Math 266: Forced Vibration and Resonances

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Mechanical vibration

We consider a mass-spring system with a simple harmonic external force

\[ m u'' + \gamma u' + ku = F_0 \cos \omega t. \]

Here

- \( u = u(t) \) is the displacement, a function of time \( t \)
- \( m > 0 \) is the mass
- \( \gamma \geq 0 \) is the damping constant
- \( k > 0 \) is the spring constant
- \( F_0 \) is the amplitude of the external force
- \( \omega \) is the frequency of the external force
Forced vibration with damping

First we consider the case with damping $\gamma > 0$. Then the general solution to

$$mu'' + \gamma u' + ku = F_0 \cos \omega t$$

is given by (using the method of undetermined coefficients)

$$u(t) = u_c(t) + U(t).$$

Here

- $u_c(t) = C_1 u_1(t) + C_2 u_2(t)$ is the general solution of the homogeneous equation $mu'' + \gamma u' + ku = 0$.
- $U(t) = A \cos \omega t + B \sin \omega t$ is a particular solution.

Note that $\cos \omega t$ and $\sin \omega t$ are not solutions to the homogeneous equation $mu'' + \gamma u' + ku = 0$. 
An example
Transient solutions

The part \( u_c(t) \) is the general solution to the homogeneous equation

\[ mu'' + \gamma u' + ku = 0. \]

From previous discussion and since \( \gamma > 0 \),

- \( u_c(t) \) converges to 0 exponentially.
- \( u_c(t) \) may oscillate or not, depending on the damping ratio \( \zeta = \gamma / 2\sqrt{km} \).
- The contribution of \( u_c(t) \) is negligible after a certain period of time.
- \( u_c(t) \) is called the transient solution.
Review: Free vibration

\[ u'' + 2\zeta u' + u = 0, \quad u(0) = 1, u'(0) = 0. \]
Steady-state solution

The main contribution now come from

\[ U(t) = A \cos \omega t + B \sin \omega t \]

which is a particular solution of

\[ m u'' + \gamma u' + ku = F_0 \cos \omega t. \]

- \( U(t) \) oscillates with the same frequency \( \omega \) as the external force.
- It only depends on the system and the external force, but not the initial conditions.
- All solutions eventually settle down to \( U(t) \).
- \( U(t) \) is called the steady-state solution or the forced response.
Explicit form of the steady-state solution

Since

\[ U(t) = A \cos \omega t + B \sin \omega t, \]

we have

\[ U'(t) = -A\omega \sin \omega t + B\omega \cos \omega t, \]
\[ U''(t) = -A\omega^2 \cos \omega t + B\omega^2 \sin \omega t. \]

Therefore

\[ mU'' + \gamma U' + kU = (-Am\omega^2 + B\gamma \omega + Ak) \cos \omega t + (-Bm\omega^2 - A\gamma \omega + Bk) \sin \omega t. \]

We have

\[
\begin{align*}
(k - m\omega^2)A + \gamma \omega B &= F_0 \\
-\gamma \omega A + (k - m\omega^2)B &= 0
\end{align*}
\Rightarrow
\begin{align*}
A &= \frac{F_0(k - m\omega^2)}{(k - m\omega^2)^2 + \gamma^2 \omega^2} \\
B &= \frac{F_0\gamma \omega}{(k - m\omega^2)^2 + \gamma^2 \omega^2}
\end{align*}
\]
Amplitude and phase of the steady-state solution

Now we write

\[ U(t) = A \cos \omega t + B \sin \omega t = R \cos(\omega t - \delta) \]

where

\[ A = \frac{F_0(k - m\omega^2)}{(k - m\omega^2)^2 + \gamma^2 \omega^2}, \quad B = \frac{F_0\gamma \omega}{(k - m\omega^2)^2 + \gamma^2 \omega^2} \]

and thus the amplitude is given by

\[ R = \sqrt{A^2 + B^2} = \frac{F_0}{\Delta} \]

where

\[ \Delta = \sqrt{(k - m\omega^2)^2 + \gamma^2 \omega^2} \]

while the phase \( \delta \) is given by

\[ \cos \delta = \frac{A}{R} = \frac{k - m\omega^2}{\Delta}, \quad \sin \delta = \frac{B}{R} = \frac{\gamma \omega}{\Delta} > 0. \]

