ERROR BOUND FOR NUMERICAL METHODS FOR THE ROF IMAGE SMOOTHING MODEL

A Dissertation

Submitted to the Faculty

of

Purdue University

by

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In Partial Fulfillment of the

Requirements for the Degree

of

Doctor of Philosophy

August 2008

Purdue University

West Lafayette, Indiana

To my parents.

ACKNOWLEDGMENTS

First of all, I would like to express my deepest sense of gratitude to my advisor, Professor Bradley J Lucier, for his guidance and inspiration throughout my PhD study. This thesis would not have been possible without his generous advice, constant support and great patience.

I would also like to thank my other committee members: Professors Zhiqiang Cai, Donatella Danielli and Monica Torres, for taking time to review my thesis and their valuable comments.

Thanks are also due to my friends, too many to be named here, for the good time we have shared in Purdue.

Finally, I take this opportunity to express my profound gratitude to my beloved parents for their love, support and encouragement during my study in Purdue.

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ABSTRACT

Wang, Jingyue Ph.D., Purdue University, August, 2008. Error Bound for Numerical Methods for the ROF Image Smoothing Model . Major Professor: Bradley J. Lucier.

The Rudin-Osher-Fatemi variational model has been extensively studied and used in image analysis. There have been several very successful numerical algorithms developed to compute the minimizer of the discrete version of the ROF energy. We study the convergence of numerical solutions of discrete total variation models to the solution of the continuous model. We use the discrete ROF energy with a symmetric discrete TV operator and obtain an error bound between the minimizer for the discrete ROF model with a symmetric TV operator and the minimizer for the continuous ROF model. Partial results are also obtained on error bounds of some non-symmetric discrete TV minimizers.

1. INTRODUCTION

1.1 Notation

One of the most influential variational models for image reconstruction is the total variation–based model developed by Rudin, Osher and Fatemi [1]. This model studies the minimizer of the following energy

$$E(u) = \int_\Omega |Du| + \frac{1}{2\lambda} \int_\Omega (u-g)^2 \, dx \,, \tag{1.1}$$

where

$$\int_\Omega |Du| := \sup_{|\phi|\le 1;\ \phi\in C_0^\infty(\Omega)} \int_\Omega u \, div\phi,$$

is the total variation of u, Ω is a bounded region in $\mathbb{R}^2$ with Lipschitz boundary. For more details on functions of bounded variation, we refer the reader to [2]. This functional is to be minimized over all $u \in L^2(\Omega)$. The function g represents the observed image, which is treated as a L^2 function. The existence and uniqueness of the minimizer have been studied by Acar and Vogel [3]. We study the case of $\Omega = [0,1]^2 := I$, the unit square.

On the computing side, the most fundamental discrete model is based on the discrete energy

$$E_k(u) = \sum_{i,j=0}^{k-1} h^2 |(\nabla u)_{i,j}| + \frac{1}{2\lambda} \sum_{i,j=0}^{k-1} h^2 (u_{i,j} - g_{i,j})^2 \,, \tag{1.2}$$

where u is defined by a 2-dimensional matrix of size $k \times k$, h is the scale factor. The space of all such discrete images is denoted by $X^k = \mathbb{R}^{k\times k}$. There are several possible choices for the discrete gradient operator ∇u ([4], [5]); one common choice is

$$(\nabla u)_{i,j} = \left((\nabla_x u)_{i,j}, (\nabla_y u)_{i,j}\right),$$

with

$$(\nabla_x u)_{i,j} = \frac{u_{i+1,j} - u_{i,j}}{h}, \quad (\nabla_y u)_{i,j} = \frac{u_{i,j+1} - u_{i,j}}{h}.$$

On the boundary, u is assumed to satisfy the discrete Neumann boundary conditions:

$$u_{-1,j} = u_{0,j}, \quad u_{k,j} = u_{k-1,j}, \tag{1.3}$$

$$u_{i,-1} = u_{i,0}, \quad u_{i,k} = u_{i,k-1}. \tag{1.4}$$

The discrete function $g_{i,j}$ is the input image.

It is not hard to show that minimizers of E_n Γ-converge to the minimizer of E, therefore, the sequence $\{u^n\}$ of minimizers of J converges to u in $L^1(I)$ and $E_n(u^n)$ converges to $E(u)$ as n tends to ∞.

In this paper, we study a slightly different version of the discrete energy, (1.2). Before we go into details, we explain the notation used throughout this paper.

We use superscripts to indicate a discrete image, for example u^k is a $k \times k$ image. When there is no ambiguity, the scale factor h in (1.2) equals $1/k$.

For a k by k discrete image u^k, we sometimes need to extend it over all indexes $-\infty \le i, j \le +\infty$. We apply the following process: first reflect u^k along the boundary $i = k - 1/2$,

$$u^k_{k+i,j} = u^k_{k-i-1,j} \qquad \text{for } 0 \le i, j \le k-1; \tag{1.5}$$

then reflect the whole image along the boundary $j = k - 1/2$,

$$u^k_{i,k+j} = u^k_{i,k-j-1} \qquad \text{for } 0 \le i \le 2k-1, 0 \le j \le k-1; \tag{1.6}$$

last, periodize it over all indexes (i, j). Notice that the extension of u^k satisfies the discrete Neumann boundary conditions (1.3) and (1.4).

Define

$$\nabla^+_x u^k_{i,j} = \frac{u^k_{i+1,j} - u^k_{i,j}}{h}, \quad \nabla^+_y u^k_{i,j} = \frac{u^k_{i,j+1} - u^k_{i,j}}{h},$$
$$\nabla^-_x u^k_{i,j} = \frac{u^k_{i,j} - u^k_{i-1,j}}{h}, \quad \nabla^-_y u^k_{i,j} = \frac{u^k_{i,j} - u^k_{i,j-1}}{h},$$

and

$$\nabla^{++}u_{i,j}^k = \begin{pmatrix} \nabla_x^+ u_{i,j}^k \\ \nabla_y^+ u_{i,j}^k \end{pmatrix}, \quad \nabla^{+-}u_{i,j}^k = \begin{pmatrix} \nabla_x^+ u_{i,j}^k \\ \nabla_y^- u_{i,j}^k \end{pmatrix},$$

$$\nabla^{-+}u_{i,j}^k = \begin{pmatrix} \nabla_x^- u_{i,j}^k \\ \nabla_y^+ u_{i,j}^k \end{pmatrix}, \quad \nabla^{--}u_{i,j}^k = \begin{pmatrix} \nabla_x^- u_{i,j}^k \\ \nabla_y^- u_{i,j}^k \end{pmatrix}.$$

We use $E_k(u^k)$ to denote the discrete energy of u^k,

$$E_k(u^k) = J_k(u^k) + \frac{1}{2\lambda}\sum_{i,j=0}^{k-1} h^2 |u_{i,j}^k - g_{i,j}^k|^2,$$

where J_k is a discrete TV operator to be chosen later.

In this paper, we study the error bound for operator J_k that can be written as a convex linear combination of J_{++}, J_{-+}, J_{+-} and J_{--}, where

$$J_{++}(u^k) = \sum_{i,j=0}^{k-1} h^2 \sqrt{(\nabla_x^+ u_{i,j}^k)^2 + (\nabla_y^+ u_{i,j}^k)^2},$$

$$J_{+-}(u^k) = \sum_{i,j=0}^{k-1} h^2 \sqrt{(\nabla_x^+ u_{i,j}^k)^2 + (\nabla_y^- u_{i,j}^k)^2},$$

$$J_{-+}(u^k) = \sum_{i,j=0}^{k-1} h^2 \sqrt{(\nabla_x^- u_{i,j}^k)^2 + (\nabla_y^+ u_{i,j}^k)^2},$$

$$J_{--}(u^k) = \sum_{i,j=0}^{k-1} h^2 \sqrt{(\nabla_x^- u_{i,j}^k)^2 + (\nabla_y^- u_{i,j}^k)^2}.$$

A special discrete TV operator is $J_\star$,

$$J_\star(u^k) = \frac{1}{4}\left(J_{++}(u^k) + J_{+-}(u^k) + J_{-+}(u^k) + J_{--}(u^k)\right);$$

this operator is invariant under horizontal or vertical reflection. In the following context we call $J_\star$ the symmetric TV operator and any other convex combination of J_{++}, J_{-+}, J_{+-} and J_{--} a non-symmetric TV operator.

The discrete input data $g_{i,j}^k$ in E_k is the discretized input image g, with

$$g_{i,j}^k = \frac{1}{|I_{i,j}|}\int_{I_{i,j}} g\,.$$

Finally, we denote the discretization of the anisotropic TV, $\int_I |D_x u| + |D_y u|$, as

$$J_a(u^k) = \sum_{i,j=0}^{k-1} h^2(|\nabla_x^+ u_{i,j}^k| + |\nabla_y^+ u_{i,j}^k|)\,. \tag{1.7}$$

1.2 Properties of the Minimizer of Continuous Energy

In this section, we assume

$$E(v) = \int_I |Dv| + \frac{1}{2\lambda}\|v-g\|^2\,.$$

We present some fundamental properties of the minimizer of E.

Lemma 1 is well known.

Lemma 1 (Contraction)

$$\|u-v\| \leq \|f-g\|\,.$$

Proof By definition of minimizer, for any w

$$(\frac{g-u}{\lambda}, w-u) \leq J(w) - J(u), \tag{1.8}$$

$$(\frac{f-v}{\lambda}, w-v) \leq J(w) - J(v), \tag{1.9}$$

put v, u for w in these inequalities respectively, and add them to get

$$(\frac{f-g+u-v}{\lambda}, u-v) \leq 0,$$

then

$$\|u-v\|^2 \leq (f-g, u-v) \leq \|f-g\|\|u-v\|,$$

the result follows. ∎

In the next lemma, we need the concept of modulus of continuity. For any function $f \in L^2(I)$, the modulus of continuity of f is defined by:

$$\omega_1(f,t)_{L^2(I)} := \sup_{0<|h|\leq t} \|f(x+h) - f(x)\|_{L^2(I_h)}, \tag{1.10}$$

where

$$I_h := \{x : x \in I, x + h \in I\}.$$

For more details on modulus of continuity and Lipschitz space, we refer the reader to [6].

Lemma 2 (Continuity of translation)

Assume u is the minimizer of E. Extend u to $\bar{u}$ over $\mathbb{R}^2$ by mirroring along each side and iterating, then

$$\|\bar{u}(x+h) - \bar{u}(x)\|_I \leq C\omega_1(g, |h|)_{L^2(I)}.$$

Proof Define $\bar{u}$ on the torus $M := (\mathbb{R}/2\mathbb{Z})^2$. It is easy to see, by this way of extending u,

$$|D\bar{u}|(M) = 4|Du|(I).$$

We do not introduce new extra total variation on the extension boundary.

The minimization problem on the torus M,

$$\min_{v \in \mathrm{BV}(M)} |Dv|(T) + \frac{1}{2\lambda} \int_M |v - \bar{g}|^2$$

is equivalent to the following minimization problem

$$\min_{v \in \mathrm{BV}(\overline{2I})} |Dv|(2I) + \int_{\gamma_1} |Dv| + \int_{\gamma_2} |Dv| + \frac{1}{2\lambda} \int_{2I} |v - \bar{g}|^2, \tag{1.11}$$

where

$$\gamma_1 = \{0\} \times (0, 2),$$
$$\gamma_2 = (0, 2) \times \{0\}.$$

The function $\bar{g}$ is the extension of g the same way as extending u.

Simple calculation shows that $\bar{u}$ is the minimizer of (1.11).

Let

$$J(v) = |Dv|(2I) + \int_{\gamma_1} |Dv| + \int_{\gamma_2} |Dv|.$$

We can easily verify that J is also convex and lower semi-continuous. Therefore, the Euler-Lagrange equation for (1.11) is also

$$\frac{g-u}{\lambda} \in \partial J(u),$$

thus we also have Lemma 1

$$\|v_1 - v_2\| \leq \|f_1 - f_2\|,$$

where v_1 is the minimizer for initial data f_1 in (1.11), v_2 is the minimizer for initial data f_2 in (1.11).

