AUTOMORPHIC DESCENT FOR SYMPLECTIC GROUPS: THE BRANCHING PROBLEMS AND L-FUNCTIONS

BAIYING LIU AND BIN XU

ABSTRACT. We study certain automorphic descent constructions for symplectic groups, and obtain results related to branching problems of automorphic representations. As a byproduct of the construction, based on the knowledge of the global Vogan packets for $\mathrm{Mp}_2(\mathbb{A})$, we give a new approach to prove the result that for an automorphic cuspidal representation of $\mathrm{GL}_2(\mathbb{A})$ of symplectic type, if there exists a quadratic twist with positive root number, then there exist quadratic twists with non-zero central L-values.

1. INTRODUCTION

1.1. **Background.** The automorphic descent method, developed by Ginzburg, Rallis and Soudry in their series of papers [27, 28, 29, 30, 31], gives rise to an inverse map of the functorial lift (see [8, 9, 10]) of automorphic representations of classical groups. More precisely, starting from irreducible generic isobaric sum automorphic representations of general linear groups with certain (symmetric) properties, this method constructs (generic) cuspidal automorphic representations of quasi-split classical groups, by taking various Fourier coefficients on certain residual representations obtained from Siegel Eisenstein series. A complete and detailed reference of this theory is [33]. With developments in related fields, automorphic descent method shows its importance in vast aspects of the study of automorphic representations and automorphic L-functions. For one thing, the representations constructed provide concrete generic members in the global (generic) Arthur packets (see [4, 51, 59]). And for another, the construction is related to the global zeta integrals of Rankin-Selberg type (or Shimura type), which represent Rankin-Selberg L-functions with respect to classical groups and general linear groups (see, for example, [26, 33]). In addition, some related subjects, such as irreducibility of the descent, are taken into account in [44, 24, 53, 60].

In the works [42, 47, 48, 41, 50], and also some early considerations in [25, 20, 21], a twisted version of automorphic descent is developed. In this case, not just starting from an irreducible generic isobaic sum automorphic representation τ of a general linear group, an irreducible cuspidal automorphic representation σ of a classical group is also involved in the initial data, and the descent is constructed by taking Fourier coefficients on certain residual representations obtained from Eisenstein series supported on maximal parabolic subgroups of non-Siegel type. The point is, guided by the endoscopic classification theory (see [4, 51, 59]), the twisted automorphic descent provides a systematic way to concretely construct more members (e.g. the non-generic ones) in the global Arthur packets of classical groups parametrized by generic global parameters. Moreover, the twisted automorphic descent method is capable of constructing representations of pure inner

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forms of classical groups, hence has potential to recover the global Vogan packets. From this point of view, based on more understandings of these global packets, more applications are expected under this framework. On the other hand, this construction is connected to more general global zeta integrals, and one important significance of them is that the Bessel and Fourier-Jacobi periods come out naturally from them by the unfolding of Eisenstein series. Thus the twisted automorphic descent has a natural relation to the well-known Gan-Gross-Prasad (GGP) conjectures ([35, 14]), which relate such periods to central values of Rankin-Selberg L-functions. Results on non-vanishing of such L-values, as well as examples of global GGP conjectures in symplectic-metaplectic, orthogonal, and unitary cases, have been obtained using this approach (see [20, 21, 22, 47, 49]). In these works, concrete constructions via twisted automorphic descent have shown notable importance.

We note that in order to provide concrete representations in global packets and show non-vanishing of global periods, it is crucial to show the non-vanishing of certain Fourier coefficients of automorphic forms. This is not easy in general, however, when the representation σ in the initial data (τ, σ) is a representation of a small size group, it is possible to obtain some definite results, as in [20, 21, 22, 42, 41]. The objective of this article is to study some new cases of twisted automorphic descent for symplectic groups, where the desired non-vanishing properties can be obtained. As applications, we can obtain some results on the GGP conjecture for the symplectic-metaplectic case, and also some results on non-vanishing of central L-values (see §1.2 below for details).

1.2. **The global GGP conjecture in symplectic-metaplectic case and related problems.** The questions under consideration in this article are about branching problems of automorphic representations and their impacts on the study of L-functions. Their formulations are based on the GGP conjecture.

We recall first the global GGP conjecture for the symplectic-metaplectic case, based on [14]. Let F be a number field and $\mathbb{A} = \mathbb{A}_F$ be its ring of adeles. Let $\psi : F\backslash\mathbb{A} \longrightarrow \mathbb{C}^\times$ be a fixed non-trivial additive character. For an integer $N \geq 1$, we denote V_N to be a symplectic space over F of dimension $2N$. Let n and r be positive integers. We denote by $G_n = \mathrm{Sp}(V_n)$ the symplectic group of rank n over F, and denote by $H_r = \mathrm{Mp}(V_r)$ the metaplectic group of rank r over F, i.e. the unique two-fold central extension of $\mathrm{Sp}(V_r)$:

$$1 \longrightarrow \{\pm 1\} \longrightarrow \mathrm{Mp}(V_r) \longrightarrow \mathrm{Sp}(V_r) \longrightarrow 1.$$

A *generic* global Arthur parameter (see [4]) for G_n is given as a formal sum

$$\phi = (\tau_1, 1) \boxplus \cdots \boxplus (\tau_t, 1), \tag{1.1}$$

where τ_i, with $i = 1, 2, \ldots, t$, is a unitary irreducible cuspidal automorphic representation of $\mathrm{GL}_{a_i}(\mathbb{A})$, and $\tau_i \ncong \tau_j$ if $i \neq j$. Moreover, each τ_i is of orthogonal type in the sense that the symmetric square L-function $L(s, \tau_i, \mathrm{Sym}^2)$ has a pole at $s = 1$. Such parameters are also called *discrete global L-parameters* in [14, §25]. Note that one must have $2n + 1 = \sum_{i=1}^t a_i$, and each generic global Arthur parameter ϕ in (1.1) parametrizes an irreducible generic isobaric sum automorphic representation of $\mathrm{GL}_{2n+1}(\mathbb{A})$. We denote by $\widetilde{\Phi}_2(G_n)$ the set of generic global Arthur parameters of G_n. For each Arthur parameter $\phi \in \widetilde{\Phi}_2(G_n)$, the associated global Arthur packet is denoted by $\widetilde{\Pi}_\phi(G_n(\mathbb{A}))$, which is also the global Vogan packet $\widetilde{\Pi}_\phi[G_n(\mathbb{A})]$ associated to ϕ, since G_n is the only pure inner form of itself. In this article we use the terminology of global Vogan packets following [14, §25].

Similarly, we have generic global Arthur parameters ϕ' for H_r

$$\phi' = (\tau'_1, 1) \boxplus \cdots \boxplus (\tau'_{t'}, 1), \tag{1.2}$$

where τ'_i's $(i = 1, \ldots, t')$ are distinct unitary irreducible cuspidal automorphic representations of $\mathrm{GL}_{a'_i}(\mathbb{A})$ $(i = 1, \ldots, t')$ respectively, which are of symplectic type in the sense that each exterior square L-function $L(s, \tau'_i, \wedge^2)$ has a pole at $s = 1$ (see [14, §11] or [16]). We denote the set of generic global Arthur parameters of H_r by $\widetilde{\Phi}_2(H_r)$. Accordingly, we have the global Arthur packet $\widetilde{\Pi}^{\psi}_{\phi'}(H_r(\mathbb{A}))$, which is also the global Vogan packet $\widetilde{\Pi}^{\psi}_{\phi'}[H_r(\mathbb{A})]$. Note that the parameters, and also the packets for metaplectic groups depend on a fixed additive character ψ (see [14, §11, §25] and [24]). With the above data, we have the global Vogan packet (see [14, §25]) for $\phi \times \phi'$:

$$\widetilde{\Pi}^{\psi}_{\phi\times\phi'}[G_n(\mathbb{A}) \times H_r(\mathbb{A})]. \tag{1.3}$$

From now on, we assume that $n \geq r$. For irreducible cuspidal automorphic representations π and $\widetilde{\sigma}$ of $G_n(\mathbb{A})$ and $H_r(\mathbb{A})$ respectively, one defines the *Fourier-Jacobi period* (see [14, §23] or §2.3 for details)

$$\mathcal{P}^{\mathrm{FJ}}_{\psi,\varphi_r}(\boldsymbol{\phi}_\pi, \boldsymbol{\phi}_{\widetilde{\sigma}}), \quad (\boldsymbol{\phi}_\pi \in \pi,\ \boldsymbol{\phi}_{\widetilde{\sigma}} \in \widetilde{\sigma},\ \varphi_r \in \mathcal{S}(\mathbb{A}^r)) \tag{1.4}$$

whose non-vanishing is used to detect whether σ occurs in the "restriction" of π to $H_r(\mathbb{A})$. On the other hand, one defines the tensor product L-function

$$L_\psi(s, \phi \times \phi') := \prod_{i=1}^{t} \prod_{j=1}^{t'} L(s, \tau_i \times \tau'_j) \tag{1.5}$$

associated to the pair of generic global Arthur parameters ϕ and ϕ' for the fixed ψ (see [14, §22]). The global GGP conjecture asserts that the central value $L_\psi(1/2, \phi \times \phi')$ is non-zero if and only if there exists a pair $(\pi_0, \widetilde{\sigma}_0)$ in the global Vogan packet $\widetilde{\Pi}^{\psi}_{\phi\times\phi'}[G_n(\mathbb{A}) \times H_r(\mathbb{A})]$ with a non-zero Fourier-Jacobi period $\mathcal{P}^{\mathrm{FJ}}_{\psi,\varphi_r}(\boldsymbol{\phi}_{\pi_0}, \boldsymbol{\phi}_{\widetilde{\sigma}_0})$. An important feature is that such a pair $(\pi_0, \widetilde{\sigma}_0) \in \widetilde{\Pi}^{\psi}_{\phi\times\phi'}[G_n(\mathbb{A}) \times H_r(\mathbb{A})]$ is *unique*, following from the local GGP conjecture ([14, §17]). When such a pair exists, we call it *the Gan-Gross-Prasad pair* (or *GGP pair* for short) for the given pair of generic global Arthur parameter (ϕ, ϕ').

The GGP conjecture is about the *branching problem*, which concerns about the decomposition of a representation when restricting to subgroups. As seen in some previous works (see [20, 21, 42, 38, 41]), constructive methods, in particular the twisted version of the automorphic descent method, can be used to study the global GGP conjecture. With this approach, besides a generic automorphic representation τ of some $\mathrm{GL}_N(\mathbb{A})$ which determines a global parameter ϕ_τ, one also takes an automorphic representation $\widetilde{\sigma}$ of the group $H_r(\mathbb{A})$ with a generic global parameter. Then one may ask the following:

Problem 1.1. *For a given irreducible cuspidal automorphic representation $\widetilde{\sigma}$ of $H_r(\mathbb{A})$ with a generic global Arthur parameter ϕ', how to find some group G_{t^*}, and an irreducible cuspidal automorphic representation π of $G_{t^*}(\mathbb{A})$ with a generic global Arthur parameter ϕ, such that π and $\widetilde{\sigma}$ have a non-zero Fourier-Jacobi period?*

This is understood as the *reciprocal branching problem* introduced in [41]. From the pair of representations $(\tau, \widetilde{\sigma})$ mentioned above, the twisted automorphic descent method gives a concrete and uniform construction of a tower of representations $\{\pi_i\}_t$ of $\{G_t(\mathbb{A})\}_t$. An important feature of this method is that, if π_t is non-zero, it has a non-zero Fourier-Jacobi period with $\widetilde{\sigma}$ (see §2.3 for some examples). Then it follows that if moreover some basic properties (such as cuspidality), as well as the global Arthur parameter ϕ_t of π_t, can be determined, one obtains $(\pi_t, \widetilde{\sigma})$ as the GGP pair for the pair of global Arthur parameters (ϕ_t, ϕ'), where ϕ' is the Arthur parameter of $\widetilde{\sigma}$. We remark here that even when ϕ_t is not generic, the construction could also provide examples for the non-tempered GGP (see [15])

We explain the general framework with more details. Assume that $\widetilde{\sigma}$ belongs to the generic global Vogan packet $\widetilde{\Pi}^{\psi}_{\phi'}[H_r(\mathbb{A})]$ (for a fixed non-trivial additive character $\psi : F\backslash\mathbb{A} \longrightarrow \mathbb{C}^{\times}$). Towards the above reciprocal branching problem, we take a *generic* global Arthur parameter $\phi \in \widetilde{\Phi}_2(G_n)$ and assume that $L_{\psi}(1/2, \phi \times \phi') \neq 0$. Assume that $(\pi_0, \widetilde{\sigma}_0)$ is the unique GGP pair in the global Vogan packet $\widetilde{\Pi}^{\psi}_{\phi\times\phi'}[G_n(\mathbb{A}) \times H_r(\mathbb{A})]$ as in the global GGP conjecture. Then, if $\widetilde{\sigma} \simeq \widetilde{\sigma}_0$, the global GGP conjecture predicts that the member π_0 in the global Vogan packet $\widetilde{\Pi}_{\phi}[G_n(\mathbb{A})]$ gives an answer to the reciprocal branching problem, and this member π_0 is expected to be constructed by twisted automorphic descent method from the parameter ϕ and the representation $\widetilde{\sigma}_0$. However, for an arbitrary $\widetilde{\sigma} \in \widetilde{\Pi}^{\psi}_{\phi'}[H_r(\mathbb{A})]$, we can not always expect $\widetilde{\sigma}$ to be part of the GGP pair for $\phi \times \phi'$ by the uniqueness property in the GGP conjecture, for a fixed parameter $\phi \in \widetilde{\Phi}_2(G_n)$. In this situation, the automorphic descent method helps us to find a (specific) group G_{n+k} for some $k \geq 1$, and an explicit irreducible cuspidal automorphic representation π_{n+k} of $G_{n+k}(\mathbb{A})$ with the property that π_{n+k} has a generic global Arthur parameter ϕ_{n+k}, and has a non-zero Fourier-Jacobi period with the given $\widetilde{\sigma}$. Moreover, it turns out that, the group G_{n+k}, and also the parameter ϕ_{n+k}, should be controlled by certain *first occurrence* property (see §2.2) of the construction. Hence, they are all determined by the initial data ϕ, ϕ' and $\widetilde{\sigma}$ via the constructive mechanism.

On the other hand, based on some known cases that the non-vanishing of GGP periods implies non-vanishing of L-functions (see, for example, [20, 48]), one may obtain some results on non-vanishing of (twists) of central L-values by showing the corresponding periods are non-zero. Under the framework suggested above, this can be obtained by proving certain automorphic descent construction is non-zero, provided that one knows enough information about the related global parameters. In particular, from the understanding of the global packet $\widetilde{\Pi}^{\psi}_{\phi'}[\mathrm{Mp}_2(\mathbb{A})]$ (see [74, 76, 63, 13]), the symplectic-metaplectic cases are naturally related to quadratic twists of L-functions (see also §6.1). Then one can obtain certain non-vanishing results for quadratic twists of L-functions for GL_2 via a more conceptional approach (see Theorem 1.5 below) which depends on the whole machinery of twisted automorphic descent. We hope this approach is applicable to more general cases.

1.3. **The cases in this article.** In this article we consider the above mentioned problems for $r = 1$. In this case, starting with an irreducible genuine cuspidal automorphic representation $\widetilde{\sigma}$ of $H_1(\mathbb{A}) = \mathrm{Mp}_2(\mathbb{A})$, we want to construct cuspidal representations with generic global Arthur parameters which have non-zero Fourier-Jacobi periods with $\widetilde{\sigma}$, and then study related cases of the global GGP conjectures and the applications to L-functions.

We fix a non-trivial additive character $\psi : F\backslash\mathbb{A} \longrightarrow \mathbb{C}^{\times}$. Without particularly notification, the parameters and Vogan packets are associated to this character. Assume that $\widetilde{\sigma}$ has a generic global Arthur parameter $\phi_{\tau_0} = (\tau_0, 1)$, where τ_0 is an irreducible unitary cuspidal representation $\mathrm{GL}_2(\mathbb{A})$ of symplectic type. For the parameter ϕ_{τ_0} parametrizing an automorphic representation $\widetilde{\sigma}$ of $\mathrm{Mp}_2(\mathbb{A})$, one requires

$$\varepsilon(\frac{1}{2}, \tau_0 \otimes \eta_0) = 1 \tag{1.6}$$

for some quadratic character $\eta_0 : F^\times\backslash\mathbb{A}^\times \longrightarrow \mathbb{C}^\times$ (see [13, Lemma 8.6]).

Let

$$\tau = \tau_1 \boxplus \tau_2 \boxplus \cdots \boxplus \tau_t \tag{1.7}$$

be an irreducible generic isobaric sum automorphic representation of $\mathrm{GL}_m(\mathbb{A})$. Here τ_i's are distinct irreducible unitary cuspidal automorphic representations of $\mathrm{GL}_{n_i}(\mathbb{A})$ such that $\sum_{i=1}^t n_i = m$. Then τ corresponds to a generic global Arthur parameter

$$\phi_\tau = (\tau_1, 1) \boxplus (\tau_2, 1) \boxplus \cdots \boxplus (\tau_t, 1),$$

and we also assume that each τ_i is of orthogonal type, i.e. the L-function $L(s, \tau_i, \mathrm{Sym}^2)$ has a pole at $s = 1$. Moreover, we require that

$$L_\psi(1/2, \tau \times \widetilde{\sigma}) \neq 0$$

as suggested by the global GGP conjecture. Here the L-function $L_\psi(s, \tau \times \widetilde{\sigma})$ is defined in [26], which also depends on the character ψ, and we always suppose the following identification of L-functions:

$$L_\psi(s, \tau \times \widetilde{\sigma}) = L(s, \tau \times \tau_0) = L_\psi(s, \phi_\tau \times \phi_{\tau_0}). \tag{1.8}$$

For the given pair of representations $(\tau, \widetilde{\sigma})$, we have a square-integrable residual automorphic representation $\widetilde{\mathcal{E}}_{\tau\otimes\widetilde{\sigma}}$ on $\mathbf{H}(\mathbb{A}) = H_{m+1}(\mathbb{A})$ generated by the residues of Eisenstein series supported at $(\widetilde{P}, \tau \otimes \widetilde{\sigma})$ (see §2.2), where $P = MN$ is the standard parabolic subgroup of Sp_{2m+2} such that the Levi subgroup $M \simeq \mathrm{GL}_m \times \mathrm{Sp}_2$, and $\widetilde{P}$ is the pull back of P to $H_{m+1}(\mathbb{A})$ with Levi subgroup $\widetilde{M} \cong \mathrm{GL}_m \times \mathrm{Mp}_2$. The representation $\widetilde{\mathcal{E}}_{\tau\otimes\widetilde{\sigma}}$, which depends on ψ again, contains information of both τ and $\widetilde{\sigma}$, and serves as a source of our descent construction. For any $\beta \in F^\times$ and $1 < \ell < 2m - 1$, a tower of *automorphic descent of* τ *twisted by* $\widetilde{\sigma}$, denoted by

$$\left\{\pi_{\ell,\beta} = \mathcal{D}^{\mathrm{FJ}}_{\psi_{\ell,\beta}}(\widetilde{\mathcal{E}}_{\tau\otimes\widetilde{\sigma}})\right\}_\ell, \tag{1.9}$$

is constructed by taking Fourier-Jacobi coefficients of depth ℓ (see §2.2) of the residual representation $\widetilde{\mathcal{E}}_{\tau\otimes\widetilde{\sigma}}$. For each ℓ, if the twisted descent $\pi_{\ell,\beta}$ is non-zero for some choice of $\beta \in F^\times$, it consists of certain automorphic functions on $G_{m-\ell}(\mathbb{A}) = \mathrm{Sp}_{2m-2\ell}(\mathbb{A})$. As in previously known cases, the descent tower $\{\pi_{\ell,\beta}\}_\ell$ satisfies certain tower property (see §2.2), and the *first occurrence* in this tower gives a cuspidal representation with desired properties. In particular, as indicated by the general framework of Rankin-Selberg method (see, for example [25, 26, 20, 21, 33, 46, 47, 48]), unfolding Eisenstein series in certain global zeta integrals will show that each irreducible cuspidal component has a non-zero Fourier-Jacobi period with $\widetilde{\sigma}$. Hence it is always crucial to determine the first occurrence in the above descent tower.

For our purpose as described in §1.2, there are some differences between the cases that $m = 2n$ and $m = 2n + 1$:

(Case I). Suppose that $m = 2n$. In this case, ϕ_τ is not relevant to a symplectic group, and hence we do not expect an element in the descent tower $\{\pi_{\ell,\beta}\}_\ell$ would be parametrized by ϕ_τ.

(Case II). Suppose that $m = 2n + 1$. In this case, we have $\phi_{\tau\otimes\omega_\tau} \in \widetilde{\Phi}_2(G_n)$, where ω_τ is the central character of τ. Under the general framework of twisted automorphic descent, one expects that, if $\pi_{n+1,\beta}$ (which is a representation of $G_n(\mathbb{A}) = \mathrm{Sp}_{2n}(\mathbb{A})$) in the descent tower is non-zero for some $\beta \in F^\times$, then it will be parametrized by $\phi_{\tau\otimes\omega_\tau}$ (see, for example, the local calculation in §3.3, in particular Proposition 3.3). Then $(\pi_{n+1,\beta}, \widetilde{\sigma})$ will be the GGP pair in the global Vogan packet

$$\widetilde{\Pi}^{\psi^\beta}_{\phi_{\tau\otimes\omega_\tau}\times\phi_{\tau_0}}[G_n(\mathbb{A}) \times H_1(\mathbb{A})], \tag{1.10}$$

here $\psi^\beta(x) = \psi(\beta x)$. However, since $\widetilde{\sigma}$ may not be a part of the GGP pair in the global Vogan packet (1.10) by the uniqueness of GGP pairs, $\pi_{n+1,\beta}$ may not always be non-vanishing. Then, to get definite results, we make moreover the following assumption to exclude the existence of $\pi_{n+1,\beta}$:

Assumption 1.2. *In* **(Case II)***, the representation $\widetilde{\sigma}$ does not occur in the GGP pair $(\pi_0, \widetilde{\sigma}_0)$ in the global Vogan packet $\widetilde{\Pi}^{\psi^\beta}_{\phi_{\tau\otimes\omega_\tau}\times\phi_{\tau_0}}[G_n(\mathbb{A}) \times H_1(\mathbb{A})]$ for the fixed positive integer n and any $\beta \in F^\times/(F^\times)^2$.*

The point is that, in both **(Case I)** and **(Case II)** (under Assumption 1.2), we can show that the first occurrence in the tower (1.9) is at the depth $\ell^* = n$ (for some choice of $\beta \in F^\times$). Denote the resulting representation by

$$\pi_\beta = \mathcal{D}^{\mathrm{FJ}}_{\psi_n,\beta}(\widetilde{\mathcal{E}}_{\tau\otimes\widetilde{\sigma}}),$$

which turns out to be a cuspidal automorphic representation of $G_{m-n}(\mathbb{A}) = \mathrm{Sp}_{2m-2n}(\mathbb{A})$ (see §4). Moreover, basic information about the global Arthur parameters of irreducible components of π_β can be obtained by calculations of the local counterparts at unramified places (see §3). We state the main results on the above construction in the following two theorems.

Theorem 1.3. *Let $\widetilde{\sigma} \in \widetilde{\Pi}^{\psi}_{\phi_{\tau_0}}[\mathrm{Mp}_2(\mathbb{A})]$ be an irreducible genuine cuspidal automorphic representation, where $\phi_{\tau_0} = (\tau_0, 1)$ with τ_0 being an irreducible unitary cuspidal representation of $\mathrm{GL}_2(\mathbb{A})$ of symplectic type. Let $\tau = \tau_1 \boxplus \cdots \boxplus \tau_t$ be an irreducible generic isobaric sum automorphic representation of $\mathrm{GL}_m(\mathbb{A})$, where τ_i's are distinct irreducible unitary cuspidal automorphic representations of $\mathrm{GL}_{n_i}(\mathbb{A})$ of orthogonal type, such that $n_i > 1$ and $\sum_{i=1}^t n_i = m$. Assume moreover that*

$$L_\psi(\frac{1}{2}, \tau \times \widetilde{\sigma}) = L(\frac{1}{2}, \tau \times \tau_0) \neq 0.$$

Then we have:

(1) *In both* **(Case I)** *and* **(Case II)** *(presuming Assumption 1.2 in* **(Case II)***), there exists $\beta \in F^\times$ such that the automorphic descent $\pi_\beta = \mathcal{D}^{\mathrm{FJ}}_{\psi_n,\beta}(\widetilde{\mathcal{E}}_{\tau\otimes\widetilde{\sigma}})$ is a non-zero cuspidal automorphic representation.*
(2) *Suppose moreover that $\eta_\beta \neq \omega_\tau$ in* **(Case II)***. Here $\beta \in F^\times$ is the one occurs in Part (1), and η_β is the quadratic character associated to the quadratic extension $F(\sqrt{\beta})/F$. Then in both cases, any irreducible component π of π_β has a generic global Arthur parameter, and has a non-zero Fourier-Jacobi period with $\widetilde{\sigma}$. In other words, irreducible components of the descent π_β give answers to the reciprocal branching problem (Problem 1.1) introduced in §1.2.*

Moreover, we can determine the generic global Arthur parameter of the descent in **(Case I)** if $\omega_\tau = \mathbf{1}$, and hence deduce the following result related to the global GGP conjecture for symplectic groups.

Theorem 1.4. *Assume that $\omega_\tau = \mathbf{1}$ in* **(Case I)***. Then any irreducible component π of π_β has a generic global Arthur parameter*

$$\phi_\beta = \phi_{\tau\otimes\eta_\beta} \boxplus \mathbf{1}_{\mathrm{GL}_1},$$

and $(\pi, \widetilde{\sigma})$ *gives the GGP pair in the global Vogan packet* $\widetilde{\Pi}^{\psi^\beta}_{\phi_\beta \times \phi_{\tau_0}}[\mathrm{Sp}_{2n}(\mathbb{A}) \times H_1(\mathbb{A})]$. *Here* $\phi_{\tau \otimes \eta_\beta}$ *is the global Arthur parameter associated to the generic automorphic representation* $\tau \otimes \eta_\beta$.

