Local descent to quasi-split even general spin groups

Eyal Kaplan*[1], Jing Feng Lau[2], and Baiying Liu†[3]

[1]Department of Mathematics, Bar Ilan University, Ramat Gan 5290002, Israel, kaplaney@gmail.com

[2]Singapore University of Social Sciences, 463 Clementi Road, Singapore 599494, Singapore, jflau@suss.edu.sg

[3]Department of Mathematics, Purdue University, 150 N. University Street, West Lafayette, IN 47907-2067, U.S.A., liu2053@purdue.edu

March 14, 2019

Abstract

Let $n \geqslant 2$ and τ be an irreducible unitary supercuspidal representation of $\mathrm{GL}(2n)$ over a local non-archimedean field. Assuming the twisted symmetric square L-function of τ has a pole at $s = 0$, we construct the local descent of τ to the corresponding general spin group of even rank (split over the base field, or over a quadratic extension). We show that this local descent is non-trivial, generic, unitary and supercuspidal. Moreover, any generic irreducible supercuspidal representation of the general spin group which lifts functorially to τ is contragradient to some constituent of the representation we construct.

Keywords: Local Descent, GSpin Groups, Local Langlands Functoriality.

2000 Mathematics Subject Classification Primary 11F70, 22E50; Secondary 11F85, 22E55.

1 Introduction

Let $\mathbf{G}$ be a connected reductive algebraic group defined over a global field K or local field F, let G be the group of rational points over that field. One of the central problems in the Langlands program is to determine the functorial image of a Langlands functorial lift corresponding to some L-homomorphism from ${}^LG$ to a suitable general linear group. To this end, when $\mathbf{G}$ is a classical group, with the exception of some low rank cases, it has been shown that the local and global descent method is an effective approach in proving that the relevant functorial image is contained in the collection of irreducible admissible representations of certain p-adic linear groups or automorphic representations of adelic linear groups satisfying a list of conditions, cf. [GRS11], [GRS99b], [JNQ10] and [ST15]. The global automorphic descents from general linear groups to GSpin groups, from GL_7 to G_2, have been

*Research of the first named author was supported by the ISRAEL SCIENCE FOUNDATION (grant No. 421/17).

†Research of the third named author is partially supported by NSF grant DMS-1702218 and by start-up funds from the Department of Mathematics at Purdue University.

carried out in [HS16] and [HL19], respectively. In this paper, we work out the local descent theory for GSpin(even) groups.

Fix a non-archimedean local field F of characteristic 0. Let $\mathbf{G}$ be either the split group GSpin$(2n)$ or quasi-split non-split group GSpin$^*(2n)$ defined over F that splits over a quadratic extension E, cf. § 2.1 for the definition of GSpin$(2n)$ and GSpin$^*(2n)$. Then ${}^L G$ is either $\mathrm{GSO}(2n)(\mathbb{C}) \times \mathrm{Gal}(\overline{F}/F)$ or $\mathrm{GSO}(2n)(\mathbb{C}) \rtimes \mathrm{Gal}(\overline{F}/F)$ where GSO is the special orthogonal similitude group, the non-trivial semi-direct product is defined by

$$(g_1,\gamma_1)(g_2,\gamma_2) = \begin{cases} (g_1g_2,\gamma_1\gamma_2) & \text{if } \gamma_1|_E = id_E\,, \\ (g_1hg_2h^{-1},\gamma_1\gamma_2) & \text{if } \gamma_1|_E \neq id_E\,, \end{cases}$$

for g_1, $g_2 \in \mathrm{GSO}(2n)(\mathbb{C})$, γ_1, $\gamma_2 \in \mathrm{Gal}(\overline{F}/F)$ and

$$h = \begin{pmatrix} I_{n-1} & & & \\ & 0 & 1 & \\ & 1 & 0 & \\ & & & I_{n-1} \end{pmatrix}.$$

Let $\iota : \mathrm{GSO}(2n)(\mathbb{C}) \times \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}(2n)(\mathbb{C}) \times \mathrm{Gal}(\overline{F}/F)$ be the canonical inclusion and $\iota^* : \mathrm{GSO}(2n)(\mathbb{C}) \rtimes \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}(2n)(\mathbb{C}) \times \mathrm{Gal}(\overline{F}/F)$ be the L-homomorphism

$$(g,\gamma) \mapsto \begin{cases} (g,\gamma) & \text{if } \gamma|_E = id_E\,, \\ (hgh^{-1},\gamma) & \text{if } \gamma|_E \neq id_E\,, \end{cases}$$

for $g \in \mathrm{GSO}(2n)(\mathbb{C})$ and $\gamma \in \mathrm{Gal}(\overline{F}/F)$. Denote the standard representation of $\mathrm{GL}(2n)(\mathbb{C})$ by ρ_{2n}. Given an irreducible unitary supercuspidal representation τ of $\mathrm{GL}(2n)(F)$ and a unitary character ω of F^* such that the local Langlands-Shahidi L-function $L(s,\tau,\mathrm{Sym}^2\rho_{2n}\otimes\omega^{-1})$ has a pole at $s=0$, the purpose of this paper is to construct a generic supercuspidal representation (descent) of G such that any generic irreducible supercuspidal representation of G which lifts functorially to τ is contragradient to some constituent of the descent.

In § 2, for a quadratic form Q defined over a vector space V, we define the group GSpin(Q) and show the existence of a canonical isomorphism of the Siegel Levi subgroup of GSpin$(2m+1)$ to $\mathrm{GL}(1) \times \mathrm{GL}(m)$ with respect to a canonical projection $\mathrm{pr} : \mathrm{GSpin}(2m+1) \twoheadrightarrow \mathrm{SO}(2m+1)$ for $m \geqslant 4$. This isomorphism enables us to reduce calculations performed on the adelic or rational points of the Siegel Levi of $\mathbf{H} = \mathrm{GSpin}(2m+1)$ to that of the Siegel Levi of $\mathrm{SO}(2m+1)$. With respect to the maximal torus of $\mathrm{SO}(2m+1)$ consisting of the diagonal matrices of $\mathrm{SO}(2m+1)$ and the Borel subgroup of upper triangular matrices in $\mathrm{SO}(2m+1)$, let $\mathbf{P}'_\mathbf{l} = \mathbf{M}'_\mathbf{l}\mathbf{N}'_\mathbf{l}$ be the standard parabolic subgroup of $\mathrm{SO}(2m+1)$ whose Levi subgroup $\mathbf{M}'_\mathbf{l}$ is the direct product of l-copies of GL(1) and $\mathrm{SO}(2(m-l)+1)$ and $\mathbf{P}_\mathbf{l} = \mathrm{pr}^{-1}(\mathbf{P}'_\mathbf{l})$ have Levi decomposition $\mathbf{M}_\mathbf{l}\mathbf{N}_\mathbf{l}$. For a fixed $\alpha \in K^*$ and anisotropic vector w_0 in V depending on α, we define a rational additive character $\chi_{l,w_0} : \mathbf{N}'_\mathbf{l} \to \mathrm{G_a}$, cf. subsection 2.2. The $\mathrm{SO}(2(m-l)+1)$ factor acts on $\mathbf{N}'_\mathbf{l}$ and the set of rational additive characters from $\mathbf{N}'_\mathbf{l}$ to $\mathrm{G_a}$ by conjugation with stabilizer $\mathbf{L}'_{\mathbf{l},\alpha}$ isomorphic to a quasi-split form of $\mathrm{SO}(2(m-l))$. Let $\mathbf{R}'_{\mathbf{l},\alpha} = \mathbf{L}'_{\mathbf{l},\alpha} \ltimes \mathbf{N}'_\mathbf{l}$ and ψ be a non-trivial character of $K\backslash\mathbb{A}$. Define $\psi_{l,\alpha} = \psi \circ \chi_{l,w_0} \circ \mathrm{pr}\,|_{\mathbf{N}_\mathbf{l}(\mathbb{A})}$. Then $\psi_{l,\alpha}$ is a character of $\mathbf{N}_\mathbf{l}(\mathbb{A})$ trivial on N_l. Label the set of simple roots of $\mathbf{H}$ with respect to the maximal torus $\mathrm{pr}^{-1}(\mathbf{M}'_\mathbf{m})$ and the Borel subgroup $\mathrm{pr}^{-1}(\mathbf{P}'_\mathbf{m})$ following [Bou68]. Denote the parabolic subgroup of $\mathbf{H}$ containing $\mathrm{pr}^{-1}(\mathbf{P}'_\mathbf{m})$ whose Levi subgroup is isomorphic to $\mathrm{GL}(j) \times \mathrm{GSpin}(2(m-j)+1)$ by $\mathbf{Q}_\mathbf{j}$

and the unipotent radical of $\mathbf{Q_j}$ by $\mathbf{U_j}$. We then define a global Rankin-Selberg integral and show that it converges absolutely in a suitable right half plane and is factorizable.

To lighten the notation, we re-denote the local representations of p-adic groups using the same notation as their global counterpart. Let $\mathbf{L_{l,\alpha}} = \mathrm{pr}^{-1}(\mathbf{L'_{l,\alpha}})$, $\mathbf{R_{l,\alpha}} = \mathrm{pr}^{-1}(\mathbf{R'_{l,\alpha}})$, ω be a character of F^* and τ be a smooth representation of $\mathrm{GL}(n)(F)$. For the Langlands quotient $\pi_{\omega\otimes\tau}$ obtained by normalized parabolic induction of $\omega \otimes |\det|^{1/2}\tau$ from the Siegel parabolic subgroup of H, we compute the finite Q_l-filtration of $J_{\psi_{l,\alpha}}(\pi_{\omega\otimes\tau}|_{R_{l,\alpha}})$, the twisted Jacquet module of the restriction of $\pi_{\omega\otimes\tau}$ to $R_{l,\alpha}$ with respect to the character $\psi_{l,\alpha}$ on N_l. This is in turn used to establish Theorem 2.12 in § 2.4. Theorem 2.12 is a generic multiplicity one result which enables us to define the Rankin-Selberg γ-factor for $\mathrm{GSpin}(2(m-l)-1) \times \mathrm{GL}(1) \times \mathrm{GL}(m)$. We then proceed to establish the fact that this γ-factor has a pole at $s = 1$ if and only if σ pairs non-trivially with $J_{\psi_{l,\alpha}}(\pi_{\omega\otimes\tau}|_{R_{l,\alpha}})$ which we define as the descent $\sigma_{\psi,l}(\tau)$ of τ to G.

§ 3 states the Exchanging Roots Lemma which proves certain twisted Jacquet modules are isomorphic as vector spaces under certain conditions. This lemma will be used to prove Theorem 3.3: tower property of descents. Denote the maximal parabolic subgroup of $\mathbf{L_{l,\alpha}}$ with Levi subgroup isomorphic to $\mathrm{GL}(p) \times \mathrm{GSpin}(2m-2l-2p)$ and containing a fixed Borel subgroup of $\mathbf{G}$ by $\mathbf{Q_p^*}$. The tower property asserts there exists a suitable vector space isomorphism from the Jacquet module of the descent $\sigma_{\psi,l}(\tau)$ associated to Q_p^* to the compact induction from N_{l+p} to P_{l+p} of the descent $\sigma_{\psi,l+p}(\tau)$ restricted to N_{l+p}.

In § 4 we prove the vanishing of the descent in the relevant range. The main ingredient we use for the proof is a class of exceptional representations, which are small in the sense that they are attached to one of the coadjoint orbits next to the minimal one. In our setting these are the representations of double covers of general linear groups constructed by Kazhdan and Patterson [KP84], or the representations of double covers of general spin groups developed in [Kap17b] (following [BFG03]). We tensor two such representations to form a representation of the linear group. Such a representation is typically quite large, and may be considered as a model (see [Kab01, Kap16a, Kap16b, Kap17a]). For example, one may prove multiplicity one results (e.g., [Kab01]), or analyze the structure of its irreducible quotients ([Kap17a]). In this spirit, we say that a representation of the linear group affords an exceptional model if it is a quotient of the tensor of two exceptional representations of the double cover of the group.

Consider a supercuspidal representation τ of $\mathrm{GL}(m)(F)$ such that its symmetric square L-function has a pole at $s = 0$. According to the results of [Kap16b], this representation affords an exceptional model and so does the representation parabolically induced from $\mathbb{1} \otimes \tau$ to a general spin group. To prove the vanishing results we use the smallness of the exceptional representations, namely that a large class of their twisted Jacquet modules vanish ([Kap17b, BFG03]).

This reasoning is parallel to the procedure for the special odd orthogonal group of Ginzburg *et. al.* [GRS99a, GRS99b]. They used the interplay between Shalika models, which are related to the pole of the exterior square L-function at $s = 0$, linear models, and symplectic models (see § 4.1 for a more precise description). The presence of exceptional representations here is expected and understood, in light of the role these representations played in the (global) work of Bump and Ginzburg on the integral representation of the symmetric square L-function, or even in the earlier low rank results [GJ78, PPS89].

To handle the twisted symmetric square L-function we use the recent construction of twisted exceptional representations for double coverings of general linear groups by Takeda [Tak14], who used them to develop an integral representation for the global partial L-function. We also rely on a result of Yamana [Yam17] who proved that if the twisted symmetric square L-function of τ has a pole at

$s = 0$, τ admits a (twisted) exceptional model.

In § 5, we prove the non-vanishing of the descent $\sigma_{\psi,n}(\tau)$. The main ingredients are the result in [JLS16] on raising of nilpotent orbits in the wave front set of representations and the result in [GGS17] on relations between degenerate Whittaker models and generalized Whittaker models of representations. More explicitly, first, by [GGS17], we show that $\pi_{\omega\otimes\tau}$ has a non-zero Fourier coefficient attached to the partition $[(2n)^2 1]$, which is not special. By [JLS16], $\pi_{\omega\otimes\tau}$ has a non-zero Fourier coefficient attached to the partition $[(2n+1)(2n-1)1]$, which is the smallest orthogonal special partition bigger than $[(2n)^2 1]$. Then by [GGS17] again, $\pi_{\omega\otimes\tau}$ has a non-zero Fourier coefficient attached to the partition $[(2n+1)1^{2n}]$, which implies that the descent $\sigma_{\psi,n}(\tau)$ is non-vanishing. In [GRS11], the non-vanishing of the descent from automorphic representations of $\mathrm{GL}(2n)$ to automorphic representations of $\mathrm{SO}(2n)$ is proven by contradiction. Similar arguments also apply here and gives another proof for the non-vanishing of $\sigma_{\psi,n}(\tau)$. But, the proof we provide in this paper is a new way of proving non-vanishing of descent, and is more conceptual and much shorter. In [HS16], the non-vanishing of the descent from automorphic representations of $\mathrm{GL}(2n)$ to automorphic representations of $\mathrm{GSpin}(2n)$ is proven by similar ideas as in this paper, with much more details since they do not have the results in [JLS16] and [GGS17] at that time.

For § 6, we state and prove the main result of this paper which relates local descent to Langlands functoriality, namely, for an irreducible unitary supercuspidal representation τ of $\mathrm{GL}(2n)(F)$ whose twisted symmetric square L-function $L(s, \tau, \mathrm{Sym}^2 \rho_{2n} \otimes \omega^{-1})$ has a pole at $s = 0$ for some unitary character ω, the descent $\sigma_{\psi,n}(\tau)$ is a non-trivial unitary supercuspidal multiplicity-free representation of G. Moreover, for any irreducible generic supercuspidal representation σ of G such that $\gamma(s, \sigma \times (\omega \otimes \tau), \psi^{-1})$ has a pole at $s = 1$, $\sigma^\vee$ is a direct summand of $\sigma_{\psi,n}(\tau)$.

Acknowledgements

We would like to thank Mahdi Asgari, Mikhail Borovoi, Wee Teck Gan, Joseph Hundley, Eitan Sayag, Freydoon Shahidi, David Soudry and Lei Zhang, for useful correspondences and helpful conversations.

2 A Uniqueness Theorem

2.1 The groups

Let K be a field of characteristic 0, V be a K-vector space, Q a quadratic form on V defined over K. Denote the special orthogonal group of Q by $\mathrm{SO}(Q)$ and its simply connected cover by $\mathrm{Spin}(Q)$. Let $\pi_1 : \mathrm{Spin}(Q) \twoheadrightarrow \mathrm{SO}(Q)$ be the canonical isogeny and c be the non-trivial element in $\ker \pi_1$. Then $(-1, c)$ generates an order 2 subgroup C of $\mathrm{GL}(1)\times\mathrm{Spin}(Q)$. Define $\mathrm{GSpin}(Q) := (\mathrm{GL}(1)\times\mathrm{Spin}(Q))/C$.

Note that this definition agrees with the one in [AS06] when $\mathrm{GSpin}(Q)$ is split, cf. Proposition 2.2 of [AS06]. Also when $\dim_K V = 2n$, 16.2.3 of [Spr98] shows that the index of our non-split quasi-split $\mathrm{GSpin}^*(2n)$ coincides with that of $\mathrm{SO}^*(2n)$ which agrees with the index of the non-split quasi-split $\mathrm{GSpin}^*(2n)$ of [AS14].

From now on, assume $\dim_K V = 2m + 1$. Let b be the corresponding symmetric bilinear form on V. Fix maximal isotropic subspaces $V^{\pm}$ in duality with respect to b and a maximal flag in V^+,

$$0 \subset V_1^+ \subset V_2^+ \subset \cdots \subset V_m^+ = V^+$$

and choose a basis $\{e_1, \cdots, e_m\}$ of V^+ over K such that $V_i^+ = \mathrm{Span}_K\{e_1, \cdots, e_i\}$. Let $\{e_{-1}, \cdots, e_{-m}\}$ be the basis of V^- which is dual to $\{e_1, \cdots, e_m\}$, i.e. $b(e_i, e_{-j}) = \delta_{i,j}$ for all $1 \leqslant i, j \leqslant m$. This choice of a maximal flag fixes a Borel subgroup $\mathbf{B}'$ and maximal torus $\mathbf{T}'$ of $\mathrm{SO}(Q)$ such that $\mathbf{T}' \subset \mathbf{B}'$. Then $\mathbf{B} = \pi_1^{-1}(\mathbf{B}')$ and $\mathbf{T} = \pi_1^{-1}(\mathbf{T}')$ is a Borel subgroup and maximal torus of $\mathrm{Spin}(Q)$ such that $\mathbf{T} \subset \mathbf{B}$.

Denote the order of $t \in \mathrm{GL}(1)$ by $o(t)$.

Lemma 2.1. *Suppose V is of dimension $2m+1$ where $m \geqslant 4$. Let Δ be the set of simple roots of $\mathrm{Spin}(Q)$ with respect to $(\mathbf{B}, \mathbf{T})$, $\mathbf{Q}$ be the Siegel parabolic subgroup of $\mathrm{Spin}(Q)$ corresponding to the subset $\Delta - \{\alpha_m\}$ of Δ. Denote the Levi subgroup of $\mathbf{Q}$ by $\mathbf{M}$.*

(i) If m is odd, there exists an isomorphism

$$\varphi_1 : \mathbf{M} \xrightarrow{\sim} (\mathrm{GL}(1) \times \mathrm{SL}(m))/\{(t, t^{-2}I_m) \mid t^m = 1\}$$

where the simple coroots $\alpha_j^\vee$ for $1 \leqslant j \leqslant m-1$ of $\mathrm{Spin}(Q)$ are identified with the simple coroots of $\mathrm{SL}(m)$ with respect to the Borel subgroup of upper triangular matrices and maximal torus of diagonal matrices in $\mathrm{SL}(m)$. We have $\varphi_1(c) = \overline{(\omega, \omega^{-2}I_m)}$ where ω is a primitive $2m$ root of unity. Moreover, if we further assume that $\pi_1 \circ \alpha_j^\vee$ for $1 \leqslant j \leqslant m$ are the usual coroots of $\mathrm{SO}(Q)$ with respect to $(\mathbf{B}', \mathbf{T}')$, then

$$\pi_1 \circ \varphi_1^{-1} : (\mathrm{GL}(1) \times \mathrm{SL}(m))/\{(t, t^{-2}I_m) \mid t^m = 1\} \twoheadrightarrow \mathrm{GL}(m)$$

is given by $\overline{(t, g)} \mapsto t^2 \cdot g$.

(ii) If m is even, there exists an isomorphism

$$\varphi_2 : \mathbf{M} \xrightarrow{\sim} (\mathrm{GL}(1) \times \mathrm{SL}(m))/\{(t, t^{-1}I_m) \mid t^{m/2} = 1\}$$

where the simple coroots $\alpha_j^\vee$ for $1 \leqslant j \leqslant m-1$ of $\mathrm{Spin}(Q)$ are identified with the simple coroots of $\mathrm{SL}(m)$ with respect to the Borel subgroup of upper triangular matrices and maximal torus of diagonal matrices in $\mathrm{SL}(m)$. We have $\varphi_2(c) = \overline{(\omega, \omega^{-1}I_m)}$ where ω is a primitive m root of unity. Moreover, if we further assume that $\pi_1 \circ \alpha_j^\vee$ for $1 \leqslant j \leqslant m$ are the usual coroots of $\mathrm{SO}(Q)$ with respect to $(\mathbf{B}', \mathbf{T}')$, then

$$\pi_1 \circ \varphi_2^{-1} : (\mathrm{GL}(1) \times \mathrm{SL}(m))/\{(t, t^{-1}I_m) \mid t^{m/2} = 1\} \twoheadrightarrow \mathrm{GL}(m)$$

is given by $\overline{(t, g)} \mapsto t \cdot g$.