It is also easy to see that J is invariant under translation for any periodic function v with period $(2,2)$, thus

$$\begin{aligned} &J(v) + \frac{1}{2\lambda}\|v - \bar{g}(x+h)\|^2_{L^2(2I)} \\ &= J(v(x-h)) + \frac{1}{2\lambda}\|v(x-h) - \bar{g}\|^2_{L^2(2I)} \\ &\geq J(\bar{u}) + \frac{1}{2\lambda}\|\bar{u} - \bar{g}\|^2_{L^2(2I)} \\ &= J(\bar{u}(x+h)) + \frac{1}{2\lambda}\|\bar{u}(x+h) - \bar{g}(x+h)\|^2_{L^2(2I)}. \end{aligned}$$

We conclude that for any initial data $\bar{g}(x+h)$, $\bar{u}(x+h)$ is the minimizer for (1.11). By Lemma 1,

$$\begin{aligned} \|\bar{u}(x+h) - \bar{u}(x)\|_{L^2(I)} &\leq \|\bar{u}(x+h) - \bar{u}(x)\|_{L^2(2I)} \\ &\leq \|\bar{g}(x+h) - \bar{g}(x)\|_{L^2(2I)} \\ &\leq \omega_1(\bar{g}, |h|)_{L^2(I')}. \end{aligned}$$

where I' is the square $[-3,3] \times [3,3]$. Because our extension satisfies Whitney's extension theorem [6],

$$\omega_1(\bar{g}, |h|)_{L^2(I')} \leq C\omega_1(g, |h|)_{L^2(I)}.$$

Thus

$$\|\bar{u}(x+h) - \bar{u}(x)\|_I \leq C\omega_1(g, |h|)_{L^2(I)}.$$

$\blacksquare$

Remark One can conclude from Lemma 2 that

$$\omega_1(u,|h|)_{L^2(I)} \le C\omega_1(g,|h|)_{L^2(I)}. \tag{1.12}$$

Remark Similar techniques allow one to show that this result also holds for the discrete case of u^k and g^k where u^k is the minimizer of the discrete energy E_k with the symmetric discrete TV operator $J_\star$, and u^k is extended on $\mathbb{Z}^2$ as in (1.5), (1.6). In fact, the corresponding discrete version is.

$$\|T_{m_1,m_2}u^k - u^k\|_{l^2(A)} \le C\omega_1(g^k,|m_1|+|m_2|)_{l^2(A)}, \tag{1.13}$$

where A is the index set $\{(i,j) : 0 \le i,j \le k-1\}$. For any discrete image v^k, the discrete modulus of continuity is

$$\omega_1(v^k,m)_{l^2(A)} := \sup_{0<|n_1|+|n_2|\le m} \|T_{n_1,n_2}v^k - v^k\|_{l^2(A_{n_1,n_2})} \tag{1.14}$$

with T_{n_1,n_2} being the shift operator:

$$\left(T_{n_1,n_2}v^k\right)_{i,j} := v^k_{i+n_1,j+n_2},$$

and

$$A_{n_1,n_2} := \{(i,j) : (i,j) \in A, (i+n_1, j+n_2) \in A\}.$$

Remark The proof of the continuity of translation depends on the symmetry of the discrete TV operator $J_\star$. In Chapter 3, we use a different technique to obtain error bounds for some non-symmetric operators, for example $(J_{++} + J_{--})/2$ with input g satisfying

$$g \in L^\infty \cap \mathrm{Lip}(\beta, L^1(I))\,,$$

where $\mathrm{Lip}(\beta, L^1(I))$ is a Lipschitz space. We will give the definition in Chapter 2.

Lemma 3 (Maximum principle)

Suppose u^k is the minimizer of E_k where E_k is the discrete energy with either symmetric TV *operator or non-symmetric* TV *operator that we have considered. If $g^k \in L^\infty$, then*

$$\|u^k\|_\infty \le \|g^k\|_\infty.$$

Proof Let

$$m = \min_{i,j} g^k,\ M = \max_{i,j} g^k.$$

Define $Tu^k_{i,j}$ to be the truncation of $u^k_{i,j}$, with

$$Tu^k_{i,j} := \begin{cases} m & u^k_{i,j} < m, \\ u^k_{i,j} & m \le u^k_{i,j} \le M, \\ M & u^k_{i,j} > M. \end{cases}$$

We have the following property:

$$|Ta - Tb| \le |a - b|. \tag{1.15}$$

In deed, it is easy to verify that

$$|a \wedge M - b \wedge M| \le |a - b|,$$
$$|a \vee m - b \vee m| \le |a - b|.$$

Therefore,

$$|Ta - Tb| = |(a \wedge M) \vee m - (b \wedge M) \vee m| \le |a - b|.$$

Thus,

$$|Tu^k_{i+1,j} - Tu^k_{i,j}| \le |u^k_{i+1,j} - u^k_{i,j}|,$$
$$|Tu^k_{i,j+1} - Tu^k_{i,j}| \le |u^k_{i,j+1} - u^k_{i,j}|,$$

Hence

$$\sqrt{(Tu^k_{i+1,j} - Tu^k_{i,j})^2 + (Tu^k_{i,j+1} - Tu^k_{i,j})^2}$$
$$\le \sqrt{(u^k_{i+1,j} - u^k_{i,j})^2 + (u^k_{i,j+1} - u^k_{i,j})^2}.$$

Add the inequality over all indexes (i, j)'s, we obtain

$$J_{++}(Tu^k) \le J_{++}(u^k).$$

In fact, we have proved that for any linear combination of J_{++}, J_{+-}, J_{-+} and J_{--},

$$J_k = \lambda_1 J_{++} + \lambda_2 J_{+-} + \lambda_3 J_{-+} + \lambda_4 J_{--}$$

with $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \geq 0$,

$$J_k(Tu^k) \leq J_k(u^k).$$

For the L^2 term, again using (1.15)

$$|Tu^k_{i,j} - g^k_{i,j}| = |Tu^k_{i,j} - Tg^k_{i,j}| \leq |u^k_{i,j} - g^k_{i,j}|,$$

so

$$\int |Tu^k - g^k|^2 \leq \int |u^k - g^k|^2,$$

collecting all these results, we have, for any w^k,

$$E_k(Tw^k) \leq E_k(w^k),$$

that implies, if u^k is a minimizer, $Tu^k = u^k$, i.e.

$$\|u^k\|_\infty \leq \|g^k\|_\infty.$$

■

1.3 Plan of Proof of the Main Result

In this section, we assume

$$E_k(v^k) = J_k(v^k) + \frac{1}{2\lambda}\|v^k - g^k\|^2,$$

where J_k may be a symmetric or non-symmetric discrete TV operator. E_k is a discrete approximation to the continuous functional E.

To study the difference between $E_k(u^k)$ and $E(u)$, it should first be noticed that E_k and E are two different functionals defined on different spaces. E is defined on the general BV(I) space while E_k is a discrete operator defined on k by k arrays.

Therefore, some connection between these two operators should be built. We use two energy bounds to bridge them.

First, given a discrete minimizer u^k of functional E_k, we construct a "smoothed" function $\bar{u} \in L^2$ with $E(\bar{u})$ less than $E_k(u^k)$ plus some error. The construction of $\bar{u}$ is done by first "smoothing" u^k, then injecting it into L^2. We shall explain the details later. Assuming u is the minimizer of E, we have

$$E(u) \leq E(\bar{u}) \leq E_k(u^k) + e_{g,h}, \tag{$\star$}$$

where $e_{g,h}$ is the error between $E_k(u^k)$ and $E(\bar{u})$, which depends on initial g and mesh size h, and tends to zero as h tends to zero.

The second energy bound is similar but taken in the opposite direction. Based on u, we construct a "smoothed" discrete function $\tilde{u}^k$ by first "smoothing" it, then projecting it onto space X^k, with $E_k(\tilde{u}^k)$ less than $E(u)$ plus an error term $e'_{g,h}$ similar to $e_{g,h}$. By the definition of u^k, we have

$$E_k(u^k) \leq E_k(\tilde{u}^k) \leq E(u) + e'_{g,h}. \tag{$\star\star$}$$

From ($\star$) we see

$$E(u) - E_k(u^k) \leq e_{g,h}\,;$$

from ($\star\star$)

$$E_k(u^k) - E(u) \leq e'_{g,h}\,;$$

then we conclude that

$$|E_k(u^k) - E(u)| \leq \max\{e_{g,h}, e'_{g,h}\}\,.$$

This will complete our error bound.

To relate a discrete image in X^k to a continuous image in L^2, we restrict our bounded region in (1.1) to the unit square $I = [0,1] \times [0,1]$. In discrete settings, we divide I into k by k grids, each grid

$$I_{i,j} = \left[\frac{i}{k}, \frac{i+1}{k}\right] \times \left[\frac{j}{k}, \frac{j+1}{k}\right], \qquad 0 \leq i, j \leq k-1;$$

then assign to the center of each grid $I_{i,j}$ a pixel value $u^k_{i,j}$.

The "smoothing and projecting" process for a continuous function v is straightforward. The final projected function $\tilde{v}^k$ is

$$\tilde{v}^k = P_h(\phi_\epsilon * v),$$

where ϕ_ϵ is a mollifier with smoothing parameter ϵ and P_h is the projection $L^2 \to X^k$,

$$(P_h f)_{i,j} = \int_{I_{i,j}} f.$$

The "smoothing and injecting" process for a discrete function v^k is a discrete analogue to the above case. We first average v^k at each pixel (i,j) over a small square with side length $(2L+1)$ and center (i,j), then inject it into L^2 by piecewise linear interpolation with grid size $1/k$ to obtain the final function $\bar{v}^L$.

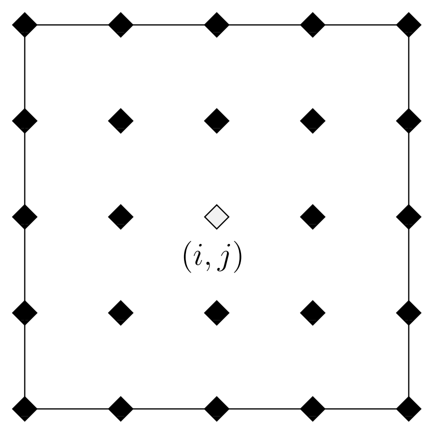

Fig. 1.1. Averaging v^k on a square with center (i,j) and $L = 2$

In our proof, the bound of the errors $e_{g,h}$ and $e'_{g,h}$ requires the following important consistency property relating the discrete TV operator $J_k(u^k)$ to the continuous operator $J(u)$:

$$J(u) = \int_I |Du|.$$

Consistency Property: For any $v^k \in X^k$ and $v \in \mathrm{BV}(I)$

$$J(\bar{v}^L) \le J_k(v^k) + \frac{Ch}{\epsilon} J_k(v^k); \qquad \text{(consistency w.r.t injection)}$$
$$J_k(\tilde{v}^k) \le J(v) + \frac{Ch}{\epsilon} J(v); \qquad \text{(consistency w.r.t projection).}$$

where $\bar{v}^L$ and $\tilde{v}^k$ are the injected and projected functions respectively and $\epsilon = Lh$.

The consistency property is used in bounding the total variation terms in $E_k(\tilde{u}^k)$ and $E(\bar{u})$ respectively. We shall show in the following chapters that our lemma both holds for the symmetric and a special class of non-symmetric discrete TV operators.

2. ERROR BOUND FOR THE SYMMETRIC DISCRETE TV OPERATOR

In this chapter, we study the error bound for the discrete energy E_k:

$$E_k(u^k) = J_\star(u^k) + \frac{1}{2\lambda}\|u^k - g^k\|^2,$$

where $J_\star$ is the symmetric discrete TV operator

$$J_\star(u^k) = \frac{1}{4}(J_{++}(u^k) + J_{+-}(u^k) + J_{-+}(u^k) + J_{--}(u^k))\,.$$

We assume throughout this paper that u is the minimizer of the continuous energy E and u^k is the minimizer of the discrete energy E_k when there is no ambiguity. The value of u^k at pixel (i,j) is denoted by $u^k_{i,j}$, with $0 \le i,j \le k-1$.

In section 2.1, we introduce the notion of injected smoothed minimizer and estimate its energy. Section 2.2 introduces the notion of projected smoothed minimizer and estimate its energy. We give the main results in section 2.3 on the bound of the energy difference $|E_k(u^k) - E(u)|$ and the error bound of $\|u^k - u\|$.

2.1 Estimate of the energy of the injected smoothed minimizer

The following Proposition is the main result of this section.