We remark that in the situation of Theorem 1.4, if we assume the corresponding local GGP conjecture in Archimedean cases (see §6), then π_β is irreducible (Proposition 6.6). More detailed discussions can be found in §6.1.

In this article, we talk about GGP conjectures in the context of cuspidal automorphic representations with generic global parameters. It is worthwhile to mention that in a recent paper [15] of Gan, Gross and Prasad, they generalize the GGP conjectures to non-tempered cases. In our construction, it is also possible to get cuspidal automorphic representations π of $\mathrm{Sp}_{2m-2n}(\mathbb{A})$ with non-generic Arthur parameters which have non-zero Fourier-Jacobi periods with $\widetilde{\sigma}$ (see local calculations in §3). Then, as predicted in [15], the representations $(\pi, \widetilde{\sigma})$ should have relevant global parameters in sense of [15, §3], and satisfy certain non-vanishing property of L-functions (see [15, Conjecture 9.1]). One may also refer to [50] and [41] for related results.

As a byproduct of the construction in **(Case I)**, we can also prove a result on non-vanishing of quadratic twists of L-functions. Note that by some known results on global GGP conjecture (see, for example, [48, Theorem 5.4] or [73] for the pair $(\mathrm{SL}_2, \mathrm{Mp}_2)$), we have

$$L_\psi(\frac{1}{2}, \phi_\beta \times \phi_{0,\beta}) \neq 0$$

for some $\beta \in F^\times$ from the non-vanishing of Fourier-Jacobi periods stated in Part (2) of Theorem 1.3 (see §6.1 for details). Here $\phi_{0,\beta} = \phi_{\tau_0 \otimes \eta_\beta}$ is the twist of ϕ_0 by the quadratic character η_β. Combining with the uniqueness property in global GGP conjectures, we obtain (Theorem 6.8):

Theorem 1.5. *Let* τ_0 *be an irreducible unitary cuspidal automorphic representation of* $\mathrm{GL}_2(\mathbb{A})$ *of symplectic type, such that*

$$\varepsilon(\frac{1}{2}, \tau_0 \otimes \eta_0) = 1$$

for some quadratic character $\eta_0 : F^\times \backslash \mathbb{A}^\times \longrightarrow \mathbb{C}^\times$. *Then there exist* $\sharp\widetilde{\Pi}^{\psi}_{\phi_{\tau_0}}[\mathrm{Mp}_2(\mathbb{A})]$ *different quadratic characters* η *such that*

$$L(\frac{1}{2}, \tau_0 \otimes \eta) \neq 0.$$

The above result is not new. In [12], using analytic methods, Friedberg and Hoffstein proved a stronger result, where the number of different quadratic twists is infinite, and the ramification of η's can be controlled. There are also some other related results on quadratic twists of automorphic L-functions for GL_2, see [37], [76] and [7], and Luo obtains a result in the case of $\mathrm{GL}_3(\mathbb{A}_\mathbb{Q})$ twisted by Dirichlet characters (see [54]). However, our approach is new, which is based on the understanding of the global packet $\widetilde{\Pi}^{\psi}_{\phi_{\tau_0}}[\mathrm{Mp}_2(\mathbb{A})]$, and the non-vanishing of certain twisted automorphic descent constructions.

We also remark here that the existence of such quadratic twists in Theorem 1.5 played an important role in the classification of automorphic discrete spectrum of $\mathrm{Mp}_2(\mathbb{A})$ (see [74, 76], also [63] and [13] for expository references). Its higher rank generalization seems quite difficult, and hence one can not follow Waldspurger's approach to get a similar description of the automorphic discrete spectrum of $\mathrm{Mp}_{2n}(\mathbb{A})$. In [16], Gan and Ichino found a novel way to get such a description for Mp_{2n} without knowing the analogous non-vanishing results of L-functions. Regarding that the framework of twisted automorphic descent is also general, it seems possible to generalize our new approach to extend Theorem 1.5 to higher rank cases. We will consider this generalization in our subsequent work.

As mentioned above, the proof of the Theorem 1.3 is based on the conventional approach of automorphic descent method (see, for example, [33]). There are both local and global arguments. The local arguments are mainly the calculations of certain twisted Jacquet modules, which give both the local information for the global Arthur parameters of the descent modules, and the vanishing results for the tower property. The global arguments are used to show the non-vanishing of the automorphic descent, which is fundamental to our main results. It is always the most technical part, and involves precise study of Fourier coefficients of automorphic forms. Our arguments use the work of Gomez, Gourevitch and Sahi ([34]), which generalizes a local result of Mœglin and Waldspurger ([55]). Applying [34], we first show a basic non-vanishing result for the residual representation $\widetilde{\mathcal{E}}_{\tau\otimes\widetilde{\sigma}}$, namely, $\widetilde{\mathcal{E}}_{\tau\otimes\widetilde{\sigma}}$ has a non-zero generalized Whittaker-Fourier coefficient attached to the partition $[m^2 1^2]$ (see Proposition 5.3 and Proposition 5.8). This result serves as a starting point of our proofs. To get the non-vanishing result for the descent, we use a result of Jiang, Liu, and Savin on raising nilpotent orbits (see [40]) in **(Case II)** (see Proposition 5.6), which is similar to the argument used in [41] to study the Fourier coefficients of Bessel type. In **(Case I)**, however, the orbits corresponding to $[m^2 1^2]$ can not be raised, we give a more technical proof by the roots exchange technique (see [33]) directly (see Proposition 5.9) and a contradiction argument similar to that used in [20].

The structure of this article is outlined in the following. We introduce the two descent constructions in details in §2, and study their local and global properties in §3 and §4. The non-vanishing results for the descent construction are proved in §5, and their applications are studied in §6.

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2. The descent construction of Fourier-Jacobi type

2.1. **Notation.** Let F be a number field and $\mathbb{A} = \mathbb{A}_F$ be its ring of adeles. Let V be a symplectic space of dimension $(2m+2)$ defined over F, with symplectic form denoted by $\langle\ ,\ \rangle$. Then V has a polarization

$$V = V^+ \oplus V^-,$$

where V^+ is a maximal totally isotropic subspace of V with dimension $(m+1)$. Fix a maximal flag

$$\mathcal{F}: \quad 0 \subset V_1^+ \subset V_2^+ \subset \cdots \subset V_{m+1}^+ = V^+$$

in V^+, and choose a basis $\{e_1, \ldots, e_{m+1}\}$ of V^+ over F such that

$$V_i^+ = \mathrm{Span}\{e_1, \ldots, e_i\}$$

for $1 \le i \le m+1$. Let $\{e_{-1}, \ldots, e_{-(m+1)}\}$ be a basis for V^-, which is dual to $\{e_1, \ldots, e_{m+1}\}$, i.e., $\langle e_i, e_{-j}\rangle = \delta_{i,j}$ for $1 \le i, j \le m+1$. For simplicity, we denote $\mathbf{G} = \mathrm{Sp}(V)$, the symplectic group over F associated to V. Note that with the above choice of basis, the form of V is associated with the skew-symmetric matrix $J_{2m+2} = \begin{pmatrix} & w_{m+1} \\ -w_{m+1} & \end{pmatrix}$, where w_i is an $(i \times i)$-matrix with 1's on the anti-diagonal and zero's elsewhere.

We fix a Borel subgroup $\mathbf{B}_0 = \mathbf{T}_0 \cdot \mathbf{U}_0$ of $\mathbf{G}$ consisting of upper-triangular matrices, and call a parabolic subgroup to be standard if it contains $\mathbf{B}_0$. For $1 \leq k \leq 2m$, we let Q_k be the maximal standard parabolic subgroup of $\mathbf{G}$ which stabilizes the partial flag $0 \subset V_k^+$. It has a Levi decomposition $Q_k = L_k \cdot U_k$ with Levi subgroup L_k isomorphic to $\mathrm{GL}_k \times \mathrm{Sp}(V^{(k)})$. Here $V^{(k)}$ is the subspace of V which sits into the decomposition

$$V = V_k^+ \oplus V^{(k)} \oplus V_k^-,$$

where $V_k^- = \mathrm{Span}\{e_{-k}, \dots, e_{-1}\}$. Hence $V^{(k)} = \mathrm{Span}\{e_{k+1}, \dots, e_{m+1}, e_{-(m+1)}, \dots, e_{-(k+1)}\}$ under the above choice of basis. It is clear that the symplectic space $V^{(k)}$ also has a polar decomposition

$$V^{(k)} = V^{(k),+} \oplus V^{(k),-},$$

where $V^{(k),+} = \mathrm{Span}\{e_{k+1}, \dots, e_{m+1}\}$ and $V^{(k),-} = \mathrm{Span}\{e_{-(m+1)}, \dots, e_{-(k+1)}\}$. For simplicity, we will denote $P = Q_m$, $M = L_m$ and $U = U_m$.

For $1 \leq \ell \leq m$, let P_ℓ be the standard parabolic subgroup of $\mathbf{G}$ which stabilizes the partial flag

$$\mathcal{F}_\ell : \quad 0 \subset V_1^+ \subset V_2^+ \subset \cdots \subset V_\ell^+ .$$

It has a Levi decomposition $P_\ell = M_\ell \cdot N_\ell$ with Levi subgroup $M_\ell \simeq \mathrm{GL}_1^\ell \times \mathrm{Sp}(V^{(\ell)})$. The unipotent subgroup N_ℓ consists of elements of the form

$$u = u_\ell(z, x, y) = \begin{pmatrix} z & z \cdot x & y \\ & I_{2m-2\ell+2} & x' \\ & & z^* \end{pmatrix},$$

where $z \in Z_\ell$, $x \in \mathrm{Mat}_{\ell \times (2m-2\ell+2)}$, $x' = J_{2m-2\ell+2} {}^t x w_\ell$, and $y \in \mathrm{Sym}_{\ell \times \ell}$. Here Z_ℓ is the maximal upper-triangular unipotent subgroup of GL_ℓ, and $\mathrm{Sym}_{\ell \times \ell}$ is the set of $(\ell \times \ell)$-symmetric matrices. Define a homomorphism $\chi_\ell : N_\ell \longrightarrow \mathbb{G}_a$ by

$$\chi_\ell(u) = \sum_{i=2}^{\ell+1} \langle u \cdot e_i, e_{-(i-1)} \rangle. \tag{2.1}$$

Here we view e_i's as column vectors, and $u \cdot e_i$ is the multiplication of matrices.

Let $\mathbf{H}$ be the metaplectic double cover of $\mathbf{G}$, i.e. $\mathbf{H} = \mathrm{Mp}(V)$. We also let $\widetilde{P}$ be the parabolic subgroup of $\mathbf{H}$ which is the inverse image of $P \subset \mathbf{G}$. Note that we have the Levi decomposition $\widetilde{P} = \widetilde{M} \cdot U$, where $\widetilde{M} \simeq \mathrm{GL}_m \times \mathrm{Mp}(V^{(m)})$ (see [17, §2.3]).

We recall some notation for the group structure of metaplectic groups. Following [65], at each local place v, one may identify $\mathrm{Mp}(V)(F_v) = \mathrm{Sp}(V)(F_v) \times \{\pm 1\}$ with multiplication given by

$$(h_1, \epsilon_1) \cdot (h_2, \epsilon_2) = (h_1 h_2, \epsilon_1 \epsilon_2 c(h_1, h_2)),$$

here $\epsilon_i \in \{\pm 1\}$ $(i = 1, 2)$ and $c(\ ,\ ) \in \{\pm 1\}$ is the Rao cocycle defined explicitly in [65]. The Rao cocycle depends on the choice of symplectic basis of V, and we inherit the choice at the beginning of this subsection. The properties of the Rao cocycle are outlined in [52], [45] and [72]. In particular, let $\mathbf{P}_0$ be the Siegel parabolic subgroup of $\mathrm{Sp}(V)(F_v)$ fixing V^+, then one has

$$c(p, h) = c(h, p) = 1, \tag{2.2}$$

and also

$$(p, \epsilon') \cdot (h, \epsilon) \cdot (p, \epsilon')^{-1} = (php^{-1}, \epsilon) \tag{2.3}$$

for any $p \in \mathbf{P}_0(F_v)$, $h \in \mathrm{Sp}(V)(F_v)$, and ϵ, $\epsilon' \in \{\pm 1\}$. With the explicit descriptions of local double covers, the global double cover $\mathrm{Mp}(V)(\mathbb{A})$ of $\mathrm{Sp}(V)(\mathbb{A})$ is defined to be compatible with the local double covers at all places (see [45, §2.2]).

In this article, without particular specification, we will always identify $h \in \mathrm{Sp}(V)$ with $(h, 1) \in \mathrm{Mp}(V)$. Note that if $\mathbf{B}_0 = \mathbf{T}_0 \cdot \mathbf{U}_0$ is a Borel subgroup of $\mathrm{Sp}(V)$, then the double covering splits uniquely over $\mathbf{U_0}(F_v)$ for each place v of F. It follows that in the adelic setting, there is a unique splitting of the double cover over $\mathbf{U}_0(\mathbb{A})$, and more generally over the adelic group of the unipotent radical of any parabolic subgroup of $\mathrm{Sp}(V)$. Hence sometimes we do not differ unipotent subgroups in $\mathrm{Sp}(V)$ and $\mathrm{Mp}(V)$ for simplicity.

We will need some notation for parabolic subgroups in the general linear group GL_N in §3 and §4. For positive integers $n_1, \ldots, n_t$ with $n_1 + \cdots + n_t = N$, we denote $P_{n_1,\ldots,n_t}$ to be the upper-block-triangular parabolic subgroup of GL_N with Levi subgroup $\mathrm{GL}_{n_1} \times \cdots \mathrm{GL}_{n_t}$.

Finally we recall some notation on Weil representations. Fix a non-trivial additive character $\psi : F\backslash\mathbb{A} \longrightarrow \mathbb{C}^\times$. For any $\alpha \in F$, let $\psi^\alpha : F\backslash\mathbb{A} \longrightarrow \mathbb{C}^\times$ be the translation of ψ by α, i.e. $\psi^\alpha(\cdot) = \psi(\alpha\,\cdot)$. For a symplectic space $(\mathbf{W}, \langle\ ,\ \rangle)$ of dimension $2r$ over F with polarization $\mathbf{W} = \mathbf{W}^+ \oplus \mathbf{W}^-$, we realize the Heisenberg group $\mathcal{H}_{\mathbf{W}}$ (or $\mathcal{H}_{2r+1}$ if we just need to emphasize the dimension) corresponding to $\big(\mathbf{W}, 2\langle\ ,\ \rangle\big)$ as $\mathcal{H}_{\mathbf{W}} = \mathbf{W} \oplus F$, with multiplication given by

$$(w_1; t_1) \cdot (w_2; t_2) = \big(w_1 + w_2; t_1 + t_2 + \langle w_1, w_2\rangle\big), \quad (w_i \in \mathbf{W}, t_i \in F, i = 1, 2).$$

We will also denote the elements in $\mathcal{H}_{\mathbf{W}}$ by

$$(w^+, w^-; t) \tag{2.4}$$

with $w^\pm \in \mathbf{W}^\pm$ and $t \in F$ if we need to specify the vectors corresponding to the polarization. For any $\alpha \in F^\times$, we denote by $\omega_{\psi^\alpha}^{(r)}$ to be the (global) Weil representation of $\mathcal{H}_{\mathbf{W}}(\mathbb{A}) \rtimes \mathrm{Mp}(\mathbf{W})(\mathbb{A})$ (see, for example, [33, §1.2]) on the Schwartz space $\mathcal{S}(\mathbf{W}^+(\mathbb{A})) \simeq \mathcal{S}(\mathbb{A}^r)$ with respect to ψ^α (the Schrödinger model). Note that the center of the Heisenberg group acts by

$$\omega_{\psi^\alpha}^{(r)}((0; t))\varphi = \psi(\alpha t)\varphi, \quad \varphi \in \mathcal{S}(\mathbf{W}^+(\mathbb{A})).$$

We define the corresponding theta series by

$$\theta_{\psi^\alpha}^{\varphi}(h \cdot \widetilde{h}) = \sum_{\xi \in \mathbf{W}^+(F)} \omega_{\psi^\alpha}^{(r)}(h \cdot \widetilde{h})\varphi(\xi),$$

here $h \in \mathcal{H}_{\mathbf{W}}(\mathbb{A})$ and $\widetilde{h} \in \mathrm{Mp}(\mathbf{W})(\mathbb{A})$. If $\alpha = 1$, we will denote $\omega_{\psi^1}^{(r)}$ by $\omega_{\psi}^{(r)}$, and $\theta_{\psi^1}^{\varphi}(h \cdot \widetilde{h})$ by $\theta_{\psi}^{\varphi}(h \cdot \widetilde{h})$.

2.2. **The twisted automorphic descent of Fourier-Jacobi type.** In this section, we describe the twisted automorphic descent construction we will use in this article. It is given by a family of Fourier-Jacobi coefficients.

For $0 < \ell < m + 1$, define a character

$$\psi_\ell : N_\ell(F)\backslash N_\ell(\mathbb{A}) \longrightarrow \mathbb{C}^\times$$

by $\psi_\ell = \psi \circ \chi_\ell$ (see (2.1)). In matrix form, we have

$$\psi_\ell(u_\ell(z, x, y)) = \psi_{Z_\ell}(z)\psi(x_{\ell,1})\,. \tag{2.5}$$

We extend ψ_ℓ trivially to $N_{\ell+1}(\mathbb{A})$. Note that $N_\ell\backslash N_{\ell+1}$ is isomorphic to the Heisenberg group $\mathcal{H}_{V^{(\ell+1)}}$. We will fix such an isomorphism

$$j_\ell : N_\ell\backslash N_{\ell+1} \simeq \mathcal{H}_{V^{(\ell+1)}}, \tag{2.6}$$

under which the center of $\mathcal{H}_{V^{(\ell+1)}}$ corresponds to the subgroup $C_\ell \subset N_{\ell+1}$ consisting of elements of the form

$$c_\ell(t) = \mathrm{diag}\left\{ I_\ell, \begin{pmatrix} 1 & & t \\ & I_{2m-2\ell} & \\ & & 1 \end{pmatrix}, I_\ell \right\}. \tag{2.7}$$

Let $\widetilde{\Pi}$ be an irreducible genuine automorphic representation of $\mathbf{H}(\mathbb{A})$. For an automorphic form $f \in \widetilde{\Pi}$ and $\beta \in F^\times$, we define *the depth-ℓ Fourier-Jacobi coefficient of f* to be (see also [33, §3.2])

$$\mathcal{FJ}^{\varphi}_{\psi_\ell,\beta}(f)(h) = \int_{N_{\ell+1}(F)\backslash N_{\ell+1}(\mathbb{A})} f(u\widetilde{h})\psi_\ell^{-1}(u)\theta^{\varphi}_{\psi^{-\beta}}(j_\ell(u)\cdot \widetilde{h})\,\mathrm{d}u\,, \tag{2.8}$$

where $\varphi \in \mathcal{S}(V^{(\ell+1),+}) \simeq \mathcal{S}(\mathbb{A}^{m-\ell})$, $\omega^{(m-\ell)}_{\psi^{-\beta}}$ is the global Weil representation of $\mathcal{H}_{V^{(\ell+1)}}(\mathbb{A}) \rtimes \mathrm{Mp}(V^{(\ell+1)})(\mathbb{A})$, $\theta^{\varphi}_{\psi^{-\beta}}$ is the corresponding theta series, and $\widetilde{h}$ is a projection pre-image of h in $\mathrm{Mp}(V^{(\ell+1)})(\mathbb{A})$. It is easy to see that $\mathcal{FJ}^{\varphi}_{\psi_\ell,\beta}(f)(h)$ is an automorphic function on $G_{m-\ell}(\mathbb{A}) = \mathrm{Sp}(V^{(\ell+1)})(\mathbb{A})$.

We define

$$\mathcal{D}^{\mathrm{FJ}}_{\psi_\ell,\beta}(\widetilde{\Pi}) = G_{m-\ell}(\mathbb{A}) - \mathrm{Span}\left\{ \mathcal{FJ}^{\varphi}_{\psi_\ell,\beta}(f)\Big|_{G_{m-\ell}(\mathbb{A})} \;\Big|\; f \in \widetilde{\Pi},\ \varphi \in \mathcal{S}(\mathbb{A}^{m-\ell}) \right\}. \tag{2.9}$$

Let the depth ℓ vary, we obtain a tower of automorphic modules of $G_{m-\ell}(\mathbb{A})$, and we call it the *automorphic descent tower of $\widetilde{\Pi}$ of Fourier-Jacobi type*. In particular, if $\beta = 1$, we denote $\mathcal{FJ}^{\varphi}_{\psi_\ell}(f) = \mathcal{FJ}^{\varphi}_{\psi_\ell,1}(f)$ and also

$$\mathcal{D}^{\mathrm{FJ}}_{\psi_\ell}(\widetilde{\Pi}) = \mathcal{D}^{\mathrm{FJ}}_{\psi_\ell,1}(\widetilde{\Pi}).$$

For our purpose, we will consider a descent tower for some specific $\widetilde{\Pi}$. Let

$$\tau = \tau_1 \boxplus \tau_2 \boxplus \cdots \boxplus \tau_t \tag{2.10}$$

be an irreducible generic isobaric sum automorphic representation of $\mathrm{GL}_m(\mathbb{A})$, and $\widetilde{\sigma} \in \widetilde{\Pi}^{\psi}_{\phi_{\tau_0}}[H_1(\mathbb{A})]$ be an irreducible unitary cuspidal automorphic representation. Here τ_0 is an irreducible unitary cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A})$ of symplectic type, and τ_i's are distinct irreducible unitary cuspidal automorphic representations of $\mathrm{GL}_{n_i}(\mathbb{A})$ such that $\sum_{i=1}^t n_i = m$. We assume that each τ_i is of orthogonal type, i.e. the symmetric square L-function $L(s, \tau_i, \mathrm{Sym}^2)$ has a pole at $s = 1$. Moreover, we assume that

$$L_\psi(\frac{1}{2}, \tau \times \widetilde{\sigma}) \neq 0,$$

which is the starting point of the whole construction.

Recall that $\mathbf{H}$ is the global double cover of $\mathbf{G} = \mathrm{Sp}(V)$, with a maximal parabolic subgroup $\widetilde{P} = \widetilde{M} \cdot U$. For $s \in \mathbb{C}$ and an automorphic function

$$\Phi_{\tau\otimes\widetilde{\sigma}} \in \mathcal{A}(\widetilde{M}(F)U(\mathbb{A})\backslash \mathbf{H}(\mathbb{A}))_{\mu_\psi(\tau\otimes\widetilde{\sigma})}\,,$$

following [56, §II.1], one defines $\lambda_s\Phi_{\tau\otimes\widetilde{\sigma}}$ to be $(\lambda_s \circ m_P)\Phi_{\tau\otimes\widetilde{\sigma}}$, where $\lambda_s \in X^{\mathbf{G}}_M \simeq \mathbb{C}$ (see [56, §I.1] for the definition of $X^{\mathbf{G}}_M$ and the map m_P), and defines the corresponding Eisenstein series

$$\widetilde{E}(s, \widetilde{h}, \Phi_{\tau\otimes\widetilde{\sigma}}) = \sum_{\gamma \in P(F)\backslash \mathbf{G}(F)} \lambda_s \Phi_{\tau\otimes\widetilde{\sigma}}(\gamma\widetilde{h}) \tag{2.11}$$

on $\mathbf{H}(\mathbb{A})$, which converges absolutely for $\mathrm{Re}(s) \gg 0$ and has a meromorphic continuation to the whole complex plane ([56, §IV]). Here $\mu_\psi = \prod_v \mu_{\psi_v}$ is a (global) genuine character, where the definition of μ_{ψ_v} will be given in §3.1 below.

We have assumed that $L(s, \tau_i, \mathrm{Sym}^2)$ has a pole at $s = 1$ for all $i = 1, \ldots, t$ and $L_\psi(1/2, \tau \times \widetilde{\sigma}) \neq 0$. Then by calculating the constant term as in [56] (see also the arguments in [20, §3.2] and [33, §2.5 – §2.9]), the Eisenstein series $\widetilde{E}(s, \widetilde{h}, \Phi_{\tau\otimes\widetilde{\sigma}})$ has a pole at $s = 1/2$ of order t. We note here that to show the existence of the pole at $s = 1/2$, we just need to use partial L-functions. Let $\widetilde{\mathcal{E}}_{\tau\otimes\widetilde{\sigma}}$ denote the automorphic representation of $\mathbf{H}(\mathbb{A})$ generated by the iterated residues $\mathrm{Res}_{s=\frac{1}{2}} \widetilde{E}(s, \widetilde{h}, \Phi_{\tau\otimes\widetilde{\sigma}})$ for all

$$\Phi_{\tau\otimes\widetilde{\sigma}} \in \mathcal{A}(\widetilde{M}(F)U(\mathbb{A})\backslash\mathbf{H}(\mathbb{A}))_{\mu_\psi(\tau\otimes\widetilde{\sigma})}\,.$$

It is square integrable by the L^2-criterion in [56]. We mention that the residual representation $\widetilde{\mathcal{E}}_{\tau\otimes\widetilde{\sigma}}$ depends on the additive character ψ, and we do not include ψ in the notation for simplicity.