Proof. Let $\mathbf{A}$ be the radical of $\mathbf{M}$ and $\mathbf{M}_D$ be the derived subgroup of $\mathbf{M}$. Then direct calculation gives

$$\begin{aligned} \mathbf{A} &= \left(\bigcap_{j=1}^{m-1} \ker \alpha_j\right)^\circ \\ &= \begin{cases} \{a(t) = \alpha_1^\vee(t^2)\alpha_2^\vee(t^4)\cdots\alpha_{m-2}^\vee(t^{2(m-2)})\alpha_{m-1}^\vee(t^{2(m-1)})\alpha_m^\vee(t^m) \mid t \in \mathrm{GL}(1)\} & \text{for odd } m\,; \\ \{a(t) = \alpha_1^\vee(t)\alpha_2^\vee(t^2)\cdots\alpha_{m-2}^\vee(t^{m-2})\alpha_{m-1}^\vee(t^{m-1})\alpha_m^\vee(t^{m/2}) \mid t \in \mathrm{GL}(1)\} & \text{for even } m\,. \end{cases} \end{aligned}$$

Since $\mathrm{Spin}(Q)$ is simply-connected, $\mathbf{M}_D$ is simply-connected. Hence $\mathbf{M}_D$ is isomorphic to $\mathrm{SL}(m)$. We may and do assume the simple coroots $\alpha_j^\vee$ for $1 \leqslant j \leqslant m-1$ of $\mathrm{Spin}(Q)$ are identified with the simple coroots of $\mathrm{SL}(m)$ with respect to the Borel subgroup of upper triangular matrices and maximal torus of diagonal matrices in $\mathrm{SL}(m)$. We have

$$\mathbf{A}\cap\mathbf{M}_D=\begin{cases}\{a(t)\,|\,t^m=1\}=\{t^2I_m\,|\,t^m=1\} & \text{for odd } m\,;\\ \{a(t)\,|\,t^{m/2}=1\}=\{tI_m\,|\,t^{m/2}=1\} & \text{for even } m\,.\end{cases}$$

The canonical isogeny $\mathbf{A}\times\mathbf{M}_D\twoheadrightarrow\mathbf{M}$ has kernel $\{(\alpha,\alpha^{-1})\,|\,\alpha\in\mathbf{A}\cap\mathbf{M}_D\}$ induces the suitable isomorphism φ_1 (resp. φ_2) when m is odd (resp. even). Note that c is the unique order 2 element in the center of $\mathrm{Spin}(Q)$. It follows from [A02, Proposition 2.2] that $c=\alpha_m^\vee(-1)$. Direct calculation shows that

$$\begin{aligned} c&=\begin{cases}a(\omega)\alpha_1^\vee(\omega^{-2})\alpha_2^\vee(\omega^{-4})\cdots\alpha_{m-2}^\vee(\omega^{-2(m-2)})\alpha_{m-1}^\vee(\omega^{-2(m-1)}) & \text{where } o(\omega)=2m \text{ for odd } m\,;\\ a(\omega)\alpha_1^\vee(\omega^{-1})\alpha_2^\vee(\omega^{-2})\cdots\alpha_{m-2}^\vee(\omega^{-(m-2)})\alpha_{m-1}^\vee(\omega^{-(m-1)}) & \text{where } o(\omega)=m \text{ for even } m\,,\end{cases}\\ &=\begin{cases}\overline{(\omega,\omega^{-2}I_m)} & \text{where } o(\omega)=2m \text{ for odd } m\,;\\ \overline{(\omega,\omega^{-1}I_m)} & \text{where } o(\omega)=m \text{ for even } m\,.\end{cases}\end{aligned}$$

We further assume that $\pi_1\circ\alpha_j^\vee$ for $1\leqslant j\leqslant m$ are the usual coroots of $\mathrm{SO}(Q)$ with respect to $(\mathbf{B}',\mathbf{T}')$. Then for odd m, we have the following commutative diagram

$$\begin{array}{ccc}\mathrm{GL}(1)\times\mathrm{SL}(m) & \xrightarrow{\qquad\qquad} & \mathrm{GL}(m)\\ \downarrow & & \uparrow\\ (\mathrm{GL}(1)\times\mathrm{SL}(m))/\{(t,t^{-2}I_m)\,|\,t^m=1\} & \twoheadrightarrow & (\mathrm{GL}(1)\times\mathrm{SL}(m))/\{(t,t^{-2}I_m)\,|\,t^{2m}=1\}\end{array}$$

where the top homomorphism is given by $(t,g)\mapsto t^2\cdot g$ with kernel $\{(t,t^{-2}I_m)\,|\,t^{2m}=1\}$ and the other two epimorphisms are the canonical epimorphisms. Thus the right vertical arrow is an isomorphism. For even m, we have the following commutative diagram

$$\begin{array}{ccc}\mathrm{GL}(1)\times\mathrm{SL}(m) & \xrightarrow{\qquad\qquad} & \mathrm{GL}(m)\\ \downarrow & & \uparrow\\ (\mathrm{GL}(1)\times\mathrm{SL}(m))/\{(t,t^{-1}I_m)\,|\,t^{m/2}=1\} & \twoheadrightarrow & (\mathrm{GL}(1)\times\mathrm{SL}(m))/\{(t,t^{-1}I_m)\,|\,t^{m}=1\}\end{array}$$

where the top homomorphism is given by $(t,g)\mapsto t\cdot g$ with kernel $\{(t,t^{-1}I_m)\,|\,t^m=1\}$ and the other two epimorphisms are the canonical epimorphisms. Thus the vertical right arrow is an isomorphism. □

Lemma 2.2. *Let* $\pi_2:\mathrm{GL}(1)\times\mathrm{Spin}(Q)$ *be the canonical projection onto* $\mathrm{Spin}(Q)$.

(i) There exists a canonical surjective K-rational map $\mathrm{pr}:\mathrm{GSpin}(Q)\twoheadrightarrow\mathrm{SO}(Q)$ *with* $\ker\mathrm{pr}$ *isomorphic to* $\mathrm{GL}(1)$. *In particular, unipotent subgroups of* $\mathrm{GSpin}(Q)$ *and* $\mathrm{SO}(Q)$ *are in bijective correspondence and corresponding unipotent subgroups are isomorphic.*

(ii) Suppose V is of dimension $2m+1$ where $m\geqslant 4$ and $\mathbf{M}$ is the Siegel Levi subgroup as in Lemma 2.1. There exists an isomorphism $\Lambda:(\mathrm{GL}(1)\times\mathbf{M})/C\to\mathrm{GL}(1)\times\mathrm{GL}(m)$ *such that* $\mathrm{pr}\circ\Lambda^{-1}$ *is given by* $(t,g)\mapsto g$ *for* $t\in\mathrm{GL}(1)$ *and* $g\in\mathrm{GL}(m)$.

Proof. Note that $\pi_1 \circ \pi_2$ is K-rational, surjective and factors through C. Hence $\pi_1 \circ \pi_2$ induces a canonical K-rational surjection $\mathrm{pr} : \mathrm{GSpin}(Q) \twoheadrightarrow \mathrm{SO}(Q)$ with $\ker \mathrm{pr} = \{\overline{(t,1)} \mid t \in \mathrm{GL}(1)\}$. This shows (i).

Assume for now the hypothesis of (ii). For odd m, we have the following commutative diagram

$$\begin{array}{ccc}
\mathrm{GL}(1) \times \mathrm{GL}(1) \times \mathrm{SL}(m) & \xrightarrow{\ \Xi_1\ } & \mathrm{GL}(1) \times \mathrm{GL}(m) \\
\downarrow & & \uparrow \\
\mathrm{GL}(1) \times (\mathrm{GL}(1) \times \mathrm{SL}(m))/\{(t, t^{-2} I_m) \mid t^m = 1\} & \twoheadrightarrow & (\mathrm{GL}(1) \times (\mathrm{GL}(1) \times \mathrm{SL}(m))/\{(t, t^{-2} I_m) \mid t^m = 1\})/(id \times \varphi_1)(C) \\
\downarrow{\scriptstyle id \times \varphi_1^{-1}} & & \uparrow \\
\mathrm{GL}(1) \times \mathbf{M} & \twoheadrightarrow & (\mathrm{GL}(1) \times \mathbf{M})/C
\end{array}$$

where the top homomorphism Ξ_1 is given by $(t_1, t_2, g) \mapsto (t_1 t_2^m, t_2^2 \cdot g)$ with kernel $\{((-1)^i, \omega^j, \omega^{-2j} \cdot I_m) \mid o(\omega) = 2m \text{ and } i \equiv j \pmod 2\}$ and the other three two head arrows are the canonical epimorphisms. Thus the two right vertical arrows are isomorphisms. Call the composition of the two vertical maps Λ. Given $(t, g) \in \mathrm{GL}(1) \times \mathrm{GL}(m)$, let α be a fixed $2m$-th root of $\det g$. Then $\Xi_1(t\alpha^{-m}, \alpha, \alpha^{-2} g) = (t, g)$. Hence

$$\mathrm{pr} \circ \Lambda^{-1}(t, g) = \mathrm{pr}\, \overline{(t\alpha^{-m}, \alpha, \alpha^{-2} g)} = \pi_1 \overline{(\alpha, \alpha^{-2} g)} = g$$

by Lemma 2.1(i). For even m, we have the following commutative diagram

$$\begin{array}{ccc}
\mathrm{GL}(1) \times \mathrm{GL}(1) \times \mathrm{SL}(m) & \xrightarrow{\ \Xi_2\ } & \mathrm{GL}(1) \times \mathrm{GL}(m) \\
\downarrow & & \uparrow \\
\mathrm{GL}(1) \times (\mathrm{GL}(1) \times \mathrm{SL}(m))/\{(t, t^{-1} I_m) \mid t^{m/2} = 1\} & \twoheadrightarrow & (\mathrm{GL}(1) \times (\mathrm{GL}(1) \times \mathrm{SL}(m))/\{(t, t^{-1} I_m) \mid t^{m/2} = 1\})/(id \times \varphi_2)(C) \\
\downarrow{\scriptstyle id \times \varphi_2^{-1}} & & \uparrow \\
\mathrm{GL}(1) \times \mathbf{M} & \twoheadrightarrow & (\mathrm{GL}(1) \times \mathbf{M})/C
\end{array}$$

where the top homomorphism Ξ_2 is given by $(t_1, t_2, g) \mapsto (t_1 t_2^{m/2}, t_2 \cdot g)$ with kernel $\{((-1)^i, \omega^j, \omega^{-j} \cdot I_m) \mid o(\omega) = m \text{ and } i \equiv j \pmod 2\}$ and the other three two head arrows are the canonical epimorphisms. Thus the two right vertical arrows are isomorphisms. Call the composition of the two vertical maps Λ. Given $(t, g) \in \mathrm{GL}(1) \times \mathrm{GL}(m)$, let β be a fixed mth root of $\det g$. Then $\Xi_2(t\beta^{-m/2}, \beta, \beta^{-1} g) = (t, g)$. Hence

$$\mathrm{pr} \circ \Lambda^{-1}(t, g) = \mathrm{pr}\, \overline{(t\beta^{-m/2}, \beta, \beta^{-1} g)} = \pi_1 \overline{(\beta, \beta^{-1} g)} = g$$

by Lemma 2.1(ii). □

2.2 Unfolding a Rankin-Selberg integral

Suppose K is a number field with ring of adeles $\mathbb{A}$. Let $\mathrm{pr} : \mathrm{GSpin}(Q) \to \mathrm{Spin}(Q)/C \cong \mathrm{SO}(Q)$ be the surjective K-rational homomorphism such that $\mathrm{pr}(s, g)C = sC$ for all $s \in \mathrm{Spin}(Q)$, $g \in \mathrm{GL}(1)$. Since the kernel of pr consists of only semisimple elements, pr restricted to a unipotent subgroup of $\mathrm{GSpin}(Q)$ is a K-rational isomorphism to its image unipotent subgroup of $\mathrm{SO}(Q)$.

Let

$$\mathbf{H} = \mathrm{GSpin}(Q) = \mathrm{GSpin}(2m+1)\,.$$

For $1 \leqslant l \leqslant m$, let φ_l be the flag

$$\varphi_l : 0 \subset V_1^+ \subset V_2^+ \subset \cdots \subset V_l^+\,.$$

Let $\mathbf{P_l'}$ be the parabolic subgroup of $\mathrm{SO}(V)$ stabilizing φ_l. Denote its Levi decomposition by $\mathbf{P_l'} = \mathbf{M_l'}\mathbf{N_l'}$. Then $\mathbf{M_l'}$ is K-isomorphic to $\mathrm{GL}(V_{\varphi_l}^+(1)) \times \cdots \times \mathrm{GL}(V_{\varphi_l}^+(l)) \times \mathrm{SO}(W)$ where $V_{\varphi_l}^+(i) = \mathrm{Span}_K\{e_i\}$, $1 \leqslant i \leqslant l$ and

$$W = (V_l^+ + V_l^-)^\perp\,.$$

Let $\alpha \in K^*$. For $0 \leqslant l < m$, choose $w_0 = y_\alpha = e_m + \frac{\alpha}{2}e_{-m}$. Then $b(w_0, w_0) = \alpha$ and $w_0 \in W$.

Define for u in $\mathbf{N_l'}$

$$\chi_{l,w_0}(u) = \sum_{i=2}^{l} b(u\cdot e_i, e_{-(i-1)}) + b(u\cdot w_0, e_{-l})\,.$$

Then χ_{l,w_0} is a K-rational homomorphism from $\mathbf{N_l'}$ to $\mathrm{G_a}$. Let ψ be a non-trivial character of $K\backslash\mathbb{A}$. Then

$$\psi'_{l,\alpha} = \psi \circ \chi_{l,w_0}$$

is a character of $\mathbf{N_l'}(\mathbb{A})$ trivial on N_l'.

Let $\mathrm{SO}(W)$ act on $\mathbf{N_l'}$ and the set of rational additive characters from $\mathbf{N_l'}$ to $\mathrm{G_a}$ by conjugation. Then the stabilizer $\mathbf{L'_{l,\alpha}}$ of χ_{l,w_0} in $\mathrm{SO}(W)$ is the stabilizer of w_0 in $\mathrm{SO}(W)$, i.e. $\mathrm{SO}(w_0^\perp \cap W)$. Denote

$$\mathbf{R'_{l,\alpha}} = \mathbf{L'_{l,\alpha}} \ltimes \mathbf{N_l'}\,.$$

Let $\mathbf{P_l} = \mathrm{pr}^{-1}(\mathbf{P_l'})$ have Levi decomposition $\mathbf{M_l}\mathbf{N_l}$, $\mathbf{L_{l,\alpha}} = \mathrm{pr}^{-1}(\mathbf{L'_{l,\alpha}})$ and $\mathbf{R_{l,\alpha}} = \mathrm{pr}^{-1}(\mathbf{R'_{l,\alpha}})$. Then $\mathrm{GSpin}(W)$ acts on $\mathbf{N_l}$ and the set of rational characters from $\mathbf{N_l}$ to $\mathrm{G_a}$ by conjugation. Also, $\mathbf{L_{l,\alpha}}$ being the stabilizer of $\chi_{l,w_0} \circ \mathrm{pr}$ in $\mathrm{GSpin}(W)$ is K-isomorphic to a GSpin group of one rank lower than $\mathrm{pr}^{-1}(\mathrm{SO}(W)) = \mathrm{GSpin}(W)$.

We choose $e_0^{(1)}, e_0^{(2)} \in w_0^\perp \cap W$ such that

$$\begin{aligned} &b(e_0^{(1)}, e_j) = 0\,, b(e_0^{(1)}, e_{-j}) = 0\,,\, b(e_0^{(1)}, e_0^{(1)}) = 1\,,\, b(e_0^{(1)}, e_0^{(2)}) = 0\,;\\ &b(e_0^{(2)}, e_j) = 0\,, b(e_0^{(2)}, e_{-j}) = 0\,,\, b(e_0^{(2)}, e_0^{(2)}) = -c \end{aligned}$$

for all $l+1 \leqslant j \leqslant m-1$ so that $\{e_{l+1}, \cdots, e_{m-1}, e_0^{(1)}, e_0^{(2)}, e_{-(m-1)}, \cdots, e_{-(l+1)}\}$ is a basis of $w_0^\perp \cap W$ and $c \in K^*$ is a square if and only if $\alpha \in K^*$ is a square. Note that $\mathrm{GSpin}(w_0^\perp \cap W)$ is quasi-split of semi-simple rank $2m-2l$ and is split if and only if $\alpha \in K^*$ is a square.

Label the set of simple roots of $\mathbf{H}$ with respect to the maximal torus $\mathrm{pr}^{-1}(\mathbf{M_m'})$ and the Borel subgroup $\mathrm{pr}^{-1}(\mathbf{P_m'})$ following [Bou68]. For $1 \leqslant j \leqslant m$, let $\mathbf{Q_j}$ be the standard parabolic subgroup of $\mathbf{H}$ which corresponds to the subset of Δ omitting the jth simple root. Its Levi subgroup is isomorphic to $\mathrm{GL}(j) \times \mathrm{GSpin}(2(m-j)+1)$ where $\mathrm{GSpin}(1)$ is taken to be $\mathrm{GL}(1)$. Denote the unipotent radical of $\mathbf{Q_j}$ by $\mathbf{U_j}$. Lemma 2.2(ii) shows that we may and shall regard the elements of the Siegel Levi as (t, g), $t \in \overline{K}$, $g \in \mathrm{GL}(m)$.

Let τ be a cuspidal automorphic representation of $\mathrm{GL}(m)(\mathbb{A})$ and ω be a Hecke character of $\mathrm{GL}(1)(\mathbb{A})$. Consider the normalized parabolic induction

$$\rho_{\omega\otimes\tau,s} = \mathrm{Ind}_{\mathbf{Q_m}(\mathbb{A})}^{\mathbf{H}(\mathbb{A})}(\omega \otimes \tau|\det|^{s-1/2})\,.$$

Let $f_{\omega\otimes\tau,s}$ be a smooth holomorphic section of $\rho_{\omega\otimes\tau,s}$. Define the Eisenstein series

$$E(h, f_{\omega\otimes\tau,s}) = \sum_{\gamma\in Q_m\backslash H} f_{\omega\otimes\tau,s}(\gamma h)\,,\ h \in \mathbf{H}(\mathbb{A})\,.$$

Let (π, V) be an automorphic representation of $\mathbf{H}(\mathbb{A})$. Define for $\xi \in V$, $h \in \mathbf{H}(\mathbb{A})$, the Gelfand-Graev coefficient of ξ with respect to $\psi_{l,\alpha} = \psi'_{l,\alpha} \circ \mathrm{pr}$ by

$$\xi^{\psi_{l,\alpha}}(h) = \int_{\mathbf{N_l}(K)\backslash\mathbf{N_l}(\mathbb{A})} \xi(vh)\psi_{l,\alpha}^{-1}(v)\,dv\,.$$

We have

$$\xi^{\psi_{l,\alpha}}(\gamma h) = \xi^{\psi_{l,\alpha}}(h)$$

for all $\gamma \in R_{l,\alpha}$. In particular, the function $\xi^{\psi_{l,\alpha}}$ restricted to $L_{l,\alpha}(\mathbb{A})$ is automorphic.

Let w_n be the $n \times n$ matrix with 1 on the anti-diagonal and 0 everywhere else. For any $1 \leqslant j \leqslant m$, denote $w_j\,{}^t\gamma^{-1}w_j$ by γ^* and let

$$\gamma^\wedge = \begin{pmatrix} \gamma & 0 & 0 \\ 0 & I_{\dim_K V-2j} & 0 \\ 0 & 0 & \gamma^* \end{pmatrix}.$$

The following proposition has been established in [ACS17].

Proposition 2.3 ([ACS17]). *Suppose that (V, Q) is a split quadratic space where $\dim V = 2m+1$. Let $0 \leqslant l < m$ be an integer and $\alpha \in K^*$. Let σ be a cuspidal automorphic representation of $\mathbf{G}(\mathbb{A})$, τ a cuspidal automorphic representation of $\mathrm{GL}(m)(\mathbb{A})$ and ω be the restriction of ω_σ^{-1} to $Z(\mathbf{G})^\circ(\mathbb{A})$. Consider the meromorphic functions*

$$I(\varphi_\sigma, f_{\omega\otimes\tau,s}) = \int_{\mathrm{GL}(1)(\mathbb{A})\mathbf{G}(K)\backslash\mathbf{G}(\mathbb{A})} \varphi_\sigma(g)E^{\psi_{l,\alpha}}(g, f_{\omega\otimes\tau,s})\,dg$$

as φ_σ varies in the space of σ and $f_{\omega\otimes\tau,s}$ varies in the space of smooth holomorphic sections in $\rho_{\omega\otimes\tau,s}$. Suppose that $I(\varphi_\sigma, f_{\omega\otimes\tau,s})$ is not identically zero (as a meromorphic function and as the data vary). Then σ is globally generic with respect to a certain Whittaker character ψ_{N_G} where $\mathbf{N_G}$ is the unipotent radical of a "standard" Borel subgroup of $\mathbf{L_{l,\alpha}}$ and

$$\psi_{N_G}(n) = \psi(z_{1,2} + \cdots + z_{m-l-2,m-l-1} + v_{m-l-1}) \text{ where}$$

$$\mathrm{pr}(n) = q(z,v,x,u) = \mathrm{diag}\left(I_l, \begin{pmatrix} z & v & x & -\frac{2}{\alpha}v & u \\ & 1 & 0 & 0 & -\frac{2}{\alpha}v' \\ & & 1 & 0 & x' \\ & & & 1 & v' \\ & & & & z^* \end{pmatrix}, I_l\right) \in SO(2m+1)\,,\ n \in N_G\,.$$

Moreover, for $\mathrm{Re}(s)$ sufficiently large, we have

$$I(\varphi_\sigma, f_{\omega\otimes\tau,s}) = \int_{\mathrm{GL}(1)(\mathbb{A})\mathbf{N_G}(\mathbb{A})\backslash\mathbf{G}(\mathbb{A})} W_{\varphi_\sigma}^{\psi}(g) \int_{\mathbf{N_l}(\mathbb{A})\cap\beta_{l,\alpha}^{-1}\mathbf{Q_m}(\mathbb{A})\beta_{l,\alpha}\backslash\mathbf{N_l}(\mathbb{A})} f_{\omega\otimes\tau,s}^{Z_m,\psi}(\beta_{l,\alpha}ug)\psi_{l,\alpha}^{-1}(u)\,du\,dg$$

where $W^{\psi}_{\varphi_\sigma}$ *is the Whittaker functional of* φ_σ*. Let* ω_b *be as in [GRS11, pp. 70–71].* $\beta_{l,\alpha}$ *is any fixed element in*

$$\mathrm{pr}^{-1}\left(\begin{pmatrix}0 & \frac{2}{\alpha}I_{m-l}\\ I_l & 0\end{pmatrix}^{\wedge}\begin{pmatrix}0 & 0 & I_l\\ 0 & I_{2m-2l+1} & 0\\ I_l & 0 & 0\end{pmatrix}\omega_b^l\right).$$

Denoting the upper triangular unipotent subgroup of $\mathrm{GL}(m)$ *by* $\mathbf{Z_m}$ *and identifying it with the isomorphic unipotent subgroup contained in* $\mathrm{pr}^{-1}(\mathbf{Z_m})$*, the superscript* (Z_m,ψ) *marks the application to* $f_{\tau\otimes\omega,s}$ *of the Whittaker coefficient along* $\mathbf{Z_m}(K)\backslash\mathbf{Z_m}(\mathbb{A})$ *with respect to the character* $\psi_{Z_m}^{-1}(z)=\psi^{-1}(z_{1,2}+\cdots+z_{m-1,m})$*,*

$$f^{Z_m,\psi}_{\omega\otimes\tau,s}(h)=\int_{\mathbf{Z_m}(K)\backslash\mathbf{Z_m}(\mathbb{A})} f_{\omega\otimes\tau,s}(z^\wedge h)\psi_{Z_m}(z)\,dz\,.$$

Let S be a finite set of places of K containing the infinite places such that for all $v\notin S$, σ_v and $\tau_v\otimes\omega_v$ are unramified. Suppose the cusp form φ_σ is a pure tensor, by the uniqueness of Whittaker models, we have

$$W^{\psi}_{\varphi_\sigma}=\prod_v W_v^{\psi_v}$$

where each $W_v^{\psi_v}$ is the local Whittaker function in the Whittaker model of σ_v and for $v\notin S$, $W_v^{\psi_v}$ is spherical such that its value at the identity is 1. Similarly, assume that $f_{\tau\otimes\omega}$ is a decomposable section. Viewing

$$f^{Z_m,\psi}_{\omega\otimes\tau,s}(h)=\int_{\mathbf{Z_m}(K)\backslash\mathbf{Z_m}(\mathbb{A})} f_{\omega\otimes\tau,s}(z^\wedge h)\psi_{Z_m}(z)\,dz$$

as the global Whittaker functional on the induced space $\rho_{\tau\otimes\omega,s}$, we have

$$f^{Z_m,\psi}_{\omega\otimes\tau,s}(h)=\prod_v f_{\omega_v\otimes\tau_v,s}(h_v;(1,I_m))$$

where $f_{\omega_v\otimes\tau_v,s}$ is a holomorphic section in $\rho_{\omega_v\otimes\tau_v,s}$ taking values in the local Whittaker model of $\omega_v\otimes\tau_v$ with respect to the character $\psi^{-1}_{Z_m,v}$; for fixed h_v, we denote the corresponding Whittaker function in the Whittaker model of $\omega_v\otimes\tau_v$ by $f_{\omega_v\otimes\tau_v,s}(h_v;-)$. For all places v outside S, $f_{\omega_v\otimes\tau_v,s}$ is spherical and the function $f_{\omega_v\otimes\tau_v,s}(e,-)$ is the unique spherical and normalized Whittaker function in the Whittaker model of $\omega_v\otimes\tau_v$.

