Project 1: Part 1

Project 1 will be to calculate orthogonal polynomials. It will have several parts.

Warmup: Solving quadratic equations

The quadratic formula says that the solutions of $ax^2 + bx + c = 0$ are given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

If $b^2 - 4ac < 0$ then $\sqrt{b^2 - 4ac}$ is imaginary, so there is no problem with round-off error.

If $b^2 - 4ac > 0$ then cancellation can occur in $-b + \sqrt{b^2 - 4ac}$ if $b > 0$ and in $-b - \sqrt{b^2 - 4ac}$ if $b < 0$. Thus, if $b > 0$ one would want to use the computations

$$x_1 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}, \quad x_2 = \frac{2c}{-b - \sqrt{b^2 - 4ac}}.$$

Similarly if $b < 0$.

Write a function `(quadratic-solver a b c)` that returns a list of the two roots of $ax^2 + bx + c = 0$ as accurately as possible. Test your code on the following problems:

```
euler-126% gsi
Gambit v4.1.2
> (sqrt -1)
+i
> (sqrt +i)
.7071067811865476+.7071067811865475i
> (load "quadratic-solver")
"/export/users/lucier/programs/615project/2007/project-1/quadratic-solver.scm"
> (quadratic-solver 1 2 5)
(-1+2i -1-2i)
> (quadratic-solver 1 -1 1)
(1/2+.8660254037844386i 1/2-.8660254037844386i)
> (quadratic-solver 1 2 -1)
(-2.414213562373095 .4142135623730951)
> (quadratic-solver 4 1 1)
(-1/8+.4841229182759271i -1/8-.4841229182759271i)
> (quadratic-solver 4 4 1)
(-1/2 -1/2)
> (quadratic-solver 4 0 1)
(+1/2i -1/2i)
> (quadratic-solver 0 0 1)
*** ERROR IN (console)@11.1 -- not a quadratic:  0 0 1
1>
> (quadratic-solver 4 0 -1)
(-1/2 1/2)
> (quadratic-solver 1 3138428376721 1)
(-3.138428376721e12 -3.186308177103568e-13)
>
*** EOF again to exit
```

Meroon

Standard Scheme (so-called R5RS Scheme, which Gambit implements) does not have an object system. We use an object system provided by the software package Meroon.

To use Gambit, you need to have `/pkgs/Gambit-C/current/bin/` in your path. The Gambit interpreter is called `gsi` and the Gambit compiler is called `gsc`.

To have Gambit load Meroon automatically, just call `gsi++` or `gsc++`.

Our system has two differences with standard Meroon:

(1) In standard Meroon, keywords begin with a colon; in our Meroon keywords end with a colon:

```
(define-class Polynomial Object
  ((= variable immutable:)
   (= terms    immutable:)))
```

(2) In standard Meroon, so-called setters begin with `set-` and end with `!`. In our Meroon, setters end with `-set!`:

```
euler-130% gsi++
[ Meroon V3 Paques2001+1 $Revision: 1.2 $ ]
Gambit v4.1.2
> (define-class Point Object (x y))
Point
> (define p (make-Point 0 1))
> (unveil p)
(a Point <------------- [Id: 1]
 x: 0
 y: 1 end Point)
#t
> (Point-x-set! p 1)
#<meroon #2>
> (unveil p)
(a Point <------------- [Id: 1]
 x: 1
 y: 1 end Point)
#t
>
```

Numerical Integration

This first part will be about numerical integration (quadrature rules).

The Gauss-Lobatto quadrature rules with n points have the form

$$\int_{-1}^{1} f(x)\,dx \approx \frac{2}{n(n-1)}[f(1)+f(-1)] + \sum_{\nu=0}^{n-3} \gamma_{n\nu} f(x_{n\nu}).$$

Here $x_{n\nu}$ are the zeros of the degree $n-2$ orthogonal polynomial over $[-1,1]$ with the weight

$$w(x) = 1 - x^2.$$

If we define

$$\ell_{n\kappa}(x) = \prod_{\substack{\nu=0\\ \nu\neq\kappa}}^{n} \frac{x - x_{n\nu}}{x_{n\kappa} - x_{n\nu}}$$

then $\ell_{n\kappa}$ has degree $n-1$ and satisfies

$$\ell_{n\kappa}(x_{n\nu}) = \begin{cases} 1, & \nu = \kappa, \\ 0, & \nu \neq \kappa. \end{cases}$$

The weights $\gamma_{n\nu}$ satisfy

$$\gamma_{n\nu} = \int_{-1}^{1} \ell_{n,\nu}(x)\,dx.$$

So, the first part of the project is to write code to manipulate polynomials. We're going to start with the code at

`http://mitpress.mit.edu/sicp/full-text/sicp/book/node49.html`

and modify it to use Meroon's framework of classes/objects and generics/methods.