Taking $\widetilde{\Pi} = \widetilde{\mathcal{E}}_{\tau\otimes\widetilde{\sigma}}$ in the above descent construction (2.9), we have a family of automorphic $G_{m-\ell}(\mathbb{A})$-modules:

$$\left\{\pi_{\ell,\beta} = \mathcal{D}^{\mathrm{FJ}}_{\psi_\ell,\beta}(\widetilde{\mathcal{E}}_{\tau\otimes\widetilde{\sigma}})\right\}_\ell\,. \tag{2.12}$$

It is expected that the tower $\{\pi_{\ell,\beta}\}_\ell$ satisfies certain *tower property* when the depth $0 < \ell < m+1$ varies, i.e., there exists $0 < \ell^* < m+1$ such that $\pi_{\ell^*,\beta} \neq 0$ for some $\beta \in F^\times$, and $\pi_{\ell,\beta} = 0$ for any $\ell^* < \ell < m+1$ and any $\beta \in F^\times$. In particular, at the *first occurrence index* $\ell^* = \ell^*(\widetilde{\mathcal{E}}_{\tau\otimes\widetilde{\sigma}})$, the $G_{m-\ell^*}(\mathbb{A})$-module

$$\pi_{\ell^*,\beta} = \mathcal{D}^{\mathrm{FJ}}_{\psi_{\ell^*},\beta}(\widetilde{\mathcal{E}}_{\tau\otimes\widetilde{\sigma}})$$

is expected to consist of *cuspidal* automorphic functions $\mathcal{FJ}^\varphi_{\psi_{\ell^*},\beta}(f)(\cdot)$. We call $\pi_{\ell^*,\beta}$ *the Fourier-Jacobi type automorphic descent of* τ, *twisted by* $\widetilde{\sigma}$, or, $\widetilde{\sigma}$*-twisted Fourier-Jacobi type automorphic descent of* τ, and call $G_{m-\ell^*}$ the *target group* of this descent construction.

The affect of $\beta \in F^\times$ in the module

$$\mathcal{D}^{\mathrm{FJ}}_{\psi_\ell,\beta}(\widetilde{\mathcal{E}}_{\tau\otimes\widetilde{\sigma}})$$

can also be interpreted in another way. As in [20, §6], for $\beta \in F^\times$, let

$$d(\beta) = \begin{pmatrix} \beta \cdot I_{m+1} & \\ & I_{m+1} \end{pmatrix} \in \mathrm{GSp}_{2m+2}(F)$$

be a diagonal similitude element. The conjugation $h^\beta = d(\beta) \cdot h \cdot d(\beta)^{-1} \in \mathrm{Sp}_{2m+2}(F)$ gives an outer automorphism of $\mathrm{Sp}_{2m+2}(\mathbb{A})$, which has a unique lift to $\mathrm{Mp}_{2m+2}(\mathbb{A})$ (see [58, Page 36]). We also denote the lift by $\widetilde{h} \longmapsto \widetilde{h}^\beta$. Define the twist by $d(\beta)$ representation $\widetilde{\Pi}^\beta$ of $\widetilde{\Pi}$ by

$$\widetilde{\Pi}^\beta(\widetilde{h}) = \widetilde{\Pi}\left(\widetilde{h}^\beta\right)\,.$$

Then one has

$$\widetilde{\mathcal{E}}^\beta_{\tau\otimes\widetilde{\sigma}} = \widetilde{\mathcal{E}}_{\tau\otimes\widetilde{\sigma}^\beta}\,,$$

where $\widetilde{\sigma}^\beta$ is a similar twist of $\widetilde{\sigma}$ by $\begin{pmatrix} \beta & \\ & 1 \end{pmatrix}$. Note that (see [24, §3], [13, §8] and [76]) if the representation $\widetilde{\sigma}$ lies in the packet $\widetilde{\Pi}^\psi_{\phi_{\tau_0}}[\mathrm{Mp}_2(\mathbb{A})]$, then we have

$$\widetilde{\sigma}^\beta \in \widetilde{\Pi}^{\psi^\beta}_{\phi_{\tau_0}}[\mathrm{Mp}_2(\mathbb{A})] = \widetilde{\Pi}^\psi_{\phi_{\tau_0\otimes\eta_\beta}}[\mathrm{Mp}_2(\mathbb{A})],$$

and also the relation of L-functions

$$L_\psi(s,\widetilde{\sigma}^\beta)=L_{\psi^\beta}(s,\widetilde{\sigma})=L(s,\tau_0\otimes\eta_\beta).$$

In particular, if $\widetilde{\sigma}$ is ψ-generic, then $\widetilde{\sigma}^\beta$ is ψ^β-generic. Now by the construction of Fourier-Jacobi coefficients (2.8), after changing of variables, we have the following equivalence of $G_{m-\ell}(\mathbb{A})$-modules

$$\mathcal{D}^{\mathrm{FJ}}_{\psi_\ell,\beta}(\widetilde{\mathcal{E}}_{\tau\otimes\widetilde{\sigma}})\simeq\mathcal{D}^{\mathrm{FJ}}_{\psi_\ell}(\widetilde{\mathcal{E}}^{\beta}_{\tau\otimes\widetilde{\sigma}})\simeq\mathcal{D}^{\mathrm{FJ}}_{\psi_\ell}(\widetilde{\mathcal{E}}_{\tau\otimes\widetilde{\sigma}^\beta})\,. \tag{2.13}$$

In this article, we will consider both the cases $m=2n$ **(Case I)** and $m=2n+1$ **(Case II)**. Moreover, in **(Case II)** we will restrict ourselves to the pair $(\tau,\widetilde{\sigma})$ that satisfies Assumption 1.2. We will show that in both cases the first occurrence index $\ell^*=n$ (see §4–§5), and the target group is $G_n(\mathbb{A})=\mathrm{Sp}_{2n}(\mathbb{A})$ in **(Case I)**, and is $G_{n+1}(\mathbb{A})=\mathrm{Sp}_{2n+2}(\mathbb{A})$ in **(Case II)**, as stated in Theorem 1.3.

2.3. **Twisted automorphic descent and Fourier-Jacobi periods.** As indicated in [20] and [48], the twisted automorphic descent construction is naturally related to GGP periods (which are Fourier-Jacobi periods in our case).

For completeness, we introduce the Fourier-Jacobi periods of automorphic forms in symplectic-metaplectic cases. We set some more general notation at first. Let κ be a positive integer. As in §2.2, for the group $G_\kappa=\mathrm{Sp}_{2\kappa}$ with $0<r<\kappa$, we have a series of parabolic subgroups

$$\left\{P^{(\kappa)}_r=M^{(\kappa)}_rN^{(\kappa)}_r\right\}_{0<r<\kappa}$$

with $M^{(\kappa)}_r\simeq\mathrm{GL}_1^r\times G_{\kappa-r}$, and also a canonical character

$$\psi^{(\kappa)}_r:N^{(\kappa)}_r(F)\backslash N^{(\kappa)}_r(\mathbb{A})\longrightarrow\mathbb{C}^\times.$$

In §2.2, we have taken $\kappa=m+1$ and use the notation M_r and N_r without upper-scripts.

Recall that we have defined the Fourier-Jacobi coefficients of automorphic forms on $H_{m+1}(\mathbb{A})=\mathrm{Mp}_{2m+2}(\mathbb{A})$ in §2.2. When $\widetilde{f}$ is an automorphic form on $H_\kappa(\mathbb{A})=\mathrm{Mp}_{2\kappa}(\mathbb{A})$, the definition of its Fourier-Jacobi coefficients is the same as (2.8), and we will use notation $\mathcal{FJ}^{\varphi}_{\psi^{(\kappa)}_r,\alpha}(\widetilde{f})(\cdot)$ (here $\varphi\in\mathcal{S}(\mathbb{A}^{\kappa-r-1})$) to indicate the size of the group we are considering. It is an automorphic form on $G_{\kappa-r-1}(\mathbb{A})=\mathrm{Sp}_{2(\kappa-r-1)}(\mathbb{A})$. On the other hand, for an automorphic form f on $G_\kappa(F)\backslash G_\kappa(\mathbb{A})$ and $\alpha\in F^\times$, similarly one can define the Fourier-Jacobi coefficient

$$\mathcal{FJ}^{\varphi}_{\psi^{(\kappa)}_r,\alpha}(f)(\widetilde{h})=\int_{N^{(\kappa)}_{r+1}(F)\backslash N^{(\kappa)}_{r+1}(\mathbb{A})}f(uh)\psi^{(\kappa)}_r{}^{-1}(v)\theta^{\varphi}_{\psi^{-\alpha}}(j^{(\kappa)}_r(u)\cdot\widetilde{h})\,\mathrm{d}u\,. \tag{2.14}$$

Here $j^{(\kappa)}_r$ denotes the isomorphism $N^{(\kappa)}_r\backslash N^{(\kappa)}_{r+1}\simeq\mathcal{H}_{2(\kappa-r)-1}$ similar to (2.6), $\varphi\in\mathcal{S}(\mathbb{A}^{\kappa-r-1})$ is the Schwartz function in the Schrödinger model of the global Weil representation $\omega^{(\kappa-r-1)}_{\psi^{-\alpha}}$ of $\mathcal{H}_{2(\kappa-r)-1}(\mathbb{A})\rtimes\mathrm{Mp}_{2(\kappa-r-1)}(\mathbb{A})$, $\theta^{\varphi}_{\psi^{-\alpha}}$ is the corresponding theta series, and h is the projection image of $\widetilde{h}$ in $\mathrm{Mp}_{2(\kappa-r-1)}(\mathbb{A})$. By construction, $\mathcal{FJ}^{\varphi}_{\psi^{(\kappa)}_r,\alpha}(f)(\cdot)$ is an automorphic function on $H_{\kappa-r-1}(\mathbb{A})=\mathrm{Mp}_{2(\kappa-r-1)}(\mathbb{A})$.

Let π be an irreducible automorphic representation of $G_\kappa(\mathbb{A})$, and $\widetilde{\sigma}$ be an irreducible automorphic representation of $H_t(\mathbb{A})$. For $\boldsymbol{\phi}_\pi\in\pi$ and $\boldsymbol{\phi}_{\widetilde{\sigma}}\in\widetilde{\sigma}$, we define the *Fourier-Jacobi period* to be the inner product

$$\widetilde{\mathcal{P}}^{\mathrm{FJ}}_{\psi,\alpha,\varphi_t}(\boldsymbol{\phi}_\pi,\boldsymbol{\phi}_{\widetilde{\sigma}})=\int_{G_t(F)\backslash G_t(\mathbb{A})}\mathcal{FJ}^{\varphi_t}_{\psi^{(\kappa)}_{\kappa-t-1},\alpha}(\boldsymbol{\phi}_\pi)(\widetilde{g})\boldsymbol{\phi}_{\widetilde{\sigma}}(\widetilde{g})\,\mathrm{d}\widetilde{g},\;\;(\varphi_t\in\mathcal{S}(\mathbb{A}^t)) \tag{2.15}$$

if $t < \kappa$; and

$$\mathcal{P}^{\mathrm{FJ}}_{\psi,\alpha,\varphi_\kappa}(\phi_{\widetilde{\sigma}}, \phi_\pi) = \int_{G_\kappa(F)\backslash G_\kappa(\mathbb{A})} \mathcal{FJ}^{\varphi_\kappa}_{\psi^{(t)}_{t-\kappa-1},\alpha}(\phi_{\widetilde{\sigma}})(g)\phi_\pi(g)\,\mathrm{d}g,\ \ (\varphi_\kappa \in \mathcal{S}(\mathbb{A}^\kappa)) \tag{2.16}$$

if $\kappa < t$. There will be no Fourier-Jacobi coefficient involved if $\kappa = t$. Note that in (2.15), same as before, $\widetilde{g}$ denotes any pre-image of $g \in G_t(\mathbb{A})$ in $H_t(\mathbb{A})$. The convergence of the above integrals can be guaranteed if one of the automorphic forms involved is cuspidal. We say that two automorphic representations π and $\widetilde{\sigma}$ *have a non-zero Fourier-Jacobi period* if (2.15) is non-zero for some choice of data. If $\alpha = 1$, we also denote (2.15) and (2.16) by $\widetilde{\mathcal{P}}^{\mathrm{FJ}}_{\psi,\varphi_t}(\phi_\pi, \phi_{\widetilde{\sigma}})$ and $\mathcal{P}^{\mathrm{FJ}}_{\psi,\varphi_\kappa}(\phi_{\widetilde{\sigma}}, \phi_\pi)$, respectively.

In our case, let π be an irreducible cuspidal automorphic representation of $G_{m-\ell}(\mathbb{A})$ $(0 < \ell < m)$. For $\phi_\pi \in \pi$, we consider the Fourier-Jacobi period

$$\mathcal{P}^{\mathrm{FJ}}_{\psi,\alpha,\varphi_{m-\ell}}(\widetilde{E}(s,\cdot,\Phi_{\tau\otimes\widetilde{\sigma}}), \phi_\pi) = \int_{G_{m-\ell}(F)\backslash G_{m-\ell}(\mathbb{A})} \mathcal{FJ}^{\varphi_{m-\ell}}_{\psi^{(m+1)}_{\ell},\alpha}(\widetilde{E}(s,\cdot,\Phi_{\tau\otimes\widetilde{\sigma}}))(g)\phi_\pi(g)\,\mathrm{d}g, \tag{2.17}$$

where $\widetilde{E}(s,\cdot,\Phi_{\tau\otimes\widetilde{\sigma}})$ is the Eisenstein series on $\mathbf{H}(\mathbb{A})$ defined in (2.11). Unfolding the Eisenstein series as in [20] and [23], one can see that the Fourier-Jacobi period

$$\widetilde{\mathcal{P}}^{\mathrm{FJ}}_{\psi,\alpha,\varphi_1}(\phi_\pi, \phi_{\widetilde{\sigma}}) = \int_{G_1(F)\backslash G_1(\mathbb{A})} \mathcal{FJ}^{\varphi_1}_{\psi^{(m-\ell)}_{m-\ell-2},\alpha}(\phi_\pi)(\widetilde{g})\phi_{\widetilde{\sigma}}(\widetilde{g})\,\mathrm{d}\widetilde{g}\,.$$

occurs as an inner integral of (2.17), where $\phi_{\widetilde{\sigma}}$ is an automorphic form on $H_1(\mathbb{A})$ which comes from the section $\Phi_{\tau\otimes\widetilde{\sigma}}$, and $\varphi_1 \in \mathcal{S}(\mathbb{A})$ is a Schwartz function which comes from $\varphi_{m-\ell}$ in (2.17) (see, for example, [20, Proposition 6.6]). Moreover, if we let $s = 1/2$ and take residues of the Eisenstein series, one can go further to get a reciprocity formula (see [48, Theorem 3.2])

$$\mathcal{P}^{\mathrm{FJ}}_{\psi,\alpha,\varphi_{m-\ell}}(\mathrm{Res}_{s=\frac{1}{2}}\widetilde{E}(s,\cdot,\Phi_{\tau\otimes\widetilde{\sigma}}), \phi_\pi) = C_{\tau,\widetilde{\sigma},\pi} \cdot \widetilde{\mathcal{P}}^{\mathrm{FJ}}_{\psi,\alpha,\varphi_1}(\phi_\pi, \phi_{\widetilde{\sigma}}) \tag{2.18}$$

relating the two Fourier-Jacobi periods, where $C_{\tau,\widetilde{\sigma},\pi}$ is a constant determined by certain local integrals and residues of certain L-functions. In particular, one obtains:

Proposition 2.1. *Suppose that $\pi_{\ell,\beta} = \mathcal{D}^{\mathrm{FJ}}_{\psi_\ell,\beta}(\widetilde{\mathcal{E}}_{\tau\otimes\widetilde{\sigma}})$ is non-vanishing and cuspidal, and let π be any of its irreducible summand. Then $(\pi, \widetilde{\sigma})$ has a non-zero Fourier-Jacobi period.*

Proof. The proof is a direct application of (2.18). Let π be any irreducible summand of $\pi_{\ell,\beta}$, which is assumed to be non-vanishing and cuspidal. By construction, there exists $\phi_\pi \in \pi$, $\Phi_{\tau\otimes\widetilde{\sigma}} \in \mathcal{A}(\widetilde{M}(F)U(\mathbb{A})\backslash\mathbf{H}(\mathbb{A}))_{\mu_\psi(\tau\otimes\widetilde{\sigma})}$, and $\varphi_{m-\ell} \in \mathcal{S}(\mathbb{A}^{m-\ell})$, such that the Fourier-Jacobi period

$$\mathcal{P}^{\mathrm{FJ}}_{\psi,\beta,\varphi_{m-\ell}}(\mathrm{Res}_{s=\frac{1}{2}}\widetilde{E}(s,\cdot,\Phi_{\tau\otimes\widetilde{\sigma}}), \phi_\pi) \neq 0.$$

Then the right hand side of (2.18) must also be non-zero, which implies that

$$\widetilde{\mathcal{P}}^{\mathrm{FJ}}_{\psi,\beta,\varphi_1}(\phi_\pi, \phi_{\widetilde{\sigma}}) \neq 0$$

for some choice of data, as desired. □

3. Local aspects of the descent

In this section, we study the local counterparts of the automorphic descent discussed in §2.2, which are certain twisted Jacquet modules. We will calculate these Jacquet modules at unramified places. These local results will be used in the study of the global properties of the automorphic descents.

3.1. **The twisted Jacquet modules.** In this section, we let $\mathbf{k}$ be a p-adic field of characteristic 0, and fix a non-trivial additive character $\psi : \mathbf{k} \longrightarrow \mathbb{C}^\times$. For $1 \leq \ell \leq m$, we define the character ψ_ℓ on $N_\ell(\mathbf{k})$ similar to (2.5) in the global setting.

We describe the twisted Jacquet modules we are considering. Let $\alpha \in \mathbf{k}^\times$. For an irreducible admissible genuine representation $\widetilde{\Pi}$ of $\mathbf{H}(\mathbf{k})$, one defines the *Jacquet module of Fourier-Jacobi type* (of depth ℓ) to be

$$\mathcal{D}^{\mathrm{FJ,loc}}_{\psi_\ell,\alpha}(\widetilde{\Pi}) := J_{N_{\ell+1},\psi_\ell}\left(\widetilde{\Pi} \otimes \omega^{(m-\ell)}_{\psi^{-\alpha}}\right), \tag{3.1}$$

the twisted Jacquet module of $\widetilde{\Pi} \otimes \omega^{(m-\ell)}_{\psi^{-\alpha}}$ with respect to the unipotent subgroup $N_{\ell+1}(\mathbf{k})$ and the character ψ_ℓ (which is extended trivially to $N_{\ell+1}(\mathbf{k})$). Here $\omega^{(m-\ell)}_{\psi^{-\alpha}}$ is the local Weil representation of $\mathcal{H}_{V^{(\ell+1)}(\mathbf{k})} \rtimes \mathrm{Mp}(V^{(\ell+1)})(\mathbf{k})$ corresponding to the additive character $\psi^{-\alpha}$ with $\alpha \in \mathbf{k}^\times$ (see §2.1 and [33, §1.2]), and $N_{\ell+1}(\mathbf{k})$ acts on the local Weil representation $\omega^{(m-\ell)}_{\psi^{-\alpha}}$ via the isomorphism $j_\ell : N_\ell(\mathbf{k})\backslash N_{\ell+1}(\mathbf{k}) \simeq \mathcal{H}_{V^{(\ell+1)}}(\mathbf{k})$. It is clear that $\mathcal{D}^{\mathrm{FJ,loc}}_{\psi_\ell,\alpha}(\widetilde{\Pi})$ is a $G_{m-\ell}(\mathbf{k})$-module.

One may also write the twisted Jacquet module $J_{N_{\ell+1},\psi_\ell}(\widetilde{\Pi} \otimes \omega^{(m-\ell)}_{\psi^{-\alpha}})$ in another form (see [33, Chapter 6]). Recall that we have defined the subgroup (see §2.2)

$$C_\ell = \{c_\ell(t) := {j_\ell}^{-1}((0;t)) \mid t \in \mathbf{k}\} \subset N_{\ell+1},$$

i.e. the pre-image of the center of the Heisenberg group $\mathcal{H}_{V^{(\ell+1)}}$ under j_ℓ. We denote $N^0_{\ell+1} = N_\ell \cdot C_\ell$. We define a character $\psi^0_{\ell,\alpha} : N^0_{\ell+1}(\mathbf{k}) \longrightarrow \mathbb{C}^\times$ by

$$\psi^0_{\ell,\alpha}(u_\ell \cdot c_\ell(t)) = \psi_\ell(u_\ell) \cdot \psi(\alpha t)\,. \tag{3.2}$$

Note that $N^0_{\ell+1}$ is the unipotent subgroup $V_2(\mathcal{O}_\ell)$ corresponding to the nilpotent orbit $\mathcal{O}_\ell$ in the Lie algebra of $\mathbf{H}$ attached to the symplectic partition $[(2\ell+2)1^{2m-2\ell}]$ in the sense of [18]. Then we have

$$J_{N_{\ell+1},\psi_\ell}\left(\widetilde{\Pi} \otimes \omega^{(m-\ell)}_{\psi^{-\alpha}}\right) \simeq J_{\mathcal{H}_{V^{(\ell+1)}}/C_\ell}\left(J_{N^0_{\ell+1},\psi^0_{\ell,\alpha}}(\widetilde{\Pi}) \otimes \omega^{(m-\ell)}_{\psi^{-\alpha}}\right). \tag{3.3}$$

We note that $c_\ell(t) \in C_\ell(\mathbf{k})$ acts through $\omega^{(m-\ell)}_{\psi^{-\alpha}}$ by $\psi(-\alpha t)$.

As in [42] and [41], we will calculate the above twisted Jacquet module using formulae in [33] in unramified situation. Generally, we will consider the parabolically induced representation

$$\mathrm{Ind}^{\mathbf{H}(\mathbf{k})}_{\widetilde{P}(\mathbf{k})}\mu_\psi(\tau|\cdot|^{1/2} \otimes \widetilde{\sigma})\,,$$

where τ is an irreducible generic unramified unitary representation of $\mathrm{GL}_m(\mathbf{k})$ of *orthogonal type*, $\widetilde{\sigma}$ is an irreducible generic unramified unitary genuine representation of $\mathrm{Mp}_2(\mathbf{k})$, and ψ is also unramified. Our restriction on τ will give some symmetry to its unramified data. Here, for $a \in \mathrm{GL}_m(\mathbf{k})$ and $g \in \mathrm{SL}_2(\mathbf{k})$,

$$\mu_\psi(\tau \otimes \widetilde{\sigma})\left(\begin{pmatrix} a & & \\ & g & \\ & & a^* \end{pmatrix}, \epsilon\right) = \epsilon\,(\det(a), x(g))_{\mathbf{k}}\,\gamma_\psi(\det(a))\tau(a) \otimes \widetilde{\sigma}(g,\epsilon)\,, \tag{3.4}$$

where $(\ ,\ )_{\mathbf{k}}$ is the Hilbert symbol over $\mathbf{k}$, $x(\cdot) \in \mathbf{k}^\times/(\mathbf{k}^\times)^2$ is the x-function (on $\mathrm{SL}_2(\mathbf{k})$), which comes from the definition of the Rao cocycle on $\mathrm{Mp}_2(\mathbf{k})$ (see [65]), and γ_ψ is the Weil index associated with ψ.

We will fix

$$\widetilde{\sigma} = \mathrm{Ind}^{\mathrm{Mp}_2(\mathbf{k})}_{\widetilde{B}_{\mathrm{SL}_2}(\mathbf{k})}\mu_\psi\xi$$

in our calculations, where ξ is an unramified character of $\mathbf{k}^\times$, and μ_ψ is defined similarly to (3.4). In the remaining parts of this section, we also keep some other notation which are used in [33, Chapter 6].

3.2. **The local unramified calculation of Jacquet modules: $m = 2n$.** We first consider the case that $m = 2n$. We treat the cases that the central character ω_τ is trivial or non-trivial separately.

(1) The trivial central character case. We can write τ as a fully induced representation from the Borel subgroup:

$$\tau = \mathrm{Ind}_{B_{\mathrm{GL}_{2n}}(\mathbf{k})}^{\mathrm{GL}_{2n}(\mathbf{k})} \mu_1 \otimes \cdots \otimes \mu_n \otimes \mu_n^{-1} \otimes \cdots \otimes \mu_1^{-1} =: \mu_1 \times \cdots \times \mu_n \times \mu_n^{-1} \times \cdots \times \mu_1^{-1}, \tag{3.5}$$

where μ_i's are unramified characters of $\mathbf{k}^\times$. Let $\widetilde{\pi}_{\tau\otimes\widetilde{\sigma}}$ be the unramified constituent of the induced representation $\mathrm{Ind}_{\widetilde{P}(\mathbf{k})}^{\mathbf{H}(\mathbf{k})} \mu_\psi(\tau|\cdot|^{1/2} \otimes \widetilde{\sigma})$, and we want to study the unramified constituents of the twisted Jacquet module

$$J_{N_{\ell+1},\psi_\ell}\left(\widetilde{\pi}_{\tau\otimes\widetilde{\sigma}} \otimes \omega_{\psi^{-\alpha}}^{(2n-\ell)}\right) \simeq J_{\mathcal{H}_{V(\ell+1)}/C_\ell}\left(J_{N^0_{\ell+1},\psi^0_{\ell,\alpha}}(\widetilde{\pi}_{\tau\otimes\widetilde{\sigma}}) \otimes \omega_{\psi^{-\alpha}}^{(2n-\ell)}\right).$$

Proposition 3.1. *Let τ be as in (3.5), then the twisted Jacquet module $J_{N_{\ell+1},\psi_\ell}(\widetilde{\pi}_{\tau\otimes\widetilde{\sigma}} \otimes \omega_{\psi^{-\alpha}}^{(2n-\ell)}) = 0$ for all $n+1 \le \ell \le 2n$. And when $\ell = n$, $J_{N_{\ell+1},\psi_\ell}(\widetilde{\pi}_{\tau\otimes\widetilde{\sigma}} \otimes \omega_{\psi^{-\alpha}}^{(2n-\ell)})$ has a unique unramified constituent which is a subquotient of*

$$\mathrm{Ind}_{B_{\mathrm{Sp}_{2n}}(\mathbf{k})}^{\mathrm{Sp}_{2n}(\mathbf{k})} \eta_\alpha(\mu_1 \otimes \cdots \otimes \mu_n). \tag{3.6}$$

Proof. We apply [33, Theorem 6.1] to calculate $J_{N^0_{\ell+1},\psi^0_{\ell,\alpha}}(\widetilde{\pi}_{\tau\otimes\widetilde{\sigma}})$ first, and we also follow the same notation there. Note that the result in [33, Theorem 6.1] also holds with slight modification for the character $\psi^0_{\ell,\alpha}$ (which can be checked directly from the proofs of [33, Proposition 6.1, Proposition 6.2 and Proposition 6.3]), whereas [33, Theorem 6.1] is stated for $\alpha = 1$.