Therefore Proposition 2.3 implies the following:

Corollary 2.4. *With notations as before, for* $\mathrm{Re}(s)$ *large enough,*

$$I(\varphi_\sigma,f_{\omega\otimes\tau,s})=\prod_v I_v(W_v^{\psi_v},f_{\omega_v\otimes\tau_v,s})$$

where

$$I_v(W_v^{\psi_v},f_{\omega_v\otimes\tau_v,s}) = \int_{\mathrm{GL}(1)(K_v)\mathbf{N_G}(K_v)\backslash\mathbf{G}(K_v)} W_v^{\psi_v}(g)\int_{\mathbf{N_l}(K_v)\cap\beta_{l,\alpha}^{-1}\mathbf{Q_m}(K_v)\beta_{l,\alpha}\backslash\mathbf{N_l}(K_v)} f^{Z_m,\psi}_{\omega_v\otimes\tau_v,s}(\beta_{l,\alpha}ug;(1,I_m))(\psi_v)^{-1}_{l,\alpha}(u)\,du\,dg\,.$$

2.3 Some twisted Jacquet modules

Suppose F is the completion of K at some finite place v. To lighten the notation, we re-denote the character ψ_v by ψ, τ_v by τ, ω_v by ω and use the same notation for the base change to F of vector spaces and algebraic groups defined over K as they were over K.

For a smooth representation (π, V_π) of H, denote the Jacquet module of π with respect to N_l and its character $\psi_{l,\alpha}$ by $J_{\psi_{l,\alpha}}(\pi)$. The representation space of $J_{\psi_{l,\alpha}}(\pi)$ is

$$V_\pi / \operatorname{Span}\{\pi(n)\xi - \psi_{l,\alpha}(n)\xi \mid n \in N_l \,,\, \xi \in V_\pi\}\,.$$

Let $\pi = i_{Q_m}^{\mathbf{H}}(\omega \otimes \tau)$ where $i_{Q_m}^{H}$ denotes normalized parabolic induction from Q_m to H. By Bruhat theory, $J_{\psi_{l,\alpha}}(\pi|_{Q_l})$ has a finite Q_l-filtration whose subquotients are indexed by elements of $Q_m\backslash H/Q_l$. If $l = 0$, we consider the double cosets $Q_m\backslash H/G$. The subquotient corresponding to a representative w is

$$\rho_w = \text{c-i}_{Q_{l,m}^{(w)}}^{Q_l}\, \delta_{Q_l}^{1/2} \delta_{Q_{l,m}^{(w)}}^{-1/2} w^{-1}(\omega \otimes \delta_{Q_m}^{1/2}\tau)$$

where δ_{Q_l}, δ_{Q_m}, $\delta_{Q_{l,m}^{(w)}}$ are the modulus characters of Q_l, Q_m, $Q_{l,m}^{(w)}$ respectively, $w^{-1}(\delta_{Q_m}^{1/2}\tau \otimes \omega)$ is the representation of $w^{-1}Q_m w$ on the same space as that of $\omega \otimes \delta_{Q_m}^{1/2}\tau$ which takes q to

$$\omega(wqw^{-1}) \otimes \delta_{Q_m}^{1/2}\tau\,,$$

$\mathbf{Q_{l,m}^{(w)}} = \mathbf{Q_l} \cap w^{-1}\mathbf{Q_m} w$ and

$$\text{c-i}_{Q_{l,m}^{(w)}}^{Q_l}$$

denotes normalized compact induction from $Q_{l,m}^{(w)}$ to Q_l. Since $J_{\psi_{l,\alpha}}(\rho_w)|_G = J_{\psi_{l,\alpha}}(\rho_w|_{R_{l,\alpha}})$, we consider $\rho_w|_{R_{l,\alpha}}$. The double coset spaces $Q_{l,m}^{(w)}\backslash Q_l/R_{l,\alpha}$ (resp. $Q_m\backslash H/Q_l$) and $\operatorname{pr}(Q_{l,m}^{(w)})\backslash \operatorname{pr}(Q_l)/R'_{l,\alpha}$ (resp. $\operatorname{pr}(Q_m)\backslash \operatorname{SO}(Q)(F)/\operatorname{pr}(Q_l)$) correspond bijectively under pr. It follows from the discussion in [GRS11, p. 82] that we may and do fix representatives η of $Q_{l,m}^{(w)}\backslash Q_l/R_{l,\alpha}$ such that

$$\operatorname{pr}(\eta) = \begin{pmatrix} \epsilon & 0 & 0 \\ 0 & \gamma & 0 \\ 0 & 0 & \epsilon^* \end{pmatrix}$$

where ϵ is a Weyl element of $\operatorname{GL}(l)(F)$ and γ is a representative for $Q'_w\backslash \operatorname{SO}(W)(F)/L'_{l,\alpha}$ where $\mathbf{Q'_w}$ is the maximal parabolic subgroup of $\operatorname{SO}(W)$ as follows. If $w = \epsilon_{r,s}$ in equation (4.15) of [GRS11], $\mathbf{Q'_w}$ is the parabolic subgroup of $\operatorname{SO}(W)$ which preserves the standard $s-r$ dimensional isotropic subspace $V^+_{l,s-r}$ of W.

Lemma 2.5. *Let $w = \epsilon_{r,s}$ correspond to (r,s). For $r > 0$, $J_{\psi,\alpha}(\rho_w|_{R_{l,\alpha}}) = 0$.*

Proof. The proof proceeds exactly as the proof of [GRS11, Proposition 5.1] by replacing the base field K with F. Note the typo in the last paragraph of [GRS11, p. 83], both occurrences of $\omega_j N_l^{(i')}\omega_0^{-1}$

should be replaced by $\omega_j N_l^{(i')} \omega_j^{-1}$. Also, the notation

$$\begin{pmatrix} I_{l-r} & & & & & & \\ & I_r & (0\, y_2 & y_3) & \omega_b^{t'}\gamma & z_3 & \\ & & I_{s-r} & & & & \\ & & & I_{m''} & & \begin{pmatrix} * \end{pmatrix} & \\ & & & & I_{s-r} & & \\ & & & & & I_r & \\ & & & & & & I_{l-r} \end{pmatrix}$$

really means

$$\begin{pmatrix} I_l & 0 & 0 \\ 0 & \gamma^{-1} & 0 \\ 0 & 0 & I_l \end{pmatrix} \omega_b^{-t'} \begin{pmatrix} I_{l-r} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I_r & 0 & y_2 & y_3 & z_3 & 0 \\ 0 & 0 & I_{s-r} & 0 & 0 & y_3' & 0 \\ 0 & 0 & 0 & I_{m''} & 0 & y_2' & 0 \\ 0 & 0 & 0 & 0 & I_{s-r} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I_r & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I_{l-r} \end{pmatrix} \omega_b^{t'} \begin{pmatrix} I_l & 0 & 0 \\ 0 & \gamma & 0 \\ 0 & 0 & I_l \end{pmatrix}$$

where $y_2' = -w_{m''}\,{}^t y_2 w_r$ and $y_3' = -w_{s-r}\,{}^t y_3 w_r$. □

From Lemma 2.5, Lemma 5.1 and the proof of Proposition 5.1 in [GRS11], it remains to consider the case $J_{\psi_l,\alpha}(\rho_w|_{R_{l,\alpha}})$ where $w = \epsilon_{r,s}$, $r = 0$,

$$\mathrm{pr}(\eta) = \begin{pmatrix} I_t & 0 & 0 \\ 0 & \gamma & 0 \\ 0 & 0 & I_t \end{pmatrix}.$$

For any such choice of η, let $\rho_{w,\gamma,t}$ be the normalized compact induction

$$\text{c-i}_{R_{l,w_0}\cap\eta^{-1}Q_{l,m}^{(w)}\eta}^{R_{l,w_0}} \pi_{l,m}^{(w,\eta)}(\omega\otimes\tau)$$

where $\pi_{l,m}^{(w,\eta)}(\omega\otimes\tau)$ sends $h \in R_{l,w_0} \cap \eta^{-1}Q_{l,m}^{(w)}\eta$ to

$$\delta_{Q_l}^{1/2} w^{-1}(\omega \otimes \delta_{Q_m}^{1/2}\tau)(\eta h \eta^{-1}) \otimes \delta^{-1/2}_{R_{l,w_0}\cap\eta^{-1}Q_{l,m}^{(w)}\eta} \delta^{1/2}_{R_{l,w_0}}(h)\,.$$

For $\gamma = I_{2(m-l)+1}$, the inducing subgroup

$$R_{l,w_0} \cap Q_{l,m}^{(w)} = \mathrm{pr}^{-1}(R'_{l,\alpha} \cap \mathrm{pr}(Q_{l,m}^{(w)}))$$

is the subgroup of elements of the form (a, b) in $w^{-1}Q_m w$ where $a \in F^*$ and

$$b = \begin{pmatrix} n_1 & 0 & 0 & y_6 & 0 \\ 0 & d & u & v & y_6' \\ 0 & 0 & e & u' & 0 \\ 0 & 0 & 0 & d^* & 0 \\ 0 & 0 & 0 & 0 & n_1^* \end{pmatrix}^{\omega_b^{t'}} \begin{matrix} t \\ s \\ 2(m-l-s)+1\,. \\ s \\ t \end{matrix} \tag{2.1}$$

We have $n_1 \in Z_t$ and $t' = 0$. Let Z'_t be the subgroup

$$\left\{ \begin{pmatrix} I_s & y \\ 0 & z \end{pmatrix} \in \mathrm{GL}(m)(F) \,\middle|\, z \in Z_t \right\} .$$

For $a \in F^*$, define the character $\psi''_{t,a} : Z'_t \to F$ by

$$\psi''_{t,a}\left(\begin{pmatrix} I_s & y \\ 0 & z \end{pmatrix}\right) = \psi^{-1}(z_{1,2} + z_{2,3} + \cdots + z_{t-1,t} + a y_{s,1}) .$$

Denote the corresponding Jacquet module $J_{Z'_t,\psi''_{t,a}}(\tau)$ by $\tau_{(t),a}$. As representations of the mirabolic subgroup $P_{s-1,1}$ of $\mathrm{GL}(s)(F)$, $\tau_{(t),a}$ and $\tau_{(t),a'}$ are isomorphic by [GRS11, Lemma 5.2]. So we let $\tau_{(t)}$ denote any of the representations $\tau_{(t),a}$ of $P_{s-1,1}$.

Lemma 2.6. *Let η_t be a representative of $Q^{(w)}_{l,m}\backslash Q_l/R_{l,\alpha}$ such that $\mathrm{pr}(\eta_t) = I_{2m+1}$ and set $\mathbf{Q'_{m-l}} = \mathbf{L_{l,\alpha}} \cap \mathbf{Q^{(w)}_{l,m}}$. For $a = -(-\frac{\alpha}{2})^{1-t'}$, define*

$$T : \rho_{w,I_{2(m-t)+1},t} \to \text{c-i}^{L_{l,\alpha}}_{Q'_{m-l}}\, \omega |\det{}_{V^+_{l,m-l}}|^{-l/2} \tau_{(l)}$$

by

$$T(f)(x) = \int_{N_l \cap \eta_t^{-1} Q^{(w)}_{l,m} \eta_t \backslash N_l} J_{Z'_t,\psi''_{t,a}}(f(nx)) \psi^{-1}_{l,\alpha}(n)\, dn$$

for each $f \in \rho_{w,I_{2(m-t)+1},t}$ and each $x \in L_{l,\alpha}$. Here $\det_{V^+_{l,m-l}}(x)$ for $x \in L_{l,w_0}$ denotes the determinant of x restricted to $V^+_{l,m-l}$. Then T is well-defined and gives an $L_{l,\alpha}$-isomorphism

$$T' : J_{\psi_{l,\alpha}}(\rho_{w,I_{2(m-t)+1},t}) \cong \text{c-i}^{L_{l,\alpha}}_{Q'_{m-l}}\, \omega |\det{}_{V^+_{l,m-l}}|^{-l/2} \tau_{(l)} . \tag{2.2}$$

Proof. Let $n \in N_l \cap Q^{(w)}_{l,m}$ take the form in equation (2.1) with

$$\begin{pmatrix} d & u & v \\ 0 & e & u' \\ 0 & 0 & d^* \end{pmatrix} = I_{2(m-l)+1} .$$

We have

$$J_{Z'_t,\psi''_{t,a}}(f(nx)) = J_{Z'_t,\psi''_{t,a}}\left(\tau\left(\begin{pmatrix} I_s & y'_6 \\ 0 & n_1^* \end{pmatrix}\right)(f(x))\right) = \psi''_{t,a}\left(\begin{pmatrix} I_s & y'_6 \\ 0 & n_1^* \end{pmatrix}\right) J_{Z'_t,\psi''_{t,a}}(f(x)) = \psi_{l,\alpha}(n) f(x) .$$

T factors through the Jacquet module $J_{\psi_{l,\alpha}}(\rho_{w,I_{2(m-t)+1},t})$ and for $(a,q') \in Q'_{m-l} \subset w^{-1}Q_m w$, direct calculation shows that

$$\delta^{1/2}_{Q_m}(wq'w^{-1}) = \delta^{1/2}_{Q'_s}(q') |\det{}_{V^+_{l,m-l}}(q')|^{\frac{1+t}{2}} .$$

Hence T gives a map T' on $J_{\psi_{l,\alpha}}(\rho_{w,I_{2(m-t)+1},t})$ which has image contained in $\text{c-i}^{L_{l,\alpha}}_{Q'_{m-l}}\, \omega |\det_{V^+_{l,m-l}}|^{-l/2} \tau_{(l)}$. By similar arguments as in the proof of [GRS11, Lemma 5.3], T' is bijective. ∎

Remark 2.7. *It follows from the paragraph right after [GRS11, Proposition 5.3] and from [Bo91, Theorem 22.6] that $\mathbf{Q'_{m-1}}$ is not a parabolic subgroup of $\mathbf{L}_{\mathbf{l},\alpha}$.*

Since $\mathbf{Q'_w}$ preserves a maximal isotropic subspace of W, [GRS11, Proposition 4.4] shows that representatives $\gamma \neq I_{2(m-t)+1}$ for $Q'_w\backslash \mathrm{SO}(W)(F)/L'_{l,\alpha}$ show up only in case (2)(c) when $\mathrm{SO}(W)$ is split. In this case, we choose

$$\gamma = \begin{pmatrix} I_{s-1} & 0 & 0 \\ 0 & \gamma_\alpha & 0 \\ 0 & 0 & I_{s-1} \end{pmatrix}$$

where there are two non-trivial representatives $\gamma_\alpha = \gamma_\pm$ given by [GRS11, (4.33)]. Let $v_\alpha \in V_0$ be such that $b(v_\alpha, v_\alpha) = -b(y_{-\alpha}, y_{-\alpha}) = (-1)^{\dim_F V+1}\alpha$. Then

$$\gamma_\alpha(e_q) = y_{-\alpha} - v_\alpha \, ; \gamma_\alpha(e_{-q}) = e_{-q}$$

and for $v \in V_0$, $\gamma_\alpha(v) - v \in Fe_{-q}$. Here, $\alpha = \beta^2$ where $\beta \in F^*$ and the two choices of v_α are $\pm\beta e_0$. Denote the corresponding representative η by $\eta_{\gamma_\alpha,t}$. We have

$$\eta_{\gamma_\alpha,t}(y_\alpha) = e_q + v_\alpha \, .$$

Redenoting $\mathbf{R}_{\mathbf{l},\alpha}$ as $\mathbf{R}_{\mathbf{l},\mathbf{y}_\alpha}$, $l < m$ implies $\eta_{\gamma_\alpha,t}\mathbf{N_l}\eta^{-1}_{\gamma_\alpha,t} = \mathbf{N_l}$ and $\eta_{\gamma_\alpha,t}\mathbf{R}_{\mathbf{l},\mathbf{y}_\alpha}\eta^{-1}_{\gamma_\alpha,t} = \mathbf{R}_{\mathbf{l},\eta_{\gamma_\alpha,\mathbf{t}}(\mathbf{y}_\alpha)}$ where $e_q + v_\alpha$ is isotropic only if $\alpha = -1$. Thus

$$R_{l,\alpha} \cap \eta^{-1}_{\gamma_\alpha,t}Q^{(w)}_{l,m}\eta_{\gamma_\alpha,t} = \eta^{-1}_{\gamma_\alpha,t}(R_{l,\eta_{\gamma_\alpha,t}(y_\alpha)} \cap Q^{(w)}_{l,m})\eta_{\gamma_\alpha,t}$$

is the subgroup of elements of the form (a,h) in $\eta^{-1}_{\gamma_\alpha,t}w^{-1}Q_m w\eta_{\gamma_\alpha,t}$ such that $\eta_{\gamma_\alpha,t}(a,h)\eta^{-1}_{\gamma_\alpha,t} = (a, \mathrm{pr}(\eta_{\gamma_\alpha,t})h\,\mathrm{pr}(\eta^{-1}_{\gamma_\alpha,t}))$ where

$$\mathrm{pr}(\eta_{\gamma_\alpha,t})h\,\mathrm{pr}(\eta^{-1}_{\gamma_\alpha,t}) = \begin{pmatrix} n_1 & 0 & 0 & y_6 & 0 \\ 0 & d & u & v & y'_6 \\ 0 & 0 & e & u' & 0 \\ 0 & 0 & 0 & d^* & 0 \\ 0 & 0 & 0 & 0 & n_1^* \end{pmatrix} \begin{matrix} t \\ s \\ 2(m-l-s)+1 \\ s \\ t \end{matrix} \, , \tag{2.3}$$

$n_1 \in Z_t$ and

$$\begin{pmatrix} d & u & v \\ 0 & e & u' \\ 0 & 0 & d^* \end{pmatrix} \gamma_\alpha(y_\alpha) = \gamma_\alpha(y_\alpha) \, . \tag{2.4}$$

The action of $\pi^{(w,\eta_{\gamma_\alpha,t})}_{l,m}(\omega \otimes \tau)$ on (a,h) satisfying equations (2.3) and (2.4) is

$$\omega(a)\delta^{1/2}_{R_{l,w_0}}(h)\delta^{-1/2}_{R_{l,w_0}\cap\eta^{-1}Q^{(w)}_{l,m}\eta}(h)\delta^{1/2}_{Q_m}(w\eta_{\gamma_\alpha,t}h\eta^{-1}_{\gamma_\alpha,t}w^{-1})\tau\left(\begin{pmatrix} d & y'_6 \\ 0 & n_1^* \end{pmatrix}\right) \, .$$

Set $\mathbf{L}_{\mathbf{l},\mathbf{y}_\alpha} = \mathbf{L}_{\mathbf{l},\alpha}$ and

$$\mathbf{Q'}_{\mathbf{s},\pm} = \mathbf{L}_{\mathbf{l},\alpha} \cap \eta^{-1}_{\gamma_\pm,t}\mathbf{Q}^{(\mathbf{w})}_{\mathbf{l},\mathbf{m}}\eta_{\gamma_\pm,t} = \eta^{-1}_{\gamma_\pm,t}(\mathbf{L}_{\mathbf{l},\eta_{\gamma_\pm,\mathbf{t}}(\mathbf{y}_\alpha)} \cap \mathbf{Q}^{(\mathbf{w})}_{\mathbf{l},\mathbf{m}})\eta_{\gamma_\pm,t}$$

for each of the choices of $\gamma_\pm$. This is a parabolic subgroup of $\mathbf{L}_{\mathbf{l},\alpha}$ whose Levi part is isomorphic to $\mathrm{GL}(s)\times\mathrm{GL}(1)$ since $\mathrm{pr}(\eta_{\gamma_\pm,t})\,\mathrm{pr}(\mathbf{Q}'_{\mathbf{s},\pm})\,\mathrm{pr}(\eta^{-1}_{\gamma_\pm,t})$ is realized as the subgroup of elements such that $d\in\mathrm{GL}(s)$, $e(v_\alpha)=v_\alpha$ and

$$u(v_\alpha)=(d-I_s)\begin{pmatrix}0\\ \vdots\\ 1\end{pmatrix}.$$

Denote the Bernstein-Zelevinski derivative [BZ76] of τ along the subgroup Z'_t corresponding to the character

$$\psi'_t\left(\begin{pmatrix}I_s & y\\ 0 & z\end{pmatrix}\right)=\psi^{-1}(z_{1,2}+z_{2,3}+\cdots+z_{t-1,t})$$

by $\tau^{(t)}$. Note that the representation $\tau^{(t)}$ of $\mathrm{GL}(s)(F)$ is acting on the Jacquet module $J_{Z'_t,\psi'_t}(V_\tau)$ by the embedding

$$d\mapsto\begin{pmatrix}d & 0\\ 0 & I_t\end{pmatrix}.$$

Lemma 2.8. *Let $\rho_{w,\gamma_\pm,t}=\rho_{w,\gamma_\alpha,t}$ where $\gamma_\alpha=\gamma_\pm$ respectively. For $r=0$, $J_{\psi_l,\alpha}(\rho_{w,\gamma_\pm,t})=0$.*

Proof. Since τ is supercuspidal, it suffices to show that

$$J_{\psi_l,\alpha}(\rho_{w,\gamma_\pm,t})\cong \text{c-i}^{L_{l,\alpha}}_{Q'_{s,\pm}}\,\omega|\det{}_{V^+_{l,s}}|^{\frac{1-t}{2}}\tau^{(t)}.$$

Define $T:\rho_{w,\gamma_\pm,t}\to \text{c-i}^{L_{l,\alpha}}_{Q'_{s,\pm}}\,\omega|\det|^{\frac{1-t}{2}}\tau^{(t)}$ by

$$T(f)(x)=\int_{N_l\cap\eta^{-1}_{\gamma_\pm,t}Q^{(w)}_{l,m}\eta_{\gamma_\pm,t}\backslash N_l}J_{Z'_t,\psi'_t}(f(nx))\psi^{-1}_{l,\alpha}(n)\,dn$$

for each $f\in\rho_{w,\gamma_\pm,t}$ and each $x\in L_{l,\alpha}$. Let $n\in N_l\cap\eta^{-1}_{\gamma_\pm,t}Q^{(w)}_{l,m}\eta_{\gamma_\pm,t}$ be such that

$$\mathrm{pr}(\eta_{\gamma_\alpha,t}n\eta^{-1}_{\gamma_\alpha,t})=\begin{pmatrix}n_1 & 0 & 0 & y_6 & 0\\ 0 & I_s & 0 & 0 & y'_6\\ 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & I_s & 0\\ 0 & 0 & 0 & 0 & n_1^*\end{pmatrix}$$

with $n_1\in Z_t$. We have

$$J_{Z'_t,\psi'_t}(f(nx))=J_{Z'_t,\psi'_t}\left(\tau\left(\begin{pmatrix}I_s & y'_6\\ 0 & n_1^*\end{pmatrix}\right)(f(x))\right)=\psi'_t\left(\begin{pmatrix}I_s & y'_6\\ 0 & n_1^*\end{pmatrix}\right)J_{Z'_t,\psi'_t}(f(x))=\psi_{l,\alpha}(n)f(x).$$

Thus T is well-defined. By similar arguments as in of [GRS11, Lemma 5.3], T gives an $L_{l,\alpha}$-isomorphism

$$T':J_{\psi_l,\alpha}(\rho_{w,\gamma_\pm,t})\cong \text{c-i}^{L_{l,\alpha}}_{Q'_{s,\pm}}\,\omega|\det|^{\frac{1-t}{2}}\tau^{(t)}. \tag{2.5}$$

The proof is complete. □

Proposition 2.9. $J_{\psi_{l,\alpha}}(\pi|_{R_{l,\alpha}}) \cong \text{c-i}_{Q'_{m-l}}^{L_{l,\alpha}} \omega|\det_{V^+_{l,m-l}}|^{-l/2}\tau_{(l)}$.