Proposition 1 *If $g \in \mathrm{Lip}(\alpha, L^2(I))$ and u^k is the minimizer of E_k, then*

$$E(u) \le E_k(u^k) + \frac{C}{\lambda}\|g\|^2_{\mathrm{Lip}(\alpha,L^2)} h^{\alpha/(\alpha+1)}\,.$$

Section 2.1.1 discusses the notion of injected discrete function and calculates its total variation, then gives the definition of the injected smoothed minimizer. Section 2.1.2 and section 2.1.3 are devoted to bounding the BV term and the L^2 term in the energy of this injected smoothed minimizer respectively. In section 2.1.4, we give the proof for Proposition 1.

2.1.1 Injected discrete function and its TV

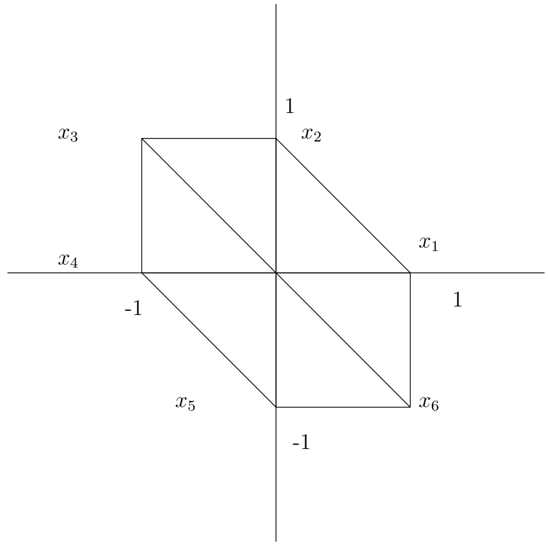

Fig. 2.1. Ω, the support of ϕ

We start by introducing some notation in discrete settings. Let Δ_i be the triangle $\{x_0, x_i, x_{i+1}\}$ as shown in figure 2.1, where $1 \leq i \leq 6$, $x_7 = x_1$, $x_0 = 0$.

Define $\Omega := \bigcup_i \Delta_i$ as shown in figure 2.1.

Let ϕ be a continuous function on $\mathbb{R}^2$, $\operatorname{supp} \phi \in \Omega$, $\phi|_{\Delta_i}$ is linear, and $\phi|_{\Delta_i}(x_i) = \delta_{i0}$.

We dilate and translate ϕ to obtain the function

$$\phi^k_{i,j}(x_1, x_2) := \phi(x_1/h - (i + 1/2), x_2/h - (j + 1/2)). \tag{2.1}$$

We can see $\operatorname{supp} \phi^k_{i,j}$ is a scaled Ω by factor h with center at $((i + 1/2)h, (j + 1/2)h)$.

For any discrete function $v^k \in X^k$, extend v^k on $\mathbb{Z}^2$ as in (1.5), (1.6) and define the injection of v^k into $\mathrm{BV}(I)$ by

$$\bar{v} = \sum_{i,j=-1}^{k} v^k_{i,j} \phi^k_{i,j} \quad \text{on } I.$$

Notice that $v_{i,j}^k$ satisfies discrete Neumann boundary conditions,

$$v_{-1,j}^k = v_{0,j}^k, \quad v_{k,j}^k = v_{k-1,j}^k,$$
$$v_{i,-1}^k = v_{i,0}^k, \quad v_{i,k}^k = v_{i,k-1}^k\,.$$

The injected function $\bar{v}$ is in the space spanned by $\{\phi_{i,j}^k\}$, $-1 \le i, j \le k$.

Now we calculate the total variation of $\bar{v}$ on I.

First notice that each triangle in our construction falls into one of two categories; see Figure 2.2.

(1) "upper-right" type triangle:

$$\Delta_{i,j}^+ := \{((i+\frac{1}{2})h,(j+\frac{1}{2})h),((i+\frac{1}{2})h,(j+\frac{3}{2})h),((i+\frac{3}{2})h,(j+\frac{1}{2})h)\}.$$

The basis functions whose supports overlap this triangle are $\phi_{i,j}^k$, $\phi_{i+1,j}^k$, $\phi_{i,j+1}^k$, and their gradients on $\Delta_{i,j}^+$ are

$$\nabla\phi_{i,j}^k = \frac{1}{h}\begin{pmatrix} -1 \\ -1 \end{pmatrix}, \ \nabla\phi_{i+1,j}^k = \frac{1}{h}\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \ \nabla\phi_{i,j+1}^k = \frac{1}{h}\begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

(2) "lower-left" type triangle:

$$\Delta_{i,j}^- := \{((i+\frac{1}{2})h,(j+\frac{1}{2})h),((i+\frac{1}{2})h,(j-\frac{1}{2})h),((i-\frac{1}{2})h,(j+\frac{1}{2})h)\}.$$

The basis functions whose supports overlap this triangle are $\phi_{i,j}^k$, $\phi_{i-1,j}^k$, $\phi_{i,j-1}^k$, and their gradients on $\Delta_{i,j}^-$ are

$$\nabla\phi_{i,j}^k = \frac{1}{h}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \ \nabla\phi_{i-1,j}^k = \frac{1}{h}\begin{pmatrix} -1 \\ 0 \end{pmatrix}, \ \nabla\phi_{i,j-1}^k = \frac{1}{h}\begin{pmatrix} 0 \\ -1 \end{pmatrix}.$$

It is easy to verify that, in each "upper-right" type triangle $\Delta_{i,j}^+$,

$$\nabla\bar{v} = \frac{1}{h}\begin{pmatrix} v_{i+1,j}^k - v_{i,j}^k \\ v_{i,j+1}^k - v_{i,j}^k \end{pmatrix},$$

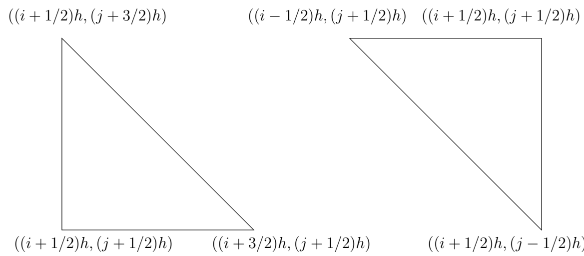

Fig. 2.2. “upper-right” and “lower-left” type triangular meshes

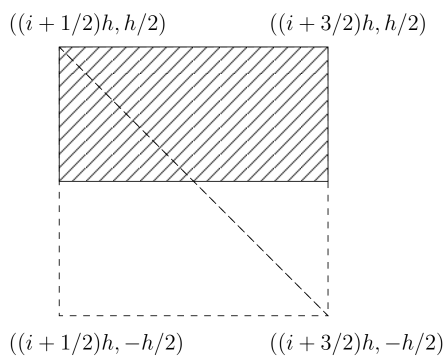

Fig. 2.3. rectangular region on the boundary

and in each "lower-left" type triangle $\Delta^-_{i,j}$,

$$\nabla \bar{v} = \frac{1}{h}\begin{pmatrix} v^k_{i,j} - v^k_{i-1,j} \\ v^k_{i,j} - v^k_{i,j-1} \end{pmatrix}.$$

On the boundary, parts of these triangles will stretch outside the region I, but notice on each rectangular region inside the boundary, for example the rectangle in Figure 2.3,

$$\{((i+\frac{1}{2})h, \frac{1}{2}h), ((i+\frac{3}{2})h, \frac{1}{2}h), ((i+\frac{3}{2})h, 0), ((i+\frac{1}{2})h, 0)\},$$

the two triangles overlapping this rectangular region are on the same plane and $\nabla \bar{v}$ is a constant vector in this rectangular region, in this example,

$$\nabla \bar{v} = \frac{1}{h}\begin{pmatrix} v^k_{i+1,0} - v^k_{i,0} \\ 0 \end{pmatrix}.$$

Finally, on the four "corner"s of region I,

$$[0, \frac{h}{2}] \times [0, \frac{h}{2}], \qquad [1-\frac{h}{2}, 1] \times [0, \frac{h}{2}],$$
$$[0, \frac{h}{2}] \times [1-\frac{h}{2}, 1], \qquad [1-\frac{h}{2}, 1] \times [1-\frac{h}{2}, 1];$$

$\bar{v}$ is a constant and $|\nabla \bar{v}| = 0$. Therefore, integrating $|\nabla \bar{v}|$ on the whole region I, we get the following relationship between the total variation of $\bar{v}$ and the discrete total variation of v^k,

$$\begin{aligned} \int_I |\nabla \bar{v}| &= \sum_{i,j=0}^{k-1} h^2(|\nabla^{++} v^k_{i,j}| + |\nabla^{--} v^k_{i,j}|)/2 \\ &= \frac{1}{2}(J_{++}(v^k) + J_{--}(v^k)). \end{aligned} \tag{2.2}$$

Now we introduce the notion of "smoothed" function in discrete settings.

Definition 2.1.1 *Define $v^{L,k}$ to be the "smoothed" function of v^k at (i,j) over a square with center (i,j) and side length $(2L+1)$ as*

$$v^{L,k}_{i,j} = \frac{1}{(2L+1)^2}\sum_{m,n=-L}^{L} v^k_{i+m,j+n}$$

for $0 \le i, j \le k-1$*. For indexes outside the range* $\{0, \cdots, k-1\}$*, we use the extension of* v^k *defined in* (1.5) *and* (1.6)*.*

We can also rewrite $v^{L,k}$ as

$$v^{L,k} = \frac{1}{(2L+1)^2} \sum_{m,n=-L}^{L} T_{m,n} v^k,$$

where $T_{m,n}$ is the shift operator

$$(T_{m,n} v^k)_{i,j} = v^k_{i+m,j+n} \, .$$

Finally we define the injected smoothed function of v^k in space $L^2(I)$.

Definition 2.1.2 *Define* $\bar{v}^L$ *to be the injected smoothed function of* v^k *if*

$$\bar{v}^L = \sum_{i,j=-1}^{k} v^{L,k}_{i,j} \phi^k_{i,j},$$

where $\phi^k_{i,j}$ *is the basis function as defined before.*

Recall that we assume u^k is the minimizer of E_k, the injected smoothed minimizer is then defined by

$$\bar{u}^L = \sum_{i,j=-1}^{k} u^{L,k}_{i,j} \phi^k_{i,j} \qquad \text{on } I,$$

where $u^{L,k}$ is the "smoothed" discrete function of u^k,

$$u^{L,k} = \frac{1}{(2L+1)^2} \sum_{m,n=-L}^{L} T_{m,n} u^k \, .$$

2.1.2 BV estimate

In this section we bound the BV term in $E(\bar{u}^L)$,

$$\int_I |D\bar{u}^L|.$$

The following lemma bounds the difference between non-symmetric discrete total variations of "smoothed" discrete function v^k.