By conjugation of certain Weyl element, it suffices to consider the unramified constituent of the induced representation $\mathrm{Ind}_{\widetilde{P}(\mathbf{k})}^{\mathbf{H}(\mathbf{k})} \mu_\psi(\tau' \otimes \widetilde{\sigma})$ instead of $\widetilde{\pi}_{\tau\otimes\widetilde{\sigma}}$, here

$$\tau' = \mathrm{Ind}_{P_{2,\ldots,2}(\mathbf{k})}^{\mathrm{GL}_{2n}(\mathbf{k})} \mu_1(\det_{\mathrm{GL}_2}) \otimes \cdots \otimes \mu_n(\det_{\mathrm{GL}_2}).$$

Note that the derivative (see [6] and [33, §5] for definition) $\tau'^{(\ell)}$ of τ' vanishes for $\ell \ge n+1$, and $\tau'_{(\ell)}$ vanishes for $\ell \ge n$. Now applying [33, Theorem 6.1 (i)] with $j = 2n$ (and hence one has $t = \ell$ in [33, (6.9)]), one can see that the corresponding twisted Jacquet module $J_{N^0_{\ell+1},\psi^0_{\ell,\alpha}}(\mathrm{Ind}_{\widetilde{P}(\mathbf{k})}^{\mathbf{H}(\mathbf{k})} \mu_\psi(\tau' \otimes \widetilde{\sigma}))$ vanishes for all $n+1 \le \ell \le 2n$.

When $\ell = n$, by [33, (6.9)] we have

$$\begin{aligned} &J_{N^0_{n+1},\psi^0_{n,\alpha}}\left(\mathrm{Ind}_{\widetilde{P}(\mathbf{k})}^{\mathbf{H}(\mathbf{k})} \mu_\psi(\tau' \otimes \widetilde{\sigma})\right) \\ \simeq\ &\mathrm{ind}_{\widetilde{P}'_n(\mathbf{k})}^{\mathrm{Mp}(V^{(n+1)})(\mathbf{k}) \ltimes \mathcal{H}_{V(n+1)}(\mathbf{k})} \mu_\psi |\det(\cdot)|^{\frac{1-n}{2}} \tau'^{(n)} \otimes J_{N^0_1,\psi^0_{0,\alpha}}(\widetilde{\sigma}), \end{aligned}$$

here P'_n is the same as being defined in [33, (6.4)]. Note that $J_{N^0_1,\psi^0_{0,\alpha}}(\widetilde{\sigma}) \simeq \mathbf{1}$ since $\widetilde{\sigma}$ is generic, then we have

$$J_{N^0_{n+1},\psi^0_{n,\alpha}}\left(\mathrm{Ind}_{\widetilde{P}(\mathbf{k})}^{\mathbf{H}(\mathbf{k})} \mu_\psi(\tau' \otimes \widetilde{\sigma})\right) \simeq \mathrm{ind}_{\widetilde{P}'_n(\mathbf{k})}^{\mathrm{Mp}(V^{(n+1)})(\mathbf{k}) \ltimes \mathcal{H}_{V(n+1)}(\mathbf{k})} \mu_\psi |\det(\cdot)|^{\frac{1-n}{2}} \tau'^{(n)}.$$

To get (3.6), we will use [33, Proposition 6.6] in our situation. By [33, (6.11)] and also the calculations in [33, Page 130], we see that

$$\begin{aligned}&J_{\mathcal{H}_{V^{(\ell+1)}}/C_\ell}\left(\mathrm{ind}_{\widetilde{P}'_n(\mathbf{k})}^{\mathrm{Mp}(V^{(n+1)})(\mathbf{k})\ltimes\mathcal{H}_{V^{(n+1)}}(\mathbf{k})}\mu_\psi|\det(\cdot)|^{\frac{1-n}{2}}\tau'^{(n)}\otimes\omega_{\psi^{-\alpha}}^{(2n-\ell)}\right)\\ &\simeq \mathrm{Ind}_{B_{\mathrm{Sp}_{2n}}(\mathbf{k})}^{\mathrm{Sp}_{2n}(\mathbf{k})}\eta_\alpha(\mu_1\otimes\cdots\otimes\mu_n)\,.\end{aligned}$$

Note that we have used the fact that $|\det(\cdot)|^{\frac{1-n}{2}}\tau'^{(n)}\simeq\mathrm{Ind}_{B_{\mathrm{GL}_n}(\mathbf{k})}^{\mathrm{GL}_n(\mathbf{k})}\mu_1\otimes\cdots\otimes\mu_n$ (see [33, (5.34)]), and the quadratic character η_α comes from the Weil representation $\omega_{\psi^{-\alpha}}^{(2n-\ell)}$. This finishes the proof of this proposition. ☐

(2) The non-trivial central character case. In this case, we let λ_0 be the unique non-trivial unramified quadratic character of $\mathbf{k}^\times$, and let

$$\tau=\mu_1\times\cdots\times\mu_{n-1}\times\mathbf{1}\times\lambda_0\times\mu_{n-1}^{-1}\times\cdots\times\mu_1^{-1}\,,\tag{3.7}$$

where μ_i's are unramified characters of $\mathbf{k}^\times$ (recall that τ is of orthogonal type). We assume that the residue characteristic of $\mathbf{k}$ is odd, this will not interfere the global results we want to obtain in §4 later. Following the notation in [33], we write $\lambda_0(x)=(x,\epsilon)_{\mathbf{k}}$ with $(\ ,\ )_{\mathbf{k}}$ being the Hilbert symbol over $\mathbf{k}$, and $\epsilon\in\mathbf{k}^\times$ being a non-square. Then we denote $\omega_{\psi^\epsilon}^{(1),+}$ the unramified piece of the Weil representation of $\mathrm{Mp}_2(\mathbf{k})$ with respect to ψ^ϵ, which is a subrepresentation of the induced representation $\mathrm{Ind}_{\widetilde{B}_{\mathrm{SL}_2}(\mathbf{k})}^{\mathrm{Mp}_2(\mathbf{k})}\mu_\psi(\lambda_0|\cdot|^{1/2})$.

Proposition 3.2. *Let τ be as in (3.7), and take $\ell=n$, then each unramified constituent of $J_{N_{n+1},\psi_n}(\widetilde{\pi}_{\tau\otimes\widetilde{\sigma}}\otimes\omega_{\psi^{-\alpha}}^{(n)})$ is a subquotient of*

$$\begin{aligned}&\mathrm{Ind}_{B_{\mathrm{Sp}_{2n}}(\mathbf{k})}^{\mathrm{Sp}_{2n}(\mathbf{k})}\eta_\alpha(\mu_1\otimes\cdots\otimes\mu_{n-1}\otimes\lambda_0)\\ \oplus\ &\delta_{[\epsilon],[\alpha]}\cdot\mathrm{Ind}_{B_{\mathrm{Sp}_{2n}}(\mathbf{k})}^{\mathrm{Sp}_{2n}(\mathbf{k})}\eta_\alpha(\mu_1\otimes\cdots\otimes\mu_{n-1}\otimes\xi)\\ \oplus\ &\delta_{[\epsilon],[\alpha]}\cdot\mathrm{Ind}_{B_{\mathrm{Sp}_{2n}}(\mathbf{k})}^{\mathrm{Sp}_{2n}(\mathbf{k})}\eta_\alpha(\mu_1\otimes\cdots\otimes\mu_{n-1}\otimes|\cdot|^{\frac12})\\ \oplus\ &\delta_{[\epsilon],[\alpha]}\cdot\left(\bigoplus_{i=1}^{n-1}\mathrm{Ind}_{P_{1,\cdots,1,2,1\cdots1}^{\mathrm{Sp}_{2n}}(\mathbf{k})}^{\mathrm{Sp}_{2n}(\mathbf{k})}\eta_\alpha(\mu_1\otimes\cdots\otimes\mu_{i-1}\otimes\mu_i(\det_{\mathrm{GL}_2})\otimes\mu_{i+1}\cdots\otimes\mu_{n-1})\right).\end{aligned}\tag{3.8}$$

Here $\delta_{[\epsilon],[\alpha]}=1$ if $[\epsilon]=[\alpha]\in\mathbf{k}^\times/(\mathbf{k}^\times)^2$, and $\delta_{[\epsilon],[\alpha]}=0$ otherwise.

Proof. By conjugation of certain Weyl element, and note that we are considering the unramified constituents, it suffices to consider the unramified constituent of the induced representation $\mathrm{Ind}_{\widetilde{Q}_{2n-2}(\mathbf{k})}^{\mathbf{H}(\mathbf{k})}\mu_\psi(\tau_1'\otimes\widetilde{\sigma}_1)$, here

$$\tau_1'=\mathrm{Ind}_{P_{2,\ldots,2}(\mathbf{k})}^{\mathrm{GL}_{2n-2}(\mathbf{k})}\mu_1(\det_{\mathrm{GL}_2})\otimes\cdots\otimes\mu_{n-1}(\det_{\mathrm{GL}_2})\,,$$

and $\widetilde{\sigma}_1=\mathrm{Ind}_{\widetilde{Q}_2^{\mathrm{Sp}_6}(\mathbf{k})}^{\mathrm{Mp}_6(\mathbf{k})}\mu_\psi(\tau_2\otimes\omega_{\psi^\epsilon}^{(1),+})$, with $\tau_2=\xi\times|\cdot|^{1/2}$ being an induced representation of $\mathrm{GL}_2(\mathbf{k})$. Note that $\tau_1'^{(\ell)}=0$ for $\ell\geq n$.

When $\ell = n$, applying [33, Theorem 6.1 (1)] with $j = 2n-2$ (hence $t = n, n-1, n-2$ in [33, (6.9)]), we have

$$\begin{aligned}
&J_{N^0_{n+1},\psi^0_{n,\alpha}}\left(\mathrm{Ind}^{\mathbf{H}(\mathbf{k})}_{\widetilde{Q}_{2n-2}(\mathbf{k})}\mu_\psi(\tau_1'\otimes\widetilde{\sigma}_1)\right)\\
\simeq\ &\mathrm{ind}^{\mathrm{Mp}(V^{(n+1)})(\mathbf{k})\ltimes\mathcal{H}_{V^{(n+1)}}(\mathbf{k})}_{\widetilde{P}'_{n-1}(\mathbf{k})}\mu_\psi|\det(\cdot)|^{\frac{2-n}{2}}\tau'^{(n-1)}\otimes J_{N_2'^0,\psi^0_{1,\alpha}}(\widetilde{\sigma}_1)\\
&\oplus\mathrm{ind}^{\mathrm{Mp}(V^{(n+1)})(\mathbf{k})\ltimes\mathcal{H}_{V^{(n+1)}}(\mathbf{k})}_{\widetilde{P}'_{n}(\mathbf{k})}\mu_\psi|\det(\cdot)|^{\frac{3-n}{2}}\tau'^{(n-2)}\otimes J_{N_3'^0,\psi^0_{2,\alpha}}(\widetilde{\sigma}_1)\,.
\end{aligned}\tag{3.9}$$

To calculate the first summand in (3.9), we apply [33, Theorem 6.1] again to calculate the Jacquet module $J_{N_2'^0,\psi^0_{1,\alpha}}(\widetilde{\sigma}_1)$, here $\ell = 1$ and $j = 2$ in [33, (6.9)] (hence $t = 1$ in [33, (6.9)]). It follows that

$$\begin{aligned}
J_{N_2'^0,\psi^0_{1,\alpha}}(\widetilde{\sigma}_1)\simeq\ &\mathrm{ind}^{\mathrm{Mp}(V_6^{(2)})(\mathbf{k})\ltimes\mathcal{H}_{V_6^{(2)}}(\mathbf{k})}_{\widetilde{P}'_1(\mathbf{k})}\mu_\psi\tau_2^{(1)}\otimes J_{N_1'^0,\psi^0_{0,\alpha}}\left(\omega^{(1),+}_{\psi^\epsilon}\right)\\
&\oplus J_{C_1,\psi^\alpha}\left(\mathrm{ind}^{\mathrm{Mp}(V_6^{(2)})(\mathbf{k})\ltimes\mathcal{H}_{V_6^{(2)}}(\mathbf{k})}_{\widetilde{P}''_1(\mathbf{k})}\omega^{(1),+}_{\psi^\epsilon}\right),
\end{aligned}$$

here V_6 is a symplectic space of dimension 6 (hence $V_6^{(2)}$ is a subspace of dimension 2), and P''_1 is the same as defined in [33, (6.8)]. Note also that $J_{N_1'^0,\psi^0_{0,\alpha}}(\omega^{(1),+}_{\psi^\epsilon}) = \mathbf{1}$ if $[\alpha] = [\epsilon] \in \mathbf{k}^\times/(\mathbf{k}^\times)^2$, and is zero otherwise. Then by [33, Proposition 6.6 and Proposition 6.7] one sees that each unramified constituent of $J_{N_{n+1},\psi_n}(\widetilde{\pi}_{\tau\otimes\widetilde{\sigma}}\otimes\omega^{(n)}_{\psi^{-\alpha}})$ is a subquotient of

$$\begin{aligned}
&\delta_{[\epsilon],[\alpha]}\cdot\mathrm{Ind}^{\mathrm{Sp}_{2n}(\mathbf{k})}_{B_{\mathrm{Sp}_{2n}}(\mathbf{k})}\eta_\alpha(\mu_1\otimes\cdots\otimes\mu_{n-1}\otimes\mathbf{1})\oplus\delta_{[\epsilon],[\alpha]}\cdot\mathrm{Ind}^{\mathrm{Sp}_{2n}(\mathbf{k})}_{B_{\mathrm{Sp}_{2n}}(\mathbf{k})}\eta_\alpha(\mu_1\otimes\cdots\otimes\mu_{n-1}\otimes\xi)\\
&\oplus\mathrm{Ind}^{\mathrm{Sp}_{2n}(\mathbf{k})}_{B_{\mathrm{Sp}_{2n}}(\mathbf{k})}\eta_\alpha\left(\mu_1\otimes\cdots\otimes\mu_{n-1}\otimes\left(\omega^{(1),+}_{\psi^\epsilon}\otimes\omega^{(1)}_{\psi^{-1}}\right)\right),
\end{aligned}$$

where $\omega^{(1)}_{\psi^{-1}}$ is the Weil representation of $\mathrm{Mp}_2(\mathbf{k})\ltimes\mathcal{H}_3(\mathbf{k})$ with respect to ψ^{-1}. To get the final result of this proposition, we compute the Jacquet module of $\omega^{(1),+}_{\psi^\epsilon}\otimes\omega^{(1)}_{\psi^{-1}}$ along the upper-triangular maximal unipotent subgroup of $\mathrm{SL}_2(\mathbf{k})$ as in the proof of [33, Theorem 6.4], which is $\lambda_0|\cdot|$.

To calculate the second summand in (3.9), we also apply [33, Theorem 6.1] to calculate the Jacquet module $J_{N_3'^0,\psi^0_{2,\alpha}}(\widetilde{\sigma}_1)$, here $\ell = 2$ and $j = 2$ in [33, (6.9)] (hence $t = 2$ in [33, (6.9)]). It follows that

$$J_{N_3'^0,\psi^0_{2,\alpha}}(\widetilde{\sigma}_1)\simeq J_{N_1'^0,\psi^0_{0,\alpha}}\left(\omega^{(1),+}_{\psi^\epsilon}\right).$$

Then as explained in the previous paragraph, we get the last part of (3.8).

This finishes the proof of the proposition. □

3.3. **The local unramified calculation of Jacquet modules: $m = 2n+1$.** Now we consider the case that $m = 2n+1$. In this case, we take

$$\tau = \mu_1\times\cdots\times\mu_n\times\lambda_0\times\mu_n^{-1}\times\cdots\times\mu_1^{-1}\,,\tag{3.10}$$

with μ_i's and λ_0 being unramified characters of $\mathbf{k}^\times$ and in addition λ_0 being quadratic. We also denote $\widetilde{\pi}_{\tau\otimes\widetilde{\sigma}}$ to be the unramified constituent of the induced representation $\mathrm{Ind}^{\mathbf{H}(\mathbf{k})}_{\widetilde{P}(\mathbf{k})}\mu_\psi(\tau|\cdot|^{1/2}\otimes\widetilde{\sigma})$. As before, we write $\lambda_0(x) = (x,\epsilon)_{\mathbf{k}}$ with $(\ ,\ )_{\mathbf{k}}$ being the Hilbert symbol over $\mathbf{k}$, and $\epsilon\in\mathbf{k}^\times$. Note that we have $\epsilon = 1$ if $\lambda_0 = \mathbf{1}$. We also assume that the residue characteristic of $\mathbf{k}$ is odd.

Proposition 3.3. *Let τ be as in (3.10), then the twisted Jacquet module*

$$J_{N_{\ell+1},\psi_\ell}(\widetilde{\pi}_{\tau\otimes\widetilde{\sigma}}\otimes\omega_{\psi^{-\alpha}}^{(2n+1-\ell)})=0$$

for all $n+2\le\ell\le 2n+1$. When $\ell=n+1$, each unramified constituent of $J_{N_{n+2},\psi_{n+1}}(\widetilde{\pi}_{\tau\otimes\widetilde{\sigma}}\otimes\omega_{\psi^{-\alpha}}^{(n)})$ is a subquotient of

$$\delta_{[\epsilon],[\alpha]}\cdot\mathrm{Ind}_{B_{\mathrm{Sp}_{2n}}(\mathbf{k})}^{\mathrm{Sp}_{2n}(\mathbf{k})}\eta_\alpha(\mu_1\otimes\cdots\otimes\mu_n)\,. \tag{3.11}$$

And when $\ell=n$, each unramified constituent of $J_{N_{n+1},\psi_n}(\widetilde{\pi}_{\tau\otimes\widetilde{\sigma}}\otimes\omega_{\psi^{-\alpha}}^{(n+1)})$ is a subquotient of

$$\begin{aligned}&\mathrm{Ind}_{B_{\mathrm{Sp}_{2n+2}}(\mathbf{k})}^{\mathrm{Sp}_{2n+2}(\mathbf{k})}\eta_\alpha(\mu_1\otimes\cdots\otimes\mu_n\otimes\lambda_0)\\ \oplus\ &\delta_{[\epsilon],[\alpha]}\cdot\mathrm{Ind}_{B_{\mathrm{Sp}_{2n+2}}(\mathbf{k})}^{\mathrm{Sp}_{2n+2}(\mathbf{k})}\eta_\alpha(\mu_1\otimes\cdots\otimes\mu_n\otimes\xi)\\ \oplus\ &\delta_{[\epsilon],[\alpha]}\cdot\left(\bigoplus_{i=1}^{n}\mathrm{Ind}_{P_{1,\cdots,1,2,1\cdots1}^{\mathrm{Sp}_{2n+2}}(\mathbf{k})}^{\mathrm{Sp}_{2n+2}(\mathbf{k})}\eta_\alpha(\mu_1\otimes\cdots\otimes\mu_{i-1}\otimes\mu_i(\det_{\mathrm{GL}_2})\otimes\mu_{i+1}\cdots\otimes\mu_n)\right)\,.\end{aligned}\tag{3.12}$$

Here $\delta_{[\epsilon],[\alpha]}$ is the same as in Proposition 3.2.

Proof. By conjugation of certain Weyl element, it suffices to consider the unramified constituent of the induced representation $\mathrm{Ind}_{\widetilde{Q}_{2n}(\mathbf{k})}^{\mathbf{H}(\mathbf{k})}\mu_\psi(\tau_1'\otimes\widetilde{\sigma}_1')$, here

$$\tau_1'=\mathrm{Ind}_{P_{2,\dots,2}(\mathbf{k})}^{\mathrm{GL}_{2n}(\mathbf{k})}\mu_1(\det_{\mathrm{GL}_2})\otimes\cdots\otimes\mu_n(\det_{\mathrm{GL}_2})\,,$$

and

$$\widetilde{\sigma}_1'=\mathrm{Ind}_{\widetilde{Q}_1^{\mathrm{Sp}_4}(\mathbf{k})}^{\mathrm{Mp}_4(\mathbf{k})}\mu_\psi(\xi\otimes\omega_{\psi^\epsilon}^{(1),+})\,.$$

Applying [33, Theorem 6.1 (1)] with $j=2n$ and $m=2n+2$ (hence $t=\ell,\ell-1$ in [33, (6.9)]), we see that if $n+2\le\ell\le 2n$, this twisted Jacquet module is always zero.

When $\ell=n+1$, again by [33, (6.9)] we have (here only $t=n$ gives possible non-zero contribution)

$$\begin{aligned}&J_{N^0_{n+2},\psi^0_{n+1,\alpha}}\left(\mathrm{Ind}_{\widetilde{Q}_{2n}(\mathbf{k})}^{\mathbf{H}(\mathbf{k})}\mu_\psi(\tau_1'\otimes\widetilde{\sigma}_1')\right)\\ \simeq\ &\mathrm{ind}_{\widetilde{P}'_{n+1}(\mathbf{k})}^{\mathrm{Mp}(V^{(n+2)})(\mathbf{k})\ltimes\mathcal{H}_{V^{(n+2)}}(\mathbf{k})}\mu_\psi|\det(\cdot)|^{\frac{1-n}{2}}\tau_1'^{(n)}\otimes J_{N_2'^0,\psi^0_{1,\alpha}}(\widetilde{\sigma}_1')\,.\end{aligned}$$

We apply [33, Theorem 6.1] to calculate $J_{N_2'^0,\psi^0_{1,\alpha}}(\widetilde{\sigma}_1')$, here $\ell=1$ and $j=1$ in [33, (6.9)] (hence $t=1$ in [33, (6.9)]). It follows that

$$J_{N_2'^0,\psi^0_{1,\alpha}}(\widetilde{\sigma}_1')\simeq J_{N_1'^0,\psi^0_{0,\alpha}}\left(\omega_{\psi^\epsilon}^{(1),+}\right).$$

Note that $J_{N_1'^0,\psi^0_{0,\alpha}}(\omega_{\psi^\epsilon}^{(1),+})=\mathbf{1}$ if $[\alpha]=[\epsilon]\in\mathbf{k}^\times/(\mathbf{k}^\times)^2$, and is zero otherwise. This proves the first part of the proposition.

Suppose that $\ell=n$. By [33, (6.9)] (with $j=2n$ and $m=2n+2$) we have

$$\begin{aligned}&J_{N^0_{n+1},\psi^0_{n,\alpha}}\left(\mathrm{Ind}_{\widetilde{Q}_{2n}(\mathbf{k})}^{\mathbf{H}(\mathbf{k})}\mu_\psi(\tau_1'\otimes\widetilde{\sigma}_1')\right)\\ \simeq\ &\mathrm{ind}_{\widetilde{P}'_{n}(\mathbf{k})}^{\mathrm{Mp}(V^{(n+1)})(\mathbf{k})\ltimes\mathcal{H}_{V^{(n+1)}}(\mathbf{k})}\mu_\psi|\det(\cdot)|^{\frac{1-n}{2}}\tau_1'^{(n)}\otimes J_{N_1'^0,\psi^0_{0,\alpha}}(\widetilde{\sigma}_1')\\ &\oplus\mathrm{ind}_{\widetilde{P}'_{n+1}(\mathbf{k})}^{\mathrm{Mp}(V^{(n+1)})(\mathbf{k})\ltimes\mathcal{H}_{V^{(n+1)}}(\mathbf{k})}\mu_\psi|\det(\cdot)|^{\frac{2-n}{2}}\tau_1'^{(n-1)}\otimes J_{N_2'^0,\psi^0_{1,\alpha}}(\widetilde{\sigma}_1').\end{aligned}\tag{3.13}$$

To calculate the first summand in (3.13), we apply [33, Theorem 6.1] again to calculate the Jacquet module $J_{N_1'^0,\psi^0_{0,\alpha}}(\widetilde{\sigma}_1')$, here $\ell=0$ and $j=1$ in [33, (6.9)] (hence $t=0$ in [33, (6.9)]). It follows that

$$J_{N_1'^0,\psi^0_{0,\alpha}}(\widetilde{\sigma}_1') \simeq \mathrm{ind}_{\widetilde{P}_1'(\mathbf{k})}^{\mathrm{Mp}(V_4^{(1)})(\mathbf{k})\ltimes\mathcal{H}_{V_4^{(1)}}(\mathbf{k})} \mu_\psi(\xi|\cdot|^{\frac{1}{2}}) \otimes J_{N_1'^0,\psi^0_{0,\alpha}}\left(\omega_{\psi^\epsilon}^{(1),+}\right)$$
$$\oplus\, J_{C_0,\psi^\alpha}\left(\mathrm{ind}_{\widetilde{P}_1''(\mathbf{k})}^{\mathrm{Mp}(V_4^{(1)})(\mathbf{k})\ltimes\mathcal{H}_{V_4^{(1)}}(\mathbf{k})}\omega_{\psi^\epsilon}^{(1),+}\right),$$

here V_4 is a symplectic space of dimension 4 (hence $V_4^{(1)}$ is a subspace of dimension 2). Then one gets the first two summands in (3.12) by arguments similar to those in the proof of Proposition 3.2.

