2r})$, the following integral is also non-vanishing

(0.40)
$$\int_{\prod_{j=1}^{r-l} \mathcal{R}_j(\mathbb{A})} \int_{[M]} \Phi^c(m, s, \phi)|\det a|^{-3r-l-\frac{1}{2}} \int_{[V^{(1)}]} \sum_{\xi_l \in F^l} \omega_{\psi^{-1}}(\ell(\bar{p}_2, \bar{z})\widetilde{m}\ell(\bar{p}_1))$$
$$\varphi_{2r+l}((0, \xi_l))\widetilde{E}_1(v^{(1)}\widetilde{m}v^-(p_1)wg, \widetilde{\phi})\psi_{r-l}(v^{(1)})dv^{(1)}dmdp_1.$$

After taking Fourier expansion of $\widetilde{E}_1$ along $[\mathcal{C}_1]$, arguing as in [GJR04, Section 4], the integral (0.40) is equal to

(0.41)
$$\int_{\prod_{j=1}^{r-l} \mathcal{R}_j(\mathbb{A})} \int_{[M]} \Phi^c(m, s, \phi)|\det a|^{-3r-l-\frac{1}{2}} \int_{[V_{r,l}]} \sum_{\xi_l \in F^l} \omega_{\psi^{-1}}(\ell(\bar{p}_2, \bar{z})\widetilde{m}\ell(\bar{p}_1))$$

$$\varphi_{2r+l}((0,\xi_l))\widetilde{E}_{1,\widetilde{P}_{2r,r}}(v\widetilde{m}v^-(p_1)wg,\widetilde{\phi})\psi_{r-l}(v)dvdmdp_1.$$

Then, following the calculations from (4.29) – (4.35) as in [GJR04, Section 4], we obtain that the following integral

$$\mathfrak{c}\int_{\mathrm{Mat}_{r-l,2r}(\mathbb{A})\times \mathrm{PM}_{2r,l}} \phi_{\pi_\psi(\widetilde{\tau})\otimes\sigma}(m)\mathcal{F}^\psi(R(g)\widetilde{\phi}_{\pi_\psi(\widetilde{\tau})\otimes\widetilde{\tau}})(mv^-(p_1)w)dmdp_1,$$

does not vanish, where $R(g)$ is the right translation operator. Hence, for $\phi_{\pi_\psi(\widetilde{\tau})\otimes\sigma} \in \mathcal{A}^{ss}_{P_{2r,l},\pi_\psi(\widetilde{\tau})\otimes\sigma}$, $R(g)\widetilde{\phi}_{\pi_\psi(\widetilde{\tau})\otimes\widetilde{\tau}} \in \widetilde{\mathcal{A}}_{\widetilde{P}_{2r,r},\pi_\psi(\widetilde{\tau})\otimes\widetilde{\tau},0}$ would be suffice, in order for the integral (0.15) to be non-vanishing.

This completes the proof of Proposition 0.3. □

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