Proof. Lemma 2.5, Lemma 2.6 and Lemma 2.8 imply that only the representation corresponding to the double coset

$$Q_m\epsilon_{0,m-l}L_{l,\alpha} \text{ (resp. } Q^{(w)}_{l,m}R_{l,\alpha})$$

in the first (resp. second) filtration gives some non-trivial representation. □

2.4 Generic Multiplicity One

From now on, we re-denote σ_v by σ. Then σ_v is locally generic with respect to the local Whittaker character ψ_{v,N_G} which we re-denote by ψ_{N_G}. Let $\beta_{l,\alpha}$ be as in Proposition 2.3 and

$$I(W^\psi, f_{\omega\otimes\tau,s}) = \int_{\mathrm{GL}(1)(F)N_G\backslash G} W^\psi(g) \int_{N_l\cap\beta^{-1}_{l,\alpha}Q_m\beta_{l,\alpha}\backslash N_l} f^{Z_m,\psi}_{\omega\otimes\tau,s}(\beta_{l,\alpha}ug;(I_m,1))\psi^{-1}_{l,\alpha}(u)\,du\,dg\,.$$

Proposition 2.10. *There exists $s_0 \in \mathbb{R}$ such that $I(W^\psi, f_{\omega\otimes\tau,s})$ is absolutely convergent for all s with $\mathrm{Re}(s) > s_0$, for all $W \in W^\psi$ and $f_{\omega\otimes\tau,s} \in i^H_{Q_m}(\omega\otimes|\det|^{s-1/2}\tau)$.*

Proof. Since we are dividing by the center of $L_{l,\alpha}$, convergence of these integrals is proved in exactly the same way as it was proved for the corresponding integrals of odd orthogonal groups, see [Sou93, § 4.4–4.6]. □

Lemma 2.11. *In the domain of absolute convergence of $I(W^\psi, f_{\omega\otimes\tau,s})$, for all $g \in L_{l,\alpha}$ and $n \in N_l$,*

$$I(\sigma(g)W^\psi, \pi(gn)f_{\omega\otimes\tau,s}) = \psi_{l,\alpha}(n)I(W^\psi, f_{\omega\otimes\tau,s})\,. \tag{2.6}$$

Proof. Note that $I(W^\psi, f_{\omega\otimes\tau,s})$ is invariant with respect to $L_{l,\alpha}$. We have

$$\begin{aligned}
&\int_{N_l\cap\beta^{-1}_{l,\alpha}Q_m\beta_{l,\alpha}\backslash N_l} f^{Z_m,\psi}_{\omega\otimes\tau,s}(\beta_{l,\alpha}ugn;(I_m,1))\psi^{-1}_{l,\alpha}(u)\,du\\
=&\int_{N_l\cap\beta^{-1}_{l,\alpha}Q_m\beta_{l,\alpha}\backslash N_l} f^{Z_m,\psi}_{\omega\otimes\tau,s}(\beta_{l,\alpha}ugng^{-1}g;(I_m,1))\psi^{-1}_{l,\alpha}(u)\,du\\
=&\int_{N_l\cap\beta^{-1}_{l,\alpha}Q_m\beta_{l,\alpha}\backslash N_l} f^{Z_m,\psi}_{\omega\otimes\tau,s}(\beta_{l,\alpha}ug;(I_m,1))\psi^{-1}_{l,\alpha}(ugn^{-1}g^{-1})\,du\\
=&\psi_{l,\alpha}(n)\int_{N_l\cap\beta^{-1}_{l,\alpha}Q_m\beta_{l,\alpha}\backslash N_l} f^{Z_m,\psi}_{\omega\otimes\tau,s}(\beta_{l,\alpha}ug;(I_m,1))\psi^{-1}_{l,\alpha}(u)\,du
\end{aligned}$$

and so $I(W^\psi, \pi(n)f_{\omega\otimes\tau,s}) = \psi_{l,\alpha}(n)I(W^\psi, f_{\omega\otimes\tau,s})$. □

Let (π_1, V_{π_1}) and (π_2, V_{π_2}) be smooth representations of a p-adic group L. Denote the set of bilinear forms on (V_{π_1}, V_{π_2}) which are invariant under L by $\mathrm{Bil}_L(\pi_1,\pi_2)$ and the contragradient of (π_1, V_{π_1}) by $(\pi_1^\vee, V^\vee_{\pi_1})$. We relabel $i^H_{Q_m}(\omega\otimes|\det|^{s-1/2}\tau)$ as $I(s-1/2,\omega\otimes\tau)$.

Theorem 2.12. *Except for a finite number of values of q^{-s}, the space of bilinear forms satisfying* (2.6) *is at most one dimensional.*

Proof. This is the local analog of the proof of Proposition 2.3. Proposition 2.9 implies that

$$\begin{aligned}\mathrm{Bil}_{L_{l,\alpha}}(\sigma, J_{\psi_l,\alpha}(I(s-1/2,\omega\otimes\tau)|_{R_{l,\alpha}})) &\cong \mathrm{Bil}_{L_{l,\alpha}}(\sigma, \text{c-i}^{L_{l,\alpha}}_{Q'_{m-l}}\,||^s\omega\otimes|\det{}_{V^+_{l,m-l}}|^{s-(l+1)/2}\tau_{(l)})\\ &\cong \mathrm{Hom}_{L_{l,\alpha}}(\sigma, i^{L_{l,\alpha}}_{Q'_{m-l}}\omega^{-1}\otimes|\det{}_{V^+_{l,m-l}}|^{(l+1)/2-s}\tau^\vee_{(l)})\,.\end{aligned}$$

Applying Frobenius reciprocity gives

$$\begin{aligned}&\mathrm{Hom}_{L_{l,\alpha}}(\sigma, i^{L_{l,\alpha}}_{Q'_{m-l}}\omega^{-1}\otimes|\det{}_{V^+_{l,m-l}}|^{(l+1)/2-s}\tau^\vee_{(l)})\\ \cong\ &\mathrm{Hom}_{Q'_{m-l}}(\sigma|_{Q'_{m-l}}, \omega^{-1}\otimes|\det{}_{V^+_{l,m-l}}|^{(l+1)/2-s}\delta^{1/2}_{Q'_{m-l}}\tau^\vee_{(l)})\,.\end{aligned}$$

Let $\mathbf{Q'_{G,m-l}}$ be the parabolic subgroup of $\mathbf{L'_{l,\alpha}}$ which preserves $W\cap y_\alpha^\perp$ and $\mathbf{Q_{G,m-l}} = \mathrm{pr}^{-1}(\mathbf{Q'_{G,m-l}})$ with Levi decomposition $\mathbf{M_{G,m-l}U_{G,m-l}}$,

$$\mathbf{C_{G,m-l}} = \{u\in\mathbf{U_{G,m-l}}\mid \mathrm{pr}(u)e_m = e_m\}\,.$$

Since $C_{G,m-l}$ acts trivially on the representation space of $\omega^{-1}\otimes|\det{}_{V^+_{l,m-l}}|^{(l+1)/2-s}\delta^{1/2}_{Q'_{m-l}}\tau^\vee_{(l)}$,

$$\begin{aligned}&\mathrm{Hom}_{Q'_{m-l}}(\sigma|_{Q'_{m-l}}, \omega^{-1}\otimes|\det{}_{V^+_{l,m-l}}|^{(l+1)/2-s}\delta^{1/2}_{Q'_{m-l}}\tau^\vee_{(l)})\\ \cong\ &\mathrm{Hom}_{Q'_{m-l}/C_{G,m-l}}(J_{C_{G,m-l}}(\sigma|_{Q'_{m-l}}), \omega^{-1}\otimes|\det{}_{V^+_{l,m-l}}|^{(l+1)/2-s}\delta^{1/2}_{Q'_{m-l}}\tau^\vee_{(l)})\,.\end{aligned}$$

Since $\mathbf{Q'_{m-l}}/\mathbf{C_{G,m-l}}\cong \mathrm{GL}(1)\times\mathbf{P_{m-l-1,1}}$ and $H^1(\mathrm{Gal}(\overline{F}/F),\mathbf{C_{G,m-l}}) = 1$, $Q'_{m-l}/C_{G,m-l}\cong F^*\times P_{m-l-1,1}$. Restriction of the last Hom space to Hom over $P_{m-l-1,1}$ has trivial kernel so

$$\mathrm{Hom}_{Q'_{m-l}/C_{G,m-l}}(J_{C_{G,m-l}}(\sigma|_{Q'_{m-l}}), \omega^{-1}\otimes|\det{}_{V^+_{l,m-l}}|^{(l+1)/2-s}\delta^{1/2}_{Q'_{m-l}}\tau^\vee_{(l)})$$

injects into

$$\mathrm{Hom}_{P_{m-l-1,1}}(J_{C_{G,m-l}}(\sigma|_{\mathrm{pr}(Q'_{m-l})}), |\det{}_{V^+_{l,m-l}}|^{(l+1)/2-s}\delta^{1/2}_{\mathrm{pr}(Q'_{m-l})}\tau^\vee_{(l)})\,.$$

For each $0\leqslant i\leqslant m-l-1$, denote by R_i the subgroup of the form

$$\left\{\begin{pmatrix} g & v\\ 0 & z\end{pmatrix}\in P_{m-l-1,1}\,\middle|\, g\in\mathrm{GL}(i)(F)\,,\ z\in Z_{m-l-i}\,,\ v\in M_{i,m-l-i}\right\}.$$

Let $\Sigma = J_{C_{G,m-l}}(\sigma|_{\mathrm{pr}(Q'_{m-l})})$, $\Pi = |\det{}_{V^+_{l,m-l}}|^{(l+1+c)/2}\tau^\vee_{(l)}$ where $\delta_{\mathrm{pr}(Q'_{m-l})} = |\det{}_{V^+_{l,m-l}}|^c$ for some constant c so that $|\det{}_{V^+_{l,m-l}}|^{(l+1)/2-s}\delta^{1/2}_{\mathrm{pr}(Q'_{m-l})}\tau^\vee_{(l)} = |\det{}_{V^+_{l,m-l}}|^{-s}\Pi$. It follows from Theorem 7.2 of [GPR87] that $\mathrm{Hom}_{P_{m-l-1,1}}(\Sigma, |\det{}_{V^+_{l,m-l}}|^{-s}\Pi)$ embeds into

$$\bigoplus_{i,j=1}^{m-l}\mathrm{Hom}_{P_{m-l-1,1}}(\text{c-i}^{P_{m-l-1,1}}_{R_{i-1}}(\Sigma^{(m-l-i+1)}\otimes\psi'), \text{c-i}^{P_{m-l-1,1}}_{R_{j-1}}(\Pi^{(m-l-j+1)}\otimes|\det|^{-s}\psi'))\,.$$

Here ψ' is a character of $Z_{m-l-i+1}$ defined by $\psi'((z_{ij})) = \psi'(\sum_{j=1}^{m-l-i} z_{j,j+1})$ and $\Sigma^{(m-l-i+1)} \otimes \psi'$ is extended from $\mathrm{GL}(i-1)(F) \times Z_{m-l-i+1}$ to R_{i-1} trivially across

$$N_{i-1} = \left\{ \begin{pmatrix} I_{i-1} & v \\ 0 & I_{m-l-i+1} \end{pmatrix} \right\} .$$

Since $\text{c-i}_{R_{i-1}}^{P_{m-l-1,1}}(\Sigma^{(m-l-i+1)} \otimes \psi')$ and $\text{c-i}_{R_{j-1}}^{P_{m-l-1,1}}(\Pi^{(m-l-j+1)} \otimes |\det|^{-s}\psi')$ are irreducible for all $1 \leqslant i, j \leqslant m-l$, a summand is non-trivial only if $i = j$ and $\Sigma^{(m-l-i+1)} \cong \Pi^{(m-l-j+1)} \otimes |\det|^{-s}$ by Schur's Lemma and Proposition 5.12*d*) of [BZ76]. For fixed $i = j$, taking central characters forces q^{-s} to be unique. Hence besides the $m-l$ values of q^{-s} corresponding to $m-l$ possibilities of $1 \leqslant i = j \leqslant m-l$, $\dim \mathrm{Bil}_{L_{l,\alpha}}(\sigma, J_{\psi_{l,\alpha}}(I(s-1/2, \omega \otimes \tau)|_{R_{l,\alpha}})) \leqslant 1$. □

Proposition 2.13. *There is a choice of $W \in W^\psi$ and $f_{\omega\otimes\tau,s} \in i_{Q_m}^H(\omega \otimes |\det|^{s-1/2}\tau)$ such that $I(W^\psi, f_{\omega\otimes\tau,s}) = 1$ for all $s \in \mathbb{C}$.*

Proof. Write $L_{l,\alpha}$ as a disjoint union of double cosets $N_G w B^-$ where $\mathbf{B}^-$ is the opposite Borel subgroup of $L_{l,\alpha}$ and w is an arbitrary Weyl group element. Since the double cosets $\mathbf{N_G} w \mathbf{B}^-$ for $w \neq 1$ as algebraic varieties is of strictly smaller dimension than the dimension of $\mathbf{L}_{\mathbf{l},\alpha}$, the measure of $N_G w B^-$ is 0. Hence writing $\beta_{l,\alpha}$ as $\kappa w'$ where w' is the long Weyl element in H such that $w' N_l w'^{-1} = N_l^-$ and $\mathbf{N}_\mathbf{l}^-$ is the unipotent radical of the parabolic opposite to $\mathbf{P}_\mathbf{l}$,

$$\begin{aligned}
&I(W^\psi, f_{\omega\otimes\tau,s}) \\
&= \int_T \int_{N_G^-} W^\psi(tv) \int_{N_l \cap \beta_{l,\alpha}^{-1} Q_m \beta_{l,\alpha} \backslash N_l} f_{\omega\otimes\tau,s}^{Z_m,\psi}(w'utv; (\kappa, 1)) \delta_{B^-}^{-1}(t) \psi_{l,\alpha}^{-1}(u)\, du\, dv\, dt \\
&= \int_T \int_{N_G^-} W^\psi(tv) \int_{N_l \cap \beta_{l,\alpha}^{-1} Q_m \beta_{l,\alpha} \backslash N_l} f_{\omega\otimes\tau,s}^{Z_m,\psi}(w'tuv; (\kappa, 1)) \delta_{R_{l,\alpha} \cap Q^0 \backslash R_{l,\alpha}}(t^{-1}) \delta_{B^-}^{-1}(t) \psi_{l,\alpha}^{-1}(u)\, du\, dv\, dt \\
&= \int_T \int_{N_G^-} W^\psi(tv) \int_{N_l \cap \beta_{l,\alpha}^{-1} Q_m \beta_{l,\alpha} \backslash N_l} f_{\omega\otimes\tau,s}^{Z_m,\psi}(w'tw'^{-1}w'uv; (\kappa, 1)) \delta_{R_{l,\alpha} \cap Q^0 \backslash R_{l,\alpha}}(t^{-1}) \delta_{B^-}^{-1}(t) \psi_{l,\alpha}^{-1}(u)\, du\, dv\, dt \\
&= \int_T \int_{N_G^-} W^\psi(tv) \int_{N_l \cap \beta_{l,\alpha}^{-1} Q_m \beta_{l,\alpha} \backslash N_l} f_{\omega\otimes\tau,s}^{Z_m,\psi}(w'uv; (\kappa w'tw'^{-1}, 1)) \delta_{R_{l,\alpha} \cap Q^0 \backslash R_{l,\alpha}}(t^{-1}) \delta_{B^-}^{-1}(t) \psi_{l,\alpha}^{-1}(u)\, du\, dv\, dt \\
&= \int_T \int_{w' N_G^- w'^{-1}} W^\psi(tw'^{-1}vw') \int_{N_l^- \cap w' \beta_{l,\alpha}^{-1} Q_m \beta_{l,\alpha} w'^{-1} \backslash N_l^-} f_{\omega\otimes\tau,s}^{Z_m,\psi}(uvw'; (\kappa w'tw'^{-1}, 1)) \delta_{R_{l,\alpha} \cap Q^0 \backslash R_{l,\alpha}}(t^{-1}) \\
&\times \delta_{B^-}^{-1}(t) \psi_{l,\alpha}^{-1}(w'^{-1}uw')\, du\, dv\, dt
\end{aligned}$$

where δ_{B^-} is the modulus character of the opposite Borel subgroup $\mathbf{B}^-$ of $\mathbf{L}_{\mathbf{l},\alpha}$ with Levi decomposition $\mathbf{B}^- = \mathbf{T}\mathbf{N}_\mathbf{G}^-$. Note that the second equality follows by making the change of variable from u to $t^{-1}ut$ and the third equality follows from the change of variable from t to $w'tw'^{-1}$.

Let $\mathfrak{n} = \mathrm{GL}(1)(F) \backslash G \cap (I + M_k(\mathfrak{p}^r))$ for some $r \gg 0$. We choose $f_{\omega\otimes\tau,s} \in i_{Q_m}^H(\omega \otimes |\det|^{s-1/2}\tau)$ such that the support of the right translate $f_{\omega\otimes\tau,s}$ by w' is contained in $\mathrm{pr}^{-1}(Q_m \mathfrak{n})$ and $f_{\omega\otimes\tau,s}(qn) =$

$W((a,x))$ for $q = (a,x)u \in Q_m$ where $(a,x) \in \mathrm{GL}(m)(F) \times F^*$ and $u \in U_m$, $n \in \mathfrak{n}$. Suppose $u \in U_m^-$ is such that $f_{\omega\otimes\tau,s}(uw') \neq 0$. Then $\mathrm{pr}(u) \in Q_m\mathfrak{n}$. Write

$$\mathrm{pr}(u) = \begin{pmatrix} I_m & 0 \\ a & I_m \end{pmatrix} = \begin{pmatrix} b & c \\ 0 & b^* \end{pmatrix} d$$

where $d \in \mathfrak{n}$. Pre-multiplying the last equation by $\begin{pmatrix} b & c \\ 0 & b^* \end{pmatrix}^{-1}$ gives

$$d = \begin{pmatrix} * & * \\ (b^*)^{-1}a & (b^*)^{-1} \end{pmatrix} \in \mathfrak{n}\,.$$

Therefore $(b^*)^{-1} \in I + M_m(\mathfrak{p}^r)$ and so $b^* \in I + M_m(\mathfrak{p}^r)$. Pre-multiplying by $\begin{pmatrix} b & 0 \\ 0 & b^* \end{pmatrix}$ to the last equality yields

$$\begin{pmatrix} * & * \\ a & I_m \end{pmatrix} = \begin{pmatrix} b & 0 \\ 0 & b^* \end{pmatrix} d \in \mathfrak{n}\,.$$

Hence $a \in M_m(\mathfrak{p}^r)$ and $\mathrm{pr}(u) \in \mathfrak{n}$. Note that conjugation by wtw^{-1} preserves $N_l \cap \beta_{l,\alpha}^{-1} Q_m \beta_{l,\alpha} \backslash N_l$ and ψ^{-1}.

$$\int\limits_{N_l \cap \beta_{l,\alpha}^{-1} Q_m \beta_{l,\alpha} \backslash N_l} f_{\omega\otimes\tau,s}^{Z_m,\psi}(\beta_{l,\alpha} ug; (I_m,1))\psi_{l,\alpha}^{-1}(u)\,du$$
$$= \int\limits_{(N_l^- \cap w\beta_{l,\alpha}^{-1} Q_m \beta_{l,\alpha} w^{-1} \backslash N_l^-)\cap\mathfrak{n}} f_{\omega\otimes\tau,s}^{Z_m,\psi}(\kappa uwg; (I_m,1))\psi_{l,\alpha}^{-1}(u)\,du$$

Similar calculations and arguments as in the proofs of [Kap13, Lemma 5.2] and [Kap13, Proposition 5.11] show that the integrations over u, v and t are positive constants. □

By Proposition 2.10 and Lemma 2.11, in a right half plane $I(W^\psi, f_{\omega\otimes\tau,s})$ can be regarded as an element of $\mathrm{Bil}_G(\sigma, J_{\psi_{l,\alpha}}(I(s-1/2, \omega\otimes\tau)|_{R_{l,\alpha}}))$. Together with Theorem 2.12 and Proposition 2.13 and using Bernstein's continuation principle ([Ban98]), we deduce the meromorphic continuation of the integral.