We'll define a polynomial class:

```
(define-class Polynomial Object
  ((= variable immutable:)
   (= terms    immutable:)))
```

and a way to check whether two Polynomial variables are the same:

```
(define (Polynomial-variable= var1 var2)
  (eq? var1 var2))
```

The terms of a polynomial is just a list of nonzero terms, in decreasing order by degree (unfortunately called "order" at that web page), so we need some code to manipulate terms and lists of terms:

```
;;; a term is a pair (order coeff) (order should really be degree, but ...)
;;; We're going to use a Meroon class for terms to aid debugging.
(define-class term Object
  ((= order)
   (= coeff)))
```

We will need some operations on lists. The functions `map` and `for-each` are built into Scheme (learn them!) but we will need some more:

```
;;; See the Haskell Wiki page
;;; http://www.haskell.org/haskellwiki/Fold
;;; for a good explanation, together with pictures, for how
;;; fold-left and fold-right work.
(define (fold-left operator initial-value list)
  (if (null? list)
      initial-value
      (fold-left operator
                 (operator initial-value (car list))
                 (cdr list))))
(define (fold-right operator initial-value list)
  (if (null? list)
      initial-value
      (operator (car list)
                (fold-right operator initial-value (cdr list)))))
;;; map is a builtin function that works on lists, but it could
;;; be defined as follows:
(define (my-map f list)
  (fold-right (lambda (v l)
                (cons (f v) l))
              '()
              list))
```

And one can use the usual functions `car`, `cdr`, `cadr`, `null?` and the usual empty list `'()`, but we need a better way to add a term to a list of terms, since we don't want any zero terms in our termlists:

```
(define (adjoin-term term term-list)
  (if (=zero? (term-coeff term))
      term-list
      (cons term term-list)))
(define (map-termlist f list)
```

```
  (fold-right (lambda (v l)
                (adjoin-term (f v) l))
              '()
              list)
```

In `map-termlist`, the function `f` must return a `term` and we let the `initialize!` method for `Polynomials` take care that the order of the terms is decreasing.

The web page has code for adding two polynomials. Putting it into our terms we define a generic function `add` that should work for everything, and we start with it working with numbers:

```
(define-generic (add (x) y)
  (if (number? x)
      (if (number? y)
          (+ x y)       ; we know how to do this
          (add y x))    ; perhaps we have a method for y
      ;; If x isn't a number, we don't have a method for it.
      (error "add: This generic is not defined on these objects: " x y)))
```

and then we define a method for adding Polynomials:

```
(define-method (add (p_1 Polynomial) p_2)
  (cond ((number? p_2)
         (add p_1 (number->Polynomial p_2 (Polynomial-variable p_1))))
        ((and (Polynomial? p_2)
              (Polynomial-variable= (Polynomial-variable p_1)
                                    (Polynomial-variable p_2)))
         (instantiate Polynomial
                      variable: (Polynomial-variable p_1)
                      terms:    (add-terms (Polynomial-terms p_1)
                                           (Polynomial-terms p_2))))
        (else
         (error "add: p_2 is neither a number nor a polynomial with the same variable as
p_1 " p_1 p_2))))
```

This method is called only when `p_1` is a polynomial; if `p_2` is a number, it converts `p_2` to a Polynomial with the same variable as `p_1` and calls `add` again with both arguments now a Polynomial.

The web page has code for `add-terms`:

```
(define (add-terms l1 l2)
  (cond ((null? l1) l2)
        ((null? l2) l1)
        (else
         (let ((t1 (car l1))
               (t2 (car l2)))
           (cond ((> (term-order t1)
                     (term-order t2))
                  (adjoin-term t1
                               (add-terms (cdr l1) l2)))
                 ((< (term-order t1)
                     (term-order t2))
                  (adjoin-term t2
                               (add-terms l1 (cdr l2))))
                 (else
                  (adjoin-term
                   (make-term (term-order t1)
                              (add (term-coeff t1)
                                   (term-coeff t2)))
                   (add-terms (cdr l1)
```

```
                              (cdr l2)))))))))
```

So you need to define `number->Polynomial`, which takes two arguments.