Thus \( \delta \in (0, \pi) \).
Dimensionless quantities

To study the dependence of the steady-state solution (amplitude $R$, frequency $\omega$ and phase $\delta$) on the system (mass $m$, damping $\gamma$, spring constant $k$) and the external force (amplitude $F_0$ and frequency $\omega$), it is preferable to reduce the all the parameters to dimensionless quantities, i.e. ones without a unit. We introduce the following quantities

- **Damping ratio**: $\Gamma = \frac{\gamma^2}{mk} = 4\zeta^2$.
- **Frequency ratio**: $\omega/\omega_0$ where $\omega_0 = \sqrt{k/m}$ is the natural frequency of the system.
- **Amplitude ratio**: $\varphi = Rk/F_0$ where $F_0/k$ is the response to the constant function $F_0$ of the system, i.e. the constant solution to
  
  $$mu'' + \gamma u' + ku = F_0.$$

- **Phase**: $\delta$. 

Formulas for $\varphi$ and $\delta$ on $\Gamma$ and $\omega/\omega_0$

We can compute

$$\varphi = \frac{k}{\Delta} = \frac{m\omega_0^2}{\sqrt{m^2(\omega_0^2 - \omega^2) + \gamma^2\omega^2}}$$

thus

$$\varphi = \left[ \left(1 - \frac{\omega^2}{\omega_0^2}\right)^2 + \Gamma \frac{\omega^2}{\omega_0^2} \right]^{-1/2}.$$

Also,

$$\delta = \arccos \frac{k - m\omega^2}{\Delta}$$

$$\delta = \arccos \frac{1 - (\omega/\omega_0)^2}{\sqrt{(1 - (\omega/\omega_0)^2)^2 + \Gamma(\omega/\omega_0)^2}}.$$
Dependence of $\varphi$ and $\delta$ on $\Gamma$ and $\omega/\omega_0$

We plot $\varphi$ and $\delta$ against $\omega/\omega_0$ for different $\Gamma$: 

![Amplitude and Phase Graphs](attachment:image.png)
Amplitude ratio

\[ \varphi = \left[ \left( 1 - \frac{\omega^2}{\omega_0^2} \right)^2 + \Gamma \frac{\omega^2}{\omega_0^2} \right]^{-1/2}. \]
Amplitude ratio

The amplitude ratio $\varphi$ is clearly decreasing in $\Gamma$. For fixed $\Gamma$,

- As $\omega \searrow 0$, $\varphi \to 1$: For low frequency excitation, the amplitude of the force response is close to the response to the static force.
- As $\omega \nearrow \infty$, $\varphi \to 0$: For high frequency excitation, the amplitude of the force response is small.
- If $\Gamma \geq 2$, then $\varphi' < 0$ for all $\omega$, thus when the damping is large, the higher the frequency, the smaller the amplitude of the response.
- If $0 < \Gamma < 2$, then $\varphi$ has a unique maximum at frequency

$$\omega_{\text{max}} = \omega_0 \sqrt{1 - \frac{\Gamma}{2}}$$

with amplitude

$$\varphi_{\text{max}} = \left[ \Gamma \left(1 - \frac{\Gamma}{4}\right) \right]^{-1/2}.$$
The phenomenon of resonances

\[ \omega_{\text{max}} = \omega_0 \sqrt{1 - \frac{\Gamma}{2}}, \quad \varphi_{\text{max}} = \left[ \Gamma \left(1 - \frac{\Gamma}{4}\right) \right]^{-1/2}. \]

- When the damping \( \Gamma \) is small,

\[ \omega_{\text{max}} \sim \omega_0 \left(1 - \frac{\Gamma}{4}\right), \quad \varphi_{\text{max}} \sim \Gamma^{-1/2} \left(1 + \frac{\Gamma}{8}\right). \]

- Thus when the damping is small and the frequency of the external force is close to the natural frequency of the system, the response could be very large even if the external force is small. This phenomenon is called resonance.