Corollary 2.14. *$I(W^\psi, f_{\omega\otimes\tau,s})$ is a rational function of q^{-s}.*

Let $\mathbf{U}$ be the unipotent radical of the standard Borel subgroup in $\mathbf{H}$ and χ be a non-degenerate character of U defined by ψ^{-1} and the F-splitting of $\mathbf{H}$. Suppose $\mathbf{Q_m} = \mathbf{MN}$ is the Levi decomposition of $\mathbf{Q_m}$ with $\mathbf{M}$ F-isomorphic to $\mathrm{GL}(1)\times\mathrm{GL}(m)$ and $\mathbf{N} \subset \mathbf{U}$. Set α to be the mth simple root, $\tilde{\alpha}$ the fundamental weight for α and w_0 the unique element in the Weyl group in $\mathbf{H}$ which sends $\Delta - \{\alpha\}$ to a subset of Δ and α to a negative root. Denote the local coefficient attached to χ, $s\tilde{\alpha}$, $\omega\otimes\tau$ and w_0 in [Sha10, § 5.1] by $C_\chi(s\tilde{\alpha}, \omega\otimes\tau, w_0)$. Here, we write w_0 as a product of simple reflections and for each simple reflection w_δ corresponding to a simple root δ, let $\mathbf{P}_\delta$ be the parabolic subgroup defined by $\theta = \{\delta\}$. The Levi subgroup $\mathbf{M}_\delta$ of $\mathbf{P}_\delta$ has semisimple rank 1 and so its commutator subgroup $(\mathbf{M}_\delta, \mathbf{M}_\delta)$ is semisimple of rank 1. Thus we have a F-algebraic homomorphism $\varphi_\delta : \mathrm{SL}(2) \to (\mathbf{M}_\delta, \mathbf{M}_\delta)$

such that the image of the upper triangular unipotent group in SL(2) is the root subgroup $\mathbf{U}_\delta$ for the root δ. We choose

$$w_\delta = \varphi_\delta\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right).$$

In this way, the choice of w_0 is unique and the local coefficient $C_\chi(s\tilde{\alpha}, \omega\otimes\tau, w_0)$ is uniquely determined.

Let $A(s, \omega\otimes\tau, w_0) : I(s, \omega\otimes\tau) \to I(w_0(s), w_0(\omega\otimes\tau))$ be the standard intertwining operator. Define the Rankin-Selberg γ-factor associated to σ, $\omega\otimes\tau$ and ψ^{-1} by

$$\gamma(s, \sigma\times(\omega\otimes\tau), \psi^{-1}) := C_\chi(s\tilde{\alpha}, \omega\otimes\tau, w_0)\frac{I(W^\psi, A(s, \omega\otimes\tau, w_0)f_{\omega\otimes\tau,s})}{I(W^\psi, f_{\omega\otimes\tau,s})}. \tag{2.7}$$

It follows from the proof of Lemma 2.11 that

$$D^\psi(f_{\omega\otimes\tau,s}) = \int\limits_{N_l\cap\beta_{l,\alpha}^{-1}Q_m\beta_{l,\alpha}\backslash N_l} f^{Z_m,\psi}_{\omega\otimes\tau,s}(\beta_{l,\alpha}ug; (I_m, 1))\psi^{-1}_{l,\alpha}(u)\,du$$

factors through the twisted Jacquet module $J_{\psi_{l,\alpha}}(\pi|_{R_{l,\alpha}})$.

Proposition 2.15. *Assume that σ and τ are supercuspidal, σ is generic with respect to the Whittaker character ψ_{N_G} with Whittaker function W^ψ in the Whittaker model of σ. Then $I(W^\psi, f_{\omega\otimes\tau,s})$ is holomorphic, and $I(W^\psi, A(s, \omega\otimes\tau, w_0)f_{\omega\otimes\tau,s})$ is holomorphic outside the poles of $A(s, \omega\otimes\tau, w_0)$.*

Proof. Since σ is supercuspidal, W^ψ has compact support modulo $\mathrm{GL}(1)(F)N_G$. This together with the fact that the inner integral

$$\int\limits_{N_l\cap\beta_{l,\alpha}^{-1}Q_m\beta_{l,\alpha}\backslash N_l} f^{Z_m,\psi}_{\omega\otimes\tau,s}(\beta_{l,\alpha}ug; (I_m, 1))\psi^{-1}_{l,\alpha}(u)\,du$$

stabilizes for large compact open subgroups of $N_l\cap\beta_{l,\alpha}^{-1}Q_m\beta_{l,\alpha}\backslash N_l$ implies that $I(W^\psi, f_{\omega\otimes\tau,s})$ is holomorphic. The statement regarding $I(W^\psi, A(s, \omega\otimes\tau, w_0)f_{\omega\otimes\tau,s})$ follows immediately. □

Theorem 2.16. *Let σ, τ be irreducible supercuspidal representations of G, $\mathrm{GL}(m)(F)$ respectively where σ is generic with respect to the Whittaker character ψ_{N_G} with Whittaker function W^ψ in the Whittaker model of σ and $\pi_{\omega\otimes\tau}$ be the image of $i^H_{Q_m}(\omega\otimes|\det|^{1/2}\tau)$ by $A(1, \omega\otimes\tau, w_0)$. Set ρ_m to be the standard m-dimensional representation of $\mathrm{GL}(m)(\mathbb{C})$, μ to be the similitude character of $\mathrm{GSO}(2m, \mathbb{C})$ and $r = \mathrm{Sym}^2\rho_m\otimes\mu^{-1}$. For the Rankin-Selberg γ-factor $\gamma(s, \sigma\times(\omega\otimes\tau), \psi^{-1})$ to have a pole at $s = 1$, it is necessary and sufficient that the Langlands-Shahidi L-function $L(s, \omega^{-1}\otimes\tau, r)$ has a pole at $s = 0$ and σ pairs non-trivially with $J_{\psi_{l,\alpha}}(\pi_{\omega\otimes\tau}|_{R_{l,\alpha}})$.*

Proof. By Proposition 2.13, we may and do choose suitable $W\in W^\psi$ and

$$f_{\omega\otimes\tau,s}\in i^H_{Q_m}(\omega\otimes|\det|^{s-1/2}\tau)$$

such that $I(W^\psi, f_{\omega\otimes\tau,s}) = 1$ for all $s\in\mathbb{C}$. Since $A(1, \omega\otimes\tau, w_0)$ is holomorphic ([CS98, Theorem 5.1], $L(2s-1, \omega^{-1}\otimes\tau, r)$ is holomorphic for $s = 1$), it follows from equation (2.7) and Proposition 2.15 that $\gamma(s, \sigma\times(\omega\otimes\tau), \psi^{-1})$ has a pole at $s = 1$ if and only if $C_\chi(s\tilde{\alpha}, \omega\otimes\tau, w_0)$ has a pole at $s = 1$ and

$$I(W^\psi, A(s, \omega\otimes\tau, w_0)f_{\omega\otimes\tau,s})$$

is non-zero at $s = 1$. Condition 2) of Theorem 8.3.2 of [Sha10] shows that $C_\chi(s\tilde{\alpha}, \omega \otimes \tau, w_0)$ having a pole at $s = 1$ is equivalent to the Langlands-Shahidi γ-factor $\gamma(s, \omega^{-1} \otimes \tau, r, \overline{\psi})$ having a pole at $s = 1$, i.e. the Langlands-Shahidi L-function $L(s, \omega^{-1} \otimes \tau, r)$ having a pole at $s = 0$. Non-vanishing of $I(W^\psi, A(s, \omega \otimes \tau, w_0) f_{\omega\otimes\tau,s})$ at $s = 1$ is equivalent to σ pairing with $J_{\psi_{l,\alpha}}(\pi_{\omega\otimes\tau}|_{R_{l,\alpha}})$. □

Note that $\pi_{\omega\otimes\tau}$ is isomorphic to the Langlands quotient of $i^H_{Q_m}(\omega \otimes |\det|^{1/2}\tau)$.

For a given irreducible unitary supercuspidal representation τ of $\mathrm{GL}(2n)(F)$ such that the local Langlands-Shahidi L-function $L(s, \tau, \mathrm{Sym}^2 \rho_{2n} \otimes \omega^{-1})$ has a pole at $s = 0$, define the descent of τ to $L_{l,\alpha}$,

$$\sigma_{\psi,l}(\tau) := J_{\psi_{l,\alpha}}(\pi_{\omega\otimes\tau}|_{R_{l,\alpha}}) .$$

Remark 2.17. *Theorem* 2.12 *implies that any representation of $L_{l,\alpha}$ that pairs up with $\sigma_{\psi,l}(\tau)$ is ψ_{N_G}-generic.*

3 Tower Property

The following lemma is a minor modification of [GRS99a, Lemma 2.2]. Their arguments for unipotent subgroups of the symplectic group apply to our setting as well, but in their case the intersections (in item (6) below) $X \cap C$ and $Y \cap C$ were trivial. See also [LM15, Appendix 1] for analogous integration formulas.

Lemma 3.1. (***Exchanging Roots***) *Let $\mathbf{C}$ be an F-subgroup of a maximal unipotent subgroup of $\mathbf{H}$ and ψ_C be a non-trivial character of C. Assume that there are two unipotent F-subgroups $\mathbf{X}$, $\mathbf{Y}$ such that the following conditions are satisfied.*

(1) *$\mathbf{X}$ and $\mathbf{Y}$ normalize $\mathbf{C}$.*

(2) *$\mathbf{X} \cap \mathbf{C}$ and $\mathbf{Y} \cap \mathbf{C}$ are normal in $\mathbf{X}$ and $\mathbf{Y}$ respectively and $(\mathbf{X} \cap \mathbf{C})\backslash\mathbf{X}$ and $(\mathbf{Y} \cap \mathbf{C})\backslash\mathbf{Y}$ are abelian.*

(3) *X and Y preserve ψ_C (when acting by conjugation).*

(4) *ψ_C is trivial on $X \cap C$ and on $Y \cap C$.*

(5) *$[\mathbf{X}, \mathbf{Y}] \subset \mathbf{C}$.*

(6) *The pairing $(X \cap C)\backslash X \times (Y \cap C)\backslash Y \to \mathbb{C}^*$ given by*

$$(x, y) \mapsto \psi_C([x, y])$$

is multiplicative in each coordinate, non-degenerate and identifies $(Y \cap C)\backslash Y$ with the dual of $(X \cap C)\backslash X$ and $(X \cap C)\backslash X$ with the dual of $(Y \cap C)\backslash Y$.

Represent the setup above by the following diagram,

$$\begin{array}{ccccc} & & A & & \\ & \nearrow & & \nwarrow & \\ B = CY & & & & D = CX \\ & \nwarrow & & \nearrow & \\ & & C & & \end{array}$$

Here, $A = BX = DY = CXY$. Extend the character ψ_C to a character ψ_B of B (resp. ψ_D of D) by making it trivial on Y (resp. X). Let π be a smooth representation of A. As representations of C,

$$J_{B,\psi_B}(\pi)|_C \cong J_{D,\psi_D}(\pi)|_C\,.$$

In particular, $J_{B,\psi_B}(\pi)$ and $J_{D,\psi_D}(\pi)$ are isomorphic as vector spaces.

Proof. Note that the first five conditions imply that for each $y \in Y$, $\psi_C([x,y])$ is a character of X which is trivial on $X \cap C$. (Similarly upon fixing x, we get a character of $(Y \cap C)\backslash Y$.) Indeed for x_1, $x_2 \in X$, since $[x_1x_2, y] = x_1[x_2, y]x_1^{-1}[x_1, y]$,

$$\psi_C([x_1x_2, y]) = \psi_C(x_1[x_2, y]x_1^{-1})\psi_C([x_1, y])\psi_C([x_2, y])\,.$$

As y preserve ψ_C,

$$\psi_C([y, c]) = \psi_C(ycy^{-1})\psi_C(c^{-1}) = \psi_C(c)\psi_C(c^{-1}) = 1\,.$$

Thus $\psi_C([y, c]) = 1$ for all $c \in C$.

The rest of the proof follows similarly as the proof of Lemma 2.2 in [GRS99a] where Sp_{4n} is replaced by $\mathbf{H}$ and two lines after equation (2.6), the typo $y^{-1}xy \in C$ should be replaced by $x^{-1}y^{-1}xy \in C$. □

Denote by $q_{l,\alpha}$ the Witt index of the restriction of the form b to $W \cap y_\alpha^\perp$. For $1 \leqslant p \leqslant q_{l,\alpha}$, let $\mathbf{Q}_\mathbf{p}^{*\prime}$ be the standard maximal parabolic subgroup of $\mathbf{L}'_{\mathbf{l},\alpha}$ which preserves the totally isotropic subspace

$$V_{l,p,+} = \mathrm{Span}_F\{e_{l+1}\,, \cdots, e_{l+p}\} \cap y_\alpha^\perp \subset W \cap y_\alpha^\perp$$

and $\mathbf{Q}_\mathbf{p}^* = \mathrm{pr}^{-1}(\mathbf{Q}_\mathbf{p}^{*\prime})$. Denote by $\mathbf{U}_\mathbf{p}^*$ the unipotent radical of $\mathbf{Q}_\mathbf{p}^*$. For $1 \leqslant i \leqslant p + l - 1$, let

$$\mathbf{U}_{\mathbf{l+p}}^{\mathbf{i}} = \left\{ \begin{pmatrix} I_{p+l-i} & * \\ 0 & z \end{pmatrix}^\wedge \,\middle|\, z \in \mathbf{Z}_\mathbf{i} \right\} \cdot \mathbf{U}_{\mathbf{l+p}} \subset \mathbf{N}_{\mathbf{l+p}}\,,\ \mathcal{L} = \left\{ \begin{pmatrix} I_p & 0 \\ * & I_l \end{pmatrix}^\wedge \right\}.$$

For $0 \leqslant i \leqslant l - 1$, let

$$\mathcal{L}^{\mathbf{i}} = \left\{ \begin{pmatrix} I_p & 0 \\ \lambda & I_l \end{pmatrix}^\wedge \in \mathcal{L} \,\middle|\, \lambda = \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_l \end{pmatrix},\ \lambda_j = 0 \text{ for all } j \neq l - i \right\},$$

$$\mathcal{L}_{\mathbf{i}} = \left\{ \begin{pmatrix} I_p & 0 \\ \lambda & I_l \end{pmatrix}^\wedge \in \mathcal{L} \,\middle|\, \lambda = \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_l \end{pmatrix},\ \lambda_{l-i} = \cdots = \lambda_l = 0 \right\}.$$

To simplify notation, we identify each of the unipotent subgroups $\mathbf{U}_\mathbf{p}^*$, $\mathbf{U}_{\mathbf{l+p}}^{\mathbf{i}}$, $\mathcal{L}$, $\mathcal{L}^{\mathbf{i}}$ and $\mathcal{L}_{\mathbf{i}}$ with the respective isomorphic unipotent subgroup contained in the respective pre-image under pr. For $1 \leqslant p \leqslant q_{l,\alpha}$, fix $\beta = \beta_{p,l}$ such that

$$\mathrm{pr}(\beta_{p,l}) = \begin{pmatrix} 0 & I_p \\ I_l & 0 \end{pmatrix}^\wedge.$$

Let

$$\mathbf{P}'_{\mathbf{l+p}} = \left\{ \begin{pmatrix} d & * & * & * & * \\ 0 & z & * & * & * \\ 0 & 0 & I_{2(m-p-l)+1} & * & * \\ 0 & 0 & 0 & z^* & * \\ 0 & 0 & 0 & 0 & d^* \end{pmatrix} \in \mathrm{SO}(V)(F) \;\middle|\; d \in \mathrm{GL}(p)\,,\ z \in \mathbf{Z_l} \right\} .$$

Since $\mathbf{P}'_{\mathbf{l+p}}$ is isomorphic to the semi-direct product of $\mathrm{GL}(p)$ with a unipotent subgroup, we may choose a suitable F-subgroup in its pre-image under pr in $\mathbf{H}$ which is isomorphic and call this group $\mathbf{P_{l+p}}$.

Also let

$$\mathbf{E}' = \left\{ \begin{pmatrix} m & x & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & I_{2(m-p-1)+1} & 0 & 0 \\ 0 & 0 & 0 & 1 & x' \\ 0 & 0 & 0 & 0 & m^* \end{pmatrix} \in \mathrm{SO}(V)(F) \;\middle|\; m \in \mathrm{GL}(p) \right\}$$

and $\mathbf{E}$ be a suitable F-subgroup of $\mathrm{pr}^{-1}(\mathbf{E}')$ which is isomorphic.

Proposition 3.2. *Let π be a smooth representation of H and π' be the representation of E on $J_{U_{l+p}^{l-1},\psi_{l+p,\alpha}}(\pi)$ via restriction. There exists a filtration*

$$0 = W_0 \subseteq W_1 \subseteq \cdots \subseteq W_{p+1} = J_{U_p^*}(J_{\psi_{l,\alpha}}(\pi))$$

of vector spaces such that each successive quotient $W_{i-1}\backslash W_i$ is isomorphic to $\text{c-i}_{R_{i-1}}^{P_{p+1}}(\pi'^{(p-i+2)} \otimes \psi)$ as vector spaces for each $i = 1, \cdots, p+1$. Identifying $\mathbf{P_{p+1}}$ with the mirabolic subgroup $\mathbf{P_{p,1}}$ of $\mathrm{GL}(p+1)$, here R_{i-1} is the subgroup of P_{p+1} of the form

$$\left\{ \begin{pmatrix} g & v \\ 0 & z \end{pmatrix} \in P_{p+1} \;\middle|\; g \in \mathrm{GL}(i-1)(F)\,,\ z \in Z_{p-i+2}\,,\ v \in M_{i-1,p-i+2} \right\}$$

and ψ is regarded as a character of Z_{p-i+2} defined by $\psi((z_{ij})) = \psi(\sum_{j=1}^{p-i+1} z_{j,j+1})$ and $\pi'^{(p-i+2)} \otimes \psi$ is extended from $\mathrm{GL}(i-1) \times Z_{p-i+2}$ to R_{i-1} trivially across

$$N_{i-1} = \left\{ \begin{pmatrix} I_{i-1} & v \\ 0 & I_{p-i+2} \end{pmatrix} \right\} .$$

Proof. Denote the subgroup $\beta \mathbf{N_l} \mathbf{U_p^*} \beta^{-1}$ by $\mathbf{S}$. The elements of $\mathbf{S}$ are identified with its image in $\mathrm{SO}(V)$ under pr which are of the form

$$s(z; u, a, d, e; x, y) = \begin{pmatrix} I_p & 0 & x & d & y \\ u & z & a & e & d' \\ 0 & 0 & I_{2(m-l-p)+1} & a' & x' \\ 0 & 0 & 0 & z^* & 0 \\ 0 & 0 & 0 & u' & I_p \end{pmatrix}$$

with $z \in \mathbf{Z_l}$ and $x \cdot y_\alpha^{(l+p)} = 0$ where $y_\alpha^{(l+p)}$ is the vector y_α regarded as a column vector in $\overline{F}^{\dim_F V - 2(l+p)}$. In this proof, we identify $\mathbf{Z_l}$ as a subgroup of $\mathbf{S}$.

We first extend $\psi_{l,\alpha}$ to $U_p^*N_l$ so that it is trivial on U_p^* and then let $\psi_S(s) = \psi_{l,\alpha}(\beta^{-1}s\beta)$. Thus

$$\psi_S(s(z;u,a,d,e;x,y)) = \psi(z_{1,2} + z_{2,3} + \cdots + z_{l-1,1} + a_l \cdot y_\alpha^{(l+p)}) .$$

Denote by π_β the restriction of π to $\beta R_{l,\alpha}\beta^{-1}$. Then the map $v \mapsto \pi(\beta)v$ for $v \in V_\pi$ induces an isomorphism

$$J_{U_p^*}(J_{\psi_{l,\alpha}}(\pi)) \cong J_{S,\psi_S}(V_{\pi_\beta})$$

of vector spaces.

To avoid confusion, we shall denote the subgroups $\mathbf{C}$, $\mathbf{X}$ and $\mathbf{Y}$ to which Lemma 3.1 is applied the $(i+1)$th time by $\mathbf{C^i}$, $\mathbf{X^i}$, $\mathbf{Y^i}$ respectively. Also, set $\mathbf{B^i} = \mathbf{C^i Y^i}$, $\mathbf{D^i} = \mathbf{C^i X^i}$ and $\mathbf{A^i} = \mathbf{C^i X^i Y^i}$.

Let $\mathbf{C^0}$ be the subgroup of $\mathbf{S}$ which consists of the elements of the form $s(z;u,a,d,e;x,y)$ with $u=0$, $z = I_l$. Note that $\mathcal{L}$ is the subgroup of elements in $\mathbf{S}$ which are lower triangular and ψ_S is trivial on $\mathcal{L}$. Let $\mathbf{X^0}$ and $\mathbf{Y^0}$ be unipotent subgroups of $\mathbf{H}$ such that $\mathrm{pr}(\mathbf{X^0}) = \{s(I_l;0,0,0,0;x,y) \in \mathrm{SO}(V)\}$ and $\mathrm{pr}(\mathbf{Y^0}) = \mathcal{L}^{\mathbf{0}}$. Also let $\mathbf{J_0} = \mathbf{X^0} \cap \mathbf{S} = \beta \mathbf{U_p^*}\beta^{-1} = \mathbf{X^0} \cap \mathbf{C^0}$. Then $J_0\backslash X^0$ is identified with F^p and so is abelian. The commutator of an element of $\mathbf{X^0}$ and an element of $\mathbf{Y^{(0)}}$ has the form

$$s\left(I_l, 0, \begin{pmatrix} 0 \\ \vdots \\ u\cdot x \end{pmatrix}, *, *, 0, 0\right)$$

and thus lies in $\mathbf{C^0}$. Note that $\mathbf{D^0} = \mathbf{C^0X^0} = \mathbf{U_{l+p}}$ and $\psi_{D^0} = \psi_{l+p,\alpha}|_{D^0}$.