Lemma 4 *Extend* v^k *over* $\mathbb{Z}^2$ *as in* (1.5) *and* (1.6). *For any non-negative integer* $L < k$,

$$|J_{++}(v^{L,k}) - J_{+-}(v^{L,k})| \le \frac{C}{L} J_a(v^k). \tag{2.3}$$

Proof For simplicity we rewrite $J_{++}(v^{L,k})$ as

$$J_{++}(v^{L,k}) = \sum_{i,j=0}^{k-1} h^2 \delta_{i,j}^{++}, \tag{2.4}$$

where

$$\begin{aligned}
\delta_{i,j}^{++} &= \frac{1}{(2L+1)^2}\sqrt{(A_{i,j}^+)^2 + (B_{i,j}^+)^2},\\
A_{i,j}^+ &= \frac{v_{i+1,j}^{L,k} - v_{i,j}^{L,k}}{h} = \sum_{\ell=-L}^{L} \frac{v_{i+L+1,j+\ell}^k - v_{i-L,j+\ell}^k}{h},\\
B_{i,j}^+ &= \frac{v_{i,j+1}^{L,k} - v_{i,j}^{L,k}}{h} = \sum_{\ell=-L}^{L} \frac{v_{i+\ell,j+L+1}^k - v_{i+\ell,j-L}^k}{h}.
\end{aligned}$$

Adding two terms

$$\begin{aligned}
&\frac{v_{i+L+1,j+L+1}^k - v_{i-L,j+L+1}^k}{h};\\
&\frac{v_{i+L+1,j+L+1}^k - v_{i+L+1,j-L}^k}{h};
\end{aligned}$$

to $A_{i,j}^+$ and $B_{i,j}^+$ respectively, we introduce:

$$\begin{aligned}
\tilde{A}_{i,j}^+ &:= \sum_{\ell=-L}^{L+1} \frac{v_{i+L+1,j+\ell}^k - v_{i-L,j+\ell}^k}{h},\\
\tilde{B}_{i,j}^+ &:= \sum_{\ell=-L}^{L+1} \frac{v_{i+\ell,j+L+1}^k - v_{i+\ell,j-L}^k}{h}.
\end{aligned}$$

In other words,

$$\begin{pmatrix} \tilde{A}_{i,j}^+ \\ \tilde{B}_{i,j}^+ \end{pmatrix} = \begin{pmatrix} A_{i,j}^+ \\ B_{i,j}^+ \end{pmatrix} + \frac{1}{h}\begin{pmatrix} v_{i+L+1,j+L+1}^k - v_{i-L,j+L+1}^k \\ v_{i+L+1,j+L+1}^k - v_{i+L+1,j-L}^k \end{pmatrix}.$$

Then

$$\begin{aligned}\left|\begin{pmatrix} A_{i,j}^{+} \\ B_{i,j}^{+} \end{pmatrix}\right| &\le \left|\begin{pmatrix} \tilde{A}_{i,j}^{+} \\ \tilde{B}_{i,j}^{+} \end{pmatrix}\right| + \frac{1}{h}\left|\begin{pmatrix} v_{i+L+1,j+L+1}^{k} - v_{i-L,j+L+1}^{k} \\ v_{i+L+1,j+L+1}^{k} - v_{i+L+1,j-L}^{k} \end{pmatrix}\right| \\ &\le \left|\begin{pmatrix} \tilde{A}_{i,j}^{+} \\ \tilde{B}_{i,j}^{+} \end{pmatrix}\right| + \frac{1}{h}\left|v_{i+L+1,j+L+1}^{k} - v_{i-L,j+L+1}^{k}\right| \\ &\qquad + \frac{1}{h}\left|v_{i+L+1,j+L+1}^{k} - v_{i+L+1,j-L}^{k}\right| .\end{aligned}$$

Expanding the second term into a collapsing sum and applying the triangle inequality gives

$$\begin{aligned}\frac{1}{h}\left|v_{i+L+1,j+L+1}^{k} - v_{i-L,j+L+1}^{k}\right| &= \left|\sum_{\ell=-L+1}^{L+1} \nabla_x^{-} v_{i+\ell,j+L+1}^{k}\right| \\ &\le \sum_{\ell=-L+1}^{L+1} |\nabla_x^{-} v_{i+\ell,j+L+1}^{k}|,\end{aligned}$$

and similarly for the third term, so

$$\begin{aligned}\sqrt{(A_{i,j}^{+})^2 + (B_{i,j}^{+})^2} &\le \sqrt{(\tilde{A}_{i,j}^{+})^2 + (\tilde{B}_{i,j}^{+})^2} \\ &\qquad + \sum_{\ell=-L+1}^{L+1} |\nabla_x^{-} v_{i+\ell,j+L+1}^{k}| + \sum_{\ell=-L+1}^{L+1} |\nabla_y^{-} v_{i+L+1,j+\ell}^{k}| . \end{aligned} \tag{2.5}$$

Let

$$\tilde{\delta}_{i,j}^{++} = \frac{1}{(2L+1)^2}\sqrt{(\tilde{A}_{i,j}^{+})^2 + (\tilde{B}_{i,j}^{+})^2},$$

we bound $J_{++}(v^{L,k})$ by the sum of $h^2\tilde{\delta}_{i,j}^{++}$ over all i,j plus some error term.

$$\begin{aligned}J_{++}(v^{L,k}) &= \frac{1}{(2L+1)^2}\sum_{i,j=0}^{k-1} h^2\sqrt{(A_{i,j}^{+})^2 + (B_{i,j}^{+})^2} \\ &\le \frac{1}{(2L+1)^2}\sum_{i,j=0}^{k-1} h^2\sqrt{(\tilde{A}_{i,j}^{+})^2 + (\tilde{B}_{i,j}^{+})^2} \\ &\quad + \frac{1}{(2L+1)^2}\sum_{i,j=0}^{k-1} h^2\left\{\sum_{\ell=-L+1}^{L+1} |\nabla_x^{-} v_{i+\ell,j+L+1}^{k}| + |\nabla_y^{-} v_{i+L+1,j+\ell}^{k}|\right\} .\end{aligned}$$

The last line follows from (2.5). Exchange the order of summation in the second term of the above line and notice that $L < k$, by the definition of extended v^k,

$$\begin{aligned}\sum_{i,j=0}^{k-1} |\nabla_x^- v^k_{i+\ell,j+L+1}| &\le \sum_{i,j=0}^{2k-1} |\nabla_x^- v^k_{i,j}| \\ &= 4\sum_{i,j=0}^{k-1} |\nabla_x^- v^k_{i,j}|.\end{aligned}$$

Similarly for the last term, so

$$\sum_{i,j=0}^{k-1} |\nabla_x^- v^k_{i+\ell,j+L+1}| + |\nabla_y^- v^k_{i+L+1,j+\ell}| \le 4\left(\sum_{i,j=0}^{k-1} |\nabla_x^- v^k_{i,j}| + |\nabla_y^- v^k_{i,j}|\right)$$

for any ℓ, L. Thus we obtain

$$J_{++}(v^{L,k}) \le \sum_{i,j=0}^{k-1} h^2 \tilde{\delta}^{++}_{i,j} + \frac{4}{(2L+1)^2} \sum_{\ell=-L+1}^{L+1} \sum_{i,j=0}^{k-1} h^2 (|\nabla_x^- v^k_{i,j}| + |\nabla_y^- v^k_{i,j}|)\,.$$

Recall that

$$J_a = \sum_{i,j=0}^{k-1} h^2 (|\nabla_x^- v^k_{i,j}| + |\nabla_y^- v^k_{i,j}|)\,.$$

Thus

$$\begin{aligned}J_{++}(v^{L,k}) &\le \sum_{i,j=0}^{k-1} h^2 \tilde{\delta}^{++}_{i,j} + \frac{4}{2L+1} J_a(v^k) \\ &\le \sum_{i,j=0}^{k-1} h^2 \tilde{\delta}^{++}_{i,j} + \frac{2}{L} J_a(v^k)\,. \end{aligned} \tag{2.6}$$

Similarly, we can bound $J_{+-}(v^{L,k})$ by

$$J_{+-}(v^{L,k}) \ge \sum_{i,j=0}^{k-1} h^2 \tilde{\delta}^{+-}_{i,j} - \frac{2}{L} J_a(v^k), \tag{2.7}$$

where

$$\begin{aligned}\tilde{\delta}^{+-}_{i,j} &= \frac{1}{(2L+1)^2}\sqrt{(\tilde{C}^+_{i,j})^2 + (\tilde{D}^-_{i,j})^2}, \\ \tilde{C}^+_{i,j} &= \sum_{\ell=-L-1}^{L} \frac{v^k_{i+L+1,j+\ell} - v^k_{i-L,j+\ell}}{h}, \\ \tilde{D}^-_{i,j} &= \sum_{\ell=-L}^{L+1} \frac{v^k_{i+\ell,j+L} - v^k_{i+\ell,j-L-1}}{h}.\end{aligned}$$

One can now make the crucial observation that

$$\tilde{\delta}^{++}_{i,j} = \tilde{\delta}^{+-}_{i,j+1}\,.$$

Therefore, subtracting (2.7) from (2.6), we obtain

$$\begin{aligned} J_{++}(v^{L,k}) - J_{+-}(v^{L,k}) &\le \sum_{i=0}^{k-1} h^2(-\tilde{\delta}^{+-}_{i,0} + \tilde{\delta}^{++}_{i,k-1}) + \frac{4}{L} J_a(v^k) \\ &\le \sum_{i=0}^{k-1} h^2(\tilde{\delta}^{+-}_{i,0} + \tilde{\delta}^{++}_{i,k-1}) + \frac{4}{L} J_a(v^k)\,. \end{aligned} \tag{2.8}$$

To bound the first term in (2.8), we need to bound

$$\sum_{i=0}^{k-1} h^2 \tilde{\delta}^{+-}_{i,0} = \frac{1}{(2L+1)^2} \sum_{i=0}^{k-1} h^2 \sqrt{(\tilde{C}^{+}_{i,0})^2 + (\tilde{D}^{-}_{i,0})^2}\,.$$

Since v^k is symmetric about the boundary,

$$\tilde{D}^{-}_{i,0} = 0,$$

again writing each $\tilde{C}^{+}_{i,0}$ as a collapsing sum, we have

$$\begin{aligned} \sum_{i=0}^{k-1} h^2 \tilde{\delta}^{+-}_{i,0} &= \frac{1}{(2L+1)^2} \sum_{i=0}^{k-1} h^2 \left| \sum_{\ell=-L-1}^{L} \frac{v^k_{i+L+1,\ell} - v^k_{i-L,\ell}}{h} \right| \\ &= \frac{1}{(2L+1)^2} \sum_{i=0}^{k-1} h^2 \left| \sum_{\ell=-L-1}^{L} \sum_{s=-L+1}^{L+1} \nabla^-_x v^k_{i+s,\ell} \right| \\ &\le \frac{1}{(2L+1)^2} \sum_{i=0}^{k-1} h^2 \sum_{\ell=-L-1}^{L} \sum_{s=-L+1}^{L+1} |\nabla^-_x v^k_{i+s,\ell}|\,. \end{aligned} \tag{2.9}$$

Again since the extended v^k is symmetric about the boundary $j = -1/2$,

$$\sum_{\ell=-L-1}^{L} |\nabla^-_x v^k_{i+s,\ell}| \le 2 \sum_{\ell=0}^{L} |\nabla^-_x v^k_{i+s,\ell}|\,.$$

Exchange the order of sum in (2.9) and notice $L < k$, we have

$$\begin{aligned} \sum_{i=0}^{k-1} h^2 \tilde{\delta}^{+-}_{i,0} &\le \frac{1}{(2L+1)^2} \sum_{i=0}^{k-1} h^2 \sum_{\ell=-L-1}^{L} \sum_{s=-L+1}^{L+1} |\nabla^-_x v^k_{i+s,\ell}| \\ &\le \frac{2}{(2L+1)^2} \sum_{s=-L+1}^{L+1} \sum_{i=0}^{k-1} \sum_{\ell=0}^{L} h^2 |\nabla^-_x v^k_{i+s,\ell}| \\ &\le \frac{2}{(2L+1)^2} \sum_{s=-L+1}^{L+1} \sum_{i=0}^{k-1} \sum_{\ell=0}^{k-1} h^2 |\nabla^-_x v^k_{i+s,\ell}|\,. \end{aligned}$$

For any $-L+1<s<L+1$,

$$\sum_{i=0}^{k-1}\sum_{\ell=0}^{k-1} h^2|\nabla_x^- v_{i+s,\ell}^k| \le 2\sum_{i=0}^{k-1}\sum_{\ell=0}^{k-1} h^2|\nabla_x^- v_{i,\ell}^k|\,,$$

thus

$$\begin{aligned}\sum_{i=0}^{k-1} h^2\tilde{\delta}_{i,0}^{+-} &\le \frac{4}{(2L+1)^2}\sum_{s=-L+1}^{L+1}\sum_{i=0}^{k-1}\sum_{\ell=0}^{k-1} h^2|\nabla_x^- v_{i,\ell}^k| \\ &\le \frac{4}{2L+1}J_a(v^k) \le \frac{2}{L}J_a(v^k)\,.\end{aligned}$$

The same result also holds for $\sum_{i=0}^{k-1} h^2\tilde{\delta}_{i,k-1}^{++}$, then we obtain

$$J_{++}(v^{L,k}) - J_{+-}(v^{L,k}) \le \frac{C}{L}J_a(v^k)\,.$$

By the symmetric nature of our proof, we also have

$$J_{+-}(v^{L,k}) - J_{++}(v^{L,k}) \le \frac{C}{L}J_a(v^k),$$

then

$$|J_{++}(v^{L,k}) - J_{+-}(v^{L,k})| \le \frac{C}{L}J_a(v^k).$$

and we complete the proof. ■

Remark With slight adaption, this result can be easily extended to bound the difference between any two of the non-symmetric discrete TV operators J_{++}, J_{+-}, J_{-+} and J_{--}.

We next give the following property of the symmetric discrete total variation $J_\star$. It states that "smoothing" a discrete function v^k doesn't increase its TV.