- Dangerous in certain circumstances, e.g. for construction of bridges, etc.

- Useful in other areas, e.g. detecting earthquakes.
Phase

\[ \delta = \arccos \frac{1 - (\omega/\omega_0)^2}{\sqrt{(1 - (\omega/\omega_0)^2)^2 + \Gamma(\omega/\omega_0)^2}}. \]
For fixed $\Gamma$,

- $\delta$ is always an increasing function in $\omega/\omega_0$
- As $\omega \downarrow 0$, $\cos \delta \to 1$, and $\delta \to 0$
- When $\omega = \omega_0$, $\cos \delta = 0$ and thus $\delta = \pi/2$
- As $\omega \uparrow \infty$, $\cos \delta \to -1$ and $\delta \to \pi$.

For fixed $\omega$,

- As $\Gamma \to \infty$, $\cos \delta \to 0$ and thus $\delta \to \pi/2$
- As $\Gamma \to 0$, $\cos \delta \to 1$ and thus $\delta \to 0$ if $\omega < \omega_0$ while $\cos \delta \to -1$ and thus $\delta \to \pi$ if $\omega > \omega_0$. 
Forced vibration without damping

Now we turn to the case $\gamma = 0$:

$$mu'' + ku = F_0 \cos \omega t.$$  

The key difference from the damped case is that the general solution

$$u_c(t) = C_1 \cos \omega_0 t + C_2 \sin \omega_0 t$$

to the homogeneous equation $mu'' + ku = 0$ is no longer transient. Therefore the initial condition matters in the long-time dynamics. We consider the case with zero initial condition

$$mu'' + ku = F_0 \cos \omega t, \quad u(0) = u'(0) = 0.$$
The case $\omega \neq \omega_0$: Beats

When $\omega \neq \omega_0$, the solution to

$$mu'' + ku = F_0 \cos \omega t, \quad u(0) = u'(0) = 0$$

is given by

$$u(t) = \frac{F_0}{m(\omega_0^2 - \omega^2)} (\cos \omega t - \cos \omega_0 t)$$

$$= \frac{2F_0}{m(\omega_0^2 - \omega^2)} \sin \frac{\omega - \omega_0}{2} t \sin \frac{\omega + \omega_0}{2} t.$$

When $|\omega - \omega_0|$ is smaller comparing to $\omega + \omega_0$, then the factor $\sin \frac{\omega + \omega_0}{2} t$ oscillates more rapidly comparing to $\sin \frac{\omega - \omega_0}{2} t$, we can think that the solution $u$ has frequency $\frac{\omega + \omega_0}{2}$ with a slowly varying amplitude

$$\frac{2F_0}{m(\omega_0^2 - \omega^2)} \sin \frac{\omega - \omega_0}{2} t.$$
An example

Consider the following example

$$u'' + u = 0.5 \cos 0.8t, \quad u(0) = u'(0) = 0.$$ 

The solution is

$$u(t) = \frac{25}{9} \sin 0.1t \sin 0.9t.$$
The case $\omega = \omega_0$: Resonances

When $\omega = \omega_0$, the solution to

$$mu'' + ku = F_0 \cos \omega t, \quad u(0) = u'(0) = 0$$

is given by

$$u(t) = \frac{F_0}{2m\omega_0} t \sin \omega_0 t.$$

The function becomes unbounded as $t \to \infty$. 
Forced Nonlinear Mass-Spring System: Chaos

\[ m u'' + \gamma u' + ku + \beta u^3 = F_0 \cos \omega t. \]

Example:

\[ m = 1, \gamma = 0.3, k = 1, \beta = 1, F_0 = 0.5, \omega = 1.2. \]