To calculate the second summand in (3.13), we use the calculation of $J_{N_2'^0,\psi^0_{1,\alpha}}(\widetilde{\sigma}_1')$ above again. Then one can also gets the third summand in (3.12) by similar arguments to those in the proof of Proposition 3.2. $\square$

Remark 3.4. *In this section, we calculate the Jacquet modules corresponding to the automorphic descent module $\mathcal{D}^{\mathrm{FJ}}_{\psi_\ell,\alpha}(\widetilde{\mathcal{E}}_{\tau\otimes\widetilde{\sigma}})$. Note that we have the equivalence (2.13), and if we consider the automorphic descent $\mathcal{D}^{\mathrm{FJ}}_{\psi_\ell}(\widetilde{\mathcal{E}}_{\tau\otimes\widetilde{\sigma}^\alpha})$, the corresponding Jacquet modules at unramified places will be*

$$J_{N_{\ell+1},\psi_{\ell,v}}\left(\widetilde{\pi}_{\tau_v\otimes\widetilde{\sigma}_v^\alpha}\otimes\omega_{\psi_v^{-1}}^{(2n-\ell)}\right),$$

where

$$\widetilde{\sigma}_v^\alpha = \mathrm{Ind}_{\widetilde{B}_{\mathrm{SL}_2}(F_v)}^{\mathrm{Mp}_2(F_v)}\mu_{\psi_v^\alpha}\xi,$$

for some unramified character $\xi : F_v^\times \longrightarrow \mathbb{C}^\times$. Then we will get the same results by similar calculations, and the twist by the quadratic character $\eta_{\alpha,v}$ in the above results will come from the genuine character $\mu_{\psi_v^\alpha}$.

4. Cuspidality and parameters

In this section, we return to the global setting and study the basic properties of the descent tower

$$\left\{\pi_{\ell,\beta} = \mathcal{D}^{\mathrm{FJ}}_{\psi_\ell,\beta}(\widetilde{\mathcal{E}}_{\tau\otimes\widetilde{\sigma}})\right\}_\ell \tag{4.1}$$

introduced in §2, using the local results obtained in §3. Recall that we have taken an isobaric sum automorphic representation

$$\tau = \tau_1 \boxplus \tau_2 \boxplus \cdots \boxplus \tau_t \tag{4.2}$$

of $\mathrm{GL}_m(\mathbb{A})$, where τ_i's are distinct irreducible unitary cuspidal automorphic representations of $\mathrm{GL}_{n_i}(\mathbb{A})$ of orthogonal type, and $\sum_{i=1}^t n_i = m$.

The following proposition is known as the *tower property*, which suggests the possible first occurrence in the descent tower (4.1).

Proposition 4.1. *The representation $\pi_{\ell,\beta} = \mathcal{D}^{\mathrm{FJ}}_{\psi_\ell,\beta}(\widetilde{\mathcal{E}}_{\tau\otimes\widetilde{\sigma}})$ in the descent tower vanishes identically for any $n+1\le\ell\le m$, here $m=2n$ in* **(Case I)**, *$m=2n+1$ in* **(Case II)**, *and we presume Assumption 1.2 in* **(Case II)**.

Proof. Write the residual representation as $\widetilde{\mathcal{E}}_{\tau\otimes\widetilde{\sigma}} = \otimes'_v \Pi_v$. Let v be a finite place such that τ_v, σ_v, Π_v and ψ_v are all unramified. In **(Case I)** we can require moreover that $\omega_{\tau_v} = \mathbf{1}$. By the conditions on τ, τ_v is of the form (3.5) in **(Case I)**, and is of the form (3.10) in **(Case II)**. Then if $m = 2n$, the result follows from Proposition 3.1.

If $m = 2n+1$, then we are in **(Case II)**, and Proposition 3.3 implies that $\pi_{\ell,\beta} = \mathcal{D}^{\mathrm{FJ}}_{\psi_{\ell,\beta}}(\widetilde{\mathcal{E}}_{\tau\otimes\widetilde{\sigma}})$ in the descent tower vanishes identically for all $n+2 \le \ell \le 2n+1$. It remains to show that $\mathcal{D}^{\mathrm{FJ}}_{\psi_{n+1},\beta}(\widetilde{\mathcal{E}}_{\tau\otimes\widetilde{\sigma}}) = 0$ for all choice of data. Recall that we pressume Assumption 1.2 for **(Case II)**. Suppose that $\mathcal{D}^{\mathrm{FJ}}_{\psi_{n+1},\beta}(\widetilde{\mathcal{E}}_{\tau\otimes\widetilde{\sigma}}) \neq 0$ for some choice of data. Let π be an irreducible component of $\mathcal{D}^{\mathrm{FJ}}_{\psi_{n+1},\beta}(\widetilde{\mathcal{E}}_{\tau\otimes\widetilde{\sigma}})$. Then we have:

(1) the representation π is cuspidal, by Proposition 3.3 and a similar argument as in the proof of Proposition 4.2 below (using vanishing of $\pi_{\ell,\beta}$ for $n+2 \le \ell \le 2n+1$);
(2) the pair of representations $(\pi, \widetilde{\sigma})$ has a non-zero Fourier-Jacobi period $\widetilde{\mathcal{P}}^{\mathrm{FJ}}_{\psi,\beta,\varphi_1}(\boldsymbol{\phi}_\pi, \boldsymbol{\phi}_{\widetilde{\sigma}})$, by Proposition 2.1;
(3) the representation π lifts almost everywhere to $\tau_v \otimes \omega_{\tau,v}$, by (3.11) in Proposition 3.3 (see also the argument in the proof of Proposition 4.3 below).

Granting (1) – (3) above, we see that $(\pi, \widetilde{\sigma})$ is a GGP pair in the given global Vogan packet

$$\widetilde{\Pi}^{\psi^\beta}_{\phi_{\tau\otimes\omega_\tau}\times\phi_{\tau_0}}[G_n(\mathbb{A}) \times H_1(\mathbb{A})],$$

which contradicts to Assumption 1.2. Then we have $\mathcal{D}^{\mathrm{FJ}}_{\psi_{n+1},\beta}(\widetilde{\mathcal{E}}_{\tau\otimes\widetilde{\sigma}}) = 0$, and this finishes the proof of the proposition. □

Now we show the cuspidality of the possible first occurrence (at $\ell = n$) in the descent tower. The following proposition is a generalization of [33, Theorem 7.11].

Proposition 4.2. *Let $\tau = \tau_1 \boxplus \cdots \boxplus \tau_t$ be the same as (4.2) with $n_i > 1$, and we allow $n_i = 1$ if τ is an automorphic representation of $\mathrm{GL}_2(\mathbb{A})$. Then if the representation $\mathcal{D}^{\mathrm{FJ}}_{\psi_n,\beta}(\widetilde{\mathcal{E}}_{\tau\otimes\widetilde{\sigma}})$ is non-zero, it is cuspidal.*

Proof. Let $f \in \widetilde{\mathcal{E}}_{\tau\otimes\widetilde{\sigma}}$ and consider the Fourier-Jacobi coefficient $\mathcal{FJ}^{\varphi}_{\psi_n,\beta}(f)$, here $\varphi \in \mathcal{S}(\mathbb{A}^{m-n})$. For $1 \le p \le m-n$, let $U_p^{(m-n)}$ be the unipotent radical of the maximal parabolic subgroup $Q_p^{(m-n)} \subset G_{m-n} = \mathrm{Sp}_{2(m-n)}$ whose Levi subgroup is isomorphic to $\mathrm{GL}_p \times G_{m-n-p}$. We will show that

$$c_p(\mathcal{FJ}^{\varphi}_{\psi_n,\beta}(f)) = 0 \tag{4.3}$$

for all $1 \le p \le m-n$, where $c_p(\mathcal{FJ}^{\varphi}_{\psi_n,\beta}(f))$ is the constant term of $\mathcal{FJ}^{\varphi}_{\psi_n,\beta}(f)$ along $U_p^{(m-n)}$.

On the other hand, for $f \in \widetilde{\mathcal{E}}_{\tau\otimes\widetilde{\sigma}}$ and $1 \le q \le m+1$, let f^{U_q} be the constant term along U_q, here U_q is the unipotent radical of the parabolic subgroup $Q_q \subset \mathrm{Sp}_{2m+2}$ whose Levi subgroup is isomorphic to $\mathrm{GL}_q \times G_{m+1-q}$ (see §2.1). Then, to show (4.3), it suffices to show that

$$\mathcal{FJ}^{\varphi_1}_{\psi_{n+k},\beta}(f^{U_{p-k}}) = 0 \tag{4.4}$$

for all $f \in \widetilde{\mathcal{E}}_{\tau\otimes\widetilde{\sigma}}$, $\varphi_1 \in \mathcal{S}(\mathbb{A}^{m-n-p})$, $1 \le p \le m-n$, and $0 \le k \le p-1$. In fact, granting (4.4), [33, Theorem 7.9] implies that $c_p(\mathcal{FJ}^{\varphi}_{\psi_n,\beta}(f))$ can be expressed by the Fourier-Jacobi coefficients $\mathcal{FJ}^{\varphi}_{\psi_{n+p},\beta}(f)$ of depth $(n+p)$, which are zero by Proposition 4.1.

To show (4.4), we only need to consider the case that $f^{U_{p-k}}$ is not identically zero, which implies that $f^{U_{p-k}}\big|_{H_{m-(p-k+1)}(\mathbb{A})}$ is a residual of Eisenstein series supported on $\tau' \otimes \widetilde{\sigma}$, where τ' is of the same form as $\tau = \tau_1 \boxplus \cdots \boxplus \tau_t$, but has less summands (one may refer to the proof of [33, Theorem 7.6]). Moreover, by Proposition 3.1, Proposition 3.2, Proposition 3.3, and the assumption that $\tau = \tau_1 \boxplus \cdots \boxplus \tau_t$ with $n_i > 1$, we see that $(n+k)$ is large enough for the vanishing of the Fourier-Jacobi coefficient $\mathcal{FJ}^{\varphi_1}_{\psi_{n+k},\beta}(f^{U_{p-k}})$ with $f^{U_{p-k}}|_{H_{m-(p-k+1)}(\mathbb{A})}$ being supported on $\tau' \otimes \widetilde{\sigma}$ (see also the proof of Proposition 4.1 above). Then (4.4) holds for all $1 \le p \le m-n$ and $0 \le k \le p-1$, and we have done. □

In the end of this section, we study the global Arthur parameter of the possible first occurrence $\mathcal{D}^{\mathrm{FJ}}_{\psi_n,\beta}(\widetilde{\mathcal{E}}_{\tau\otimes\widetilde{\sigma}})$ in the descent tower.

Proposition 4.3. *Let π be an irreducible component of $\mathcal{D}^{\mathrm{FJ}}_{\psi_n,\beta}(\widetilde{\mathcal{E}}_{\tau\otimes\widetilde{\sigma}})$. We have the followings*

(1) *Consider in* **(Case I)**, *and assume that $\omega_\tau = \mathbf{1}$. Then π lifts almost everywhere to*

$$\mathrm{Ind}^{\mathrm{GL}_{2n+1}(F_v)}_{P_{n_1,\ldots,n_t}(F_v)} \eta_\beta(\tau_{1,v} \otimes \cdots \otimes \tau_{t,v}) \otimes \mathbf{1}\,.$$

(2) *Suppose that $\eta_\beta \ne \omega_\tau$ in* **(Case II)**, *and make no more assumptions in* **(Case I)**. *Then π has a generic global Arthur parameter.*

Proof. Looking at the places v of F where τ_v, $\widetilde{\sigma}_v$ and ψ_v are all unramified, then τ_v is of the form (3.5) or (3.7) in **(Case I)**, and of the form (3.10) in **(Case II)**.

We consider **(Case II)** at first. Since $\eta_\beta \ne \omega_\tau$, there exists infinitely many places v such that $\eta_{\beta,v} \ne \omega_{\tau,v}$. Then there exist infinitely many places v such that τ_v, $\widetilde{\sigma}_v$ and ψ_v are all unramified, and $\eta_{\beta,v} \ne \omega_{\tau,v}$. At these places, π_v is a subquotient of a representation fully induced from unramified characters, which is given by the first summand of (3.12) in Proposition 3.3. Then by Arthur's endoscopic classification theory (see [4]), the representation π must have a generic global Arthur parameter, otherwise π_v will have a non-generic local parameter at almost all places v.

Now we turn to **(Case I)**. Note that ω_τ is quadratic, hence is trivial at infinitely many places v. By (3.6) in Proposition 3.1 and (3.8) in Proposition 3.2, one sees that, π_v is a subquotient of a representation fully induced from unramified characters at the local places v such that τ_v, $\widetilde{\sigma}_v$ and ψ_v are all unramified, and $\omega_{\tau,v} = \mathbf{1}$. Again, π has a generic global Arthur parameter since there are infinitely many such places. Moreover, if $\omega_\tau = \mathbf{1}$, by (3.6) we see that π_v is a subquotient of

$$\mathrm{Ind}^{\mathrm{Sp}_{2n}(F_v)}_{B_{\mathrm{Sp}_{2n}}(F_v)} \eta_\beta(\mu_{1,v} \otimes \cdots \otimes \mu_{n,v})$$

with

$$\tau_v = \mu_{1,v} \times \cdots \times \mu_{n,v} \times \mu_{n,v}^{-1} \times \cdots \times \mu_{1,v}^{-1}$$

at almost all places v. This shows that for almost all v, π_v lifts to

$$\mathrm{Ind}^{\mathrm{GL}_{2n+1}(F_v)}_{P_{n_1,\ldots,n_t}(F_v)} \eta_\beta(\tau_{1,v} \otimes \cdots \otimes \tau_{t,v}) \otimes \mathbf{1}\,.$$

□

5. The non-vanishing of the descent construction

In this section, we show that the descent $\mathcal{D}^{\mathrm{FJ}}_{\psi_n,\beta}(\widetilde{\mathcal{E}}_{\tau\otimes\widetilde{\sigma}})$ is non-vanishing for some choice of data.

5.1. **Generalized and degenerate Whittaker-Fourier coefficients.** First, we recall the generalized and degenerate Whittaker-Fourier coefficients attached to nilpotent orbits, following the formulation in [34]. Let G be a reductive group defined over a number field F. Let $\mathfrak{g}$ be the Lie algebra of $\mathrm{G}(F)$, and u be a nilpotent element in $\mathfrak{g}$. Given any semisimple element $s \in \mathfrak{g}$, under the adjoint action, $\mathfrak{g}$ is decomposed into a direct sum of eigenspaces $\mathfrak{g}^s_i$ corresponding to eigenvalues i. The element s is called *rational semisimple* if all its eigenvalues are in $\mathbb{Q}$. Given a nilpotent element u and a semisimple element s in $\mathfrak{g}$, the pair (s,u) is called a *Whittaker pair* if s is a rational semisimple element, and $u \in \mathfrak{g}^s_{-2}$. The element s in a Whittaker pair (s,u) is called a *neutral element* for u if there is a nilpotent element $v \in \mathfrak{g}$ such that (v,s,u) is an $\mathfrak{sl}_2$-triple. A Whittaker pair (s,u) with s being a neutral element is called a *neutral pair*.

Given any Whittaker pair (s,u), define an anti-symmetric form ω_u on $\mathfrak{g}\times\mathfrak{g}$ by

$$\omega_u(X,Y) := \kappa(u,[X,Y])\,,$$

where κ is the Killing form on $\mathfrak{g}\times\mathfrak{g}$. For any rational number $r \in \mathbb{Q}$, let $\mathfrak{g}^s_{\geq r} = \oplus_{r'\geq r}\mathfrak{g}^s_{r'}$. Let $\mathfrak{u}_s = \mathfrak{g}^s_{\geq 1}$ and let $\mathfrak{n}_{s,u}$ be the radical of $\omega_u|_{\mathfrak{u}_s}$, then $[\mathfrak{u}_s,\mathfrak{u}_s] \subset \mathfrak{g}^s_{\geq 2} \subset \mathfrak{n}_{s,u}$. For any $X \in \mathfrak{g}$, let $\mathfrak{g}_X$ be the centralizer of X in $\mathfrak{g}$. By [34, Lemma 3.2.6], one has $\mathfrak{n}_{s,u} = \mathfrak{g}^s_{\geq 2} + \mathfrak{g}^s_1 \cap \mathfrak{g}_u$. Note that if the Whittaker pair (s,u) comes from an $\mathfrak{sl}_2$-triple (v,s,u), then $\mathfrak{n}_{s,u} = \mathfrak{g}^s_{\geq 2}$. Let $N_{s,u} = \exp(\mathfrak{n}_{s,u})$ be the corresponding unipotent subgroups of G, we define a character of $N_{s,u}$ by

$$\psi_u(n) = \psi(\kappa(u,\log(n))).$$

Here $\psi : F\backslash\mathbb{A} \to \mathbb{C}^\times$ is a fixed non-trivial additive character, and we extend the killing form κ to $\mathfrak{g}(\mathbb{A})\times\mathfrak{g}(\mathbb{A})$. Let $N'_{s,u} = N_{s,u}\cap\ker(\psi_u)$. Then $U_s/N'_{s,u}$ is a Heisenberg group with center $N_{s,u}/N'_{s,u}$, here $U_s = \exp(\mathfrak{u}_s)$.

Let π be an irreducible automorphic representation of $\mathrm{G}(\mathbb{A})$. For any $\boldsymbol{\phi} \in \pi$, the *degenerate Whittaker-Fourier coefficient* of $\boldsymbol{\phi}$ attached to a Whittaker pair (s,u) is defined to be

$$\mathcal{F}_{s,u}(\boldsymbol{\phi})(g) := \int_{[N_{s,u}]} \boldsymbol{\phi}(ng)\psi_u^{-1}(n)\,\mathrm{d}n\,. \tag{5.1}$$

If (s,u) is a neutral pair, then $\mathcal{F}_{s,u}(\boldsymbol{\phi})$ is also called a *generalized Whittaker-Fourier coefficient* of $\boldsymbol{\phi}$. Let

$$\mathcal{F}_{s,u}(\pi) = \{\mathcal{F}_{s,u}(\boldsymbol{\phi}) \mid \boldsymbol{\phi}\in\pi\}\ .$$

The *wave-front set* $\mathfrak{n}(\pi)$ of π is defined to be the set of nilpotent orbits $\mathcal{O}$ such that $\mathcal{F}_{s,u}(\pi)$ is non-zero for some neutral pair (s,u) with $u \in \mathcal{O}$. Note that if $\mathcal{F}_{s,u}(\pi)$ is non-zero for some neutral pair (s,u) with $u\in\mathcal{O}$, then it is non-zero for any such neutral pair (s,u), since the non-vanishing property of such Fourier coefficients does not depend on the choices of representatives of $\mathcal{O}$. Moreover, we let $\mathfrak{n}^m(\pi)$ be the set of maximal elements in $\mathfrak{n}(\pi)$ under the natural ordering of nilpotent orbits (i.e., the dominance ordering). We recall [34, Theorem C] as follows.

Proposition 5.1. *Let π be an automorphic representation of $\mathrm{G}(\mathbb{A})$. Given a neutral pair (s,u) and a Whittaker pair (s',u), if $\mathcal{F}_{s',u}(\pi)$ is non-zero, then $\mathcal{F}_{s,u}(\pi)$ is non-zero.*

When G is a quasi-split classical group, it is known that the nilpotent orbits are parametrized by pairs $(\underline{p},\underline{q})$, where $\underline{p}$ is a partition and $\underline{q}$ is a set of non-degenerate quadratic forms (see [77, Section I.6]). When $\mathrm{G}=\mathrm{Sp}_{2N}$, $\underline{p}$ is a symplectic partition, namely, odd parts occur with even multiplicities. Suppose that $\underline{p} = [p_1^{e_1}p_2^{e_2}\cdots p_r^{e_r}]$ is a symplectic partition with $p_1 > p_2 > \cdots > p_r$, and $\{p_{i_1}^{e_{i_1}}, p_{i_2}^{e_{i_2}}, \ldots, p_{i_t}^{e_{i_t}}\}$ are all its even parts. Then the pairs parametrizing the nilpotent orbits associated to $\underline{p}$ have the form $(\underline{p}, \underline{q})$, where

$$\underline{q} = \{q_{i_1}, q_{i_2}, \ldots, q_{i_t}\} \tag{5.2}$$

with each q_{i_j} being a non-degenerate quadratic form of dimension e_{i_j}. Given an automorphic representation π of $\mathrm{G}(\mathbb{A})$, the set of partitions corresponding to nilpotent orbits in $\mathfrak{n}^m(\pi)$ is denoted by $\mathfrak{p}^m(\pi)$, which is expected to be a singleton (see [19, §4]). In this section, for any symplectic partition $\underline{p}$, by a generalized Whittaker-Fourier coefficient of π attached to $\underline{p}$, we mean a generalized Whittaker-Fourier coefficient $\mathcal{F}_{s,u}(\boldsymbol{\phi})$ attached to a nilpotent orbit $\mathcal{O}$ parametrized by a pair $(\underline{p}, \underline{q})$ for some $\underline{q}$, with $\boldsymbol{\phi} \in \pi$, $u \in \mathcal{O}$ and (s, u) being a neutral pair.

For $\mathrm{G} = \mathrm{Sp}_{2N}$, a symplectic partition $\underline{p}$ is called *symplectic special* if it has an even number of even parts between two consecutive odd parts, and an even number of even parts greater than the largest odd part (see [11, Section 6.3]). For $\mathrm{H}(\mathbb{A}) = \mathrm{Mp}_{2N}(\mathbb{A})$, a symplectic partition $\underline{p}$ is called *metaplectic special* if it has an even number of even parts between two consecutive odd parts, and an odd number of even parts greater than the largest odd part (see [11, Section 6.3]). By the main results of [40], any $\underline{p} \in \mathfrak{p}^m(\pi)$ is special. This will play an important role in the proof of Proposition 5.9.

The non-vanishing of the generalized Whittaker-Fourier coefficients of automorphic forms is related to the non-vanishing of the Fourier-Jacobi coefficients we have defined in §2.2. Such relationship has been studied in [32] and [38]. For simplicity, we recall the result for representations of $\mathrm{Mp}_{2N}(\mathbb{A})$ that we are considering in this article, which can be deduced from [32, Lemma 2.6] (or [38, Lemma 3.1]) and [32, Lemma 1.1].

Lemma 5.2. *Let $\widetilde{\Pi}$ be an automorphic representation of $\mathrm{Mp}_{2N}(\mathbb{A})$. Suppose that $\boldsymbol{\phi} \in \widetilde{\Pi}$ has a non-zero generalized Whittaker-Fourier coefficient attached to the nilpotent orbit corresponding to a partition $[(2k)p_2 \cdots p_r](2k \geq p_2 \geq \cdots \geq p_r)$ and a set of quadratic forms $\underline{q} = \{\beta\} \cup \underline{q}'$ (see (5.2)), where*

- *the square class $[\beta] \in F^\times/(F^\times)^2$ determining a quadratic form of dimension one corresponding to the part $(2k)$ in the partition above;*
- *the set of quadratic forms $\underline{q}'$ is associated to the partition $[p_2 \cdots p_r]$.*

Then the Fourier-Jacobi coefficient (2.8) is non-zero for $\boldsymbol{\phi}$ at depth $\ell = k - 1$ with respect to $\beta \in F^\times$ and some $\varphi \in \mathcal{S}(\mathbb{A}^{N-k})$.

5.2. **Non-vanishing of the descent: Case I.** Now we come back to the global situation where the groups and representations are the same as in §2. First we prove the following proposition.

Proposition 5.3. *The residual representation $\widetilde{\mathcal{E}}_{\tau\otimes\widetilde{\sigma}}$ has a non-zero generalized Whittaker-Fourier coefficient attached to the partition $[(2n)^2 1^2]$.*

Proof. By [77, I.6] (see also the discussion after Proposition 5.1), the F-rational nilpotent orbits in each F-stable nilpotent orbit in the Lie algebra $\mathfrak{sp}_{4n+2}(F)$ attached to the partition $[(2n)^2 1^2]$ are parametrized by square classes (dimension one quadratic forms) $\{\alpha, \beta\}$. Let $\{\alpha_i = \mathbf{e}_i - \mathbf{e}_{i+1}, 1 \leq i \leq 2n, \alpha_{2n+1} = 2\mathbf{e}_{2n+1}\}$ be the set of simple roots of $\mathfrak{sp}_{4n+2}$. For any root γ for $\mathfrak{sp}_{4n+2}$, we denote by x_γ the one-dimensional root subgroup in $\mathfrak{sp}_{4n+2}$ corresponding to γ, and let $X_\gamma = \exp(x_\gamma)$. We consider one such F-rational nilpotent orbit $\mathcal{O}$ parametrized by square classes $\{1, -1\}$ and choose the following representative

$$u = \frac{1}{4n+4}\left(\sum_{i=1}^{2n-2} x_{\mathbf{e}_{i+2}-\mathbf{e}_i}\left(\frac{1}{2}\right) + x_{-2\mathbf{e}_{2n-1}}(1) + x_{-2\mathbf{e}_{2n}}(-1)\right).$$

Here the multiplication of $\dfrac{1}{4n+4}$ is due to the difference between the Killing form and the trace form for symplectic Lie algebras. Let

$$s = \mathrm{diag}(2n-1, 2n-1, 2n-3, 2n-3, \ldots, 1, 1, 0, 0, -1, -1, \ldots, 1-2n, 1-2n).$$

Then it is clear that (s, u) is a neutral pair, and

$$\mathcal{F}_{s,u}(\boldsymbol{\phi})(\widetilde{g}) = \int_{[N_{s,u}]} \boldsymbol{\phi}(n\widetilde{g})\psi_u^{-1}(n)\,\mathrm{d}n \tag{5.3}$$

is a generalized Whittaker-Fourier coefficient of $\boldsymbol{\phi} \in \widetilde{\mathcal{E}}_{\tau\otimes\widetilde{\sigma}}$ attached to the nilpotent orbit $\mathcal{O}$. Here we mean $n = (n, 1) \in \mathrm{Mp}_{4n+2}(\mathbb{A})$. In the following we will show that $\mathcal{F}_{s,u}(\boldsymbol{\phi})$ is not identically zero.