Lemma 5

$$J_\star(v^{L,k}) \le J_\star(v^k).$$

Proof Extend v^k over the torus $(\mathbb{Z}/2k\mathbb{Z})^2$, Let $\bar{J}_\star$ be the symmetric discrete TV on the torus,

$$\bar{J}_\star(v^k) = \frac{1}{4}\sum_{i,j=0}^{2k-1}(|\nabla^{++}v_{i,j}^k| + |\nabla^{+-}v_{i,j}^k| + |\nabla^{-+}v_{i,j}^k| + |\nabla^{--}v_{i,j}^k|).$$

One can verify that

$$\bar{J}_\star(v^k) = 4J_\star(v^k),$$

and $\bar{J}_\star$ is invariant under translation on the torus. Therefore,

$$\begin{aligned}\bar{J}_\star(v^{L,k}) &= \bar{J}_\star\left(\sum_{m,n=-L}^{L} T_{m,n}v^k\right)\\ &\le \sum_{m,n=-L}^{L} \bar{J}_\star(T_{m,n}v^k)\\ &= \bar{J}_\star(v^k).\end{aligned}$$

This gives

$$J_\star(v^{L,k}) \le J_\star(v^k).$$

■

Now we prove the first part of the consistency property for the symmetric discrete TV operator.

Lemma 6 (Consistency with respect to injection)

$$\int_I |D\bar{v}^L| \le J_\star(v^k) + \frac{C}{L}J_\star(v^k). \tag{2.10}$$

Proof By (2.2), we have

$$\begin{aligned}\int_I |D\bar{v}^L| &= \frac{1}{2}(J_{++}(v^{L,k}) + J_{--}(v^{L,k}))\\ &= J_\star(v^{L,k}) + \frac{1}{4}\left[J_{+-}(v^{L,k}) - J_{++}(v^{L,k}) + J_{-+}(v^{L,k}) - J_{--}(v^{L,k})\right].\end{aligned}$$

By Lemma 5,

$$J_\star(v^{L,k}) \le J_\star(v^k),$$

therefore,

$$\int_I |D\bar{v}^L| \le J_\star(v^k) + \frac{1}{4}\left|J_{++}(v^{L,k}) - J_{+-}(v^{L,k})\right| + \frac{1}{4}\left|J_{-+}(v^{L,k}) - J_{--}(v^{L,k})\right|.$$

Now apply Lemma 4 and its remark,

$$\int_I |D\bar{v}^L| \leq J_\star(v^k) + \frac{C}{L} J_a(v^k)\,.$$

Since $J_\star$ is equivalent to J_a, we obtain

$$\int_I |D\bar{v}^L| \leq J_\star(v^k) + \frac{C}{L} J_\star(v^k)\,.$$

■

2.1.3 L^2 estimate

In this section we treat g^k (u^k, $\bar{u}^{L,k}$ resp.) as a piecewise constant function on the unit square I with constant value $g^k_{i,j}$ ($u^k_{i,j}$, $\bar{u}^{L,k}_{i,j}$ resp.) on each grid

$$I_{i,j} = \left[\frac{i}{k}, \frac{i+1}{k}\right] \times \left[\frac{j}{k}, \frac{j+1}{k}\right], \qquad 0 \leq i, j \leq k-1;$$

.

The aim of this section is to bound the L^2 term

$$\frac{1}{2\lambda}\|\bar{u}^L - g\|^2$$

in $E(\bar{u}^L)$ by

$$\frac{1}{2\lambda}\|\bar{u}^k - g^k\|^2$$

plus some error term. Therefore with the help of the consistency property we proved in last section, we can get

$$\begin{aligned} E(\bar{u}^L) &= \int_I |D\bar{u}^L| + \frac{1}{2\lambda}\|\bar{u}^L - g\|^2 \\ &\leq J_\star(u^k) + \frac{1}{2\lambda}\|u^k - g^k\|^2 + \text{some error} \\ &= E_k(u^k) + \text{some error}, \end{aligned}$$

and prove Proposition 1.

The L^2 term can be expanded into several terms,

$$
\begin{aligned}
\|\bar{u}^L - g\|^2 &= \|\bar{u}^L - u^k + u^k - g^k + g^k - g\|^2 \\
&\leq \|u^k - g^k\|^2 + 2\|u^k - g^k\|\|\bar{u}^L - u^k + g^k - g\| \\
&\quad + \|\bar{u}^L - u^k + g^k - g\|^2. \qquad (2.11)
\end{aligned}
$$

By the triangle inequality,

$$
\begin{aligned}
\|\bar{u}^L - u^k + g^k - g\| &= \|\bar{u}^L - u^{L,k} + u^{L,k} - u^k + g^k - g\| \\
&\leq \|\bar{u}^L - u^{L,k}\| + \|u^{L,k} - u^k\| + \|g^k - g\|,
\end{aligned}
$$

and

$$
\begin{aligned}
\|\bar{u}^L - u^k + g^k - g\|^2 &= \|\bar{u}^L - u^{L,k} + u^{L,k} - u^k + g^k - g\|^2 \\
&\leq 3(\|\bar{u}^L - u^{L,k}\|^2 + \|u^{L,k} - u^k\|^2 + \|g^k - g\|^2).
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\|\bar{u}^L - g\|^2 \leq \|u^k - g^k\|^2 &+ 2\|u^k - g^k\| \cdot (\|\bar{u}^L - u^{L,k}\| + \|u^{L,k} - u^k\| \\
&+ \|g^k - g\|) + 3(\|\bar{u}^L - u^{L,k}\|^2 + \|u^{L,k} - u^k\|^2 + \|g^k - g\|^2). \qquad (2.12)
\end{aligned}
$$

and we shall bound terms: $\|\bar{u}^L - u^{L,k}\|$, $\|u^{L,k} - u^k\|$, $\|g^k - g\|$ and $\|u^k - g^k\|$. We first look at $\|g^k - g\|$.

Recall that the Lipschitz space $\mathrm{Lip}(\alpha, L^2(I))$, $0 < \alpha \leq 1$, is the set of all L^2 functions f on I that satisfy

$$
\left\{ \int_{I_h} |f(x+h)| - f(x)|^2 \, dx \right\}^{1/2} \leq M t^\alpha, \quad t > 0
$$

for all h with $|h| \leq t$ where $I_h := \{x : x \in I, x + h \in I\}$. Recall the definition of modulus of smoothness ω_1 in (1.10). We define the semi-norm of $\mathrm{Lip}(\alpha, L^2(I))$ as

$$
|f|_{\mathrm{Lip}(\alpha, L^2)} := \sup_{t>0} (t^{-\alpha} \omega_1(f, t)_{L^2(I)}).
$$

the Lipschitz norm of f is $\|f\|_{\mathrm{Lip}(\alpha, L^2)} := |f|_{\mathrm{Lip}(\alpha, L^2)} + \|f\|$.

For a general exposition of the theory of Lipschitz space, we refer to [6].

Lemma 7 *If g is in* $\mathrm{Lip}(\alpha, L^2(I))$, *then*

$$\|g^k - g\| \le C|g|_{\mathrm{Lip}(\alpha,L^2)} h^\alpha.$$

Proof

$$\begin{aligned}\int_I |g^k - g|^2 &= \sum_{i,j} \int_{I_{i,j}} |g^k_{i,j} - g|^2 \\ &= \sum_{i,j} \int_{I_{i,j}} \left| \frac{1}{|I_{i,j}|} \int_{I_{i,j}} g(y) - g(x)\, dy \right|^2 dx\,.\end{aligned}$$

By Holder's inequality, we have for any $p \ge 1$,

$$\left| \frac{1}{|I_{i,j}|} \int_{I_{i,j}} g(y) - g(x)\, dy \right|^p \le \frac{1}{|I_{i,j}|} \int_{I_{i,j}} |g(y) - g(x)|^p\,. \tag{2.13}$$

Let $p = 2$ and apply (2.13) to the inner integral in the previous line, then substitute the integral variable y by $x + \tau$ and exchange the order of integration, we obtain:

$$\begin{aligned}\int_I |g^k - g|^2 &\le \sum_{i,j} \int_{I_{i,j}} \frac{1}{h^2} \int_{|\tau| \le \sqrt{2}h} |g(x+\tau) - g(x)|^2\, d\tau\, dx \\ &\le \frac{1}{h^2} \int_{|\tau| \le \sqrt{2}h} \int_I |g(x+\tau) - g(x)|^2 dx\, d\tau \\ &\le C \sup_{|\tau| \le \sqrt{2}h} \int_I |g(x+\tau) - g(x)|^2\, dx \\ &\le C\omega_1^2(g, \sqrt{2}h)_{L^2} \le C|g|^2_{\mathrm{Lip}(\alpha,L^2)} h^{2\alpha}\,.\end{aligned}$$

On taking square root on both sides, we get the result. ∎

The following lemma bounds the smoothness of g^k in the discrete setting by the smoothness of g in the continuous setting. We use it later to bounding $\|u^{L,k} - u^k\|$ and $\|\bar{u}^L - u^{L,k}\|$.

Lemma 8 *For any $p \ge 1$ and any multi-index $\ell = (m, n)$,*

$$\|T_\ell g^k - g^k\|_{L^p(I)} \le C\omega_1(g, |\ell|h)_{L^p(I)},$$

where $|\ell| = |m| + |n|$.

Proof Similar to the proof of Lemma 7, apply the inequality (2.13),

$$\begin{aligned}\sum_{i,j} h^2 |g^k_{i+m,j+n} - g^k_{i,j}|^p &= \sum_{i,j} h^2 \left| \frac{1}{|I_{i,j}|} \int_{I_{i,j}} g(x+h\ell) - g(x)\,dx \right|^p \\ &\le \sum_{i,j} h^2 \frac{1}{|I_{i,j}|} \int_{I_{i,j}} |g(x+h\ell) - g(x)|^p\,dx \\ &= \int_I |g(x+h\ell) - g(x)|^p\,dx \\ &\le \omega_1(g, |\ell|h)^p_{L^p(2I)} \\ &\le C\omega_1(g, |\ell|h)^p_{L^p(I)}.\end{aligned}$$

■

Remark We have in fact proved that

$$\omega_1(g^k, |\ell|)_{l^p(I)} \le C\omega_1(g, |\ell|h)_{L^p(I)}. \tag{2.14}$$

Remark Specifically, we have

$$\begin{aligned}J_a(g^k) &= \frac{1}{h}(\|T_{1,0}g^k - g^k\|_{L^1} + \|T_{0,1}g^k - g^k\|_{L^1}) \\ &\le \frac{C}{h}\omega_1(g, h)_{L^1}\,.\end{aligned} \tag{2.15}$$

Now we estimate $\|u^{L,k} - u^k\|$ and $\|\bar{u}^L - u^{L,k}\|$.

Lemma 9 *If g is in* $\mathrm{Lip}(\alpha, L^2(I))$, *then*

$$\|u^{L,k} - u^k\| \le C|g|_{\mathrm{Lip}(\alpha, L^2)}(Lh)^\alpha. \tag{2.16}$$

Proof Because of the convexity and 1-homogeneity of $\|\cdot\|$,

$$\begin{aligned}\|u^{L,k} - u^k\| &= \left\| \frac{1}{(2L+1)^2} \sum_{m,n=-L}^{L} (T_{m,n}u^k - u^k) \right\| \\ &\le \frac{1}{(2L+1)^2} \sum_{m,n=-L}^{L} \|T_{m,n}u^k - u^k\|\,.\end{aligned}$$

Apply the translation continuity property (1.13) for u^k to each term in the sum of the above inequality and use (2.14) to see that

$$\begin{aligned}
\|u^{L,k} - u^k\| &\leq \frac{C}{(2L+1)^2} \sum_{m,n=-L}^{L} \omega_1(g^k, |m|+|n|)_{l^2} \\
&\leq \frac{C}{(2L+1)^2} \sum_{m,n=-L}^{L} \omega_1(g, Lh)_{L^2} \\
&\leq \frac{C}{(2L+1)^2} \sum_{m,n=-L}^{L} |g|_{\mathrm{Lip}(\alpha,L^2)} (Lh)^\alpha \\
&\leq C|g|_{\mathrm{Lip}(\alpha,L^2)} (Lh)^\alpha.
\end{aligned}$$

■

Lemma 10 *If g is in* $\mathrm{Lip}(\alpha, L^2(I))$, *then*

$$\|\bar{u}^L - u^{L,k}\| \leq C|g|_{\mathrm{Lip}(\alpha,L^2)} h^\alpha. \tag{2.17}$$

Proof The estimate of $\|\bar{u}^L - u^{L,k}\|$ is similar to the estimate of $\|u^{L,k} - u^k\|$. We first bound it by the discrete difference of u^k, then use the property of translation continuity and Lemma 8.

