MA 442: Quiz 5

Name: ____________________________

Question 1 (10 points): Show that every linear map $\phi\colon E^n \to E$ for any normed space E is continuous.

Answer: The continuous linear maps are the bounded linear maps, so we have with $x = (x_1, \ldots, x_n)$,

$$\begin{aligned}
|\phi(x)| &= \left|\phi\left(\sum_{k=1}^{n} x_k \vec{e}_k\right)\right| \\
&= \left|\sum_{k=1}^{n} x_k \phi(\vec{e}_k)\right| \quad \text{by linearity} \\
&\leq \sum_{k=1}^{n} |x_k||\phi(\vec{e}_k)| \quad \text{by the triangle inequality} \\
&\leq \left(\sum_{k=1}^{n} |x_k|^2\right)^{1/2} \left(\sum_{k=1}^{n} |\phi(\vec{e}_k)|^2\right)^{1/2} \quad \text{by Cauchy–Schwartz} \\
&= |x| \left(\sum_{k=1}^{n} |\phi(\vec{e}_k)|^2\right)^{1/2} \quad \text{by definition of } |x|.
\end{aligned}$$

So $\|\phi\| \leq \left(\sum_{k=1}^{n} |\phi(\vec{e}_k)|^2\right)^{1/2}$ and ϕ is continuous.

Question 2 (10 points): Define $f\colon E^2 \to E^2$ by

$$f(\vec{x}) = \begin{pmatrix} x_2 \\ -\sin(x_1) - x_2 \end{pmatrix}, \quad \vec{x} = (x_1, x_2).$$

Determine the points $\vec{x}$ where $df(\vec{x})$ is one-to-one.

Answer: We have

$$[df(\vec{x})] = \begin{pmatrix} 0 & 1 \\ -\cos x_1 & -1 \end{pmatrix}.$$

The determinant of this matrix is $\cos x_1$, which is nonzero iff $x_1 \neq \pi/2 \pm n\pi$, n an integer, so $[df(\vec{x})]$ is one-to-one for these values.

Question 3 (10 points): Let E be the space of all real-valued polynomials on $(0,1)$ with norm $|f| = \sup_{x\in(0,1)} |f(x)|$, and let the linear mapping $\phi\colon E \to E$ be defined by $\phi(f) = f'$, the derivative of f.

Show that ϕ is not a bounded map.

Answer: If $f(x) = x^n$ then $f'(x) = nx^{n-1}$, $|f| = \sup_{x\in(0,1)} |x^n| = 1$ and $|\phi(f)| = \sup_{x\in(0,1)} |nx^{n-1}| = n$.

So $\|\phi\| \geq |\phi(f)|/|f| = n$ for all n, so $\|\phi\|$ is not finite.

Definitions

(1) An *inner product space* is a vector space E over E^1 and a mapping $\cdot\colon E\times E \to E^1$ that satisfies: $x \cdot x \geq 0$ and $x \cdot x = 0$ iff $x = 0$; $x \cdot y = y \cdot x$; $(ax) \cdot y = a(x \cdot y)$; $(x + y) \cdot z = x \cdot z + y \cdot z$.

(2) The *Cauchy–Schwartz inequality* states that in any inner product space $|x \cdot y| \leq (x \cdot x)^{1/2}(y \cdot y)^{1/2}$.

(3) A *normed linear space* is a vector space with scalar field E^1 and a function $|\cdot|\colon E \to E^1$ that satisfies: $|x| \geq 0$ and $|x| = 0$ iff $x = 0$; for all $a \in E^1$ and all $x \in E$ $|ax| = |a|\,|x|$; for all $x, y \in E$, $|x + y| \leq |x| + |y|$.

(4) A *metric space* is a set S with a function $\rho\colon S \times S \to E^1$ such that for all $x, y, z \in S$:

  (1) $\rho(x, y) \geq 0$, and $\rho(x, y) = 0$ iff $x = y$.

  (2) $\rho(x, y) = \rho(y, x)$.

  (3) $\rho(x, z) \leq \rho(x, y) + \rho(y, z)$.

(5) The usual metric on E^n is given by $\rho(x, y) = |x - y|$ with the norm given by $|x| = \sqrt{x \cdot x}$ and the dot product $x \cdot y = \sum_{k=1}^{n} x_k y_k$.

(6) A sequence in a metric space (S, ρ) is Cauchy iff $(\forall \epsilon > 0)$ $(\exists K > 0)$ $(\forall m, n > K)$ $\rho(x_m, x_n) < \epsilon$.

(7) A metric space (S, ρ) is *complete* iff every Cauchy sequence converges.

(8) A function $f\colon A \to (T, \rho')$ with $A \subset (S, \rho)$ is continuous at $p \in A$ iff $(\forall \epsilon > 0)$ $(\exists \delta > 0)$ $(\forall x \in G_p(\delta))$ $f(x) \in G_{f(p)}(\epsilon)$.

(9) A function $f\colon A \to (T, \rho')$ with $A \subset (S, \rho)$ is uniformly continuous on $B \subseteq A$ iff $(\forall \epsilon > 0)$ $(\exists \delta > 0)$ $(\forall x, p \in B \mid \rho(x, p) < \delta)$ $\rho'(f(x), f(p)) < \epsilon$.

(10) A function $\phi\colon E' \to E$, with E' and E two vector spaces over E^1, is *linear* iff $\phi(ax + by) = a\phi(x) + b\phi(y)$ for all $a, b \in E^1$ and $x, y \in E'$.

(11) A linear mapping $\phi\colon E' \to E$ is *bounded* iff $\|\phi\| = \sup_{x \in E'} |\phi(x)|/|x| < \infty$.

(12) A function $f\colon E' \to E$ is *differentiable* at $\vec{p} \in E'$ iff there is a bounded linear mapping $\phi\colon E' \to E$ such that $\lim_{|\vec{t}| \to \vec{0}} |f(\vec{p} + \vec{t}) - f(\vec{p}) - \phi(\vec{t})|/|\vec{t}| = 0$.

(13) If $f\colon E^n \to E^m$ is differentiable at $\vec{p}$, the matrix $[df(\vec{p})]$ is given by $[D_j f_i(\vec{p})]_{i=1,\ldots m; j=1,\ldots,n}$.