Remark 5.4. *Note that any unipotent subgroup of* $\mathrm{Mp}_{4n+2}(\mathbb{A})$ *has a unique splitting. If no more explanation is needed, we will identify each unipotent subgroup with its unique splitting in* $\mathrm{Mp}_{4n+2}(\mathbb{A})$*, and just use the notation for the linear group. We also note that a character on the elements of metaplectic cover is always defined to be its restriction on the linear group.*

Let $A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ and let

$$\varrho = \mathrm{diag}(A, \ldots, A; I_2; A^*, \ldots, A^*).$$

Conjugating the integral in (5.3) from left by ϱ, it becomes

$$\int_{[V_1]} \boldsymbol{\phi}(v\,\widetilde{g})\psi_{u,\varrho}^{-1}(v)\,\mathrm{d}v, \tag{5.4}$$

where $V_1 = \varrho N_{s,u}\varrho^{-1} = N_{s,u}$, and $\psi_{u,\varrho}(v) = \psi_u(\varrho^{-1}v\varrho)$. Note that

$$\psi_{u,\varrho}(v) = \psi\left(\sum_{i=1}^{2n-2} v_{i,i+2} + v_{2n-1,2n+3} + v_{2n,2n+4}\right).$$

Let ω be the Weyl element sending the following torus element

$$\mathrm{diag}\left(t_1^{(1)}, t_1^{(2)}, t_2^{(1)}, t_2^{(2)}, \ldots, t_n^{(1)}, t_n^{(2)}, t_{2n+1}, t_{2n+2}, (t_n^{(2)})^{-1}, (t_n^{(1)})^{-1}, \ldots, (t_1^{(2)})^{-1}, (t_1^{(1)})^{-1}\right)$$

to

$$\mathrm{diag}\Big(t_1^{(1)}, t_2^{(1)}, \ldots, t_n^{(1)}, (t_n^{(2)})^{-1}, (t_{n-1}^{(2)})^{-1}, \ldots, (t_1^{(2)})^{-1}, t_{2n+1}, t_{2n+2},$$
$$t_1^{(2)}, t_2^{(2)}, \ldots, t_n^{(2)}, (t_n^{(1)})^{-1}, \ldots, (t_2^{(1)})^{-1}, (t_1^{(1)})^{-1}\Big).$$

Conjugating the integral (5.4) from left by ω which is identified with $(\omega, 1)$, it becomes

$$\int_{[V_2]} \boldsymbol{\phi}(v\,\widetilde{g})\psi_{u,\varrho,\omega}^{-1}(v)\,\mathrm{d}v, \tag{5.5}$$

where $V_2 = \omega V_1 \omega^{-1}$, and $\psi_{u,\varrho,\omega}(v) = \psi_{u,\varrho}(\omega^{-1}v\omega)$. Note that

$$\psi_{u,\varrho,\omega}(v) = \psi\left(\sum_{i=1}^{n} v_{i,i+1} - \sum_{j=n+1}^{2n-1} v_{j,j+1}\right).$$

Remark 5.5. *Here we are using the convention that if $n+1 > 2n-1$, then there are no such terms indexed by j. We will follow this convention in the rest of this section.*

Let

$$u_1 = \sum_{i=1}^{n} x_{-\alpha_i}\left(\frac{1}{2(4n+4)}\right) + \sum_{i=n+1}^{2n-1} x_{-\alpha_i}\left(-\frac{1}{2(4n+4)}\right),$$

and let s_1 be the following semisimple element

$$s_1 = \mathrm{diag}(2n-1, 2n-3, \dots, 1-2n, 0, 0, 2n-1, 2n-3, \dots, 1-2n).$$

Then (s_1, u_1) is also a neutral pair, and one can check directly that $V_2 = N_{s_1,u_1}$ and the integral (5.5) is exactly the generalized Whittaker-Fourier coefficient $\mathcal{F}_{s_1,u_1}(\boldsymbol{\phi})$.

Take another semisimple element

$$s_1' = \mathrm{diag}(4n+1, 4n-1, \dots, 1, -1, \dots, -4n+1, -4n-1).$$

It is clear that (s_1', u_1) is a Whittaker pair. We consider $\mathcal{F}_{s_1',u_1}(\boldsymbol{\phi})$ for $\boldsymbol{\phi} \in \widetilde{\mathcal{E}}_{\tau\otimes\widetilde{\sigma}}$. Recall that $\widetilde{Q}_{2n}(\mathbb{A})$ is the parabolic subgroup of $\mathrm{Mp}_{4n+2}(\mathbb{A})$ with Levi subgroup isomorphic to $\mathrm{GL}_{2n}(\mathbb{A}) \times \mathrm{Mp}_2(\mathbb{A})$ and unipotent radical subgroup $U_{2n}(\mathbb{A})$. Then, by definition, for any $\boldsymbol{\phi} \in \widetilde{\mathcal{E}}_{\tau\otimes\widetilde{\sigma}}$, $\mathcal{F}_{s_1',u_1}(\boldsymbol{\phi})$ is the constant term integral over $U_{2n}(F)\backslash U_{2n}(\mathbb{A})$ combined with a non-degenerate Whittaker-Fourier coefficient of τ. Since $\widetilde{\mathcal{E}}_{\tau\otimes\widetilde{\sigma}}$ is constructed from data $\tau\otimes\widetilde{\sigma}$ on the Levi subgroup $\mathrm{GL}_{2n}(\mathbb{A}) \times \mathrm{Mp}_2(\mathbb{A})$ with τ being generic, the constant term integral over $U_{2n}(F)\backslash U_{2n}(\mathbb{A})$, and also the non-degenerate Whittaker-Fourier coefficient of τ are both non-zero. Then $\mathcal{F}_{s_1',u_1}(\boldsymbol{\phi})$ is not identically zero for $\boldsymbol{\phi} \in \widetilde{\mathcal{E}}_{\tau\otimes\widetilde{\sigma}}$, and by Proposition 5.1 $\mathcal{F}_{s_1,u_1}(\boldsymbol{\phi})$ is also not identically zero for $\boldsymbol{\phi} \in \widetilde{\mathcal{E}}_{\tau\otimes\widetilde{\sigma}}$. Hence $\mathcal{F}_{s,u}(\boldsymbol{\phi})$ is also not identically zero for $\boldsymbol{\phi} \in \widetilde{\mathcal{E}}_{\tau\otimes\widetilde{\sigma}}$. This completes the proof of the proposition. □

Proposition 5.6. *The residual representation $\widetilde{\mathcal{E}}_{\tau\otimes\widetilde{\sigma}}$ has a non-zero generalized Whittaker-Fourier coefficient attached to the partition $[(2n+2)1^{2n}]$.*

Proof. We prove this proposition by contradiction. Assume that the residual representation $\widetilde{\mathcal{E}}_{\tau\otimes\widetilde{\sigma}}$ has no non-zero Fourier coefficients attached to the partition $[(2n+2)1^{2n}]$. In the following, we inherit the notation used in the proof of Proposition 5.3 and follow the conventions in Remark 5.4 and Remark 5.5.

Note that in Proposition 5.3, we have already shown that the integral (5.5)

$$\int_{[V_2]} \boldsymbol{\phi}(v\,\widetilde{g})\psi_{u,\varrho,\omega}^{-1}(v)\,\mathrm{d}v$$

is non-zero. Recall that it is exactly the generalized Whittaker-Fourier coefficient $\mathcal{F}_{s_1,u_1}(\boldsymbol{\phi})$ where

$$u_1 = \sum_{i=1}^{n} x_{-\alpha_i}\left(\frac{1}{2(4n+4)}\right) + \sum_{i=n+1}^{2n-1} x_{-\alpha_i}\left(-\frac{1}{2(4n+4)}\right),$$

and

$$s_1 = \mathrm{diag}(2n-1, 2n-3, \dots, 1-2n, 0, 0, 2n-1, 2n-3, \dots, 1-2n).$$

Recall also that $V_2 = N_{s_1,u_1}$.

Let $Y = X_{\mathbf{e}_n+\mathbf{e}_{2n+1}} X_{\mathbf{e}_{2n+1}-\mathbf{e}_{n+1}}$, and $V_3 = V_2 Y$. Recall that for a root γ, the unipotent element $X_\gamma = \exp(x_\gamma)$, where x_γ is the one-dimensional root subgroup in $\mathfrak{sp}_{4n+2}$ corresponding to γ. By

[34, Lemma 6.0.2] or [32, Lemma 1.1], $\mathcal{F}_{s_1,u_1}(\boldsymbol{\phi})$ is non-vanishing if and only if the integral

$$f(\widetilde{g}) := \int_{[V_3]} \boldsymbol{\phi}(v\,\widetilde{g})\psi_{u_1}^{-1}(v)\,\mathrm{d}v \tag{5.6}$$

is non-vanishing. Let

$$\omega_0 = \begin{pmatrix} 0 & 0 & -I_{2n} \\ 0 & I_2 & 0 \\ I_{2n} & 0 & 0 \end{pmatrix}. \tag{5.7}$$

Note that V_3 is stable under the conjugation of ω_0^{-1}, then it is easy to see that $f(\widetilde{g}) = f(\omega_0\widetilde{g})$. Here we identify ω_0 with $(\omega_0, 1)$.

Let V_3' be the subgroup of V_3 consisting of elements v with $v_{2n+1,n+1} = 0$. Then one can see that the quadruple

$$(V_3', \psi_{u_1}, X_{\mathbf{e}_n - \mathbf{e}_{2n+1}}, X_{\mathbf{e}_{2n+1}-\mathbf{e}_{n+1}}) \tag{5.8}$$

satisfies all the conditions for [33, Lemma 7.1]. Applying [33, Lemma 7.1], one has

$$f(\widetilde{g}) = \int_{X_{\mathbf{e}_{2n+1}-\mathbf{e}_{n+1}}(\mathbb{A})} \int_{[V_3' X_{\mathbf{e}_n - \mathbf{e}_{2n+1}}]} \boldsymbol{\phi}(vx\widetilde{g})\psi_{u_1}^{-1}(v)\,\mathrm{d}v\mathrm{d}x. \tag{5.9}$$

Let $W = V_3' X_{\mathbf{e}_n - \mathbf{e}_{2n+1}}$, the elements in W are of the following form

$$w = \begin{pmatrix} z & z\cdot q_1 & q_2 \\ 0 & I_2 & q_1' \\ 0 & 0 & z^* \end{pmatrix} \begin{pmatrix} I_{2n} & 0 & 0 \\ p_1' & I_2 & 0 \\ p_2 & p_1 & I_{2n} \end{pmatrix} \in \mathrm{Sp}_{4n+2}, \tag{5.10}$$

where $z \in Z_{2n}$, the standard maximal unipotent subgroup of GL_{2n}; $q_1 \in \mathrm{Mat}_{2n\times 2}$ with $q_1(i,j) = 0$ for $n+1 \le i \le 2n$ and $1 \le j \le 2$; $q_2 \in \mathrm{Mat}_{2n\times 2n}$ with $q_2(i,j) = 0$ for $i \ge j$ (and certain additional properties which we do not specify here); $p_1 \in \mathrm{Mat}_{2n\times 2}$ with $p_1(i,j) = 0$ for $n \le i \le 2n$ and $1 \le j \le 2$; and $p_2 \in \mathrm{Mat}_{2n\times 2n}$ with $p_2(i,j) = 0$ for $i \ge j$ (similarly, with certain additional properties which we do not specify here). Define a character $\psi_W(w) := \psi_{u_1}(v)$ for $w = vy \in W$, where $v \in V_3'$ and $y \in X_{\mathbf{e}_n - \mathbf{e}_{2n+1}}$. For $w \in W$ of form in (5.10), one has

$$\psi_W(w) = \psi\left(\sum_{i=1}^{n} w_{i,i+1} - \sum_{j=n+1}^{2n-1} w_{j,j+1}\right).$$

To continue, we define a sequence of unipotent subgroups as follows. For $1 \le i \le n$ and $1 \le j \le i$, define $X_j^i = X_{\mathbf{e}_i + \mathbf{e}_{2n-i+j}}$ and also $Y_j^i = X_{-\mathbf{e}_{2n-i+j} - \mathbf{e}_{i+1}}$. For $n+1 \le i \le 2n-1$ and $1 \le j \le 2n-i$, define $X_j^i = X_{\mathbf{e}_i + \mathbf{e}_{i+j+1}}$ and also $Y_j^i = X_{-\mathbf{e}_{i+j+1} - \mathbf{e}_{i+1}}$. Moreover, for $n+1 \le i \le 2n-1$, define $X_i = X_{\mathbf{e}_i - \mathbf{e}_{2n+1}} X_{\mathbf{e}_i + \mathbf{e}_{2n+1}}$ and also $Y_i = X_{\mathbf{e}_{2n+1} - \mathbf{e}_{i+1}} X_{-\mathbf{e}_{2n+1} - \mathbf{e}_{i+1}}$.

Let $\mathcal{W}$ be the subgroup of W with elements of the form being as in (5.10), but with the p_1 and p_2 parts being zero. We will apply [33, Lemma 7.1] (exchanging unipotent subgroups) to a sequence of quadruples. For i going from 1 to n, the following sequence of quadruples satisfy all the conditions for [33, Lemma 7.1]:

$$\begin{aligned}
&\left(\mathcal{W}_i \prod_{j=2}^{i} Y_j^i, \psi_{\mathcal{W}}, X_1^i, Y_1^i\right),\\
&\left(X_1^i\mathcal{W}_i \prod_{j=3}^{i} Y_j^i, \psi_{\mathcal{W}}, X_2^i, Y_2^i\right),\\
&\qquad\vdots\\
&\left(\prod_{j=1}^{k-1} X_j^i\mathcal{W}_i \prod_{j=k+1}^{i} Y_j^i, \psi_{\mathcal{W}}, X_k^i, Y_k^i\right),\\
&\qquad\vdots\\
&\left(\prod_{j=1}^{i-1} X_j^i\mathcal{W}_i, \psi_{\mathcal{W}}, X_i^i, Y_i^i\right),
\end{aligned} \tag{5.11}$$

where

$$\mathcal{W}_i = \prod_{s=1}^{i-1}\prod_{j=1}^{s} X_j^s \mathcal{W} \prod_{t=n+1}^{2n-1} Y_t \prod_{j=1}^{2n-t} Y_j^t \prod_{k=i+1}^{n}\prod_{j=1}^{k} Y_j^k,$$

and $\psi_{\mathcal{W}}$ denotes the character on the first entry of each quadruple in (5.11), which is extended from the character $\psi_W|_{\mathcal{W}}$ on $\mathcal{W}$ trivially. Here we note that during the verification of the conditions of [33, Lemma 7.1], we need to use the properties of Rao cocycle (for example, (2.2) and (2.3)), as in the proof of [33, Lemma 8.3]. In particular, since $\mathcal{W}_i$'s and X_j^i's are contained in the Siegel parabolic subgroup $\mathbf{P}_0$ (see §2.1), the (unique) splittings of the first entries of the quadruples in (5.11) in the metaplectic cover are determined by the splittings of Y_j^i-parts. Applying [33, Lemma 7.1] repeatedly to the above sequence of quadruples, with i going from 1 to n, one obtains that

$$f(\widetilde{g}) = \int_{\left(\prod_{s=1}^{n}\prod_{j=1}^{s} Y_j^s X_{\mathbf{e}_{2n+1}-\mathbf{e}_{n+1}}\right)(\mathbb{A})} \int_{[\mathcal{W}_n']} \boldsymbol{\phi}(wx\widetilde{g})\psi_{\mathcal{W}_n'}^{-1}(w)\,\mathrm{d}w\mathrm{d}x, \tag{5.12}$$

where

$$\mathcal{W}_n' = \prod_{s=1}^{n}\prod_{j=1}^{s} X_j^s \mathcal{W} \prod_{t=n+1}^{2n-1} Y_t \prod_{j=1}^{2n-t} Y_j^t, \tag{5.13}$$

and $\psi_{\mathcal{W}_n'}$ is extended from $\psi_{\mathcal{W}}$ trivially.

Next, we take the Fourier expansion of f along $X_{2\mathbf{e}_{n+1}}$. Under the action of GL_1, we get two kinds of Fourier coefficients corresponding to the two orbits of the dual of $[X_{2\mathbf{e}_{n+1}}] \simeq F\backslash\mathbb{A}$: the trivial one and the non-trivial one. Since we have assumed that $\widetilde{\mathcal{E}}_{\tau\otimes\widetilde{\sigma}}$ has no non-zero Fourier coefficients attached to the partition $[(2n+2)1^{2n}]$, by Lemma 5.7 below (which will be stated and proved right after the proof of Proposition 5.6), all the Fourier coefficients corresponding to the non-trivial orbit are identically zero. Hence, we obtain that

$$f(\widetilde{g}) = \int_{\left(\prod_{s=1}^{n}\prod_{j=1}^{s} Y_j^s X_{\mathbf{e}_{2n+1}-\mathbf{e}_{n+1}}\right)(\mathbb{A})} \int_{[X_{2\mathbf{e}_{n+1}}]} \int_{[\mathcal{W}_n']} \boldsymbol{\phi}(wxy\widetilde{g})\psi_{\mathcal{W}_n'}^{-1}(w)\,\mathrm{d}w\mathrm{d}x\mathrm{d}y. \tag{5.14}$$

To continue, note that the following sequence of quadruples satisfy all the conditions for [33, Lemma 7.1]:

$$\begin{aligned}
&\left(X_{2\mathbf{e}_{n+1}}\mathcal{W}_{n+1}\prod_{j=2}^{n-1}Y_j^{n+1},\psi_{\mathcal{W}},X_1^{n+1},Y_1^{n+1}\right),\\
&\left(X_1^nX_{2\mathbf{e}_{n+1}}\mathcal{W}_{n+1}\prod_{j=3}^{n-1}Y_j^{n+1},\psi_{\mathcal{W}},X_2^{n+1},Y_2^{n+1}\right),\\
&\qquad\vdots\\
&\left(\prod_{j=1}^{k-1}X_j^{n+1}X_{2\mathbf{e}_{n+1}}\mathcal{W}_{n+1}\prod_{j=k+1}^{n-1}Y_j^{n+1},\psi_{\mathcal{W}},X_k^{n+1},Y_k^{n+1}\right),\\
&\qquad\vdots\\
&\left(\prod_{j=1}^{n-2}X_j^{n+1}X_{2\mathbf{e}_{n+1}}\mathcal{W}_{n+1},\psi_{\mathcal{W}},X_{n-1}^{n+1},Y_{n-1}^{n+1}\right)\\
&\left(\prod_{j=1}^{n-1}X_j^{n+1}X_{2\mathbf{e}_{n+1}}\mathcal{W}_{n+1},\psi_{\mathcal{W}},X_{n+1},Y_{n+1}\right),
\end{aligned}\tag{5.15}$$

where

$$\mathcal{W}_{n+1}=\prod_{t=1}^{n}\prod_{j=1}^{t}X_j^t\mathcal{W}\prod_{k=n+2}^{2n-1}Y_k\prod_{j=1}^{2n-k}Y_j^k,$$

and $\psi_{\mathcal{W}}$ has a similar definition to that in (5.11). Applying [33, Lemma 7.1] repeatedly to the above sequence of quadruples, one can see that

$$f(\widetilde{g})=\int_{\left(Y_{n+1}\prod_{j=1}^{n-1}Y_j^{n+1}\prod_{s=1}^{n}\prod_{j=1}^{s}Y_j^sX_{\mathbf{e}_{2n+1}-\mathbf{e}_{n+1}}\right)(\mathbb{A})}\int_{[\mathcal{W}'_{n+1}]}\boldsymbol{\phi}(wx\widetilde{g})\psi^{-1}_{\mathcal{W}'_{n+1}}(w)\,\mathrm{d}w\mathrm{d}x,\tag{5.16}$$

where

$$\mathcal{W}'_{n+1}=X_{n+1}X_{2\mathbf{e}_{n+1}}\prod_{j=1}^{n-1}X_j^{n+1}\prod_{t=1}^{n}\prod_{j=1}^{t}X_j^t\mathcal{W}\prod_{k=n+2}^{2n-1}Y_k\prod_{j=1}^{2n-k}Y_j^k,\tag{5.17}$$

and $\psi_{\mathcal{W}'_{n+1}}$ is extended from $\psi_{\mathcal{W}}$ trivially.

For i going from $n+2$ to $2n-1$, define

$$\mathcal{W}'_i=\prod_{s=n+1}^{i}X_s\prod_{j=1}^{2n-s}X_j^s\prod_{\ell=n+1}^{i}X_{2\mathbf{e}_\ell}\prod_{t=1}^{n}\prod_{j=1}^{t}X_j^t\mathcal{W}\prod_{k=i+1}^{2n-1}Y_k\prod_{j=1}^{2n-k}Y_j^k,\tag{5.18}$$

and also a character $\psi_{\mathcal{W}'_i}$ which is extended from $\psi_{\mathcal{W}}$ trivially. We claim that the integral

$$\int_{[\mathcal{W}'_{i-1}]}\boldsymbol{\phi}(w\widetilde{g})\psi^{-1}_{\mathcal{W}'_{i-1}}(w)\,\mathrm{d}w\tag{5.19}$$

is equal to the integral

$$\int_{\left(Y_i \prod_{j=2}^{2n-i} Y_j^i\right)(\mathbb{A})} \int_{[\mathcal{W}_i']} \phi(wx\widetilde{g})\psi_{\mathcal{W}_i'}^{-1}(w)\,\mathrm{d}w\mathrm{d}x. \tag{5.20}$$

To show the above equality, we first take the Fourier expansion of the integral in (5.19) along $X_{2\mathbf{e}_i}$. Note that $[X_{2\mathbf{e}_i}] \simeq F\backslash\mathbb{A}$, we again consider the trivial and the non-trivial orbits separately. By Lemma 5.7 below, all the Fourier coefficients corresponding to the non-trivial orbit are identically zero. Therefore only the Fourier coefficient attached to the trivial orbit survives. When $i = 2n-1$, the Fourier coefficient attached to the trivial orbit is exactly the integral in (5.20). When $n+2 \le i \le 2n-2$, one can see that the following sequence of quadruples satisfy all the conditions for [33, Lemma 7.1]:

$$\left(X_{2\mathbf{e}_i}\mathcal{W}_i Y_i \prod_{j=2}^{2n-i} Y_j^i, \psi_{\mathcal{W}}, X_1^i, Y_1^i\right),$$

$$\left(X_1^i X_{2\mathbf{e}_i}\mathcal{W}_i Y_i \prod_{j=3}^{2n-i} Y_j^i, \psi_{\mathcal{W}}, X_2^i, Y_2^i\right),$$

$$\vdots$$

$$\left(\prod_{j=1}^{k-1} X_j^i X_{2\mathbf{e}_i}\mathcal{W}_i Y_i \prod_{j=k+1}^{2n-i} Y_j^i, \psi_{\mathcal{W}}, X_k^i, Y_k^i\right),$$

$$\vdots$$

$$\left(\prod_{j=1}^{2n-i-1} X_j^i X_{2\mathbf{e}_i}\mathcal{W}_i Y_i, \psi_{\mathcal{W}}, X_{2n-i}^i, Y_{2n-i}^i\right),$$

$$\left(\prod_{j=1}^{2n-i} X_j^i X_{2\mathbf{e}_i}\mathcal{W}_i, \psi_{\mathcal{W}}, X_i, Y_i\right),$$

where

$$\mathcal{W}_i = \prod_{s=n+1}^{i-1} X_s \prod_{j=1}^{2n-s} X_j^s \prod_{\ell=n+1}^{i-1} X_{2\mathbf{e}_\ell} \prod_{t=1}^{n}\prod_{j=1}^{t} X_j^t \mathcal{W} \prod_{k=i+1}^{2n-1} Y_k \prod_{j=1}^{2n-k} Y_j^k,$$

and $\psi_{\mathcal{W}}$ has a similar definition to that in (5.11). Applying [33, Lemma 7.1] repeatedly to the above sequence of quadruples, we deduce that the Fourier coefficient attached to the trivial orbit above is equal to the integral (5.20). This proves the claim.

One can see that elements of $\mathcal{W}_{2n-1}'$ have the following form:

$$w = \begin{pmatrix} z & z\cdot q_1 & q_2 \\ 0 & I_2 & q_1' \\ 0 & 0 & z^* \end{pmatrix}, \tag{5.21}$$

where $z \in Z_{2n}$; $q_1 \in \mathrm{Mat}_{2n\times 2}$ with $q_1(2n,j) = 0$ for $1 \le j \le 2$; and $q_2 \in \mathrm{Mat}_{2n\times 2n}$. As before, q_1 and q_2 also have certain additional properties on symmetry we do not specify here. We also define a character $\psi_{\mathcal{W}_{2n-1}'}$ by $\psi_{\mathcal{W}_{2n-1}'}(w) = \psi\left(\sum_{i=1}^{2n-1} z_{i,i+1}\right)$ for $w \in \mathcal{W}_{2n-1}'$ being of the form in (5.21).

Now we need to take the Fourier expansion of the integral over $\mathcal{W}'_{2n-1}$ in (5.20) along

$$X_{\mathbf{e}_{2n}-\mathbf{e}_{2n+1}} X_{\mathbf{e}_{2n}+\mathbf{e}_{2n+1}}.$$

In this situation, the Fourier coefficients corresponding to the non-trivial orbit are generic Fourier coefficients. Since $\widetilde{\mathcal{E}}_{\tau\otimes\widetilde{\sigma}}$ is not generic, only the Fourier coefficient corresponding to the trivial orbit survives. Therefore, the integral over $\mathcal{W}'_{2n-1}$ in (5.20) becomes

$$\begin{aligned} &\int_{[X_{\mathbf{e}_{2n}-\mathbf{e}_{2n+1}} X_{\mathbf{e}_{2n}+\mathbf{e}_{2n+1}}]} \int_{[\mathcal{W}'_{2n-1}]} \boldsymbol{\phi}(wx\widetilde{g})\psi^{-1}_{\mathcal{W}'_{2n-1}}(w)\,\mathrm{d}w\mathrm{d}x \\ =&\int_{[N_{2n}]} \boldsymbol{\phi}(n\widetilde{g})\psi^{-1}_{N_{2n}}(n)\,\mathrm{d}n, \end{aligned} \tag{5.22}$$

where N_{2n} is the unipotent radical of the parabolic subgroup with Levi subgroup isomorphic to $\mathrm{GL}_1^{2n} \times \mathrm{Mp}_2$ as defined in §2.1, and $\psi_{N_{2n}}(n) = \psi(\sum_{i=1}^{2n-1} n_{i,i+1})$.