Let $\tilde{u}^L = \sum_{i',j'} u_{i,j}^{L,k} \phi_{i',j'}^k$, the sum of $\phi_{i',j'}^k$ over all (i', j') such that for each (i', j'), supp $\phi_{i',j'}^k$ intersects supp $\phi_{i,j}^k$ non-trivially, It's easy to see that in the support of $\phi_{i,j}^k$, $\tilde{u}^L$ is a constant $u_{i,j}^{L,k}$.

$$\begin{aligned}
\int_{I_{i,j}} |u^{L,k} - \bar{u}^L|^2 &= \int_{I_{i,j}} |\tilde{u}^L - \bar{u}^L|^2 \\
&\leq \int_{I_{i,j}} \left| \sum_{i',j'} (u_{i,j}^{L,k} \phi_{i',j'}^k - u_{i',j'}^{L,k} \phi_{i',j'}^k) \right|^2 \\
&\leq 6 \sum_{i',j'} \int_{I_{i,j}} (u_{i,j}^{L,k} - u_{i',j'}^{L,k})^2 .
\end{aligned}$$

Notice that, for the "diagonal" type terms in above expression, for example, $(u_{i,j}^{L,k} - u_{i-1,j-1}^{L,k})^2$, we have

$$(u_{i,j}^{L,k} - u_{i-1,j-1}^{L,k})^2 \leq 2[(u_{i,j}^{L,k} - u_{i,j-1}^{L,k})^2 + (u_{i,j-1}^{L,k} - u_{i-1,j-1}^{L,k})^2],$$

then, sum all these terms and re-index terms when necessary, we get

$$
\begin{aligned}
\int_I |u^{L,k} - \bar{u}^L|^2 &= \sum_{i,j} \int_{I_{i,j}} |u^{L,k} - \bar{u}^L|^2 \\
&\le \sum_{i,j} 6 \sum_{i',j'} \int_{I_{i,j}} (u^{L,k}_{i,j} - u^{L,k}_{i',j'})^2 \\
&\le 36 \sum_{i,j} (|u^{L,k}_{i+1,j} - u^{L,k}_{i,j}|^2 + |u^{L,k}_{i,j+1} - u^{L,k}_{i,j}|^2) h^2 \\
&= C(\|T_{1,0}u^{L,k} - u^{L,k}\|^2 + \|T_{0,1}u^{L,k} - u^{L,k}\|^2)\,. \qquad (2.18)
\end{aligned}
$$

Then

$$
\begin{aligned}
\|u^{L,k} - \bar{u}^L\| &= \left\{ \int_I |u^{L,k} - \bar{u}^L|^2 \right\}^{1/2} \\
&\le C(\|T_{1,0}u^{L,k} - u^{L,k}\| + \|T_{0,1}u^{L,k} - u^{L,k}\|) \\
&\le C\Bigg(\left\| \frac{1}{(2L+1)^2} \sum_{m,n=-L}^{L} T_{m,n}(T_{1,0}u^k - u^k) \right\| + \\
&\qquad \left\| \frac{1}{(2L+1)^2} \sum_{m,n=-L}^{L} T_{m,n}(T_{0,1}u^k - u^k) \right\| \Bigg)\,.
\end{aligned}
$$

Again by the convexity and 1-homogeneity of $\|\cdot\|$,

$$
\begin{aligned}
\|u^{L,k} - \bar{u}^L\| \le C \frac{1}{(2L+1)^2} \sum_{m,n=-L}^{L} (&\|T_{m,n}(T_{1,0}u^k - u^k)\| + \\
&\|T_{m,n}(T_{0,1}u^k - u^k)\|).
\end{aligned}
$$

By the property of translation continuity of u^k (1.13) and the definition of $\omega_1(g^k, m)_{l^2}$ in (1.14),

$$
\begin{aligned}
\|T_{m,n}(T_{1,0}u^k - u^k)\| &\le C\omega_1(g^k, 1)_{l^2}; \\
\|T_{m,n}(T_{0,1}u^k - u^k)\| &\le C\omega_1(g^k, 1)_{l^2};
\end{aligned}
$$

therefore by Lemma 8,

$$
\begin{aligned}
\|u^{L,k} - \bar{u}^L\| &\le C(\omega_1(g^k, 1)_{l^2} + \omega_1(g^k, 1)_{l^2}) \\
&\le C|g|_{\mathrm{Lip}(\alpha, L^2)} h^{\alpha}.
\end{aligned}
$$

$\blacksquare$

Now we are at the position to bound the last L^2 term $\|u^k - g^k\|$ in (2.12).

Lemma 11

$$\|u^k - g^k\| \leq \|g\|.$$

Proof The proof is straightforward, since u^k is the minimizer of E_k,

$$\frac{1}{2\lambda}\|u^k - g^k\|^2 \leq E_k(u^k) \leq E_k(0) = \frac{1}{2\lambda}\|g^k\|^2 \leq \frac{1}{2\lambda}\|g\|^2.$$

■

2.1.4 Proof of Proposition 1

We prove Proposition 1 in this section.

By the BV estimate (2.10), we have

$$\int_I |D\bar{u}^L| \leq J_\star(u^k) + \frac{C}{L} J_\star(u^k)\,.$$

Then

$$\begin{aligned} E(\bar{u}^L) &= \int_I |D\bar{u}^L| + \frac{1}{2\lambda}\|\bar{u}^L - g\|^2 \\ &= J_\star(u^k) + \frac{C}{L} J_\star(u^k) + \frac{1}{2\lambda}\|\bar{u}^L - g\|^2\,. \end{aligned}$$

Notice that

$$J_\star(u^k) \leq E_k(u^k) \leq E_k(0) \leq \frac{C}{\lambda}\|g\|^2\,. \tag{2.19}$$

Thus

$$E(\bar{u}^L) \leq J_\star(u^k) + \frac{C}{L\lambda}\|g\|^2 + \frac{1}{2\lambda}\|\bar{u}^L - g\|^2\,. \tag{2.20}$$

Recall (2.12), we rewrite the expending of the last L^2 term here,

$$\begin{aligned} \|\bar{u}^L - g\|^2 &= \|\bar{u}^L - u^k + u^k - g^k + g^k - g\|^2 \\ &\leq \|u^k - g^k\|^2 + 2\|u^k - g^k\|\|\bar{u}^L - u^k + g^k - g\| \\ &\quad + \|\bar{u}^L - u^k + g^k - g\|^2 \\ &\leq \|u^k - g^k\|^2 + 2\|u^k - g^k\| \cdot (\|\bar{u}^L - u^{L,k}\| + \|u^{L,k} - u^k\| \\ &\quad + \|g^k - g\|) + 3(\|\bar{u}^L - u^{L,k}\|^2 + \|u^{L,k} - u^k\|^2 + \|g^k - g\|^2). \end{aligned}$$

Now applying Lemma 7, Lemma 9, and Lemma 10 to $\|g^k - g\|$, $\|u^{L,k} - u^k\|$, and $\|\bar{u}^L - u^{L,k}\|$, respectively, in the above inequality, we obtain

$$\|\bar{u}^L - g\|^2 \le \|u^k - g^k\|^2 + C(|g|^2_{\mathrm{Lip}(\alpha,L^2)} h^{2\alpha} + |g|^2_{\mathrm{Lip}(\alpha,L^2)} h^\alpha (Lh)^\alpha + \\ |g|^2_{\mathrm{Lip}(\alpha,L^2)} (Lh)^{2\alpha} + \|g\| |g|_{\mathrm{Lip}(\alpha,L^2)} (Lh)^\alpha + \|g\| |g|_{\mathrm{Lip}(\alpha,L^2)} h^\alpha).$$

Then put it back to (2.20),

$$\begin{aligned} E(\bar{u}^L) \le J_\star(u^k) + \frac{1}{2\lambda}\|u^k - g^k\|^2 + \frac{C}{\lambda}(\|g\|^2 L^{-1} + |g|^2_{\mathrm{Lip}(\alpha,L^2)} h^{2\alpha} + \\ + |g|^2_{\mathrm{Lip}(\alpha,L^2)} h^\alpha (Lh)^\alpha + |g|^2_{\mathrm{Lip}(\alpha,L^2)} (Lh)^{2\alpha} \\ + \|g\| |g|_{\mathrm{Lip}(\alpha,L^2)} (Lh)^\alpha + \|g\| |g|_{\mathrm{Lip}(\alpha,L^2)} h^\alpha). \end{aligned}$$

Setting $L = \lfloor h^{-\gamma} \rfloor$, the smallest powers of h in the above expression are

$$h^\gamma, \quad h^{\alpha(1-\gamma)}.$$

Setting them equal each other, we get

$$\gamma = \frac{\alpha}{\alpha+1},$$

and

$$\begin{aligned} E(\bar{u}^L) &\le J_\star(u^k) + \frac{1}{2\lambda}\|u^k - g^k\|^2 + \frac{C}{\lambda}(\|g\|^2 + \|g\| |g|_{\mathrm{Lip}(\alpha,L^2)}) h^{\alpha/(\alpha+1)} \\ &\le E_k(u^k) + \frac{C}{\lambda}\|g\|^2_{\mathrm{Lip}(\alpha,L2)} h^{\alpha/(\alpha+1)}. \end{aligned} \tag{2.21}$$

Since $E(u) \le E(\bar{u}^L)$, we have immediately that Proposition 1 is true.

2.2 Estimate of the energy of the projected smoothed minimizer

We prove in this section the following proposition.

Proposition 2 *If $g \in \mathrm{Lip}(\alpha, L^2(I))$ and u^k is the minimizer of E_k, then*

$$E_k(u^k) \le E(u) + \frac{C}{\lambda}\|g\|^2_{\mathrm{Lip}(\alpha,L^2)} h^{\alpha/(\alpha+1)}.$$

The structure of this section is similar to section 2.1. Section 2.2.1 introduces the notion of projected smoothed function. In Section 2.2.2 and section 2.2.3, we bound the BV term and the L^2 term in the energy of the projected smoothed minimizer respectively. In section 2.2.4, we give the proof for Proposition 2.

2.2.1 Projected smoothed function

To study the energy estimate in the other direction, we first briefly recall the definition of mollified function v^ϵ; here we assume v is a function in $\mathrm{BV}(I)$.

Define $\hat{v}$ be the extension of v over R^2 by symmetrizing v across the boundaries of I and periodizing it (with period 2 in each direction).

Let η be a non-negative symmetric mollifier, with

$$\int \eta = 1;$$

for example,

$$\eta = \begin{cases} 0, & |x| \geq 1, \\ Ce^{\frac{1}{|x|^2-1}}, & |x| < 1. \end{cases}$$

Define for each $\epsilon > 0$

$$\eta_\epsilon = \epsilon^{-2}\eta(\frac{x}{\epsilon}) \quad \text{and}$$
$$v^\epsilon = \eta_\epsilon * \hat{v}\,.$$

Now we introduce the notion of projected smoothed function. We call $\tilde{v}^k$ a projected smoothed function for $v \in \mathrm{BV}$ if $\tilde{v}^k$ is a discrete function with value $\tilde{v}^k_{i,j}$ at index (i, j) for $0 \leq i, j \leq k-1$,

$$\tilde{v}^k_{i,j} = \frac{1}{h^2}\int_{I_{i,j}} v^\epsilon\, dx,$$

where v^ϵ is the mollified function of v.

Recall that we assume u is the minimizer of E; then $\tilde{u}^k$ is denoted as the projected smoothed function for u. We also call $\tilde{u}^k$ the projected smoothed minimizer for E.

2.2.2 BV estimate

We estimate in this section the BV term of the discrete energy $E_k(\tilde{v}^k)$, i.e. we prove the consistency with respect to projection for the discrete TV operator $J_\star$.