Conditions (1) – (6) of Lemma 3.1 are satisfied and invoking it gives $J_{B^0,\psi_{B^0}}(\pi) \cong J_{D^0,\psi_{D^0}}(\pi)$ as vector spaces. Since $J_{S,\psi_S}(V_{\pi_\beta})$ and $J_{Z_l,\psi_{l+p,\alpha}}(J_{\mathcal{L}_0}(J_{B^0,\psi_{B^0}}(\pi)))$ are isomorphic as vector spaces,

$$J_{U_p^*}(J_{\psi_{l,\alpha}}(\pi)) \cong J_{Z_l,\psi_{l,\alpha}}(J_{\mathcal{L}_0}(J_{U_{l+p},\psi_{l+p,\alpha}}(\pi)))$$

as vector spaces. Write

$$\mathcal{L}_{\mathbf{0}} = \mathcal{L}^{\mathbf{1}}\mathcal{L}_{\mathbf{1}} \, , \, \mathbf{Z_1} = \mathbf{Z^1 Z_{l-1}}$$

where

$$\mathbf{Z^1} = \left\{ \begin{pmatrix} I_{l-1} & * \\ 0 & 1 \end{pmatrix} \right\} \, , \, \mathbf{Z_{l-1}} = \left\{ \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} \in \mathbf{Z_l} \right\} .$$

Let $\mathbf{C^1} = \mathbf{U_{l+p}Z^1}$, $\mathbf{Y^1} = \mathcal{L}^{\mathbf{1}}$,

$$\mathbf{X^1} = \left\{ \begin{pmatrix} I_p & 0 & x \\ 0 & I_{l-1} & 0 \\ 0 & 0 & 1 \end{pmatrix}^\wedge \right\} .$$

Then $\psi_{C^1} = \psi_{l+p,\alpha}$ and we have the set-up in Lemma 3.1. It follows from Lemma 3.1 that

$$J_{\mathcal{L}^1}(J_{U_{l+p}Z^{(1)},\psi_{l+p,\alpha}}(\pi)) \cong J_{\mathcal{L}_1}(J_{U^1_{l+p},\psi_{l+p,\alpha}}(\pi))$$

as vector spaces. Hence $J_{U_p^*}(J_{\psi_{l,\alpha}}(\pi)) \cong J_{Z_{l-1},\psi_{l+p,\alpha}}(J_{\mathcal{L}_1}(J_{U^1_{l+p},\psi_{l+p,\alpha}}(\pi)))$ as vector spaces. Assume by induction for $1 \leqslant i \leqslant l-2$, $J_{U_p^*}(J_{\psi_{l,\alpha}}(\pi)) \cong J_{Z_{l-i},\psi_{l+p,\alpha}}(J_{\mathcal{L}_i}(J_{U^i_{l+p},\psi_{l+p,\alpha}}(\pi)))$ as vector spaces. Write

$$\mathcal{L}_{\mathbf{i}} = \mathcal{L}^{\mathbf{i+1}}\mathcal{L}_{\mathbf{i+1}} \, , \, \mathbf{Z_{l-i}} = \mathbf{Z^{i+1}Z_{l-i-1}}$$

where

$$\mathbf{Z^{i+1}} = \left\{ \begin{pmatrix} I_{l-i-1} & x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & I_i \end{pmatrix} \right\} , \mathbf{Z_{l-i-1}} = \left\{ \begin{pmatrix} z & 0 \\ 0 & I_{i+1} \end{pmatrix} \in \mathbf{Z_l} \right\} .$$

Let $\mathbf{C^{i+1}} = \mathbf{U^i_{l+p}} \mathbf{Z^{i+1}}$, $\mathbf{Y^{i+1}} = \mathcal{L}^{\mathbf{i+1}}$ and

$$\mathbf{X^{i+1}} = \left\{ \begin{pmatrix} I_p & 0 & x & 0 \\ 0 & I_{l-i-1} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & I_i \end{pmatrix}^{\wedge} \right\} .$$

Then $\psi_{C^{i+1}} = \psi_{l+p,\alpha}$ and conditions $(1) - (6)$ of Lemma 3.1 are satisfied. Thus Lemma 3.1 implies

$$J_{\mathcal{L}^{i+1}}(J_{U^i_{l+p}Z^{(i+1)},\psi_{l+p,\alpha}}(\pi)) \cong J_{\mathcal{L}_{i+1}}(J_{U^{i+1}_{l+p},\psi_{l+p,\alpha}}(\pi))$$

as vector spaces. Therefore $J_{U_p^*}(J_{\psi_{l,\alpha}}(\pi)) \cong J_{Z_{l-i-1},\psi_{l+p,\alpha}}(J_{\mathcal{L}_{i+1}}(J_{U^{i+1}_{l+p},\psi_{l+p,\alpha}}(\pi)))$ as vector spaces. Taking $i = l-2$,

$$J_{U_p^*}(J_{\psi_{l,\alpha}}(\pi)) \cong J_{Z_1,\psi_{l+p,\alpha}}(J_{U^{l-1}_{l+p},\psi_{l+p,\alpha}}(\pi)) \tag{3.1}$$

as vector spaces. By Theorem 7.2 of [GPR87], we see that π' has a filtration

$$0 = W_0 \subseteq W_1 \subseteq \cdots \subseteq W_{p+1} = J_{U^{l-1}_{l+p},\psi_{l+p,\alpha}}(\pi)$$

such that each successive quotient

$$W_{i-1}\backslash W_i \cong \text{c-i}^{P_{p+1}}_{R_{i-1}}(\pi'^{(p-i+2)} \otimes \psi)$$

for each $i = 1, \cdots, p+1$. Transporting the vector space structure of $J_{U^{l-1}_{l+p},\psi_{l+p,\alpha}}(\pi)$ to $J_{U_p^*}(J_{\psi_{l,\alpha}}(\pi))$, the conclusion follows. □

Theorem 3.3. (***Tower Property***) *Let τ be a supercuspidal representation of $\mathrm{GL}(m)(F)$ where $m = 2n$ and $\pi^{\omega\otimes\tau}$ be a subquotient of $i^H_{Q_m}(\omega \otimes |\det|^{1/2}\tau)$. We have a vector space isomorphism*

$$J_{U_p^*}(J_{\psi_{l,\alpha}}(\pi^{\omega\otimes\tau})) \cong \text{c-i}^{P_{l+p}}_{N_{l+p}}(J_{\psi_{l+p,\alpha}}(\pi^{\omega\otimes\tau})|_{N_{l+p}}) .$$

Proof. Replacing π in Proposition 3.2 by $\pi^{\omega\otimes\tau}$, $\pi'^{(p-i+2)}$ is trivial for $2 \leqslant i \leqslant p+1$ since τ is supercuspidal and $\pi^{\omega\otimes\tau}$ is a subquotient of $i^H_{Q_m}(\omega \otimes |\det|^{1/2}\tau)$. The conclusion then follows from the fact that

$$\pi' \cong J_{U^{l-1}_{l+p},\psi_{l+p,\alpha}}(\pi) \qquad \text{and} \qquad \pi'^{(p+1)} \otimes \psi \cong J_{\psi_{l+p,\alpha}}(\pi^{\omega\otimes\tau})|_{N_{l+p}}$$

as vector spaces. □

We know that $i^H_{Q_m}(\omega \otimes |\det|^{-1/2}\tau)$ has two constituents: one irreducible sub-representation which is not generic and an irreducible quotient which is generic. The irreducible sub-representation is isomorphic to $\pi_{\omega\otimes\tau}$.

4 Vanishing of Descents

4.1 Background

In this section we prove the vanishing of the descent map for $m/2 < l \leqslant m$. We briefly recall the analogous result in the descent construction for SO_{2n+1}. For details see Ginzburg *et. al.* [GRS99a, GRS99b]. Let τ be an irreducible unitary supercuspidal representation of $\mathrm{GL}(m)(F)$ such that $L(s,\tau,\wedge^2\rho_m)$ has a pole at $s=0$ (in particular, m is even). Then τ affords a Shalika model, hence also a linear model, i.e., τ embeds into $C^\infty(\mathrm{GL}(m/2)(F)\times\mathrm{GL}(m/2)(F)\backslash\mathrm{GL}(m)(F))$. It then follows that the representation parabolically induced from $\tau|\det|^{1/2}$ to Sp_{2m} injects into $C^\infty(\mathrm{Sp}_m(F)\times\mathrm{Sp}_m(F)\backslash\mathrm{Sp}_{2m}(F))$. Granted these results, the vanishing of the tower of local descent maps up to and except the top level, is effectively proved by showing that the corresponding twisted Jacquet modules vanish on the latter space. This is precisely [GRS99b, Theorem 17].

The integral representation for the symmetric square L-function was first developed by Shimura for $m=2$ (see also [GJ78]). Patterson and Piatetski-Shapiro [PPS89] extended his result to $m=3$ and Bump and Ginzburg [BG92] developed the general case. The twisted version was constructed by Gelbart and Jacquet [GJ78] for $m=2$, Banks [Ban97] for $m=3$, and recently for any m by Takeda [Tak14].

We consider a representation τ such that the symmetric or twisted symmetric square L-function has a pole at $s=0$. Here and throughout, these L-functions are the ones defined by Shahidi ([Sha90]). Let us begin with the case of $L(s,\tau,\mathrm{Sym}^2\rho_m)$. As exhibited by Bump and Ginzburg [BG92], the (global and local) study of this L-function involves the theory of exceptional representations of Kazhdan and Patterson [KP84]. Globally, the period of a cusp form against a pair of theta functions characterizes the pole of the partial symmetric square L-function at $s=1$ ([BG92, Theorem 7.6]). Let θ and θ' be a pair of exceptional representations in the sense of [KP84] of the double cover $\widetilde{\mathrm{GL}}(m)$ of $\mathrm{GL}(m)$, over a local p-adic field. The tensor $\theta\otimes\theta'$ is a well defined representation of $\mathrm{GL}(m)$ and, in light of several results ([Kap15, Kap16a, Kap16b, Kap17b, Kap17a]), is expected to play the role of the linear model in an analog of the aforementioned construction.

In more detail, one can find a pair (θ,θ') such that τ is a quotient of $\theta\otimes\theta'$ ([Kap16b, Theorem1.3]). Then in [Kap16b, Proposition 4.1] it was proved that the representation parabolically induced from $\mathbb{1}\otimes\tau|\det|^{1/2}$ to $\mathrm{GSpin}(2m+1)$ is a quotient of $\Theta\otimes\Theta'$, where Θ and Θ' are exceptional representations of a double cover $\widetilde{\mathrm{GSpin}}(2m+1)$ of $\mathrm{GSpin}(2m+1)$, defined in [Kap17b]. The remaining step, proving the vanishing of the twisted Jacquet modules of $\Theta\otimes\Theta'$, has already been partially worked out in [Kap16b, Theorem 1.1], but only for the "ground level", i.e., the generic case ($l=m$). Here we complete the proof for all $l>m/2$.

Now assume that ω is a unitary character of F^* and $L(s,\tau,\mathrm{Sym}^2\rho_m\otimes\omega^{-1})$ has a pole at $s=0$. The integral representation for the global version of this L-function involved a variant of the aforementioned representations θ, which we call an extended exceptional representation (exceptional ones are included in the definition). For $m=2$, this representation is an extension of the Weil representation ([Gel76, GPS80]). Banks [Ban94] constructed it over p-adic fields of odd residual characteristic, and Takeda [Tak14] developed the general case (under the name "twisted exceptional representations").

Yamana proved that if $L(s,\tau,\mathrm{Sym}^2\rho_m\otimes\omega^{-1})$ has a pole at $s=0$, τ (or a twist of τ for odd m) is a quotient of a tensor of two extended exceptional representations ([Yam17, Theorem 3.19]). To proceed, we need to define an extended version of the exceptional representation of $\widetilde{\mathrm{GSpin}}(2m+1)$. This can be done along the line of arguments of [Tak14, Kap17b], and [Kap16b, Proposition 4.1] will now apply to extended exceptional representations. Thus we can treat both $\mathrm{Sym}^2 \rho_m$ and $\mathrm{Sym}^2 \rho_m \otimes \omega^{-1}$ uniformly.

4.2 Exceptional representations

Put $\mathbf{H} = \mathbf{GSpin(2m+1)}$ and fix the Borel subroup $\mathbf{B_H} = \mathbf{T_H} \ltimes \mathbf{N_H}$ of $\mathbf{H}$ as described above (i.e., using pr^{-1}). Recall that $\mathbf{Q_k} = \mathbf{M_k} \ltimes \mathbf{U_k}$ is the standard maximal parabolic subgroup of $\mathbf{H}$, whose Levi part $\mathbf{M_k}$ is isomorphic to $\mathbf{GL(k)} \times \mathbf{GSpin(2(m-k)+1)}$. Let Υ denote the "canonical" character of $\mathbf{H}$ constructed in [Kap17b, § 1.2]. Its restriction to the $\mathbf{GL}(k)$ part of $\mathbf{M_k}$ is det. Let $\widetilde{H}$ be the double cover of H, constructed in [Kap17b] by restricting the double cover of $\mathrm{Spin}(2m+3)$ of Matsumoto [Mat69] and using the cocycle σ of Banks *et. al.* [BLS99] (in [Kap17b] we showed that σ is block-compatible). We fix a section $\mathfrak{s} : H \to \widetilde{H}$ such that $\sigma(h, h') = \mathfrak{s}(h)\mathfrak{s}(h')\mathfrak{s}(hh')^{-1}$. This section is a homomorphism of N_H. For any $X \subset H$ let $\widetilde{X}$ be its preimage in $\widetilde{H}$. For any group L, $Z(L)$ denotes its center. Then we have $Z(\widetilde{H}) = \widetilde{Z(H)}$ (in contrast with double coverings of $\mathrm{GL}(2m)$).

The exceptional representations of $\widetilde{H}$ were developed (locally and globally) in [Kap17b], by adapting the construction of Bump *et. al.* [BFG03, BFG06] for a covering of SO_{2m+1}. For a convenient summary see [Kap16b, § 2.8]. In the group $\mathbf{GL(m)}$ let $\mathbf{T_{GL(m)}} \ltimes \mathbf{Z_m}$ be the Borel subgroup of upper triangular matrices, where $\mathbf{T_{GL(m)}}$ is the diagonal torus. Regard $\mathbf{T_{GL(m)}}$ as a subgroup of $\mathbf{H}$ by identifying it with the natural subgroup of $\mathbf{M_m}$. Let ξ be a genuine character of $Z(\widetilde{T}_H)$, whose restriction to $Z(\widetilde{T}_{\mathrm{GL}(m)})$ and $Z(\widetilde{H})$ is a genuine lift of $\delta_{B_{\mathrm{GL}(m)}}^{1/4} \cdot |\det|^{(m+1)/4}$ and the trivial character, respectively (note that $Z(\widetilde{T}_{\mathrm{GL}(m)}) < Z(\widetilde{T}_H)$). This determines ξ uniquely when m is even, in the odd case there is an additional choice of a Weil factor. Let $\rho(\xi)$ denote the corresponding genuine irreducible representation of $\widetilde{T}_H$ (see e.g., [KP84, McN12]). Then $i_{\widetilde{B}_H}^{\widetilde{H}}(\rho(\xi))$ has a unique irreducible quotient Θ_0 (normalized induction). An exceptional representation Θ of $\widetilde{H}$ is then any twist of Θ_0 by a non-genuine character of H, i.e., $\Theta = (\chi \circ \Upsilon) \cdot \Theta_0$ where χ is a quasi-character of F^*.

The main property of Θ is that the Jacquet functor along a radical of a parabolic subgroup is, essentially, an exceptional representation of the stabilizer. See [Kap17b, Proposition 2.19] for a more precise statement (see also [BFG03, Theorem 2.3]). This result and the fact that exceptional representations of $\widetilde{\mathrm{GL}}(m)$ do not afford a Whittaker functional for $m > 2$ ([KP84, Kap17a], see also [Yam17]), imply through a series of intermediate results, that Θ is "small" in the sense that it is attached to one of the unipotent orbits next to the minimal one (see [BFG03]). The following theorem encapsulates all the vanishing properties of Θ.

We may regard the elements of $U_1 (= \boldsymbol{U_1}(F))$ as row vectors $(u_1, \ldots, u_{2m-1})$. Fix an additive character ψ of F. Then any character λ of U_1 takes the form

$$\lambda(u) = \psi\left(\sum_{i=1}^{2m-1} \beta_i u_i\right),$$

with $\beta_i \in F$. Define the length of λ by

$$2\sum_{i=1}^{m-1} \beta_i \beta_{2m-i} + \beta_m^2.$$

While the length depends on ψ, we are only interested in the case when it is non-zero, and this does not depend on ψ.

Theorem 4.1. *([BFG03, Theorem 2.6], [BFG06, Proposition 3] and [Kap17b, Lemma 2.25]) For any λ with non-zero length, $J_{U_1,\lambda}(\Theta) = 0$.*

Corollary 4.2. *Let V be a subgroup of U_1 and λ be a character of V, such that any extension of λ to a character of U_1 has non-zero length. Then $J_{V,\lambda}(\Theta) = 0$.*

Proof. Since U_1 is abelian, the representation $J_{V,\lambda}(\Theta)$ is filtered by $J_{U_1,\lambda'}(\Theta)$, where λ' is a character of U_1 extending λ. Since λ' has non-zero length, the latter module vanishes. □

For example when $m = 2$, if V is defined by $u_1 = 0$, the character $\lambda(v) = \psi(\beta u_2)$ satisfies the requirement of the corollary, for any $\beta \neq 0$.

Corollary 4.3. *Let λ be a character of U_1 defined by the vector $(\beta, 0, \ldots, 0)$, where $\beta \neq 0$. The subgroup U_2 acts trivially on $J_{U_1,\lambda}(\Theta)$.*

Proof. We argue exactly as in the proof of [BFG06, Proposition 4]. □

As mentioned above, the Jacquet functor takes exceptional representations into exceptional representations. We describe the particular case of J_{U_m}. Let θ_0 be the unique irreducible quotient of $i_{\widetilde{B}_{\mathrm{GL}(m)}}^{\widetilde{\mathrm{GL}}(m)}(\rho(\xi_0))$, where ξ_0 is a lift of $\delta_{B_{\mathrm{GL}(m)}}^{1/4}$ to a genuine character of $Z(\widetilde{T}_{\mathrm{GL}(m)})$. This lift is unique when m is even and depends on a Weil factor in the odd case; still, if we fix one Weil factor, the representations θ_0 corresponding to the different lifts are twists of one another by a square-trivial character (see [Kap17a, Claim 2.6]). The exceptional representations of $\widetilde{\mathrm{GL}}(m)$ are thus θ_0 and its twists $\theta = (\chi \circ \det) \cdot \theta_0$ (see [KP84, BG92, Kab01]). Then

$$\delta_{Q_m}^{1/2} J_{U_m}(\Theta_0) = \mathbb{1} \otimes |\det|^{(m-1)/4}\theta_0 \tag{4.1}$$

([Kap16b, (2.8)], the Jacquet functor there was not normalized; see also [Kap17b, Claim 2.21]). Note that the direct factors of M_m commute in the cover, but this is a special phenomenon, which does not hold for M_k with $k < m$. Equality (4.1) implies (almost formally) that when we take a unitary quotient τ of $\theta \otimes \theta'$, there is a suitable unitary character ω of F^* (depending on θ and θ') and a pair of exceptional representations (Θ, Θ') of $\widetilde{H}$, such that the representation parabolically induced from $\omega^{-1} \otimes \tau|\det|^{1/2}$ to H is a quotient of $\Theta \otimes \Theta'$ ([Kap16b, Proposition 4.1]).

As explained in § 4.1, to handle the twisted symmetric square L-function we need to consider a wider class of exceptional representations, which we call extended exceptional representations. These will only be used here for even m, so assume this is the case. We begin with a brief description of the construction of Takeda [Tak14, § 2.2–2.4] (following [GPS80, Ban94]) of these representations for $\widetilde{\mathrm{GL}}(m)$. Let χ be a unitary character of F^* such that $\chi(-1) = -1$. Denote by ω_ψ^- the irreducible summand of the Weil representation ω_ψ of $\widetilde{\mathrm{Sp}}_2$ consisting of odd functions. One can extend ω_ψ^- to a representation of the subgroup $\widetilde{\mathrm{GL}}^{(2)}(2)$ of $\widetilde{\mathrm{GL}}(2)$, where $\mathrm{GL}^{(2)}(2)$ is the subgroup of matrices whose determinant is a square, by letting $Z(\mathrm{GL}(2))$ act by χ. More precisely if $\mathfrak{s} : \mathrm{GL}(2) \to \widetilde{\mathrm{GL}}(2)$ is the chosen section, the action is given by $\mathfrak{s}(aI_2) \mapsto \chi(a)\gamma_{\psi'}(a)$, where $\gamma_{\psi'}$ is the Weil factor corresponding to an additive character ψ' of F. The extended exceptional representation θ_2^χ of $\widetilde{\mathrm{GL}}(2)$ is obtained by inducing from $\widetilde{\mathrm{GL}}^{(2)}(2)$ to $\widetilde{\mathrm{GL}}(2)$. It is an irreducible representation, and independent of the choice of ψ'. Moreover, it is unitary and supercuspidal (as opposed to the exceptional representations of [KP84]).

Now let $\mathbf{R} = \mathbf{M_R} \ltimes \mathbf{U_R}$ be the standard parabolic subgroup of $\mathbf{GL(m)}$ whose Levi part $\mathbf{M_R}$ is isomorphic to $\mathbf{GL(2)} \times \ldots \times \mathbf{GL(2)}$, where $\mathbf{GL(2)}$ appears $m/2$ times (m is even). Consider the representation

$$i_{\widetilde{R}}^{\widetilde{\mathrm{GL}}(m)}((\theta_2^{\chi}\widetilde{\otimes}\ldots\widetilde{\otimes}\theta_2^{\chi})\delta_R^{1/4}).$$

Here $\widetilde{\otimes}$ is the metaplectic tensor ([Kab01, Mez04]), which in this case is canonical (see [Tak16, Remark 4.3]).

The inducing data is tempered, hence the Langlands Quotient Theorem, proved for metaplectic groups by Ban and Jantzen [BJ13], implies that it has a unique irreducible quotient θ_m^{χ}, which is an extended exceptional representation of $\widetilde{\mathrm{GL}}(m)$. The representation θ_m^{χ} is also the image of the intertwining operator with respect to the longest Weyl element relative to R. Furthermore, the "periodicity result" [Tak14, Proposition 2.36] reads

$$J_{U_R}(\theta_m^{\chi}) = (\theta_2^{\chi}\widetilde{\otimes}\ldots\widetilde{\otimes}\theta_2^{\chi})\delta_R^{-1/4}$$

(J_{U_R} was not normalized there). See [KP84, Theorem I.2.9] for this statement on exceptional representations. As above, we can twist θ_m^{χ} by $\chi_1 \circ \det$ for a quasi-character χ_1.

We follow a similar paradigm, to construct extended exceptional representations of $\widetilde{H}$. Let $\mathbf{Q} = \mathbf{M} \ltimes \mathbf{U}$ be the standard parabolic subgroup of $\mathbf{H}$ whose Levi part $\mathbf{M}$ is isomorphic to $\mathbf{M_R} \times \mathrm{GL}(\mathbf{1})$. Consider the representation

$$\Pi^* = i_{\widetilde{Q}}^{\widetilde{H}}((\mathbb{1} \otimes (\theta_2^{\chi}\widetilde{\otimes}\ldots\widetilde{\otimes}\theta_2^{\chi}))\delta_Q^{1/4}).$$

Again, according to the Langlands Quotient Theorem [BJ13] this representation has a unique irreducible quotient $\Theta^* = \Theta^{\chi}$. Since the inducing data is supercuspidal, according to [BZ77, Corollary 2.13c], $J_U(\Pi^*)$ is glued from

$$^{w}((\mathbb{1} \otimes (\theta_2^{\chi}\widetilde{\otimes}\ldots\widetilde{\otimes}\theta_2^{\chi}))\delta_Q^{1/4}),$$

where w varies over the Weyl elements of H, which satisfy $wMw^{-1} = M$ and are reduced modulo the Weyl group of M. The periodicity result becomes

$$J_U(\Theta^*) = (\mathbb{1} \otimes (\theta_2^{\chi}\widetilde{\otimes}\ldots\widetilde{\otimes}\theta_2^{\chi}))\delta_Q^{-1/4}. \tag{4.2}$$

See [Kap17b, Proposition 2.16] for this statement on Θ. A family of extended exceptional representations can be obtained by varying χ, and twisting using $\chi_1 \circ \Upsilon$.