Recall that we have defined the parabolic subgroup $\widetilde{P}(\mathbb{A}) = \widetilde{M}(\mathbb{A})U(\mathbb{A})$ with Levi subgroup $\widetilde{M}(\mathbb{A})$ isomorphic to $\mathrm{GL}_{2n}(\mathbb{A}) \times \mathrm{Mp}_2(\mathbb{A})$. Write $N_{2n}(\mathbb{A}) = U(\mathbb{A})N'_{2n}(\mathbb{A})$ with $N'_{2n}(\mathbb{A}) = \widetilde{M}(\mathbb{A}) \cap N_{2n}(\mathbb{A})$. Based on the above discussion, we obtain that

$$f(\widetilde{g}) = \int_{\left(\prod_{s=n+1}^{2n-1} Y_s \prod_{j=1}^{2n-s} Y_j^s \prod_{k=1}^{n}\prod_{\ell=1}^{k} Y_\ell^k X_{\mathbf{e}_{2n+1}-\mathbf{e}_{n+1}}\right)(\mathbb{A})} \int_{[N'_{2n}]} \boldsymbol{\phi}^U(nx\widetilde{g})\psi^{-1}_{N'_{2n}}(n)\,\mathrm{d}n\mathrm{d}x, \tag{5.23}$$

where $\boldsymbol{\phi}^U$ is the constant term of $\boldsymbol{\phi}$ along U, and $\psi_{N'_{2n}} = \psi_{N_{2n}}|_{N'_{2n}}$. Note that in (5.23), $\prod_{s=n+1}^{2n-1} Y_s \prod_{j=1}^{2n-s} Y_j^s \prod_{k=1}^{n}\prod_{\ell=1}^{k} Y_\ell^k X_{\mathbf{e}_{2n+1}-\mathbf{e}_{n+1}}$ is equal to $V_2 \cap U^-$, where U^- is the unipotent radical of the parabolic subgroup opposite to $\widetilde{P}$. By a similar calculation as in [43, Lemma 2,2], we deduce that

$$\boldsymbol{\phi}^U \in \mathcal{A}(U(\mathbb{A})\widetilde{M}(F)\backslash \mathrm{Mp}_{4n+2}(\mathbb{A}))_{\mu_\psi(\tau|\cdot|^{-\frac{1}{2}}\otimes\widetilde{\sigma})}. \tag{5.24}$$

For $t \in \mathbb{A}^\times$, let

$$D(t) = \begin{pmatrix} tI_{2n} & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & t^{-1}I_{2n} \end{pmatrix} \in \mathrm{Sp}_{4n+2}(\mathbb{A}).$$

Then it is easy to see that $\omega_0 D(t)\omega_0^{-1} = D(t^{-1})$, where ω_0 is the Weyl element defined in (5.7). Consider $f(D(t)\widetilde{g}) = f((D(t),1)\widetilde{g})$. Note that $f(\widetilde{g})$ has the form in (5.23). Conjugating $D(t)$ to the left, by changing of variables on

$$\left(\prod_{s=n+1}^{2n-1} Y_s \prod_{j=1}^{2n-s} Y_j^s \prod_{k=1}^{n}\prod_{\ell=1}^{k} Y_\ell^k X_{\mathbf{e}_{2n+1}-\mathbf{e}_{n+1}}\right)(\mathbb{A}),$$

we get a factor $|t|_{\mathbb{A}}^{-(2n^2+2n)+1}$. By (5.24), we have

$$\boldsymbol{\phi}^U(D(t)nx\widetilde{g}) = \delta_P(D(t))^{\frac{1}{2}}|D(t)|^{-\frac{1}{2}}\omega_\tau(t)\gamma_\psi(t^{2n})\boldsymbol{\phi}^U(nx\widetilde{g}) = |t|_{\mathbb{A}}^{2n^2+2n}\omega_\tau(t)\boldsymbol{\phi}^U(nx\widetilde{g}),$$

here $\gamma_\psi(t^{2n}) = 1$ by the properties of γ_ψ (see, for example, [52, Lemma 4.1]). Therefore, one can get $f(D(t)\widetilde{g}) = |t|_{\mathbb{A}}\omega_\tau(t)f(\widetilde{g})$. On the other hand,

$$f(D(t)\widetilde{g}) = f(\omega_0 D(t)\widetilde{g}) = f((D(t^{-1}),\epsilon)\omega_0\widetilde{g}) = \epsilon|t^{-1}|_{\mathbb{A}}\omega_\tau(t^{-1})f(\omega_0\widetilde{g}) = \epsilon|t|_{\mathbb{A}}^{-1}\omega_\tau(t^{-1})f(\widetilde{g}),$$

here $\epsilon = \pm 1$ comes from the conjugation $\omega_0 D(t)\omega_0^{-1}$ in the metaplectic group. Hence, we have $|t|_{\mathbb{A}}\omega_\tau(t)f(\widetilde{g}) = \epsilon|t|_{\mathbb{A}}^{-1}\omega_\tau(t^{-1})f(\widetilde{g})$, and $|t|_{\mathbb{A}}f(\widetilde{g}) = \epsilon|t|_{\mathbb{A}}^{-1}f(\widetilde{g})$ since ω_τ is quadratic. Since $t \in \mathbb{A}^\times$ can be arbitrary, we get that $f(\widetilde{g})$ is identically zero, which is a contradiction.

It follows that the assumption in the beginning of the proof is false, i.e., the residual representation $\widetilde{\mathcal{E}}_{\tau\otimes\widetilde{\sigma}}$ must have a non-zero Fourier coefficient attached to the partition $[(2n+2)1^{2n}]$. This completes the proof of the proposition. $\square$

Now we prove the following lemma used in the proof of the above proposition.

Lemma 5.7. *Assume that the residual representation $\widetilde{\mathcal{E}}_{\tau\otimes\widetilde{\sigma}}$ has no non-zero Fourier coefficients attached to the partition $[(2n+2)1^{2n}]$, which is the assumption in the beginning of the proof of Proposition 5.6. Then for any $\phi \in \widetilde{\mathcal{E}}_{\tau\otimes\widetilde{\sigma}}$, the following integral*

$$\int_{[X_{2\mathbf{e}_{i+1}}]}\int_{[\mathcal{W}_i']} \phi(wx\widetilde{g})\psi_{\mathcal{W}_i'}^{-1}(w)\psi^{-1}(ax)\,\mathrm{d}w\mathrm{d}x \tag{5.25}$$

is zero, where $a \in F^\times$, $n \le i \le 2n-2$, $\mathcal{W}_i'$ is as in (5.13) when $i = n$, $\mathcal{W}_i'$ is as in (5.17) when $i = n+1$, and $\mathcal{W}_i'$ is as in (5.18) when $n+2 \le i \le 2n-2$.

Proof. We continue using the notation introduced in Proposition 5.3. Note that elements in $\mathcal{W}_i'$ have the following form

$$w = \begin{pmatrix} z & z\cdot q_1 & q_2 \\ 0 & I_2 & q_1' \\ 0 & 0 & z^* \end{pmatrix}\begin{pmatrix} I_{2n} & 0 & 0 \\ p_1' & I_2 & 0 \\ p_2 & p_1 & I_{2n} \end{pmatrix} \in \mathrm{Sp}_{4n+2}, \tag{5.26}$$

where $z \in Z_{2n}$, the standard maximal unipotent subgroup of GL_{2n}; $q_1 \in \mathrm{Mat}_{2n\times 2}$ with $q_1(k,j) = 0$ for $i+1 \le k \le 2n$ and $1 \le j \le 2$; $q_2 \in \mathrm{Mat}_{2n\times 2n}$ with $q_2(k,j) = 0$ for $i+1 \le k \le 2n$ and $1 \le j \le 2n-i$; $p_1 \in \mathrm{Mat}_{2n\times 2}$ with $p_1(k,j) = 0$ for $2n-i \le k \le 2n$ and $1 \le j \le 2$; $p_2 \in \mathrm{Mat}_{2n\times 2n}$ with $p_2(k,j) = 0$ for $2n-i \le k \le 2n$ or $1 \le j \le i+1$. We also have a character $\psi_{\mathcal{W}_i'}$ defined by $\psi_{\mathcal{W}_i'}(w) = \psi\left(\sum_{i=1}^{2n-1} z_{i,i+1}\right)$ for $w \in \mathcal{W}_i'$ being of the form in (5.26).

In the rest of this proof, we set $\mathcal{W}_i^{(1)} := \mathcal{W}_i'$, and denote by $\mathcal{W}_i^{(2)}$ the subgroup of $\mathcal{W}_i^{(1)}$ consisting of the elements of the following form

$$\begin{pmatrix} I_{i+1} & 0 & 0 & 0 & 0 \\ 0 & z & 0 & 0 & 0 \\ 0 & 0 & I_2 & 0 & 0 \\ 0 & 0 & 0 & z^* & 0 \\ 0 & 0 & 0 & 0 & I_{i+1} \end{pmatrix},$$

where $z \in Z_{2n-i-1}$, the standard maximal unipotent subgroup of GL_{2n-i-1}. Write

$$\mathcal{W}_i^{(1)} = \mathcal{W}_i^{(2)}\mathcal{W}_i^{(3)},$$

where $\mathcal{W}_i^{(3)}$ is a subgroup of $\mathcal{W}_i^{(1)}$ consisting of elements with $\mathcal{W}_i^{(2)}$-part being trivial. Let $\psi_{\mathcal{W}_i^{(2)}} = \psi_{\mathcal{W}_i^{(1)}}|_{\mathcal{W}_i^{(2)}}$ and $\psi_{\mathcal{W}_i^{(3)}} = \psi_{\mathcal{W}_i^{(1)}}|_{\mathcal{W}_i^{(3)}}$. Then the integral in (5.25) can be written as

$$\int_{[\mathcal{W}_i^{(2)}]}\int_{[X_{2\mathbf{e}_{i+1}}]}\int_{[\mathcal{W}_i^{(3)}]} \phi(w_1xw_2\widetilde{g})\psi_{\mathcal{W}_i^{(2)}}^{-1}(w_2)\psi^{-1}(ax)\psi_{\mathcal{W}_i^{(3)}}^{-1}(w_1)\,\mathrm{d}w_1\mathrm{d}x\mathrm{d}w_2. \tag{5.27}$$

Therefore, to show the integral in (5.25) is identically zero, it is suffices to show the following integral is identically zero

$$\int_{[X_{2\mathbf{e}_{i+1}}]}\int_{[\mathcal{W}_i^{(3)}]}\phi(wx\widetilde{g})\psi^{-1}(ax)\psi^{-1}_{\mathcal{W}_i^{(3)}}(w)\,\mathrm{d}w\mathrm{d}x. \tag{5.28}$$

It is straightforward to see that the integral in (5.28) contains an inner integral which is exactly a Fourier coefficient of $\phi\in\widetilde{\mathcal{E}}_{\tau\otimes\widetilde{\sigma}}$ attached to the partition $[(2i+2)1^{4n-2i}]$. When $i=n$, by assumption, any Fourier coefficient attached to the partition $[(2n+2)1^{2n}]$ will be identically zero. And when $n+1\le i\le 2n-2$, by Proposition 4.1, any Fourier coefficient attached to the partition $[(2i+2)1^{4n-2i}]$ will be identically zero. Therefore, the integral in (5.28) is identically zero, and hence the integral in (5.25) is identically zero. This completes the proof of the lemma. □

5.3. **Non-vanishing of the descent: Case II.** We turn to the case $m=2n+1$. As in §5.2, we prove the following proposition at first.

Proposition 5.8. *The residual representation $\widetilde{\mathcal{E}}_{\tau\otimes\widetilde{\sigma}}$ has a non-zero generalized Whittaker-Fourier coefficient attached to the partition $[(2n+1)^2 1^2]$.*

Proof. As in Proposition 5.3, let $\{\alpha_i=\mathbf{e}_i-\mathbf{e}_{i+1},\alpha_{2n+2}=2\mathbf{e}_{2n+2}\mid 1\le i\le 2n+1\}$ be the set of simple roots for $\mathfrak{sp}_{4n+4}$. By [77, Section I.6], there is only one nilpotent orbit $\mathcal{O}$ corresponding to the partition $[(2n+1)^2 1^2]$. A representative of the nilpotent orbit $\mathcal{O}$ can be taken to be

$$u=\sum_{i=1}^{2n}x_{-\alpha_i}\left(\frac{1}{2(4n+6)}\right).$$

Let s be the following semisimple element

$$s=\mathrm{diag}(2n,2n-2,\dots,-2n,0,0,2n,2n-2,\dots,-2n).$$

Then it is clear that (s,u) is a neutral pair.

We want to show that $\mathcal{F}_{s,u}(\phi)$ is not identically zero for $\phi\in\widetilde{\mathcal{E}}_{\tau\otimes\widetilde{\sigma}}$. To this end, we take another semisimple element

$$s'=\mathrm{diag}(4n+3,4n+1,\dots,1,-1,\dots,-4n-1,-4n-3).$$

It is clear that (s',u) is a Whittaker pair. We consider $\mathcal{F}_{s',u}(\phi)$ for $\phi\in\widetilde{\mathcal{E}}_{\tau\otimes\widetilde{\sigma}}$. Recall that $\widetilde{Q}_{2n+1}(\mathbb{A})$ is the parabolic subgroup of $\mathrm{Mp}_{4n+4}(\mathbb{A})$ with Levi subgroup isomorphic to $\mathrm{GL}_{2n+1}(\mathbb{A})\times\mathrm{Mp}_2(\mathbb{A})$ and unipotent radical subgroup $U_{2n+1}(\mathbb{A})$. Then, for any $\phi\in\widetilde{\mathcal{E}}_{\tau\otimes\widetilde{\sigma}}$, $\mathcal{F}_{s',u}(\phi)$ is the constant term integral over $U_{2n+1}(F)\backslash U_{2n+1}(\mathbb{A})$ combined with a non-degenerate Whittaker-Fourier coefficient of τ. Since $\widetilde{\mathcal{E}}_{\tau\otimes\widetilde{\sigma}}$ is constructed from data $\tau\otimes\widetilde{\sigma}$ on the Levi subgroup $\mathrm{GL}_{2n+1}(\mathbb{A})\times\mathrm{Mp}_2(\mathbb{A})$ with τ being generic, the constant term integral over $U_{2n+1}(F)\backslash U_{2n+1}(\mathbb{A})$, and the non-degenerate Whittaker-Fourier coefficients of τ are both non-zero. Then $\mathcal{F}_{s',u}(\phi)$ is not identically zero for $\phi\in\widetilde{\mathcal{E}}_{\tau\otimes\widetilde{\sigma}}$, and by Proposition 5.1 $\mathcal{F}_{s,u}(\phi)$ is also not identically zero for $\phi\in\widetilde{\mathcal{E}}_{\tau\otimes\widetilde{\sigma}}$. This completes the proof of the proposition. □

Proposition 5.9. *The residual representation $\widetilde{\mathcal{E}}_{\tau\otimes\widetilde{\sigma}}$ has a non-zero generalized Whittaker-Fourier coefficient attached to a nilpotent orbit parametrized by the pair $([(2n+2)(2n)1^2],\{\beta,\alpha\})$, for some square classes $[\beta],[\alpha]\in F^\times/(F^\times)^2$.*

Proof. By Proposition 5.8, we know that $\widetilde{\mathcal{E}}_{\tau\otimes\widetilde{\sigma}}$ has a non-zero generalized Whittaker-Fourier coefficient attached to the partition $[(2n+1)^2 1^2]$. As a symplectic partition, $[(2n+1)^2 1^2]$ is not metaplectic special, and the smallest metaplectic special partition which is greater than $[(2n+1)^2 1^2]$ is $[(2n+2)(2n)1^2]$, which is called the metaplectic special expansion of the partition $[(2n+1)^2 1^2]$. Then by [40, Theorem 11.2], we have that $\widetilde{\mathcal{E}}_{\tau\otimes\widetilde{\sigma}}$ has a non-zero generalized Whittaker-Fourier coefficient attached to a nilpotent orbit parametrized by the pair $([(2n+2)(2n)1^2], \{\beta, \alpha\})$, for some square classes $[\beta], [\alpha] \in F^\times/(F^\times)^2$. $\square$

Remark 5.10. *The arguments in this subsection also work if we take $\widetilde{\sigma}$ to be an irreducible representation of a general size metaplectic group. More precisely, Let $\tau = \tau_1 \boxplus \cdots \boxplus \tau_t$ be an isobaric sum automorphic representation of $\mathrm{GL}_{2n+1}(\mathbb{A})$ with τ_i's being distinct irreducible unitary cuspidal automorphic representations of $\mathrm{GL}_{n_i}(\mathbb{A})$ of orthogonal type such that $\sum_{i=1}^t n_i = m$. Let $\widetilde{\sigma}$ be an irreducible unitary genuine cuspidal automorphic representation of $H_r(\mathbb{A}) = \mathrm{Mp}_{2r}(\mathbb{A})$ $(1 \le r \le n)$ with generic global Arthur parameter ϕ_0 (see [16] and [14, §11]) such that*

$$L_\psi(\frac{1}{2}, \phi_\tau \times \phi_0) \neq 0.$$

Then the residual representation $\widetilde{\mathcal{E}}_{\tau\otimes\widetilde{\sigma}}$ of $\mathrm{Mp}_{4n+2r+2}(\mathbb{A})$ has a non-zero generalized Whittaker-Fourier coefficient attached to the partition $[(2n+2)(2n)1^{2r}]$.

To conclude this section, we apply Lemma 5.2 to the above results (Proposition 5.6 and Proposition 5.9), and obtain the first Part of Theorem 1.3.

Theorem 5.11. *In both* **(Case I)** *and* **(Case II)***, there exists $\beta \in F^\times$, such that the automorphic descent space $\mathcal{D}^{\mathrm{FJ}}_{\psi_n,\beta}(\widetilde{\mathcal{E}}_{\tau\otimes\widetilde{\sigma}})$ is non-zero.*

Proof. The theorem follows immediately from Proposition 5.6, Proposition 5.9 and Lemma 5.2. $\square$

6. Applications

We have constructed non-zero cuspidal automorphic representations

$$\pi_\beta = \mathcal{D}^{\mathrm{FJ}}_{\psi_n,\beta}(\widetilde{\mathcal{E}}_{\tau\otimes\widetilde{\sigma}}) \quad (\beta \in F^\times)$$

in both **(Case I)** and **(Case II)**. Recall that we have $m = 2n$ or $m = 2n+1$ in the two cases respectively (see the notation in §2), and π_β is an automorphic representation of $G_{m-n}(\mathbb{A}) = \mathrm{Sp}_{2(m-n)}(\mathbb{A})$. In this section, we give some applications of these descent constructions.

Recall that in both cases, we begin with representations $(\tau, \widetilde{\sigma})$, where:

- τ is an isobaric sum automorphic representation

$$\tau = \tau_1 \boxplus \cdots \boxplus \tau_t \tag{6.1}$$

  of $\mathrm{GL}_m(\mathbb{A})$ such that the τ_i's are distinct irreducible unitary cuspidal automorphic representations of $\mathrm{GL}_{n_i}(\mathbb{A})$ of orthogonal type, and $\sum_{i=1}^t n_i = m$;
- $\widetilde{\sigma}$ is an irreducible unitary genuine cuspidal automorphic representation of $\mathrm{Mp}_2(\mathbb{A})$ with a generic Arthur parameter ϕ_0 (which is of symplectic type), such that

$$L_\psi(1/2, \tau \times \widetilde{\sigma}) \neq 0.$$

The important thing is that, under suitable conditions, these descent constructions produce representations in certain global Vogan packets parametrized by generic Arthur parameters. As a result, they provide some answers to the questions we have mentioned in §1.

6.1. **On the global GGP conjecture and quadratic twists of L-function.** We consider the **(Case I)**, where $m = 2n$. In this case, the descent π_β is an automorphic representation of $\mathrm{Sp}_{2n}(\mathbb{A})$, and this construction is related to the global GGP conjecture for the groups $\mathrm{Sp}_{2n}(\mathbb{A}) \times \mathrm{Mp}_2(\mathbb{A})$.

We state the global GGP conjecture for the pair $(\mathrm{Sp}_{2n}(\mathbb{A}), \mathrm{Mp}_2(\mathbb{A}))$ here. Let $\phi \in \widetilde{\Phi}_2(\mathrm{Sp}_{2n})$ and $\phi_0 \in \widetilde{\Phi}_2(\mathrm{Mp}_2)$ be global generic Arthur parameters. Note that the parameter ϕ is of dimension $(2n+1)$, and the parameter ϕ_0 is of dimension 2. Recall also that both the global Vogan packets and L-functions associated to ϕ_0 depend on choices of additive characters $\psi : F\backslash\mathbb{A} \longrightarrow \mathbb{C}^\times$, and we have presumed the identification (1.8) of L-functions.

Conjecture 6.1. *The central value of the L-function*

$$L_\psi(s, \phi \times \phi_0)$$

is non-zero if and only if there exists a pair $(\pi_0, \widetilde{\sigma}_0)$ in the global Vogan packet $\widetilde{\Pi}^\psi_{\phi\times\phi_0}[\mathrm{Sp}_{2n}(\mathbb{A}) \times \mathrm{Mp}_2(\mathbb{A})]$ such that the Fourier-Jacobi period

$$\mathcal{P}^{\mathrm{FJ}}_{\psi,\varphi_1}(\boldsymbol{\phi}_\pi, \boldsymbol{\phi}_{\widetilde{\sigma}})$$

is non-zero for some $\boldsymbol{\phi}_\pi \in \pi$, $\boldsymbol{\phi}_{\widetilde{\sigma}} \in \widetilde{\sigma}$ and $\varphi_1 \in \mathcal{S}(\mathbb{A})$.

In [48], using their approach in [47], Jiang and Zhang showed that if the period $\mathcal{P}^{\mathrm{FJ}}_{\psi,\varphi_1}(\boldsymbol{\phi}_\pi, \boldsymbol{\phi}_{\widetilde{\sigma}}) \neq 0$ for some choice of data, then $L_\psi(1/2, \phi \times \phi_0) \neq 0$. For the other direction of Conjecture 6.1, in the framework of the constructive approach introduced in §1.2, one needs to guarantee an explicit construction exactly at certain depth. See the paragraph right before Assumption 1.2. However, as discussed in §1.2, this seems not easy in general, which makes this direction a harder problem. It is also worthwhile to mention that in [73], Xue proves a refined version of the above conjecture for $n = 1$ and $n = 2$ (under certain conditions) via theta correspondence.

On the other hand, if we start from a generic automorphic representation τ of $\mathrm{GL}_{2n}(\mathbb{A})$ above (see (6.1)), twisting by an automorphic member $\widetilde{\sigma} \in \Pi^\psi_{\phi_0}[\mathrm{Mp}_2(\mathbb{A})]$, we obtain a non-zero cuspidal descent construction $\pi_\beta = \mathcal{D}^{\mathrm{FJ}}_{\psi_n,\beta}(\widetilde{\mathcal{E}}_{\tau\otimes\widetilde{\sigma}})$ on $\mathrm{Sp}_{2n}(\mathbb{A})$, which has a non-zero Fourier-Jacobi period with respect to $\widetilde{\sigma}$. Note that the global Arthur parameter for irreducible components of π_β have dimension $(2n+1)$, which is given by the descent construction from ϕ_τ (which is of dimension $2n$). Moreover, from the local calculations in §3.2, under an additional assumption that $\omega_\tau = \mathbf{1}$, the global Arthur parameter for irreducible components of π_β can be determined, hence leads to a result related to global GGP conjecture in this direction.

More precisely, for $\beta \in F^\times$, we form the global Arthur parameter

$$\phi_\beta = \phi_{\tau\otimes\eta_\beta} \boxplus \mathbf{1}_{\mathrm{GL}_1}\,, \tag{6.2}$$

which is of dimension $(2n+1)$. Note that if the central character of τ is trivial, the central character of $\tau \otimes \eta_\beta$ is also trivial. Then we have the following result related to the global GGP conjecture:

Theorem 6.2. *Let τ be an isobaric sum automorphic representation $\tau = \tau_1 \boxplus \cdots \boxplus \tau_t$ of $\mathrm{GL}_{2n}(\mathbb{A})$ such that the τ_i's are distinct irreducible unitary cuspidal automorphic representations of $\mathrm{GL}_{n_i}(\mathbb{A})$ of orthogonal type with $n_i > 1$, and assume that $\omega_\tau = \mathbf{1}$. We also allow $n_i = 1$ if $n = 1$. Let $\phi_0 = (\tau_0, 1)$ be a generic global Arthur parameter of symplectic type of dimension 2 corresponding to a unitary cuspidal automorphic representation τ_0 of $\mathrm{GL}_2(\mathbb{A})$, and assume that*

$$L(\frac{1}{2}, \tau \times \tau_0) \neq 0.$$

Then for any automorphic member $\widetilde{\sigma} \in \widetilde{\Pi}^{\psi}_{\phi_0}[\mathrm{Mp}_2(\mathbb{A})]$, *there exist a* $[\beta] \in F^\times/(F^\times)^2$ *and a representation* $\pi \in \widetilde{\Pi}_{\phi_\beta}[\mathrm{Sp}_{2n}(\mathbb{A})]$ *such that the Fourier-Jacobi period*

$$\mathcal{P}^{\mathrm{FJ}}_{\psi,\varphi_1}(\boldsymbol{\phi}_\pi, \boldsymbol{\phi}_{\widetilde{\sigma}^\beta}) \neq 0 \quad (\boldsymbol{\phi}_\pi \in \pi,\ \boldsymbol{\phi}_{\widetilde{\sigma}^\beta} \in \widetilde{\sigma}^\beta)$$

for some choice of data. Here the notation of Fourier-Jacobi periods are introduced in §2.3, *and the twist* $\widetilde{\sigma}^\beta$ *is introduced in* §2.2.