Lemma 12

$$J_\star(\tilde{v}^k) \le \int_I |Dv^\epsilon| + Ch\int_I (|D_{xx}v^\epsilon| + |D_{yy}v^\epsilon|). \tag{2.22}$$

Proof Recall

$$\nabla_x^+ \tilde{v}_{i,j}^k = \frac{\tilde{v}_{i+1,j}^k - \tilde{v}_{i,j}^k}{h}, \qquad \nabla_y^+ \tilde{v}_{i,j}^k = \frac{\tilde{v}_{i,j+1}^k - \tilde{v}_{i,j}^k}{h},$$
$$\nabla_x^- \tilde{v}_{i,j}^k = \frac{\tilde{v}_{i,j}^k - \tilde{v}_{i-1,j}^k}{h}, \qquad \nabla_y^- \tilde{v}_{i,j}^k = \frac{\tilde{v}_{i,j}^k - \tilde{v}_{i,j-1}^k}{h}.$$

Then

$$\begin{aligned}
&\nabla_x^+ \tilde{v}_{i,j}^k - \frac{1}{h^2}\int_{I_{i,j}} D_x v^\epsilon \\
=&\frac{\tilde{v}_{i+1,j}^k - \tilde{v}_{i,j}^k}{h} - \frac{1}{h^2}\int_{I_{i,j}} D_x v^\epsilon \\
=&\frac{1}{h}\frac{1}{|I_{i,j}|}\int_{I_{i,j}} [v^\epsilon(x+h,y) - v^\epsilon(x,y)]\,dx\,dy - \frac{1}{h^2}\int_{I_{i,j}} D_x v^\epsilon\,.
\end{aligned}$$

The integrand of the first integral can be rewritten as an integral of Dv^ϵ, then combine these two integrals and once again rewrite the integrand as an integral of the second derivative of v^ϵ, we have,

$$\begin{aligned}
&\nabla_x^+ \tilde{v}_{i,j}^k - \frac{1}{h^2}\int_{I_{i,j}} D_x v^\epsilon \\
=&\frac{1}{h^3}\int_{I_{i,j}} \left(\int_0^h D_x v^\epsilon(x+t,y)\,dt - hD_x v^\epsilon(x,y)\right)\,dx\,dy \\
=&\frac{1}{h^3}\int_{I_{i,j}}\int_0^h (D_x v^\epsilon(x+t,y) - D_x v^\epsilon(x,y))\,dt\,dx\,dy \\
=&\frac{1}{h^3}\int_{I_{i,j}}\int_0^h\int_0^t D_{xx} v^\epsilon(x+s,y)\,ds\,dt\,dx\,dy\,.
\end{aligned}$$

Therefore

$$\begin{aligned}
\nabla_x^+ \tilde{v}_{i,j}^k = \frac{1}{h^2}\int_{I_{i,j}} D_x v^\epsilon + \\
\frac{1}{h^3}\int_{I_{i,j}}\int_0^h\int_0^t D_{xx} v^\epsilon(x+s,y)\,ds\,dt\,dx\,dy\,.
\end{aligned}$$

Similarly,

$$\nabla_y^+ \tilde{v}_{i,j}^k = \frac{1}{h^2}\int_{I_{i,j}} D_y v^\epsilon + \\ \frac{1}{h^3}\int_{I_{i,j}}\int_0^h\int_0^t D_{yy}v^\epsilon(x, y+s)\,ds\,dt\,dx\,dy\,.$$

So we can bound the norm of $\nabla^+\tilde{v}_{i,j}^k$ by

$$\begin{aligned} |\nabla^+\tilde{v}_{i,j}^k| \le \frac{1}{h^2}&\left|\begin{pmatrix} \int_{I_{i,j}} D_x v^\epsilon \\ \int_{I_{i,j}} D_y v^\epsilon \end{pmatrix}\right| + \\ &\frac{1}{h^3}\left|\begin{pmatrix} \int_{I_{i,j}}\int_0^h\int_0^t D_{xx}v^\epsilon(x+s, y)\,ds\,dt\,dx\,dy \\ \int_{I_{i,j}}\int_0^h\int_0^t D_{yy}v^\epsilon(x, y+s)\,ds\,dt\,dx\,dy \end{pmatrix}\right| \\ \le \frac{1}{h^2}&\int_{I_{i,j}} |Dv^\epsilon| + \frac{1}{h^3}\int_{I_{i,j}}\int_0^h\int_0^t |D_{xx}v^\epsilon(x+s, y)|\,ds\,dt\,dx\,dy \\ &\frac{1}{h^3}\int_{I_{i,j}}\int_0^h\int_0^t |D_{yy}v^\epsilon(x, y+s)|\,ds\,dt\,dx\,dy\,. \end{aligned} \tag{2.23}$$

The last line follows from the fact that

$$\left|\begin{pmatrix} \int f \\ \int g \end{pmatrix}\right| \le \int \sqrt{f^2+g^2}$$

by Jensen's inequality, and

$$\left|\begin{pmatrix} a \\ b \end{pmatrix}\right| \le |a| + |b|\,.$$

To bound the discrete total variation $J_{++}(\tilde{v}^k)$, we should sum (2.23) over all indexes (i, j) with weight h^2 at each index. We obtain

$$J_{++}(\tilde{v}^k) \le \int_I |Dv^\epsilon| + e_1 + e_2,$$

where

$$
\begin{aligned}
e_1 &= \sum_{i,j} h^2 \frac{1}{h^3} \int_{I_{i,j}} \int_0^h \int_0^t |D_{xx} v^\epsilon (x+s, y)| \, ds \, dt \, dx \, dy \\
&\leq \frac{1}{h} \int_0^h \int_0^t \left\{ \int_I |D_{xx} v^\epsilon (x+s, y)| \, dx \, dy \right\} ds \, dt \\
&\leq \frac{1}{h} \int_0^h \int_0^t \left\{ \int_{2I} |D_{xx} v^\epsilon| \, dx \, dy \right\} ds \, dt \\
&\leq Ch \int_I |D_{xx} v^\epsilon| \, .
\end{aligned}
$$

We also have

$$
\begin{aligned}
e_2 &= \sum_{i,j} h^2 \frac{1}{h^3} \int_{I_{i,j}} \int_0^h \int_0^t |D_{yy} v^\epsilon (x, y+s)| \, ds \, dt \, dx \, dy \\
&\leq Ch \int_I |D_{yy} v^\epsilon| \, .
\end{aligned}
$$

Therefore

$$
J_{++}(\tilde{v}^k) \leq \int_I |Dv^\epsilon| + Ch \int_I (|D_{xx} v^\epsilon| + |D_{yy} v^\epsilon|) \, .
$$

By the same argument, we have the same bound for J_{+-}, J_{-+}, and J_{--},

$$
J(\tilde{v}^k) \leq \int_I |Dv^\epsilon| + Ch \int_I (|D_{xx} v^\epsilon| + |D_{yy} v^\epsilon|),
$$

where $J \in \{J_{+-}, J_{-+}, J_{--}\}$. Thus, we complete the proof. ■

Remark In fact, we have proved that for any convex linear combination of the operators J_{++}, J_{+-}, J_{-+}, J_{--},

$$
\begin{aligned}
& J = \lambda_1 J_{++} + \lambda_2 J_{+-} + \lambda_3 J_{-+} + \lambda_4 J_{--}, \\
& \lambda_1, \lambda_2, \lambda_3, \lambda_4 \geq 0, \\
& \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 1,
\end{aligned}
$$

the same result holds:

$$
J(\tilde{v}^k) \leq \int_I |Dv^\epsilon| + Ch \int_I (|D_{xx} v^\epsilon| + |D_{yy} v^\epsilon|) \, .
$$

Similar to the discrete case, we also have the following result.

Lemma 13 *For any* $v \in \mathrm{BV}(I)$,

$$\int_I |Dv^\epsilon| \leq \int_I |Dv|.$$

Proof The proof follows directly the one given in the discrete case in Lemma 5. ■

Now we prove the consistency with respect to projection.

Lemma 14 (Consistency with respect to projection)

$$J_\star(\tilde{v}^k) \leq \int_I |Dv| + \frac{Ch}{\epsilon} \int_I |Dv|.$$

Proof It is clear we only need to bound

$$\int_I (|D_{xx}v^\epsilon| + |D_{yy}v^\epsilon|)$$

in (2.22). We now prove

$$\int_I |D_{xx}v^\epsilon| \leq \frac{C}{\epsilon} \int_I |Dv|\,.$$

In fact

$$\begin{aligned}
\int_I |D_{xx}v^\epsilon| &= \sup_{|\phi|\leq 1;\ \phi\in C_0^\infty(I)} \int D_{xx}v^\epsilon \cdot \phi \\
&= \sup_{|\phi|\leq 1;\ \phi\in C_0^\infty(I)} \int D_x v^\epsilon D_x\phi \\
&= \sup_{|\phi|\leq 1;\ \phi\in C_0^\infty(I)} \int (D_x\eta_\epsilon * v) D_x\phi \\
&= \sup_{|\phi|\leq 1;\ \phi\in C_0^\infty(I)} \int v D_x(-D_x\eta_\epsilon(x,y) * \phi).
\end{aligned}$$

Notice

$$\begin{aligned}
|-D_x\eta_\epsilon(x,y) * \phi| &\leq \|D_x\eta_\epsilon(x,y)\|_{L^1}\|\phi\|_\infty \\
&\leq \frac{C}{\epsilon},
\end{aligned}$$

and $-D_x\eta_\epsilon(x,y) * \phi \in C_0^\infty(I^\epsilon)$, where

$$I^\epsilon := \{x : dist(x, \bar{I}) \leq \epsilon\},$$

therefore

$$\begin{aligned}\int_I |D_{xx}v^\epsilon| &\leq \frac{C}{\epsilon}\int_{I^\epsilon} |Dv| \\ &\leq \frac{C}{\epsilon}\int_{9I} |Dv| \leq \frac{C}{\epsilon}\int_I |Dv|\,.\end{aligned}$$

The same result also holds for $\int_I |D_{yy}v^\epsilon|$. Applying these results and Lemma 13 to (2.22), we have the result. ■

Recall that we assume $\tilde{u}^k$ to be the discretization of u^ϵ, where u^ϵ is the mollified minimizer u of the continuous energy E. Applying Lemma 14, we have

$$J_\star(\tilde{u}^k) \leq \int_I |Du| + \frac{Ch}{\epsilon}\int_I |Du|\,. \tag{2.24}$$

Noticing that

$$\int_I |Du| \leq E(u) \leq E(0) \leq \frac{1}{2\lambda}\|g\|^2,$$

we obtain

$$J_\star(\tilde{u}^k) \leq \int_I |Du| + \frac{Ch}{\lambda\epsilon}\|g\|^2\,. \tag{2.25}$$

This completes the estimation of the total variation term in $E_k(\tilde{u}^k)$.

2.2.3 L^2 estimate

Finally we bound the L^2 term

$$\|\tilde{u}^k - g^k\|$$

in

$$E_k(\tilde{u}^k) = J_\star(u^k) + \frac{1}{2\lambda}\|\tilde{u}^k - g^k\|^2\,.$$

Similar to the first part, we prove

$$\|\tilde{u}^k - g^k\|^2 \leq \|u - g\|^2 + \text{some error}.$$

Expand the L^2 term into a sum of collapsing terms,

$$\begin{aligned}\|\tilde{u}^k - g^k\|^2 &= \|\tilde{u}^k - u + u - g + g - g^k\|^2 \\ &\leq \|u - g\|^2 + 2\|u - g\|\|\tilde{u}^k - u + g - g^k\| + \|\tilde{u}^k - u + g - g^k\|^2.\end{aligned}$$

Notice

$$\begin{aligned}\|\tilde{u}^k - u + g - g^k\| &= \|\tilde{u}^k - u^\epsilon + u^\epsilon - u + g - g^k\| \\ &\leq \|\tilde{u}^k - u^\epsilon\| + \|u^\epsilon - u\| + \|g - g^k\|,\end{aligned}$$

and

$$\begin{aligned}\|\tilde{u}^k - u + g - g^k\|^2 &= \|\tilde{u}^k - u^\epsilon + u^\epsilon - u + g - g^k\|^2 \\ &\leq 3(\|\tilde{u}^k - u^\epsilon\|^2 + \|u^\epsilon - u\|^2 + \|g - g^k\|^2).\end{aligned}$$

Thus

$$\begin{aligned}\|\tilde{u}^k - g^k\|^2 \leq \|u - g\|^2 &+ 2\|u - g\| \cdot (\|\tilde{u}^k - u^\epsilon\| + \|u^\epsilon - u\| + \|g - g^k\|) \\ &+ 3(\|\tilde{u}^k - u^\epsilon\|^2 + \|u^\epsilon - u\|^2 + \|g - g^k\|^2).\end{aligned} \tag{2.26}$$

We shall bound four terms:

$$\|\tilde{u}^k - u^\epsilon\|, \|u^\epsilon - u\|, \|u - g\|, \|g - g^k\|\,.$$

We have already obtained the bound for $\|g - g^k\|$ previously,

$$\|g - g^k\| \leq C|g|_{\mathrm{Lip}(\alpha, L^2)} h^\alpha\,. \tag{2.27}$$

Also,

$$\frac{1}{2\lambda}\|u - g\|^2 \leq E(u) \leq E(0) = \frac{1}{2\lambda}\|g\|^2,$$

thus

$$\|u - g\| \leq \|g\|\,. \tag{2.28}$$

We only need to bound the first two terms.