You need to define a `multiply` generic that works with numbers by default, and a method for `multiply` that works on Polynomials; follow ths same pattern as for `add`. The web page has the guts of the code:

```
(define (multiply-terms l1 l2)
  (if (null? l1)
      l1
      (add-terms (multiply-term-by-all-terms (car l1) l2)
                 (multiply-terms (cdr l1) l2))))
(define (multiply-term-by-all-terms t1 L)
  (if (null? L)
      L
      (let ((t2 (car L)))
        (adjoin-term
         (make-term (+ (term-order t1)
                       (term-order t2))
                    (multiply (term-coeff t1)
                              (term-coeff t2)))
         (multiply-term-by-all-terms t1 (cdr L))))))
```

So that's pretty much the code that comes on the web page. Meroon defines a generic function `show` that we can specialize for Polynomials as such:

```
(define-method (show (p Polynomial) . stream)
  (let ((port (if (null? stream)
                  (current-output-port)
                  (car stream))))
    (if (=zero? p)
        (display 0)
        (show-terms (Polynomial-variable p)
                    (Polynomial-terms p)
                    port))
    (newline port)))
(define (show-terms variable terms port)
  (show-first-term variable (car terms) port)
  (for-each (lambda (term)
              (show-term variable term port))
            (cdr terms)))
(define (show-first-term variable term port)
  (let ((coeff (term-coeff term))
        (order (term-order term)))
    (print port: port
           (list (if (and (= coeff 1)
                          (positive? order))
                     '()
                     coeff)
                 (cond ((zero? order) '())
                       ((= order 1) variable)
                       (else
                        (list variable "^" order)))))))
(define (show-term variable term port)
  (let ((coeff (term-coeff term))
        (order (term-order term)))
    (print port: port
```

```
            (list (if (negative? coeff)
                      "-"
                      "+")
                  (let ((abs-coeff (abs coeff)))
                    (if (and (eq? coeff 1)
                             (< 0 order))
                        '()
                        (abs coeff)))
                  (cond ((zero? order) '())
                        ((= order 1) variable)
                        (else
                         (list variable "^" order)))))))
```

It will probably help your debugging.

So, here are some problems.

(1) The above code uses a function `=zero?`. Define a generic function `=zero?` that handles numbers. Define a method that works with Polynomials.

(2) Define a generic function `(negate (x))` that handles numbers by default. Define a method for `negate` that works with Polynomials. Use the generic `negate` to define a regular function `(subtract x y)`. (Remember that a polynomial in `x` may have coefficients that are polynomials in `y`, so write `negate` so it works with these types of polynomials.)

(3) Define a function `(exponentiate x n)` that uses `multiply` to exponentiate anything that `multiply` can multiply. Use the discussion of exponentiation on page

`http://mitpress.mit.edu/sicp/full-text/book/book-Z-H-11.html#%_sec_1.2`

as your model. You should be able to exponentiate a polynomial.

(4) Define a function `(variable->Polynomial x)` that takes a symbol `x` and returns a Polynomial that represents the polynomial x, i.e., a single term with coefficient 1 and order 1.

(5) Define a generic function `(evaluate f x)` that evaluates the function `f` at `x`. If `f` is a number, assume that it means a function that constantly returns `f` (so the generic is supposed to work with both numbers `v` and Scheme functions `f`). Define a method for Polynomials. If `p` is a polynomial, then you should be able to say `(evaluate p p)`, i.e., evaluate a polynomial with another polynomial as an argument.

If you've done the exercises until now, something like the following should work.

```
;;;  evaluation
(define-generic (evaluate (f) x)
  (cond ((number? f) f)
        ((procedure? f) (f x))
        (else (error "evaluate: unknown argument types " f x))))
(define-method (evaluate (p Polynomial) x)
  (evaluate-terms (Polynomial-terms p) x))
(define (evaluate-terms terms x)
  (if (null? terms)
      0
      (add (evaluate-term (car terms) x)
           (evaluate-terms (cdr terms) x))))
(define (evaluate-term term x)
  (multiply (exponentiate x (term-order term))
            (term-coeff term)))
```

Can you write a method that uses Horner's rule for evaluating Polynomials in our representation?

Changes made 2012/02/27

(1) The project will just be about orthogonal polynomials.
(2) Corrected the formula for $\gamma_{n\nu}$.
(3) Changed the definition of `add` so it can add polynomials to numbers in either order.

I added some comments to the problems:

(1) Added `map-termlist` as an exercise.
(2) Made more explicit my expectations for `negate`, `exponentiate`, and `evaluate`.
(3) Redid the definition of the `evaluate` generic to match its specification.

Changes made 2012/03/07

(1) Make a `term` a Meroon object for easier debugging.
(2) `(Polynomial-terms p)` was a regular list, so I'm just going to use the regular list operations, except for `adjoin-term` and `map-termlist`, which check that the term is nonzero before adding it to the list. Got rid of `first-term`, `rest-terms`, `empty-termlist?`, etc., and changed their uses to regular list operations.
(3) Since `map-termlist` is already defined, don't have it as an exercise.
(4) Defined `fold-left` and `fold-right` earlier, so I could use them in `map-termlist`.