MA 442: Quiz 1

Note: Definitions are given on the last page.

Name: ____________________________

Question 1. (10 points) Assume that A and B are nonempty subsets of E^*.

Show that if $(\forall x \in A)\ (\exists y \in B)\ x \leq y$, then $\sup A \leq \sup B$.

Answers

First, it is *not true* that $(\exists y \in B)\ (\forall x \in A)\ x \leq y$. This has the "exists" statement and the "for all" statement *reversed*. Consider $A = B = (0, 1)$, for which *this* statement is false, but the assumptions of the Question are true.

Direct proof: We have that $(\forall y \in B)\ y \leq \sup B$, by definition of supremum.

By the hypothesis, for each x in A, there is a y in B, such that $x \leq y$; combining with the previous statement, we have $x \leq y \leq \sup B$.

So, $(\forall x \in A)\ x \leq \sup B$. So $\sup B$ is an upper bound of A; so $\sup A$, the *least* upper bound of A, satisfies $\sup A \leq \sup B$.

Proof by contradiction: $\sup A$ is the *least* upper bound of A; so if $\sup B < \sup A$, then $\sup B$ is *not* an upper bound of A, i.e., there exists an $x \in A$ such that $\sup B < x$, which means that $(\exists x \in A)\ (\forall y \in B)\ y \leq \sup B < x$, which contradicts our assumption.

Question 2. (10 points) Assume that (X, ρ') and (Y, ρ'') are metric spaces, and let $S = X \times Y = \{(x, y) \mid x \in X,\ y \in Y\}$.
Show that if $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$ are in S, then $\rho\colon S \times S \to E^1$ defined by

$$\rho(z_1, z_2) = \rho'(x_1, x_2) + \rho''(y_1, y_2)$$

is a metric on S.

Answer: We have $\rho(z_1, z_2) = \rho'(x_1, x_2) + \rho''(y_1, y_2) \geq 0$ because $\rho'(x_1, x_2) \geq 0$ and $\rho''(y_1, y_2) \geq 0$. Furthermore, $\rho(z_1, z_2) = 0$ iff $\rho'(x_1, x_2) + \rho''(y_1, y_2) = 0$ iff $\rho'(x_1, x_2) = 0$ and $\rho''(y_1, y_2) = 0$ iff $x_1 = x_2$ and $y_1 = y_2$ iff $z_1 = z_2$.
We have $\rho(z_1, z_2) = \rho'(x_1, x_2) + \rho''(y_1, y_2) = \rho'(x_2, x_1) + \rho''(y_2, y_1) = \rho(z_2, z_1)$.
And for all z_1, z_2, and $z_3 = (x_3, y_3)$ in S,

$$\begin{aligned} \rho(z_1, z_3) &= \rho'(x_1, x_3) + \rho''(y_1, y_3) \\ &\leq \rho'(x_1, x_2) + \rho'(x_2, x_3) + \rho''(y_1, y_2) + \rho''(y_2, y_3) \\ &= \rho(z_1, z_2) + \rho(z_2, z_3). \end{aligned}$$

So it's a metric.

Question 3. (10 points) Let $S = [0, 1] \subset E^1$ with the metric $\rho(x, y) = |x - y|$ for $x, y \in S$.

Determine, with proof, whether the set $A = [0, 1/2)$ is open or closed (or possibly both) in S.

Answer: The set is open.

If $p \in [0, 1/2)$, then $0 \leq p < 1/2$.

Let $\epsilon < 1/2 - p$.

Then $G_p(\epsilon) = \{x \in S \mid |x - p| < \epsilon\}$ and each $x \in G_p(\epsilon)$ satisfies (1) $x \in S$, so $0 \leq x \leq 1$, and (2) $|x - p| < \epsilon$ so $p - \epsilon < x < p + \epsilon$. But by the choice of ϵ, $p + \epsilon < 1/2$, so $x \in [0, 1] \cap (p - \epsilon, 1/2) \subseteq A$.

So $G_p(\epsilon) \subset A$, so p is interior to A.

So A is open.

It is not closed, because $1/2 \in S - A$ and each $G_{1/2}(\epsilon)$ contains a number $\max(0, 1/2 - \epsilon) < x < 1/2$, so $x \in A$. So $1/2$ is not interior to $S - A$, so $S - A$ is not open, so A is not closed.

Axioms of the Real Numbers:

(1) (Closure)
$(\forall x, y \in E^1)$ $x + y \in E^1$ and $x \cdot y \in E^1$.

(2) (Commutativity)
$(\forall x, y \in E^1)$ $x + y = y + x$ and $x \cdot y = y \cdot x$.

(3) (Associativity)
$(\forall x, y, z \in E^1)$ $x + (y + z) = (x + y) + z$ and $x \cdot (y \cdot z) = (x \cdot y) \cdot z$.

(4) (Neutral elements)
$(\exists! 0 \in E^1)$ $(\forall x \in E^1)$ $x + 0 = x$.
$(\exists! 1 \in E^1)$ $(\forall x \in E^1)$ $x \cdot 1 = x$.

(5) (Inverses)
$(\forall x \in E^1)$ $(\exists! (-x) \in E^1)$ $x + (-x) = 0$.
$(\forall x \in E^1 \mid x \neq 0)$ $(\exists! x^{-1} \in E^1)$ $x \cdot x^{-1} = 1$.

(6) (Distributivity)
$(\forall x, y, z \in E^1)$ $(x + y) \cdot z = x \cdot z + y \cdot z$.

(7) (Trichotomy)
$(\forall x, y \in E^1)$ precisely one of $x = y$, $x < y$, or $y < x$ is true.

(8) (Transitivity)
$(\forall x, y, z \in E^1)$ $x < y$ and $y < z$ implies $x < z$.

(9) (Monotonicity)
$(\forall x, y, z \in E^1)$ $x < y$ implies $x + z < y + z$.
$(\forall x, y, z \in E^1 \mid z > 0)$ $x < y$ implies $x \cdot z < y \cdot z$.

(10) (Completeness)
Every right-bounded set $A \subseteq E^1$ has a least upper bound denoted by $\sup A$ or l.u.b. A.

Any object F that satisfies (1)–(6) is a *field*; any object F that satisfies (1)–(9) is an *ordered field*.

Definitions

(1) A *metric space* is a set $S \neq \emptyset$ with a function $\rho \colon S \times S \to E^1$ such that for all $x, y, z \in S$:
   (1) $\rho(x, y) \geq 0$, and $\rho(x, y) = 0$ iff $x = y$.
   (2) $\rho(x, y) = \rho(y, x)$.
   (3) $\rho(x, z) \leq \rho(x, y) + \rho(y, z)$.

(2) The *open ball* of radius ϵ about $p \in (S, \rho)$ is $G_p(\epsilon) = \{x \in S \mid \rho(x, p) < \epsilon\}$.

(3) A set $A \subseteq (S, \rho)$ is *open* iff $(\forall p \in A)$ $(\exists \epsilon > 0)$ $G_p(\epsilon) \subseteq A$.

(4) A set $A \subseteq (S, \rho)$ is *closed* iff $S - A$ is open.