# MA 442: Quiz 4

Name: ____________________

**Question 1**

PART 1 (5 POINTS): Determine all compact subsets of $E^1$ with the discrete metric.

**Answer:** If a set has infinitely many points, then there's an infinite sequence of distinct points in the set, and each point is distant one from the other points, so it can't cluster. So it's not compact.

If a set has finitely many points, then each infinite sequence in that set must repeat at least one of the points infinitely many times, so it has a convergent subsequence.

So the compact sets are the finite sets.

PART 2 (5 POINTS): Give an example of metric spaces $(S, \rho)$ and $(T, \rho')$ and a one-to-one, onto function $f\colon S \to T$ such that $f$ is uniformly continuous and $f^{-1}$ is continuous nowhere.

**Answer:** Let $(S, \rho)$ be $E^1$ with the discrete metric, $(T, \rho')$ be $E^1$ with the usual metric, and $f(x) = x$.

Then $(\forall x \in G_p(1/2) = \{p\})\ f(x) = f(p) \in G_{f(p)}(\epsilon)$ for any $\epsilon > 0$, so $f$ is uniformly continuous.

On the other hand, if $y \neq q$, $\rho(f^{-1}(y), f^{-1}(q)) = 1$ no matter how small $\rho'(y, q)$ is, so $f^{-1}$ is nowhere continuous.

**Question 2** You know that a subset $A$ of $E^n$ with the standard metric is compact if and only if it's closed and bounded.

PART 1 (5 POINTS): Give an example of a closed subset of $E^1$ that is not compact. (*Prove* that it's closed, and that it's not compact.)

**Answer:** Let $A = [0, +\infty)$. Then $A$ is closed, because if $p \in -A = (-\infty, 0)$ we have $G_p(|p|/2) = (3p/2, p/2) \subset -A$, and it's not compact because the sequence $\{x_n\} \subset A$ with $x_n = n$ doesn't cluster.

PART 2 (5 POINTS): Give an example of a bounded subset of $E^1$ that is not compact. (*Prove* that it's bounded, and that it's not compact.)

**Answer:** Let $A = (0, 1)$. It's bounded ($A \subset G_0(1)$) and the sequence $\{x_n\} \subset A$ with $x_n = 1/n$ doesn't cluster in $A$.

**Question 3 (10 points):** Prove using the definition that the function $f\colon E^1 \to E^1$ given by $f(x) = x^2$ is uniformly continuous on $[0, a]$ for any $a > 0$.

**Answer:** There are multiple ways to do this.

I'll just note that for $0 \leq x, p \leq a$ we have $|x|, |p| \leq a$, so $|x^2 - p^2| = |(x-p)(x+p)| \leq 2a|x-p|$.

So for any $\epsilon > 0$ choose $\delta = \epsilon/2a$; for $|x-p| < \delta$ we have $|f(x) - f(p)| = |x^2 - p^2| \leq 2a|x-p| < 2a\epsilon/2a = \epsilon$.

**Definitions**

(1) A *metric space* is a set $S$ with a function $\rho\colon S \times S \to E^1$ such that for all $x, y, z \in S$:
   (1) $\rho(x,y) \geq 0$, and $\rho(x,y) = 0$ iff $x = y$.
   (2) $\rho(x,y) = \rho(y,x)$.
   (3) $\rho(x,z) \leq \rho(x,y) + \rho(y,z)$.
(2) The *open ball* of radius $\epsilon$ about $p \in (S,\rho)$ is $G_p(\epsilon) = \{x \in S \mid \rho(x,p) < \epsilon\}$.
(3) A set $A \subseteq (S,\rho)$ is *open* iff $(\forall p \in A)$ $(\exists \epsilon > 0)$ $G_p(\epsilon) \subseteq A$.
(4) A set $A \subseteq (S,\rho)$ is *closed* iff $S - A$ is open.
(5) The usual metric on $E^n$ is given by $\rho(x,y) = |x - y|$ with the norm given by $|x| = \sqrt{x \cdot x}$ and the dot product $x \cdot y = \sum_{k=1}^{n} x_k y_k$.
(6) Given any sequence $\{x_n\} \subset E^*$, we define $\limsup\limits_{n\to\infty} x_n = \inf\limits_{n} \sup\limits_{k>n} x_k$ and $\liminf\limits_{n\to\infty} x_n = \sup\limits_{n} \inf\limits_{k>n} x_k$.
(7) A sequence in a metric space $(S,\rho)$ is Cauchy iff $(\forall \epsilon > 0)$ $(\exists K > 0)$ $(\forall m, n > K)$ $\rho(x_m, x_n) < \epsilon$.
(8) A metric space $(S,\rho)$ is *complete* iff every Cauchy sequence converges.
(9) A sequence $\{x_m\} \subset (S,\rho)$ *clusters* at $p \in S$ iff every ball $G_p(\delta)$ contains infinitely many members of the sequence.
(10) A set $A \subset (S,\rho)$ *clusters* at $p \in S$ iff every ball $G_p(\delta)$ contains infinitely many elements of $A$.
(11) A set $A \subset (S,\rho)$ is *compact* iff every sequence $\{x_n\} \subset A$ has a cluster point *in* $A$.
(12) A function $f\colon A \to (T,\rho')$ with $A \subset (S,\rho)$ is continuous at $p \in A$ iff $(\forall \epsilon > 0)$ $(\exists \delta > 0)$ $(\forall x \in G_p(\delta))$ $f(x) \in G_{f(p)}(\epsilon)$.
(13) A function $f\colon A \to (T,\rho')$ with $A \subset (S,\rho)$ is uniformly continuous on $B \subseteq A$ iff $(\forall \epsilon > 0)$ $(\exists \delta > 0)$ $(\forall x, p \in B \mid \rho(x,p) < \delta)$ $\rho'(f(x), f(p)) < \epsilon$.

**Corollaries**

(1) Every bounded sequence in $E^n$ has a cluster point.
(2) $E^n$ is complete.
(3) A set $B \subset E^n$ is compact iff it's closed and bounded.