MATH 266 PROJECT 2: RLC CIRCUIT

In a series RLC circuit,

- $R$ represents the resistance
- $L$ represents the inductance
- $C$ represents the capacitance
- $Q(t)$ is the charge on the capacitor
- $E(t)$ is the applied voltage
- $I(t) = Q'(t)$ is the current

We can write down the differential equations using Kirchhoff’s Laws:

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{1}{C}Q = E(t).$$

Assume that $L = 1$, $C = 1/5$, $R = 4$ and $E(t) = 10\cos\omega t$, we obtain the equation

$$Q'' + 4Q' + 5Q = 10\cos\omega t$$

and we take the initial condition $Q(0) = 0$, $Q'(0) = 0$.

1. Numerical data

Let $A(\omega)$ be the maximum of $|Q(t)|$ for $t \in [30, 80]$. Then we have the following table from numerical solution of $Q$.

<table>
<thead>
<tr>
<th>$\omega$</th>
<th>$A$</th>
<th>$\omega$</th>
<th>$A$</th>
<th>$\omega$</th>
<th>$A$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>2.0</td>
<td>2.5</td>
<td>0.99</td>
<td>6</td>
<td>0.25</td>
</tr>
<tr>
<td>0.5</td>
<td>1.94</td>
<td>3</td>
<td>0.79</td>
<td>7</td>
<td>0.19</td>
</tr>
<tr>
<td>1</td>
<td>1.77</td>
<td>3.5</td>
<td>0.63</td>
<td>8</td>
<td>0.15</td>
</tr>
<tr>
<td>1.5</td>
<td>1.51</td>
<td>4</td>
<td>0.51</td>
<td>12</td>
<td>0.07</td>
</tr>
<tr>
<td>2</td>
<td>1.24</td>
<td>5</td>
<td>0.35</td>
<td>16</td>
<td>0.04</td>
</tr>
</tbody>
</table>

2. Analytic solutions

The general solution for (1) is given by

$$Q = Q_c + Q_p, \quad Q_c = C_1e^{-2t}\cos t + C_2e^{-2t}\sin t, \quad Q_p = A\cos\omega t + B\sin\omega t.$$
To find $A$ and $B$, we compute that

$$Q''_p + 4Q'_p + 5Q_p = (-A\omega^2 + 4B\omega + 5A)\cos\omega t + (-B\omega^2 - 4A\omega + 5B)\sin\omega t.$$ 

Therefore we have

$$(5 - \omega^2)A + 4\omega B = 10, \quad -4\omega A + (5 - \omega^2)B = 0$$

and thus

$$A = \frac{10(5 - \omega^2)}{(5 - \omega^2)^2 + 16\omega^2}, \quad B = \frac{40\omega}{(5 - \omega^2)^2 + 16\omega^2}.$$ 

Thus the steady-state solution to the equation (1) is given by

$$Q_p = \frac{10(5 - \omega^2)\cos\omega t + 40\omega\sin\omega t}{(5 - \omega^2)^2 + 16\omega^2} = \frac{10}{\sqrt{(5 - \omega^2)^2 + 16\omega^2}}\cos(\omega t - \delta)$$

where

$$\delta = \arccos\frac{5 - \omega^2}{\sqrt{(5 - \omega^2)^2 + 16\omega^2}}.$$ 

Moreover, using the initial condition, we have

$$Q(0) = C_1 + A = 0, \quad Q'(0) = -2C_1 + C_2 + B\omega = 0.$$ 

Therefore

$$C_1 = -A = -\frac{10(5 - \omega^2)}{(5 - \omega^2)^2 + 16\omega^2}, \quad C_2 = 2C_1 - B\omega = -2A - B\omega = -\frac{20(5 + \omega^2)}{(5 - \omega^2)^2 + 16\omega^2}.$$ 

**Amplitude of the response.** The amplitude of the response is

$$R(\omega) = \frac{10}{\sqrt{(5 - \omega^2)^2 + 16\omega^2}} = \frac{10}{\sqrt{\omega^4 + 6\omega^2 + 25}}.$$ 

The following table of numeric values of $R$ matches the table for $A$.

<table>
<thead>
<tr>
<th>$\omega$</th>
<th>$R$</th>
<th>$\omega$</th>
<th>$R$</th>
<th>$\omega$</th>
<th>$R$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>2</td>
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<td>0.99</td>
<td>6</td>
<td>0.26</td>
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<td>0.5</td>
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</tr>
</tbody>
</table>

- As $\omega \to 0^+$, $R(\omega) \to 2$.
- As $\omega \to +\infty$, $R(\omega) \to 0$.
- $R(\omega)$ is strictly decreasing for $\omega \in (0, \infty)$.
Figure 1. $Q'' + 4Q' + 5Q = 10 \cos \omega t, \ Q(0) = Q'(0) = 0, \ \omega = 0$

Figure 2. $Q'' + 4Q' + 5Q = 10 \cos \omega t, \ Q(0) = Q'(0) = 0, \ \omega = 0.5$

Notice that in analogue to the mass-spring system, the damping ratio is $\zeta = 2/\sqrt{5}$. The system is under-damped but the damping ratio is above the threshold $\zeta_0 = \sqrt{2}/2$ that $R(\omega)$ has a maximum in $(0, \infty)$. 
Figure 3. $Q'' + 4Q' + 5Q = 10 \cos \omega t$, $Q(0) = Q'(0) = 0$, $\omega = 1$

Figure 4. $Q'' + 4Q' + 5Q = 10 \cos \omega t$, $Q(0) = Q'(0) = 0$, $\omega = 2$

Figure 5. $Q'' + 4Q' + 5Q = 10 \cos \omega t$, $Q(0) = Q'(0) = 0$, $\omega = 4$
Figure 6. $Q'' + 4Q' + 5Q = 10 \cos \omega t$, $Q(0) = Q'(0) = 0$, $\omega = 8$

Figure 7. $Q'' + 4Q' + 5Q = 10 \cos \omega t$, $Q(0) = Q'(0) = 0$, $\omega = 16$