Proof. By assumption, $L_\psi(1/2, \phi_\tau \times \phi_0) = L(1/2, \tau \times \tau_0) \neq 0$. We take an automorphic representation $\widetilde{\sigma} \in \widetilde{\Pi}^{\psi}_{\phi_0}[\mathrm{Mp}_2(\mathbb{A})]$, and consider the pair $(\tau, \widetilde{\sigma})$, this is under the situation of **(Case I)** with the additional condition $\omega_\tau = \mathbf{1}$. In this case, we have a non-zero residual representation $\widetilde{\mathcal{E}}_{\tau\otimes\widetilde{\sigma}}$, and hence get a non-zero cuspidal automorphic representation

$$\pi_\beta = \mathcal{D}^{\mathrm{FJ}}_{\psi_n,\beta}(\widetilde{\mathcal{E}}_{\tau\otimes\widetilde{\sigma}}) \simeq \mathcal{D}^{\mathrm{FJ}}_{\psi_n}(\widetilde{\mathcal{E}}_{\tau\otimes\widetilde{\sigma}^\beta})$$

of $\mathrm{Sp}_{2n}(\mathbb{A})$ for some choice of $[\beta] \in F^\times/(F^\times)^2$ (see (2.13)). Let π be any irreducible component of π_β, then $\pi \in \widetilde{\Pi}_{\phi_\beta}[\mathrm{Sp}_{2n}(\mathbb{A})]$ by Proposition 4.3, and it has a non-zero Fourier-Jacobi period $\mathcal{P}^{\mathrm{FJ}}_{\psi,\varphi_1}(\boldsymbol{\phi}_\pi, \boldsymbol{\phi}_{\widetilde{\sigma}^\beta})$ for some choice of data by Proposition 2.1. This proves the theorem. □

Moreover, if we do not twist $\widetilde{\sigma}$ by $\beta \in F^\times$ as above and just use the descent module $\mathcal{D}^{\mathrm{FJ}}_{\psi_n,\beta}(\widetilde{\mathcal{E}}_{\tau\otimes\widetilde{\sigma}})$, we will get a non-zero Fourier-Jacobi period

$$\mathcal{P}^{\mathrm{FJ}}_{\psi,\beta,\varphi_1}(\boldsymbol{\phi}_\pi, \boldsymbol{\phi}_{\widetilde{\sigma}}) \quad (\boldsymbol{\phi}_\pi \in \pi, \boldsymbol{\phi}_{\widetilde{\sigma}} \in \widetilde{\sigma})$$

by Proposition 2.1. Then we get Theorem 1.4 in §1.3, which asserts that $(\pi, \widetilde{\sigma})$ gives a GGP pair in global Vogan packet $\widetilde{\Pi}^{\psi^\beta}_{\phi\times\phi_0}[\mathrm{Sp}_{2n}(\mathbb{A}) \times \mathrm{Mp}_2(\mathbb{A})]$ for any irreducible component π of π_β.

We remark also that the choice of $[\beta] \in F^\times/(F^\times)^2$ in the above theorem depends on the choice of $\widetilde{\sigma} \in \widetilde{\Pi}^{\psi}_{\phi_0}[\mathrm{Mp}_2(\mathbb{A})]$, and may not be unique. Furthermore, in this case, if we assume the uniqueness part of the local GGP conjecture (which is known for non-Archimedean cases, see the statements below), we can also show that π_β is irreducible, and hence the pair $(\pi_\beta, \widetilde{\sigma})$ gives the GGP pair in the global Vogan packet $\widetilde{\Pi}^{\psi^\beta}_{\phi_\beta\times\phi_0}[\mathrm{Sp}_{2n}(\mathbb{A}) \times \mathrm{Mp}_2(\mathbb{A})]$.

We introduce the local GGP conjecture in our particular case briefly, one may refer to [14] for more details. Let $\mathbf{k}$ be a local field of characteristic 0, and fix a non-trivial additive character $\psi : \mathbf{k} \longrightarrow \mathbb{C}^\times$. Let $\phi' \in \widetilde{\Phi}(\mathrm{Sp}_{2n}(\mathbf{k}))$ be a generic local L-parameter of orthogonal type of dimension $(2n+1)$, and $\phi'_0 \in \widetilde{\Phi}(\mathrm{Mp}_2(\mathbf{k}))$ be a generic local L-parameter of symplectic type of dimension 2. The local GGP conjecture in this case asserts:

Conjecture 6.3. *There exists a unique pair* $(\pi', \widetilde{\sigma}')$ *in the local Vogan L-packet*

$$\widetilde{\Pi}^{\psi}_{\phi'\times\phi'_0}[\mathrm{Sp}_{2n}(\mathbf{k}) \times \mathrm{Mp}_2(\mathbf{k})]$$

such that the Hom-space

$$\mathrm{Hom}_{N^{(n)}_{n-1}(\mathbf{k})\rtimes\mathrm{Mp}_2(\mathbf{k})}(\pi' \otimes \widetilde{\sigma}' \otimes \overline{\psi^{(n)}_{n-2} \otimes \omega^{(1)}_\psi}, \mathbb{C}) \tag{6.3}$$

is non-zero. Here $\omega^{(1)}_\psi$ *is the local Weil representation of* $\mathcal{H}_3(\mathbf{k}) \rtimes \mathrm{Mp}_2(\mathbf{k})$ *with respect to* ψ, *and the other notation is the same as in* §2.3 *and* §3.1.

The above conjecture is known (in general) if $\mathbf{k}$ is non-Archimedean (see [5]). For later use, we give a remark on the uniqueness part of Conjecture 6.3 in the case of $n = 1$.

Remark 6.4. *Let $\mathbf{k}$ be an Archimedean local field, and $\psi : \mathbf{k} \longrightarrow \mathbb{C}^\times$ be a fixed non-trivial additive character. Based on the knowledge of local theta correspondence in this low-rank case, the uniqueness part of local GGP conjecture for the pair $(\mathrm{SL}_2(\mathbf{k}), \mathrm{Mp}_2(\mathbf{k}))$ can be deduced from the see-saw identity and the uniqueness in the orthogonal case for the pair $(\mathrm{SO}_3(\mathbf{k}), \mathrm{SO}_2(\mathbf{k}))$ (see Lemma 6.5 below).*

Lemma 6.5. *Let $\mathbf{k}$ be an Archimedean local field. Let $\phi = \phi_1 \boxplus \eta \in \widetilde{\Phi}(\mathrm{SL}_2(\mathbf{k}))$ and $\phi_0 \in \widetilde{\Phi}(\mathrm{Mp}_2(\mathbf{k}))$ be generic local parameters, where η is a quadratic character. Then there exists at most one pair $(\pi, \widetilde{\sigma}) \in \widetilde{\Pi}^{\psi}_{\phi\times\phi_0}[\mathrm{SL}_2(\mathbf{k}) \times \mathrm{Mp}_2(\mathbf{k})]$ such that the* Hom*-space*

$$\mathrm{Hom}_{\mathrm{Mp}_2(\mathbf{k})}(\pi \otimes \widetilde{\sigma} \otimes \overline{\omega}_{\psi}^{(1)}, \mathbb{C}) \tag{6.4}$$

is non-zero.

Proof. Consider the see-saw diagram

$$\begin{tikzcd}
\mathrm{SL}_2(\mathbf{k}) \times \mathrm{Mp}_2(\mathbf{k}) \arrow[drr, dash] & & \mathrm{O}(\mathbf{V}_3^\bullet) \\
\mathrm{Mp}_2(\mathbf{k}) \arrow[u] \arrow[urr, dash] & & \mathrm{O}(\mathbf{V}_2^\bullet) \times \mathrm{O}(\langle 1 \rangle). \arrow[u]
\end{tikzcd}$$

Here $\mathbf{V}_2^\bullet$ is a quadratic space over $\mathbf{k}$ of dimension 2, and $\mathbf{V}_3^\bullet$ is a relevant quadratic space over $\mathbf{k}$ of dimension 3.

Let $(\pi, \widetilde{\sigma}) \in \widetilde{\Pi}^{\psi}_{\phi\times\phi_0}[\mathrm{SL}_2(\mathbf{k}) \times \mathrm{Mp}_2(\mathbf{k})]$ such that the Hom-space (6.4) is non-zero. By [61, Corollary 23], [71, §7] and [2], there exist a quadratic space $\mathbf{V}_2^\bullet$ with $\eta_{\mathbf{V}_2^\bullet} = \eta$, and an irreducible admissible representation σ' of $\mathrm{O}(\mathbf{V}_2^\bullet)$ such that $\pi = \theta_\psi(\sigma')$. Here $\eta_{\mathbf{V}_2^\bullet}$ is the quadratic character associated to the quadratic space $\mathbf{V}_2^\bullet$. Then the see-saw identity implies

$$\begin{aligned} 0 \neq \mathrm{Hom}_{\mathrm{Mp}_2(\mathbf{k})}(\pi \otimes \widetilde{\sigma} \otimes \overline{\omega}_{\psi}^{(1)}, \mathbb{C}) &\simeq \mathrm{Hom}_{\mathrm{Mp}_2(\mathbf{k})}(\pi \otimes \overline{\omega}_{\psi}^{(1)}, \widetilde{\sigma}^\vee) \\ \simeq \mathrm{Hom}_{\mathrm{O}(\mathbf{V}_2^\bullet)}(\theta_\psi(\widetilde{\sigma}^\vee), \sigma') &\subset \mathrm{Hom}_{\mathrm{SO}(\mathbf{V}_2^\bullet)}(\theta_\psi(\widetilde{\sigma}^\vee), \sigma'). \end{aligned} \tag{6.5}$$

Moreover, we have $(\theta_\psi(\widetilde{\sigma}^\vee)|_{\mathrm{SO}(\mathbf{V}_3^\bullet)}, \sigma'|_{\mathrm{SO}(\mathbf{V}_2^\bullet)}) \in \widetilde{\Pi}_{\phi_0\times\phi_1}[\mathrm{SO}_3(\mathbf{k}) \times \mathrm{SO}_2(\mathbf{k})]$ (see [1], [76] and [62]). Suppose that there exist distinct pairs $(\pi_i, \widetilde{\sigma}_i) \in \widetilde{\Pi}^{\psi}_{\phi\times\phi_0}[\mathrm{SL}_2(\mathbf{k}) \times \mathrm{Mp}_2(\mathbf{k})]$ $(i = 1, 2)$ such that (6.4) is non-zero. By (6.5), there exist different pairs $(\theta_\psi(\widetilde{\sigma}_i^\vee), \sigma_i') \in \widetilde{\Pi}_{\phi_0\times\phi_1}[\mathrm{SO}_3(\mathbf{k}) \times \mathrm{SO}_2(\mathbf{k})]$ $(i = 1, 2)$ such that

$$\mathrm{Hom}_{\mathrm{SO}(\mathbf{V}_2^\bullet)}(\theta_\psi(\widetilde{\sigma}_i^\vee), \sigma_i') \neq 0.$$

Here we have used the injectivity of the local theta correspondence (see [36, Theorem 1]). But this contradicts to the uniqueness of GGP pair for $(\mathrm{SO}_3(\mathbf{k}), \mathrm{SO}_2(\mathbf{k}))$, which is known by the work of Waldspurger [75] (see also [64, Theorem 4]). □

In general, if the uniqueness part of Conjecture 6.3 is known for Archimedaen cases, then one obtains the uniqueness of the global GGP pair (for fixed parameters).

Proposition 6.6. *Let $(\tau, \widetilde{\sigma})$ be as in* **(Case I)** *with $\omega_\tau = \mathbf{1}$. Assume the uniqueness part of Conjecture 6.3 is true for Archimedean cases. Then the descent π_β is irreducible, and lies in the global Vogan packet $\widetilde{\Pi}_{\phi_\beta}[\mathrm{Sp}_{2n}(\mathbb{A})]$.*

Proof. By Proposition 4.3, we know that all irreducible components of π_β are parametrized by the global Arthur parameter $\phi_\beta = \phi_{\tau\otimes\eta_\beta} \boxplus \mathbf{1}$. It remains to show that π_β is irreducible. We already know that π_β is cuspidal by Proposition 4.2. By multiplicity one theorems for Fourier-Jacobi models (see [3, 69, 70]), we have a multiplicity free direct sum decomposition of irreducible representations:

$$\pi_\beta = \pi_1 \oplus \pi_2 \oplus \cdots , \tag{6.6}$$

with $\pi_i \in \widetilde{\Pi}_{\phi_\beta}[\mathrm{Sp}_{2n}(\mathbb{A})]$. Moreover, each π_i has a non-zero Fourier-Jacobi period with $\widetilde{\sigma}$. Then at each place v of F, $(\pi_{i,v}, \widetilde{\sigma}_v)$ gives a pair in the local Vagan packet

$$\widetilde{\Pi}^{\psi_v}_{\phi_{\beta,v}\times\phi_{0,v}}[\mathrm{Sp}_{2n}(F_v)\times \mathrm{Mp}_2(F_v)]$$

such that the Hom-space (6.3) is non-zero. By the uniqueness part of the local GGP conjecture, we must have $\pi_{i,v} \simeq \pi_{j,v}$ for all indices i, j and all places v. This implies that all π_i's are equivalent, and hence the multiplicity free property shows that there would be only one summand in the direct sum decomposition (6.6), i.e. π_β is irreducible. □

Remark 6.7. *We remark here that the above results are based on the complete determination of the global Arthur parameter of the descent. However, if $\omega_\tau \neq \mathbf{1}$ (in* **(Case I)***), the unramified calculations in §3 can not provide enough information to determine the global parameters, and the irreducibility may not be guaranteed since there might be more possibilities of global parameters.*

Another application to our construction is the non-vanishing of quadratic twists of L-functions. Let

$$\phi_{0,\beta} = \phi_{\tau_0\otimes\eta_\beta} \tag{6.7}$$

be the twist of the global Arthur parameter ϕ_0 by the quadratic character η_β. Then one has $\widetilde{\sigma}^\beta \in \widetilde{\Pi}^\psi_{\phi_{0,\beta}}[\mathrm{Mp}_2(\mathbb{A})]$. Since we have shown that

$$(\pi, \widetilde{\sigma}^\beta) \in \widetilde{\Pi}^\psi_{\phi_\beta\times\phi_{0,\beta}}[\mathrm{Sp}_{2n}(\mathbb{A})\times \mathrm{Mp}_2(\mathbb{A})]$$

has a non-zero Fourier-Jacobi period, then by the main result of [73] (for the case $n=1$) (or [48, Theorem 5.4]) and also (2.18)) one also have

$$L_\psi(\frac{1}{2}, \phi_\beta\times\phi_{0,\beta}) = L_\psi(\frac{1}{2}, (\tau\otimes\eta_\beta)\times\widetilde{\sigma}^\beta)L_\psi(\frac{1}{2}, \widetilde{\sigma}^\beta) \neq 0. \tag{6.8}$$

Moreover, by [26, §3.1, Page 198], we have

$$L_\psi(\frac{1}{2}, (\tau\otimes\eta_\beta)\times\widetilde{\sigma}^\beta)L_\psi(\frac{1}{2}, \widetilde{\sigma}^\beta) = L(\frac{1}{2}, \tau\times\tau_0)L_{\psi^\beta}(\frac{1}{2}, \widetilde{\sigma}) \neq 0.$$

Note that the above non-vanishing properties of L-functions show that $\widetilde{\sigma}$ must be ψ^β-generic.

Recall that we have taken $\phi_0 = \phi_{\tau_0}$, where τ_0 is an irreducible unitary cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A})$ of symplectic type. We can obtain the following:

Theorem 6.8. *Let τ_0 be an irreducible unitary cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A})$ of symplectic type, such that*

$$\varepsilon(\frac{1}{2}, \tau_0\otimes\eta_0) = 1 \tag{6.9}$$

for some quadratic character $\eta_0 : F^\times\backslash\mathbb{A}^\times \longrightarrow \mathbb{C}^\times$. Then there exist $\sharp\widetilde{\Pi}^\psi_{\phi_{\tau_0}}[\mathrm{Mp}_2(\mathbb{A})]$ different quadratic characters η such that

$$L(\frac{1}{2}, \tau_0\otimes\eta) \neq 0.$$

Proof. The proof has two steps. First, we take a quadratic extension $E_\delta = F(\sqrt{\delta})$ ($\delta \in F^\times - (F^\times)^2$) over F, and consider the restriction of τ_0 to $E^1_\delta(\mathbb{A})$ which is isomorphic to the anisotropic $\mathrm{SO}^\delta_2(\mathbb{A})$. By construction and the spectral decomposition, there exists an irreducible component χ of $\tau_0|_{E^1_\delta(\mathbb{A})}$ such that the global Bessel period

$$\int_{E^1_\delta\backslash E^1_\delta(\mathbb{A})} \boldsymbol{\phi}_{\tau_0}(t)\chi^{-1}(t)\,\mathrm{d}t$$

is non-zero for some $\boldsymbol{\phi}_{\tau_0} \in \tau_0$. Then Waldspurger's work [75] (see also [42, Theorem 5.3] or [48, Theorem 6.9]) gives that $L(1/2, \tau_0 \times \chi) \neq 0$. Moreover, χ lifts to an irreducible generic automorphic representation τ_δ of $\mathrm{GL}_2(\mathbb{A})$ (see [10]), which is of orthogonal type, and we have $L(1/2, \tau_\delta \times \tau_0) \neq 0$. Here $\omega_{\tau_\delta} = \eta_\delta$ is non-trivial.

Let $\widetilde{\sigma}$ be an irreducible cuspidal automorphic representation in $\widetilde{\Pi}^\psi_{\phi_{\tau_0}}[\mathrm{Mp}_2(\mathbb{A})]$, whose existence is guaranteed by (6.9). Starting from the data $(\tau_\delta, \widetilde{\sigma})$, one construct the descent $\mathcal{D}^{\mathrm{FJ}}_{\psi_1,\alpha}(\widetilde{\mathcal{E}}_{\tau_\delta\otimes\widetilde{\sigma}})$ for some $\alpha \in F^\times$, which is cuspidal by Proposition 4.2. By Proposition 3.2 (for $n = 1$) and apply Arthur's classification theory as in the proof of Proposition 4.3, any irreducible component π' of $\mathcal{D}^{\mathrm{FJ}}_{\psi_1,\alpha}(\widetilde{\mathcal{E}}_{\tau_\delta\otimes\widetilde{\sigma}})$ has a generic Arthur parameter ϕ'_α. Note that ϕ'_α has the form $\phi_{\tau'} \boxplus \eta_\gamma$ for some generic automorphic representation τ' of $\mathrm{GL}_2(\mathbb{A})$ with $\omega_{\tau'} = \eta_\gamma$ ($\gamma \in F^\times$). Moreover, we have

$$L_\psi(\frac{1}{2}, \phi'_\alpha \times \phi_{0,\alpha}) \neq 0$$

by the main result of [73] or [48]. Here $\phi_{0,\alpha}$ is similar to (6.7). Then it follows from the identities in (6.8) that there exists a quadratic character η_0, possibly trivial, such that

$$L(\frac{1}{2}, \tau_0 \otimes \eta_0) \neq 0. \tag{6.10}$$

The second step is to show the statement in the theorem granting that there is a quadratic character $\eta_0 : F^\times\backslash\mathbb{A}^\times \longrightarrow \mathbb{C}^\times$ (possibly trivial) such that (6.10) holds. Under this condition, we can find an irreducible generic automorphic representation τ of $\mathrm{GL}_2(\mathbb{A})$ of orthogonal type with trivial central character, such that

$$L(\frac{1}{2}, \tau \times \tau_0) \neq 0.$$

In fact, we can view η_0 as a representation of F-split $\mathrm{SO}_2(\mathbb{A})$, and lift η_0 to an irreducible generic automorphic representation τ of $\mathrm{GL}_2(\mathbb{A})$ (see [8, 10]), which is of orthogonal type and has trivial central character. Moreover, the condition $L(1/2, \tau \otimes \eta_0) \neq 0$ implies that $L(1/2, \tau \times \tau_0) \neq 0$.

For any irreducible cuspidal automorphic representation $\widetilde{\sigma} \in \widetilde{\Pi}^\psi_{\phi_{\tau_0}}[\mathrm{Mp}_2(\mathbb{A})]$, by Theorem 5.11 and Proposition 6.6, one can construct an element $\pi_\beta \in \widetilde{\Pi}_{\phi_\beta}[\mathrm{SL}_2(\mathbb{A})]$ for some $\beta \in F^\times/(F^\times)^2$ via automorphic descent, such that $(\pi_\beta, \widetilde{\sigma}^\beta)$ gives the unique GGP pair in the packet $\widetilde{\Pi}^\psi_{\phi_\beta\times\phi_{0,\beta}}[\mathrm{SL}_2(\mathbb{A})\times \mathrm{Mp}_2(\mathbb{A})]$. Then (6.8) tells that there exists a quadratic character η_β such that

$$L(\frac{1}{2}, \tau_0 \otimes \eta_\beta) \neq 0. \tag{6.11}$$

Recall that for any $\widetilde{\sigma} \in \Pi^\psi_{\phi_{\tau_0}}[\mathrm{Mp}_2(\mathbb{A})]$, one has $L_\psi(s, \widetilde{\sigma}^\beta) = L(s, \tau_0 \otimes \eta_\beta)$.

It suffices to show that there are $\sharp\widetilde{\Pi}^\psi_{\phi_{\tau_0}}[\mathrm{Mp}_2(\mathbb{A})]$ different such quadratic twists. If we take $\widetilde{\sigma}_1, \widetilde{\sigma}_2 \in \widetilde{\Pi}^\psi_{\phi_0}[\mathrm{Mp}_2(\mathbb{A})]$ with $\widetilde{\sigma}_1 \not\cong \widetilde{\sigma}_2$, we get $\pi_{\beta_1} \in \widetilde{\Pi}_{\phi_{\beta_1}}[\mathrm{SL}_2(\mathbb{A})]$ and $\pi_{\beta_2} \in \widetilde{\Pi}_{\phi_{\beta_2}}[\mathrm{SL}_2(\mathbb{A})]$ for some $[\beta_1], [\beta_2] \in F^\times/(F^\times)^2$. Since we have $\widetilde{\sigma}_1 \not\cong \widetilde{\sigma}_2$, we must have $[\beta_1] \neq [\beta_2]$, otherwise we will have $\phi_{\beta_1} = \phi_{\beta_2}$ and $\phi_{0,\beta_1} = \phi_{0,\beta_2}$, and this contradicts to the uniqueness of global GGP pairs (see Remark 6.4 and Lemma 6.5). Then it follows from (6.8) that different elements in the global Vogan packet $\widetilde{\Pi}^{\psi}_{\phi_{\tau_0}}[\mathrm{Mp}_2(\mathbb{A})]$ give different η_β's such that (6.11) holds, and hence there are $\sharp\widetilde{\Pi}^{\psi}_{\phi_{\tau_0}}[\mathrm{Mp}_2(\mathbb{A})]$ different twists. □

The above proof gives a new approach to show the existence of different quadratic twists of automorphic L-functions of PGL_2 such that their special values at $s = 1/2$ are non-zero. This approach makes use of the information of the global packet $\widetilde{\Pi}^{\psi}_{\phi_{\tau_0}}[\mathrm{Mp}_2(\mathbb{A})]$, and decodes the non-vanishing of L-values from non-vanishing of automorphic descent constructions. We expect that it could shed some light on the higher rank cases.

6.2. **On the reciprocal branching problem.** In this last section, as another application, we record the result that the automorphic descent π_β provides answers to the reciprocal branching problem for automorphic representations of symplectic groups introduced in §1.2.

Theorem 6.9. *Let τ be an isobaric sum automorphic representation $\tau = \tau_1 \boxplus \cdots \boxplus \tau_t$ of $\mathrm{GL}_m(\mathbb{A})$ such that the τ_i's are distinct irreducible unitary cuspidal automorphic representations of $\mathrm{GL}_{n_i}(\mathbb{A})$ of orthogonal type with $n_i > 1$; and $\widetilde{\sigma} \in \widetilde{\Pi}^{\psi}_{\phi_0}[\mathrm{Mp}_2(\mathbb{A})]$ be an irreducible unitary genuine cuspidal automorphic representation with a generic Arthur parameter ϕ_0, such that*

$$L_\psi(\frac{1}{2}, \tau \times \widetilde{\sigma}) \neq 0.$$

We assume in addition that $\eta_\beta \neq \omega_\tau$ if $m = 2n+1$, here $\beta \in F^\times$ is the one occurring in Proposition 5.11. Then any irreducible component π of the automorphic descent $\pi_\beta = \mathcal{D}^{\mathrm{FJ}}_{\psi_n,\beta}(\widetilde{\mathcal{E}}_{\tau\otimes\widetilde{\sigma}})$ has a generic global Arthur parameter, and has a non-zero Fourier-Jacobi period with $\widetilde{\sigma}$. In particular, π gives an answer to the reciprocal branching problem for the pair $(\mathrm{Sp}_{2(m-n)}(\mathbb{A}), \mathrm{Mp}_2(\mathbb{A}))$.

Proof. The theorem is a direct corollary of the results we have obtained in §2 – §5. More precisely, we know that π is cuspidal by Proposition 4.2, π has a generic global Arthur parameter by Proposition 4.3, and π has a non-zero Fourier-Jacobi period with respect to $\widetilde{\sigma}$ by Proposition 2.1. □

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Department of Mathematics, Purdue University, 150 N. University St, West Lafayette, IN, 47907, USA

E-mail address: liu2053@purdue.edu

School of Mathematics, Sichuan University, No. 29 Wangjiang Road, Chengdu 610064, P. R. China

E-mail address: binxu@scu.edu.cn