Let $\mathbf{Q'} = \mathbf{M'} \ltimes \mathbf{U'}$ be a standard parabolic subgroup of $\mathbf{H}$. The following statements follow from [BZ77, Corollary 2.13]: if $\mathbf{U'} \cap \mathbf{M}$ is non-trivial, $J_{U'}(\Theta^*) = 0$; and if $\mathbf{M'}$ strictly contains $\mathbf{M}$, $J_{U'}(\Theta^*)$ is irreducible (also use the transitivity of the Jacquet functor and (4.2)).

Utilizing the above observations, the results of [Kap17b, § 2.3.1] for Θ are applicable to Θ^* as well. Also note that Yamana [Yam17] proved that θ_m^{χ} does not afford a Whittker functional when $m \geqslant 3$. Therefore the arguments of [Kap17b, § 2.3.2] are valid as well, in particular [Kap17b, Lemma 2.25], and we deduce that Theorem 4.1 and its corollaries are applicable also to Θ^*.

Additionally, the analog of (4.1) holds as well (see [Kap17b, Claim 2.21]), where θ_0 is replaced by θ_m^{χ}, and thus the proof of [Kap16b, Proposition 4.1] extends to Θ^*. For example if τ is a quotient of $\theta_m^{\chi^{-1}} \otimes \theta_m$, the representation parabolically induced from $\chi \otimes |\det|^{1/2}\tau$ to H is a quotient of two extended exceptional representations of $\widetilde{H}$.

For specific choices of parameters, the construction we described produces the (non-extended) exceptional representations. To conclude, since the extended exceptional representation Θ^* enjoys the same properties of Θ, that are relevant for the proof of the vanishing theorem in the following section, we omit references to χ from the notation and simply write Θ in all cases.

4.3 Vanishing theorem

Let ψ be a non-trivial additive character of F, τ be an irreducible unitary supercuspidal representation of $\mathrm{GL}(m)$, and ω be a unitary character of F^*.

Our assumption is that $L(s, \tau, \mathrm{Sym}^2 \rho_m \otimes \omega^{-1})$ has a pole at $s = 0$. If $\omega = 1$, according to [Kap16b, Theorem 1.3] the representation τ is a quotient of $\theta \otimes \theta'$ for some pair of exceptional representations of $\widetilde{\mathrm{GL}}(m)$. For the case $\omega \neq 1$, by [Yam17, Theorem 3.19(1)] when m is even τ is a quotient of $\theta_m^{\omega^{-1}} \otimes \theta_m$. The extended exceptional representations are determined (non-uniquely) by τ and ω. When $\omega \neq 1$ and m is odd, a twist of τ, namely $\omega_\tau^{-1}\omega^{(m-1)/2}\tau$ where ω_τ is the central character of τ, is a quotient of $\theta \otimes \theta'$ [Yam17, Theorem 3.19(2)]. The vanishing property does not depend on this twist, hence we can assume in all cases that τ is such a quotient.

By virtue of [Kap16b, Proposition 4.1], there is a pair of extended exceptional representations (Θ, Θ') of $\widetilde{H}$ such that the representation $i_{Q_m}^{H}(\omega \otimes |\det|^{1/2}\tau)$ is a quotient of $\Theta \otimes \Theta'$. Therefore, the vanishing result follows from the following theorem.

Theorem 4.4. *For any Θ, Θ' and $m/2 < l \leqslant m$, $J_{\psi_{l,\alpha}}(\Theta \otimes \Theta') = 0$. In particular, $\sigma_{\psi,l}(\tau) = 0$ for $m/2 < l \leqslant m$.*

Remark 4.5. *We do not assume anything on α, except that it is non-zero. i.e., the proof is valid whether the connected component of the stabilizer of $\psi_{l,\alpha}$ is $\mathrm{GSpin}(2(m-l))$ or $\mathrm{GSpin}^*(2(m-l))$. This is an incarnation of the fact that Theorem 4.1 applies to any character of non-zero length.*

Proof. For $l = m$ this is [Kap16b, Theorem 1.1]. The general case is not very different. For brevity, we focus on the differences.

Put $Q = Q_l$ and write $Q = M \ltimes U$ (i.e., $U = U_l$). Denote $N = N_l$,

$$N = \left\{ \begin{pmatrix} z & v & c \\ & I_{2(m-l)+1} & v' \\ & & z^* \end{pmatrix} : z \in Z_l \right\}. \tag{4.3}$$

Let $C = Z(U)$ (the center of U). The Levi subgroup M acts on the characters of C with $\lfloor l/2 \rfloor$ orbits. We choose representatives for these orbits: $\psi_0(c) = 1$ and

$$\psi_j(c) = \psi\left(\sum_{i=1}^{j} c_{l-2j+i,i}\right), \qquad 0 < j \leqslant \lfloor l/2 \rfloor.$$

Here c is regarded as an $l \times l$ matrix. When $j = l/2$ (in particular, l is even), any character in the orbit of ψ_j is called generic. Denote the stabilizer of ψ_j in M by St_j. By virtue of the Geometric

Lemma of Bernstein and Zelevinsky ([BZ77, Theorem 5.2] and [BZ76, 5.9–5.12]), as a representation of $\widetilde{Q}$, Θ is glued from

$$\text{c-i}_{\widetilde{\mathrm{St}}_j U}^{\widetilde{Q}}(J_{C,\psi_j}(\Theta)), \qquad 0 \leqslant j \leqslant \lfloor l/2 \rfloor$$

(compact normalized induction). A similar result applies to Θ', where we use ψ^{-1} for the representatives of the orbits. In turn $\Theta \otimes \Theta'$ is glued from tensors of such representations, with indices j, j'. According to [Kap17a, Lemma 2.3], when we apply the Jacquet functor with respect to N and $\psi_{l,\alpha}$ only those with $j = j'$ remain. Since the tensor of two genuine representations is a non-genuine representation, we need to show that for all j,

$$J_{\psi_{l,\alpha}}(\text{c-i}_{\mathrm{St}_j U}^{Q}(J_{C,\psi_j}(\Theta) \otimes J_{C,\psi_j^{-1}}(\Theta'))) = 0. \tag{4.4}$$

First consider $j = 0$. Then $\text{c-i}_{\widetilde{\mathrm{St}}_j U}^{\widetilde{Q}}(J_{C,\psi_j}(\Theta)) = J_C(\Theta)$ and the left hand side of (4.4) becomes

$$J_{\psi_{l,\alpha}}(J_C(\Theta) \otimes J_C(\Theta')).$$

We apply a second filtration argument, according to the orbits of characters of $C\backslash U$, with respect to the action of M. Note that if ψ_U is a character of U which is trivial on C, $J_{C\backslash U,\psi_U} J_C(\Theta) = J_{U,\psi_U}(\Theta)$.

Let $u \in U$ be written as in (4.3). Consider the restriction of ψ_U to the last row b of v. Using the action of M, we can assume it takes the form $b \mapsto \psi(b_1 + \beta b_{2(m-l)+1})$, for some $\beta \in F$. If $\beta \neq 0$, this character can be conjugated into a character λ of a subgroup of U_1, such that any extension of λ to a character of U_1 is of non-zero length. But then by Corollary 4.2, the Jacquet module vanishes. Hence $\beta = 0$. Re-denote by b the $(l-1)$-th row of v, on this row the character takes the form $b \mapsto \psi(b_2 + \beta b_{2(m-l)})$ and again we deduce $\beta = 0$. Proceeding similarly, it follows that ψ_U can be non-trivial on at most $m - l$ rows, which we assume are the last ones. Indeed, otherwise it can be conjugated into a character whose restriction to one of the rows is $b \mapsto \psi(b_1 + \beta b_{2(m-l)+1})$ for $\beta \neq 0$, and as above Corollary 4.2 implies that the Jacquet module vanishes.

Therefore, we only need to consider $m - l + 1$ orbits, with representatives $\psi_{U,0} = 1$ and

$$\psi_{U,k}(u) = \psi\left(\sum_{i=1}^{k} v_{l-i+1,i}\right), \qquad 0 < k \leqslant m - l.$$

As above, we begin with quotients $J_{U,\psi_{U,k}}(\Theta) \otimes J_{U,\psi_{U,k'}^{-1}}(\Theta')$, but when we apply $J_{\psi_{l,\alpha}}$, only those with $k = k'$ remain ([Kap17a, Lemma 2.3]).

For $k > 0$, let $\mathbf{Z_{l-k,k}}$ be the unipotent radical of the standard maximal parabolic subgroup of $\mathbf{GL(l)}$ whose Levi part is isomorphic to $\mathbf{GL(l-k)} \times \mathbf{GL(k)}$. Since $l > m/2$, $Z_{l-k,k}$ is non-trivial. The group $Z_{l-k,k}$ normalizes U and stabilizes $\psi_{U,k}$ and we prove that its action on $J_{U,\psi_{U,k}}(\Theta)$ is trivial. To this end we show that for any non-trivial character μ of $Z_{l-k,k}$,

$$J_{Z_{l-k,k} \ltimes U, \mu\, \psi_{U,k}}(\Theta) = 0. \tag{4.5}$$

Indeed, as in [Kap16b, Claim 3.3] applying Lemma 3.1 (the version of [GRS99a, Lemma 2.2]) and another conjugation, we see that $J_{Z_{l-k,k} \ltimes U, \mu\, \psi_{U,k}}(\Theta)$ is a quotient of $J_{U_1,\lambda}(\Theta)$, where $\lambda(u) = \psi(u_1)$, and the action of the $(l-k+1)$-th row of U, which is given by the restriction of ψ_k to this row, transforms into a non-trivial action of U_2. This contradicts Corollary 4.3, unless (4.5) holds.

We deduce that for all $k \geqslant 0$, there is a root subgroup of N on which $\psi_{l,\alpha}$ is non-trivial, but its action on

$$J_{U,\psi_{U,k}}(\Theta) \otimes J_{U,\psi_{U,k}^{-1}}(\Theta')$$

is trivial. Note that for $k = 0$, U itself acts trivially on this space while $\psi_{l,\alpha}$ is non-trivial on U. This proves (4.4) for $j = 0$.

Next we state the generic case.

Lemma 4.6. *Equality* (4.4) *holds for* $j = l/2$.

The proof is deferred to § 4.4 below, we now explain how to reduce the remaining cases to the generic one. Assume $0 < j < l/2$. Consider the standard parabolic subgroup $\mathbf{Q_{l-2j}}$ whose Levi part is isomorphic to $\mathbf{GL(l-2j)} \times \mathbf{H^j}$, where $\mathbf{H^j} = \mathrm{GSpin}(\mathbf{2(m-l+2j)+1})$. Let $\mathbf{Q^j} = \mathbf{M^j} \ltimes \mathbf{U^j}$ be the standard parabolic subgroup of $\mathbf{H^j}$ with $\mathbf{M^j} \cong \mathbf{GL(2j)}$, and $C^j = Z(U^j)$. Note that $\psi_j|_{C^j}$ is a generic character of C^j. Also set $Q_0 = Q \cap Q_{l-2j}$.

Claim 4.7. *The representation* $J_{C,\psi_j}(\Theta)$ *is trivial on* U_{l-2j}. *As a representation of* $\widetilde{M}_{l-2j}$ *it is a finite direct sum of representations* $\vartheta \otimes J_{C^j,\psi_j}(\Theta^j)$, *where* ϑ *is a finite direct sum of irreducible representations of* $\widetilde{\mathrm{GL}}(l-2j)$ *and* Θ^j *is an extended exceptional representation of* $\widetilde{H}^j$.

Remark 4.8. *The representation* ϑ *is essentially an exceptional representation, its description was given in the similar result [Kap16b, Proposition 3.1], but will not be needed here.*

Using the claim and transitivity of induction, the left-hand side of (4.4) becomes a finite sum of representations

$$J_{\psi_{l,\alpha}}\left(\text{c-i}_{Q_0}^{Q}\,\text{c-i}_{\mathrm{St}_k U}^{Q_0}\left((\vartheta \otimes \vartheta') \otimes (J_{C^j,\psi_j}(\Theta^j) \otimes J_{C^j,\psi_j^{-1}}(\Theta'^j))\right)\right).$$

We can compute this module using the Geometric Lemma [BZ77, Theorem 5.2]. Specifically, choose a set of Weyl elements w representing the double cosets $Q_0 \backslash Q/N$. We see that $\psi_{l,\alpha}|_{wUw^{-1} \cap N} \neq 1$ unless $w = \begin{pmatrix} & I_{2j} \\ I_{l-2j} & \end{pmatrix}$, regarded as an element of Q via its embedding in M. The corresponding quotient is, up to a modulus character,

$$J_{Z_{l-2j},\psi_{l,\alpha}}(\vartheta \otimes \vartheta') \otimes J_{\psi_{l,\alpha}}(J_{C^j,\psi_j}(\Theta^j) \otimes J_{C^j,\psi_j^{-1}}(\Theta'^j)).$$

Since $\psi_{l,\alpha}$ restricts to a similar character on the subgroup $Z_{2j} \ltimes U^j$ of Q^j, i.e., a generic character on Z_{2j} and the coordinates of U where $\psi_{l,\alpha}$ is non-trivial belong to U^j, Lemma 4.6 (the generic case) implies

$$J_{\psi_{l,\alpha}}(J_{C^j,\psi_j}(\Theta^j) \otimes J_{C^j,\psi_j^{-1}}(\Theta'^j)) = 0.$$

This completes the proof of (4.4) for all j. □

Proof of Claim 4.7. As in [Kap16b, Proposition 3.1], the claim follows once we show that the action of U_{l-2j} on $J_{C,\psi_j}(\Theta)$ is trivial. The main difference is that when $l = m$, there are only two orbits for the characters of U_m, namely the orbit of a character defined by a vector of non-zero length (restriction of the generic character of N_H to U_m), and the trivial orbit. When $l < m$ there are also non-trivial characters defined by vectors of zero length to consider. (For a character defined by a vector in F^{2r+1}, the length of the character is the length of the vector with respect to the symmetric bilinear form defining SO_{2r+1}.)

Write $U_{l-2j} = V \cdot (C \cap U_{l-2j})$. Clearly $C \cap U_{l-2j}$ acts trivially on $J_{C,\psi_j}(\Theta)$. It remains to show $J_{V,\mu}(J_{C,\psi_j}(\Theta)) = 0$ for all non-trivial characters μ of V.

Let μ be such a character. Using the action of $\mathrm{GL}(l-2j) \times M^j$, we may assume that its restriction to the last row of $v \in V$ is given by

$$b \mapsto \psi(b_1 + b_{2j+1} + \beta b_{2(m-l+j)+1}),$$

where b is a row of length $2(m-l+j)+1$ and $\beta \in F$. If $\beta \neq 0$, $J_{V,\mu}(J_{C,\psi_j}(\Theta)) = 0$ because as in the case $j = 0$, we can conjugate this character into a character of U_1 and apply Corollary 4.2. Hence $\beta = 0$.

Now let $g \in M_{l-2j}$ be such that ${}^g\mu(v) = \mu(g^{-1}vg)$ restricts to the character $b \mapsto \psi(b_1)$ on b. The element g does not normalize U^j and does not stabilize $\psi_j|_{C^j}$, but $g^{-1}C^jg < U^j$. Let $X < C^j$ be consisting of matrices whose only non-zero coordinates are the $(1,1)$ and $(2j,2j)$-th ones. We can regard g as the matrix

$$\mathrm{diag}\left(I_{l-2j}, \begin{pmatrix} 1 & & \\ & I_{2j-1} & \\ 1 & & 1 \end{pmatrix}, I_{2(m-l)-1}, \begin{pmatrix} 1 & & \\ & I_{2j-1} & \\ -1 & & 1 \end{pmatrix}, I_{l-2j}\right).$$

Since U^j normalizes U_{l-2j}, we deduce that $g^{-1}Xg$ normalizes U_{l-2j} and $J_{V,\mu}(J_{C,\psi_j}(\Theta))$ is a quotient of

$$J_{g^{-1}Xg \ltimes U_{l-2j},\mu}(\Theta).$$

Conjugating by g, we obtain $J_{X \ltimes U_{l-2j},{}^g\mu}(\Theta)$, where ${}^g\mu$ is non-trivial on X (g normalizes U_{l-2j}). Now we can proceed exactly as in [Kap16b, Claim 3.3]: applying Lemma 3.1 ([GRS99a, Lemma 2.2]) and another conjugation, $J_{X \ltimes U_{l-2j},{}^g\mu}(\Theta)$ is seen to be a quotient of $J_{U_1,\lambda}(\Theta)$, where $\lambda(u) = \psi(u_1)$, and the action of X becomes a non-trivial action of U_2, contradicting Corollary 4.3. Thus $J_{X \ltimes U_{l-2j},{}^g\mu}(\Theta) = 0$ whence $J_{V,\mu}(J_{C,\psi_j}(\Theta))$ vanishes. □

4.4 Proof of Lemma 4.6

Let $j = l/2$ and $r = 2j(2(m-l)+1)$. The subgroup U is a generalized Heisenberg group $\mathcal{H}$ of rank $r+1$. Idenfity $\mathcal{H}$ with the set of elements $(a, b; c)$, where a and b are rows in $F^{r/2}$ and $c \in F$, with the product given by

$$(a, b; c) \cdot (a', b'; c') = \left(a + a', b + b', c + c' + \tfrac{1}{2}(a, b) \begin{pmatrix} & J_{r/2} \\ -J_{r/2} & \end{pmatrix} {}^t(a', b')\right).$$

Here $J_{r/2}$ is the $r/2 \times r/2$ permutation matrix having 1 on its anti-diagonal and ${}^t(a', b')$ is the transpose of (a', b'). We have the epimorphism $\ell : U \to \mathcal{H}$ defined by

$$\ell(u) = (a_1, \ldots, a_j, b_1, \ldots, b_j, \tfrac{1}{2}(\sum_{i=1}^{j} c_{i,i} - c_{l+i,l+i})),$$

where $b_1,\dots,b_j$ are the first j rows of v and $a_1,\dots,a_j$ are the last and we recall that u is written with the notation of (4.3) (with $z=I_l$). Also let $R<\mathcal{H}$ be the subgroup consisting of elements $(0,b;0)$.

Since ψ_j is trivial on the kernel of ℓ, we may regard $J_{C,\psi_j}(\Theta)$ as a smooth representation of $\mathcal{H}$, and as such it is the direct sum of irreducible Weil representations ω_ψ, where ψ is our fixed character of F.

The representation ω_ψ extends to a representation of $\widetilde{\mathrm{Sp}}_r\ltimes\mathcal{H}$, where $\widetilde{\mathrm{Sp}}_r$ is the metaplectic double cover of Sp_r. Since $\mathrm{St}_j=\mathrm{Sp}_l$, using the action of St_j on each of the $2(m-l)+1$ columns of v we construct an embedding of St_j in Sp_r. Moreover, the covering $\widetilde{\mathrm{St}}_j$ obtained by restricting $\widetilde{H}$ does not split over St_j, hence it is the metaplectic double cover, therefore the embedding extends to an embedding of the coverings, also denoted ℓ (one may also apply the strong block compatibility property of the cocycle [BLS99, Theorem 7] to deduce this). The action of $g\in\widetilde{\mathrm{Sp}}_l$ is now given by $\varpi_\psi(\ell(g))$.

As a smooth representation of a generalized Jacobi group, $J_{C,\psi_j}(\Theta)$ is isomorphic to a representation $\kappa\otimes\omega_\psi$, where $\kappa\otimes\omega_\psi(\ell(g)h)=\kappa(g)\otimes\omega_\psi(\ell(g)h)$ for $g\in\widetilde{\mathrm{Sp}}_l$ and $h\in\mathcal{H}$, and κ is a non-genuine representation. The following claim proves that κ is trivial.

Claim 4.9. *The representation $J_{C,\psi_j}(\Theta)$ is isomorphic to a (possibly infinite) direct sum of the representation ω_ψ.*

The proof appears below. Now we proceed to prove (4.4), exactly as in [Kap16b, Claim 4.3]. Observe that by Claim 4.9,

$$J_{U,\psi_{l,\alpha}}(J_{C,\psi_j}(\Theta)\otimes J_{C,\psi_j^{-1}}(\Theta'))$$

is a direct sum of

$$J_{\mathcal{H},\psi_{l,\alpha}\circ\ell^{-1}}(\omega_\psi\otimes\omega_{\psi^{-1}}).$$

Note that $\psi_{l,\alpha}\circ\ell^{-1}$ is well defined because $\psi_{l,\alpha}$ is trivial on C and the coordinates of v in the preimage of ℓ are uniquely defined.

Applying the theory of l-sheafs of Bernstein and Zelevinsky ([BZ76, 1.13, § 6, Theorem 6.9]), it suffices to show that for all representatives $g\in\mathrm{GL}(l)$ such that the last row of g is $(0,\dots,0,1)$,

$$\mathrm{Hom}_{\mathrm{Sp}_l^g}({}^{g^{-1}}J_{\mathcal{H},\psi_{l,\alpha}\circ\ell^{-1}}(\omega_\psi\otimes\omega_\psi^{-1}),\psi_{l,\alpha}). \tag{4.6}$$

Here $\mathrm{Sp}_l^g=g^{-1}\,\mathrm{Sp}_l\,g\cap Z_l$ and for $x\in\mathrm{Sp}_l^g$, ${}^{g^{-1}}J_{\dots}(\cdots)(x)=J_{\dots}(\cdots)(\ell({}^gx))$. Note that since the last row of g is $(0,\dots,0,1)$, ${}^g(\mathrm{Sp}_l^g)$ stabilizes the restriction of $\psi_{l,\alpha}$ to U.

According to [Kap16b, Claim 2.5], if λ is the character of $\mathcal{H}$ given by $(x,y;z)\mapsto\psi(x_1)$, $J_{\mathcal{H},\lambda}(\omega_\psi\otimes\omega_{\psi^{-1}})$ is the trivial one-dimensional representation of its stabilizer in Sp_r. Since Sp_r acts transitively on the non-trivial characters of $Z(\mathcal{H})\backslash\mathcal{H}$, the same applies to any such character, in particular to the character $\psi_{l,\alpha}\circ\ell^{-1}$. Thus ${}^{g^{-1}}J_{\mathcal{H},\psi_{l,\alpha}\circ\ell^{-1}}(\omega_\psi\otimes\omega_{\psi^{-1}})$ is trivial.

We conclude that (4.6) vanishes, using the results of Offen and Sayag [OS08, Proposition 2] (we use $\mathcal{H}^{r,r'}$ with $r=0$ and $r'=l$, in their notation), namely that for any generic character ψ of Z_l, $\psi|_{\mathrm{Sp}_l^g}\neq 1$ for any $g\in\mathrm{GL}(l)$. The proof of the lemma is complete.

Proof of Claim 4.9. We proved this result in [Kap16b, Theorem 1.4] when $l=m$ (whence $j=m/2$). The proof carries over to $l<m$, we briefly describe the argument.

We need to show that κ is a trivial representation. Let

$$Y = \left\{ \begin{pmatrix} 1 & & y \\ & I_{l-2} & \\ & & 1 \end{pmatrix} \right\} < \mathrm{Sp}_l \, .$$

It is enough to show $J_{\ell(Y),\psi_\beta}(\kappa) = 0$, where $\psi_\beta(y) = \psi(\beta y)$, for all $\beta \neq 0$. To this end consider the subgroup

$$V = \left\{ \begin{pmatrix} 1 & & y & b_1 & & & * \\ & I_{l-2} & & & & & \\ & & 1 & & & & \\ & & & I_{2(m-l)+1} & & & b' \\ & & & & 1 & & y' \\ & & & & & I_{l-2} & \\ & & & & & & 1 \end{pmatrix} \right\} < U_1.$$

The mapping ℓ is an isomorphism of V onto the direct product $\ell(Y) \cdot R_1$, where $R_1 < R$ consists of elements $(0, (b, 0, \dots, 0); 0)$ with $b \in F^{2(m-l)+1}$.

First observe that

$$J_{U,\psi_\beta \circ \ell}(J_{C,\psi_j}(\Theta)) = 0. \tag{4.7}$$

This follows from Corollary 4.2, because this space is a quotient of $J_{V\cdot(C \cap U_1),(\psi_\beta \circ \ell)\psi_j}(\Theta)$ and since for $c \in C \cap U_1$, $\psi_j(c) = \psi(c_{1,1})$, any extension of $(\psi_\beta \circ \ell)\psi_j$ to a character of U_1 is a character of non-zero length.

Since $J_R(\omega_\psi)$ is one-dimensional ([Kap16b, Claim 2.4]), there is a vector φ in the space of ω_ψ such that the Jacquet integrals

$$\varphi^{\mathcal{Y},\mathcal{R}} = \int\limits_{\mathcal{Y}} \int\limits_{\mathcal{R}} \omega_\psi(yr)\varphi \, dr \, dy$$

do not vanish for all compact subgroups $\mathcal{Y} < Y$ and $\mathcal{R} < R$ (see [BZ76, 2.33]). Then given ξ in the space of κ, using a change of variables, again the fact that $J_R(\omega_\psi)$ is one-dimensional, and (4.7), one shows that for sufficiently large $\mathcal{Y}$ and $\mathcal{R}$,

$$\int\limits_{\mathcal{Y}} \kappa(y)\psi_\beta^{-1}(y)\xi \, dy \otimes \varphi^{\mathcal{Y},\mathcal{R}} = 0.$$

This implies that ξ vanishes in $J_{\ell(Y),\psi_\beta}(\kappa)$ ([BZ76, 2.33]). □

5 Non-vanishing of the Descent for $l = n$ and $m = 2n$

In this section, let $m = 2n$, we show that $\sigma_{\psi,n}(\tau) := J_{\psi_{n,\alpha}}(\pi_{\omega\otimes\tau}|_{R_{n,\alpha}})$ is non-zero, for some $\alpha \in F^*$.

5.1 Generalized and degenerate Whittaker models

We recall the generalized and degenerate Whittaker models attached to nilpotent orbits, following the formulation in [GGS17]. Let $\mathbf{H}$ be a reductive group defined over F or a central extension of finite degree. Fix a non-trivial additive character $\psi : F \to \mathbb{C}^\times$. Let $\mathfrak{h}$ be the Lie algebra of $H = \mathbf{H}(F)$ and u be a nilpotent element in $\mathfrak{h}$. The element u defines a function on $\mathfrak{h}$:

$$\psi_u : \mathfrak{h} \to \mathbb{C}^\times$$

by $\psi_u(x) = \psi(\kappa(u, x))$, where κ is the Killing form on $\mathfrak{h}$.

Given any semisimple element $s \in \mathfrak{h}$, under the adjoint action, $\mathfrak{h}$ is decomposed to a direct sum of eigenspaces $\mathfrak{h}_i^s$ of h corresponding to eigenvalues i. The element s is called *rational semisimple* if all its eigenvalues are rational. Given a nilpotent element u, a *Whittaker pair* is a pair (s, u) with $s \in \mathfrak{h}$ being a rational semisimple element and $u \in \mathfrak{h}_{-2}^s$. The element s in a Whittaker pair (s, u) is called a *neutral element* for u if there exists $v \in \mathfrak{h}_2^s$ such that (v, s, u) is an $\mathfrak{sl}_2$-triple. A Whittaker pair (s, u) is called a *neutral pair* if s is a neutral element for u. For any $X \in \mathfrak{h}$, let $\mathfrak{h}_X$ be the centralizer of X in $\mathfrak{h}$.

Given any Whittaker pair (s, u), define an anti-symmetric form ω_u on $\mathfrak{h}$ by $\omega_u(X, Y) := \kappa(u, [X, Y])$. For any rational number r, let $\mathfrak{h}_{\geqslant r}^s = \oplus_{r' \geqslant r} \mathfrak{h}_{r'}^s$. Let $\mathfrak{u}_s = \mathfrak{h}_{\geqslant 1}^s$ and let $\mathfrak{n}_{s,u}$ be the radial of $\omega_u|_{\mathfrak{u}_s}$. Then $[\mathfrak{u}_s, \mathfrak{u}_s] \subset \mathfrak{g}_{\geqslant 2}^s \subset \mathfrak{n}_{s,u}$. By [GGS17, Lemma 3.2.6], $\mathfrak{n}_{s,u} = \mathfrak{h}_{\geqslant 2}^s + \mathfrak{h}_1^s \cap \mathfrak{h}_u$. Note that if the Whittaker pair (s, u) comes from an $\mathfrak{sl}_2$-triple (v, s, u), then $\mathfrak{n}_{s,u} = \mathfrak{h}_{\geqslant 2}^s$. Let $U_s = \exp(\mathfrak{u}_s)$ and $N_{s,u} = \exp(\mathfrak{n}_{s,u})$ be the corresponding unipotent subgroups of H. Define a character of $N_{s,u}$ by $\psi_u(n) = \psi(\kappa(u, \log(n)))$. Let $N'_{s,u} = N_{s,u} \cap \ker(\psi_u)$. Then $U_s/N'_{s,u}$ is a Heisenberg group with center $N_{s,u}/N'_{s,u}$.

Let π be any irreducible admissible representation of H and (s, u) be a Whittaker pair, call the twisted Jacquet module $J_{N_{s,u},\psi_u}(\pi)$ a *degenerate Whittaker model* of π, denoted by $\pi_{s,u}$. If $(s.u)$ is a neutral pair, then $\pi_{s,u}$ is also called a *generalized Whittaker model* of π. The *wave-front set* $\mathfrak{n}(\pi)$ of π is defined to the set of nilpotent orbits $\mathcal{O}$ such that $\pi_{s,u}$ is non-zero, for some neutral pair (s, u) with $u \in \mathcal{O}$. Note that if $\pi_{s,u}$ is non-zero for some neutral pair (s, u) with $u \in \mathcal{O}$, then it is non-zero for any such neutral pair (s, u), since the non-vanishing property of such Jacquet modules does not depend on the choices of representatives of $\mathcal{O}$. Let $\mathfrak{n}^m(\pi)$ be the set of maximal elements in $\mathfrak{n}(\pi)$ under the natural order of nilpotent orbits. We recall [GGS17, Theorem A] as follows.

Theorem 5.1 (Theorem A, [GGS17]). *Let π be an irreducible admissible representation of G. Given two Whittaker pairs (s, u) and (s', u') with s being a neutral element for u, if $u \in \overline{G_{s'}(F)u'}$ where $G_{s'}$ is the centralizer of s' in G and $\pi_{s',u'}$ is non-zero, then $\pi_{s,u}$ is non-zero.*

Note that a particular case of Theorem 5.1 is that $u = u'$. In this case, the condition $u \in \overline{G_{s'}(F)u'}$ is automatically satisfied and hence Theorem 5.1 asserts in this case that if $\pi_{s',u}$ is non-zero for some Whittaker pair (s', u), then $\pi_{s,u}$ is non-zero for any neutral pair (s, u).

When H is a quasi-split classical group, it is known that the nilpotent orbits are parametrized by pairs $(\underline{p}, \underline{q})$, where $\underline{p}$ is a partition and $\underline{q}$ is a set of non-degenerate quadratic forms (see [Wp01, Section I.6]). When $H = \mathrm{Sp}(2n)(F)$, $\underline{p}$ is a symplectic partition, namely, odd parts occur with even multiplicities. When $H = \mathrm{SO}^\alpha(2n)(F)$, $\mathrm{SO}(2n+1)(F)$, $\underline{p}$ is a orthogonal partition, namely, even parts occur with even multiplicities. In these cases, given any irreducible admissible representation π of H, let $\mathfrak{p}^m(\pi)$ be the partitions corresponding to nilpotent orbits in $\mathfrak{n}^m(\pi)$. For any symplectic/orthogonal partition $\underline{p}$, by a generalized Whittaker model attached to $\underline{p}$, we mean a generalized Whittaker model $\pi_{s,u}$ attached to an orbit $\mathcal{O}$ parametrized by a pair $(\underline{p}, \underline{q})$ for some $\underline{q}$, where $u \in \mathcal{O}$ and (s, u) is a neutral pair. Sometimes, for convenience, we also write a generalized Whittaker model attached to $\underline{p}$ as $\pi_{\psi_{\underline{p}}}$, without specifying the F-rational orbit $\mathcal{O}$ and neutral pairs.

Let $\mathbf{H} = \mathrm{GSpin}(4n+1)$, as for $\mathrm{SO}(4n+1)$, an orthogonal partition $\underline{p}$ is called *special* if it has an even number of odd parts between two consecutive even parts and an odd number of odd parts greater than the largest even part (see [CM93, Section 6.3]). By the main results of [JLS16], given any irreducible admissible representation π of H, any $\underline{p} \in \mathfrak{p}^m(\pi)$ is special.

5.2 Non-vanishing of $\sigma_{\psi,n}(\tau)$

First we prove the following theorem.

Theorem 5.2. *$\pi_{\omega\otimes\tau}$ has a non-zero generalized Whittaker model attached to the partition $[(2n)^2 1]$.*

Proof. Let $\alpha_i = e_i - e_{i+1}$, $1 \leqslant i \leqslant 2n-1$, $\alpha_{2n} = e_{2n}$ be the set of roots for $\mathrm{GSpin}(4n+1)$. Let x_{α_i} be the one-dimensional root subgroup in $\mathfrak{h}$ corresponding to α_i, $1 \leqslant i \leqslant 2n$. By [Wp01, Section I.6], there is only one nilpotent orbit $\mathcal{O}$ corresponding to the partition $[(2n)^2 1]$. A representative of the nilpotent orbit $\mathcal{O}$ can be taken as follows:

$$u = \sum_{i=1}^{2n-1} x_{-\alpha_i}(1).$$

Let s be the following semi-simple element

$$s = \mathrm{diag}(2n-1, 2n-3, \ldots, 1-2n, 0, 2n-1, 2n-3, \ldots, 1-2n)\,.$$

Then it is clear that (s, u) is a neutral pair.

Claim: the generalized Whittaker model $(\pi_{\omega\otimes\tau})_{s,u}$ is non-zero.

To show the above claim, we take another semisimple element

$$s' = \mathrm{diag}(4n, 4n-2, \ldots, 2, 0, -2, \ldots, -4n)\,.$$

It is clear that (s', u) is a Whittaker pair. We consider the corresponding degenerate Whittaker model of $\pi_{\omega\otimes\tau}$. Recall that $\mathbf{Q_{2n}}$ is the parabolic subgroup of $\mathbf{H}$ with Levi subgroup isomorphic to $\mathrm{GL}(1) \times \mathrm{GL}(2n)$ and unipotent radical subgroup $\mathbf{U_{2n}}$. Then, by definition, $(\pi_{\omega\otimes\tau})_{s',u}$ is equivalent to first taking the Jacquet module with respect to U_{2n} then taking Whittaker model of τ. Since $\pi_{\omega\otimes\tau}$ is isomorphic to the Langlands quotient of $i_{Q_{2n}}^{H}(\omega \otimes |\det|^{1/2}\tau)$ and τ is an irreducible supercuspidal representation of $\mathrm{GL}(2n)(F)$ which is automatically generic, both the Jacquet module of $\pi_{\omega\otimes\tau}$ with respect to U_{2n} and the Whittaker model of τ are non-zero. Hence the generalized Whittaker model $(\pi_{\omega\otimes\tau})_{s',u}$ is non-zero. By Theorem 5.1, we know that the generalized Whittaker model $(\pi_{\omega\otimes\tau})_{s,u}$ is non-zero, hence we proved the Claim above.

This completes the proof of the theorem. □

Next we prove the following result.

Theorem 5.3. *$\pi_{\omega\otimes\tau}$ has a non-zero generalized Whittaker model attached to the partition $[(2n+1)(2n-1)1]$.*

Proof. By Theorem 5.2, we know that $\pi_{\omega\otimes\tau}$ has a non-zero generalized Whittaker model attached to the partition $[(2n)^2 1]$. But it is known that as an orthogonal partition, $[(2n)^2 1]$ is not special, and the smallest special partition which is greater than it is $[(2n+1)(2n-1)1]$, by definition, this is called the special expansion of the partition $[(2n)^2 1]$. By [JLS16, Theorem 11.1], we know that $\pi_{\omega\otimes\tau}$ has a non-zero generalized Whittaker model attached to the partition $[(2n+1)(2n-1)1]$.

This completes the proof of the theorem. □

Theorem 5.4. *There exists $\alpha \in F^*$, such that $\sigma_{\psi,n}(\tau) := J_{\psi_{n,\alpha}}(\pi_{\omega\otimes\tau}|_{R_{n,\alpha}})$ is non-zero.*

Proof. By Theorem 5.3, we know that $\pi_{\omega\otimes\tau}$ has a non-zero generalized Whittaker model attached to the partition $[(2n+1)(2n-1)1]$. By [Wp01, Section I.6], nilpotent orbits corresponding to the partition $[(2n+1)(2n-1)1]$ are parametrized by certain one-dimensional quadratic forms, i.e., certain square-classes $\{\alpha_{2n+1}, \alpha_{2n-1}, \alpha_1\}$, corresponding to the parts $(2n+1), (2n-1), 1$. By [JLS16, Proposition 8.1], actually, $\pi_{\omega\otimes\tau}$ has a non-zero generalized Whittaker model attached to the nilpotent $\mathcal{O}$, corresponding to the partition $[(2n+1)(2n-1)1]$ and parametrized by square-classes $\{\alpha, -\alpha, 1\}$ for some $\alpha \in F^*$.

Claim: for the above α, $J_{\psi_{n,\alpha}}(\pi_{\omega\otimes\tau}|_{R_{n,\alpha}})$ is non-zero.

Indeed, for the nilpotent orbit $\mathcal{O}$ above which is parametrized by square-classes $\{\alpha, -\alpha, 1\}$, one can take a representative $u = u_1 + u_2$, where $u_1 = \sum_{i=1}^{n-1} x_{-\alpha_i}(1) + x_{e_{2n}-e_n}(1) + x_{-e_{2n}-e_n}(\alpha/2)$, u_2 is any representative of the nilpotent orbit in the Levi part of the stabilizer of u_1 which is $\mathrm{GSpin}(2n)(Q)$ for certain quadratic form Q, corresponding to the partition $[(2n-1)1]$, parametrized by square-classes $\{-\alpha, 1\}$. Let s, s_1 be semi-simple elements such that s is neutral element for u and s_1 is a neutral element for u_1. Then, from above discussion, we know that $(\pi_{\omega\otimes\tau})_{s,u}$ is non-zero. On the other hand, it is easy to see that $u_1 \in \overline{\mathrm{GSpin}(4n+1)_s(F)u}$. By Theorem 5.1, we know that $(\pi_{\omega\otimes\tau})_{s_1,u_1}$ is also non-zero. Note that $(\pi_{\omega\otimes\tau})_{s_1,u_1}$ is exactly $J_{\psi_{n,\alpha}}(\pi_{\omega\otimes\tau}|_{R_{n,\alpha}})$, i.e., $\sigma_{\psi,n}(\tau)$. Hence, the Claim has been proved.

This completes the proof of the theorem. □

Remark 5.5. *Similarly, we can also show that if $m = 2n+1$, then there exists some $\alpha \in F^*$, such that $J_{\psi_{n,\alpha}}(\pi_{\omega\otimes\tau}|_{R_{n,\alpha}})$ is non-zero. We briefly sketch the main steps here: first, as an analogue of Theorem 5.2, we can show that $\pi_{\omega\otimes\tau}$ has a non-zero generalized Whittaker model attached to the partition $[(2n+1)^2 1]$, which is already a special orthogonal partition. Then, following a similar argument as in the proof of Theorem 5.4, we can obtain the non-vanishing result for the case of $m = 2n+1$.*

Theorem 5.6. *Let τ be an irreducible unitary supercuspidal representation of $\mathrm{GL}(2n)(F)$ and ω be a unitary character of F^* such that the Langlands-Shahidi L-function $L(s, \tau, \mathrm{Sym}^2 \rho_{2n} \otimes \omega^{-1})$ has a pole at $s = 0$. Then there exists cuspidal automorphic representation $\mathcal{T}$ and a finite place v such that $\mathcal{T}_v \cong \tau$ and the partial L-function $L^S(s, \mathcal{T}, \mathrm{Sym}^2 \rho_{2n} \otimes \omega^{-1})$ has a pole at $s = 1$. Here S is a finite set of places containining v and the archimedean places.*

6 Local Descent and Langlands Functoriality

Theorem 6.1. *Set ρ_{2n} to be the standard $2n$-dimensional representation. Let τ be an irreducible unitary supercuspidal representation of $\mathrm{GL}(2n)(F)$ and ω be a unitary character of F^* such that the Langlands-Shahidi L-function $L(s, \tau, \mathrm{Sym}^2 \rho_{2n} \otimes \omega^{-1})$ has a pole at $s = 0$.*

(1) *There exists some $\alpha \in F^*$ such that $\sigma_{\psi,n}(\tau) := J_{\psi_{n,\alpha}}(\pi_{\omega\otimes\tau}|_{R_{n,\alpha}})$ is a non-trivial unitary supercuspidal representation of $G = L_{n,\alpha}$. Each of its irreducible subrepresentations is $\psi_{N_G}^{-1}$-generic.*

(2) *The representation $\sigma_{\psi,n}(\tau)$ is multiplicity free.*

(3) *Let σ be an irreducible supercuspidal representation of G which is generic with respect to the Whittaker character ψ_{N_G} with Whittaker function W^ψ in the Whittaker model of σ. Then the Rankin-Selberg γ-factor $\gamma(s, \sigma \times (\omega \otimes \tau), \psi^{-1})$ has a pole at $s = 1$ if and only if $\sigma^\vee$ is a direct summand of $\sigma_{\psi,n}(\tau)$.*

Proof. By Theorem 5.3, there exists some $\alpha \in F^*$ such that $\sigma_{\psi,n}(\tau) = J_{\psi_{n,\alpha}}(\pi_{\omega\otimes\tau}|_{R_{n,\alpha}}) \neq 0$. It follows from Theorem 3.3 and Theorem 4.4 that $\sigma_{\psi,n}(\tau)$ is supercuspidal. Theorem 8.1b) of [Sha90] asserts that $\pi_{\omega\otimes\tau}$ is unitary. Thus the central character of $\pi_{\omega\otimes\tau}$ restricted to the connected component of the center of H is unitary. Denote the center of the split $\mathrm{Spin}(2n)$ or respectively quasi-split $\mathrm{Spin}^*(2n)$ by $\mathbf{Z}$. Then $(1 \times Z)/C$ being isomorphic to the Klein 4-group decomposes the representation space V' of $\pi_{\omega\otimes\tau}$ into a direct sum of at most 4 subspaces, each on which $(1 \times Z)/C$ acts via a unique quadratic character. Thus the center of G acts on each such subspace as a unitary character. Suppose there is one such subspace V_1 which is not invariant under the action of G. Then there exists $v_1 \in V_1$ and $g \in G$ such that $\sigma_{\psi,n}(\tau)(g)v_1 \notin V_1$. Since the representation space is the direct sum of such subspaces, there exists another subspace V_2 and a projection map $\pi : V' \to V_2$ such that $\pi[\sigma_{\psi,n}(\tau)(g)v_1] = v_2$ is a non-zero element of V_2. Choose $z \in (1 \times Z)/C$ such that $\pi : \sigma_{\psi,n}(\tau)(z)v_1 = \pm v_1$ and $\sigma_{\psi,n}(\tau)(z)v_2 = \mp v_2$. We have

$$\pm v_2 = \pi[\sigma_{\psi,n}(\tau)(gz)v_1] = \pi[\sigma_{\psi,n}(\tau)(zg)v_1] = \mp v_2\,,$$

contradiction. Hence each of the subspace for which the center of G acts via a unique unitary character is invariant under the action of G and so being supercuspidal is a countable direct sum of irreducible unitary supercuspidal representations. Remark 2.17 implies that each of its irreducible subrepresentations is $\psi_{N_G}^{-1}$-generic. This proves (1).

The proof of Proposition 2.12 and Proposition 2.13 further implies that for any $\psi_{N_G}^{-1}$-generic supercuspidal representation σ, the space of bilinear forms satisfying equation (2.6) is one dimensional for all s. This proves (2).

(3) follows from Theorem 2.16 and the supercuspidality of σ. □

Remark 6.2. (1) *The proofs of vanishing and non-vanishing of the descent $\sigma_{\psi,n}(\tau)$ also work for the case of $\tau = \tau_1 \times \tau_2 \times \cdots \times \tau_r$, where τ_i is a supercuspidal representation of $\mathrm{GL}(m_i)$ and $L(s, \tau_i, Sym^2 \otimes \omega^{-1})$ has a pole at $s = 0$, for $i = 1, 2, \ldots, r$. Regarding the vanishing, if $\omega = 1$, by [Kap16a, Theorem 1.3] the representation τ is a quotient of exceptional representations, since each τ_i is. Moreover, the proof in [Kap16a] applies also to the case $\omega \neq 1$, given that all m_i are even, or $m_1 = \ldots = m_r = 2m_0+1$, so that τ is a quotient of extended exceptional representations (to treat the remaining cases we may need to consider twisted exceptional representations of $\widetilde{\mathrm{GL}}(2m+1)$). Then by [Kap16b, Proposition 4.1] we deduce that the representation parabolically induced from $\omega \otimes \tau|\det|^{1/2}$ to H is a quotient of a pair of extended exceptional representations of $\widetilde{H}$. Now we may apply Theorem 4.4. For the non-vanishing part, the proof is applicable with basically the same arguments.*

(2) *The analogous statement of Theorem 3.3 still holds for the case of $\tau = \tau_1 \times \cdots \times \tau_r$ with suitable modification of proof.*

(3) *The proof of supercupidality of $\sigma_{\psi,n}(\tau)$ when $\tau = \tau_1 \times \cdots \times \tau_r$ follows similarly along the same lines from (1) and (2).*

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