Lemma 15 *If $g \in \mathrm{Lip}(\alpha, L^2)$, then*

$$\|\tilde{u}^k - u^\epsilon\| \le C|g|_{\mathrm{Lip}(\alpha,L^2)} h^\alpha.$$

Proof We have,

$$\begin{aligned}
\|\tilde{u}^k - u^\epsilon\|^2 &= \sum_{i,j} \int_{I_{i,j}} |\tilde{u}^k_{i,j} - u^\epsilon|^2 \\
&\le \sum_{i,j} \int_{I_{i,j}} \left| \frac{1}{|I_{i,j}|} \int_{I_{i,j}} |u^\epsilon(y) - u^\epsilon(x)|\, dy \right|^2 dx \\
&\text{(Apply Holder's inequality)} \\
&\le \sum_{i,j} \int_{I_{i,j}} \frac{1}{|I_{i,j}|} \int_{|z| \le \sqrt{2}h} |u^\epsilon(x+z) - u^\epsilon(x)|^2 dx\, dz \\
&\le \frac{1}{h^2} \int_{|z| \le \sqrt{2}h} \int_I |u^\epsilon(x+z) - u^\epsilon(x)|^2 dx \\
&\le C\omega_1^2(u^\epsilon, \sqrt{2}h)_{L^2} \\
&\le C\omega_1^2(u, \sqrt{2}h)_{L^2}\,.
\end{aligned}$$

By continuity of translation property (1.12),

$$\omega_1(u,h)_{L^2} \le C\omega_1(g,h)_{L^2}\,.$$

Thus

$$\begin{aligned}
\|\tilde{u}^k - u^\epsilon\| &\le C\omega_1(g,h)_{L^2} \\
&\le C|g|_{\mathrm{Lip}(\alpha,L^2)} h^\alpha.
\end{aligned}$$

■

Lemma 16 *If $g \in \mathrm{Lip}(\alpha, L^2)$, then*

$$\|u^\epsilon - u\| \le C|g|_{\mathrm{Lip}(\alpha,L^2)} \epsilon^\alpha.$$

Proof We have,

$$
\begin{aligned}
\|u^\epsilon - u\|^2 &= \int_I |u^\epsilon - u|^2 \\
&= \int_I \left| \int_{B(0,\epsilon)} \eta_\epsilon(y)(u(x-y) - u(x))\, dy \right|^2 dx \\
&\le \int_I \left(\int_{B(0,\epsilon)} \eta_\epsilon^2(y)\, dy \cdot \int_{B(0,\epsilon)} |u(x-y) - u(x)|^2\, dy \right) dx \\
&\le \frac{1}{\epsilon^2} \int_{B(0,\epsilon)} \int_I |u(x-y) - u(x)|^2 dx\, dy \\
&\le C\omega_1^2(u,\epsilon)_{L^2} \le C\omega_1^2(g,\epsilon)_{L^2} \le C|g|^2_{\mathrm{Lip}(\alpha,L^2)}\epsilon^{2\alpha}.
\end{aligned}
$$

■

With all the estimates given, we can bound the L^2 term.

Lemma 17

$$
\|\tilde{u}^k - g^k\|^2 \le \int_I |u-g|^2 + C|g|^2_{\mathrm{Lip}(\alpha,L^2)}h^{2\alpha} + C|g|^2_{\mathrm{Lip}(\alpha,L^2)}\epsilon^{2\alpha} +
$$
$$
C|g|^2_{\mathrm{Lip}(\alpha,L^2)}h^\alpha\epsilon^\alpha + C\|g\||g|_{\mathrm{Lip}(\alpha,L^2)}h^\alpha + C\|g\||g|_{\mathrm{Lip}(\alpha,L^2)}\epsilon^\alpha.
$$

Proof We collect the inequalities we need here.

From (2.27),

$$
\|g - g^k\| \le C|g|_{\mathrm{Lip}(\alpha,L^2)}h^\alpha\,;
$$

From (2.28),

$$
\|u - g\| \le \|g\|\,;
$$

From Lemma 15,

$$
\|\tilde{u}^k - u^\epsilon\| \le C|g|_{\mathrm{Lip}(\alpha,L^2)}h^\alpha;
$$

From Lemma 16,

$$
\|u^\epsilon - u\| \le C|g|_{\mathrm{Lip}(\alpha,L^2)}\epsilon^\alpha.
$$

Apply them to (2.26), the result follows.

■

2.2.4 Proof of Proposition 2

At this point, we can establish the second main energy bound.

Proof of Proposition 2 By (2.25) and Lemma 17,

$$
\begin{aligned}
E_k(\tilde{u}^k) &= J_\star(\tilde{u}^k) + \frac{1}{2\lambda}\|\tilde{u}^k - g^k\|^2 \\
&\leq \int_I |Du| + \frac{C\|g\|^2 h}{\lambda\epsilon} + \frac{1}{2\lambda}\int_I |u-g|^2 + \\
&\qquad \frac{C}{2\lambda}(|g|^2_{\mathrm{Lip}(\alpha,L^2)} h^{2\alpha} + |g|^2_{\mathrm{Lip}(\alpha,L^2)}\epsilon^{2\alpha} + |g|^2_{\mathrm{Lip}(\alpha,L^2)} h^\alpha\epsilon^\alpha + \\
&\qquad \|g\||g|_{\mathrm{Lip}(\alpha,L^2)} h^\alpha + \|g\||g|_{\mathrm{Lip}(\alpha,L^2)}\epsilon^\alpha).
\end{aligned}
$$

The terms with the smallest powers in h and ϵ (without considering coefficients) are

$$
\frac{h}{\epsilon}, \quad h^\alpha, \quad \epsilon^\alpha .
$$

Let $\epsilon = h^\gamma$, these terms become

$$
h^{1-\gamma}, \quad h^\alpha, \quad \epsilon^{\gamma\alpha} .
$$

Noticing $\gamma\alpha \leq \alpha$, let $1-\gamma = \gamma\alpha$ to get

$$
\gamma = \frac{1}{\alpha+1} .
$$

Then

$$
\begin{aligned}
E_k(u^k) &\leq E_k(\tilde{u}^k) \\
&\leq \int_I |Du| + \frac{1}{2\lambda}\int_I |u-g|^2 + \\
&\qquad \frac{C}{\lambda}(|g|^2_{\mathrm{Lip}(\alpha,L^2)} + \|g\||g|_{\mathrm{Lip}(\alpha,L^2)} + \|g\|^2) h^{\alpha/(\alpha+1)} \\
&\leq E(u) + \frac{C}{\lambda}\|g\|^2_{\mathrm{Lip}(\alpha,L^2)} h^{\alpha/(\alpha+1)}.
\end{aligned}
$$

2.3 Error estimate of the discrete minimizer

Our main theorem follows immediately from Proposition 1 and Proposition 2.

Theorem 2.3.1 *If $g \in \mathrm{Lip}(\alpha, L^2(I))$, u^k is the minimizer of E_k and u is the minimizer of E, then*

$$|E(u) - E_k(u^k)| \leq \frac{C}{\lambda}\|g\|^2_{\mathrm{Lip}(\alpha,L^2)} h^{\alpha/(\alpha+1)}.$$

Remark For $g \in \mathrm{BV}(I)$, we have $\alpha = 1/2$, so $\alpha/(\alpha+1) = 1/3$; thus the order of energy difference is at most $h^{1/3}$.

Finally we give the estimate of error between discrete minimizer and the minimizer in L^2. We need the following lemma.

Lemma 18 *If u is the minimizer of the functional E, then for any $v \in \mathrm{BV}$,*

$$\|v - u\|^2 \leq 2\lambda(E(v) - E(u)). \tag{2.29}$$

Proof Let $J(u)$ be the total variation of u,

$$J(u) = \int_I |Du|.$$

By the definition of E,

$$E(v) - E(u) = J(v) - J(u) + \frac{1}{2\lambda}(\|v - g\|^2 - \|u - g\|^2).$$

Since u is the minimizer, $(g - u)/\lambda \in \partial J(u)$, i.e. for any v,

$$\left(\frac{g - u}{\lambda}, v - u\right) \leq J(v) - J(u),$$

then

$$\begin{aligned} E(v) - E(u) \geq \ & \big(\frac{g-u}{\lambda}, v - u\big) + \frac{1}{2\lambda}(\|v - g\|^2 - \|u - g\|^2) \\ = \ & \big(\frac{g-u}{\lambda}, v - u\big) + \frac{1}{2\lambda}(\|v - u\|^2 + 2(v - u, u - g)) \\ = \ & \frac{1}{2\lambda}\|v - u\|^2. \end{aligned}$$

■

Theorem 2.3.2 *If u is the minimizer of the functional E and u^k is the minimizer of the functional E_k, then*

$$\|u^k - u\|^2 \leq C\|g\|^2_{\mathrm{Lip}(\alpha,L^2)} h^{\alpha/(\alpha+1)}.$$

Proof Let $\bar{u}^L$ be the piecewise linear function defined in the first approach,

$$\begin{aligned}\|\bar{u}^L - u\|^2 &\leq 2\lambda(E(\bar{u}^L) - E(u)) \\ &\leq 2\lambda\big[(E_k(u^k) + \frac{C}{\lambda}\|g\|^2_{\mathrm{Lip}(\alpha,L^2)}h^{\alpha/(\alpha+1)}) \\ &\quad + (-E_k(u^k) + \frac{C}{\lambda}\|g\|^2_{\mathrm{Lip}(\alpha,L^2)}h^{\alpha/(\alpha+1)})\big].\end{aligned}$$

The substitution for the first term is by (2.21); the substitution for the second term is by Proposition 2. Then clearly

$$\|\bar{u}^L - u\|^2 \leq C\|g\|^2_{\mathrm{Lip}(\alpha,L^2)}h^{\alpha/(\alpha+1)}.$$

Thus by Lemma 9 and Lemma 10,

$$\begin{aligned}\|u^k - u\|^2 &= \|u^k - u^{L,k} + u^{L,k} - \bar{u}^L + \bar{u}^L - u\|^2 \\ &\leq 3(\|u^{L,k} - u^k\|^2 + \|u^{L,k} - \bar{u}^L\|^2 + \|\bar{u}^L - u\|^2) \\ &\leq C\|g\|^2_{\mathrm{Lip}(\alpha,L^2)}h^{\alpha/(\alpha+1)}.\end{aligned}$$

■

3. ERROR BOUND FOR A NON-SYMMETRIC DISCRETE TV OPERATOR

In this chapter, we study the error bound for the discrete energy E_k,

$$E_k(u^k) = J_k(u^k) + \frac{1}{2\lambda}\|u^k - g^k\|^2, \tag{3.1}$$

where J_k is a special case of non-symmetric discrete TV operator

$$J_k(u^k) = \frac{1}{2}(J_{++}(u^k) + J_{--}(u^k)).$$

The consistency property for this discrete TV operator is particularly easy to prove based on the following observation.

$$\begin{aligned}\int_I |D\bar{u}| &= \frac{1}{2}(J_{++}(u^k) + J_{--}(u^k)) \\ &= J_k(u^k),\end{aligned} \tag{3.2}$$

where $\bar{u}$ is the piecewise linear interpolation of u^k,

$$\bar{u} = \sum_{i,j=0}^{k-1} u^k_{i,j}\phi^k_{i,j},$$

$\phi^k_{i,j}$ is the basis function defined in (2.1) in Chapter 2. By (3.2), we immediately have consistency with respect to injection for J_k,

$$J(\bar{u}) \leq J_k(u^k).$$

However, it is difficult to prove the consistency property for the non-symmetric operator J_{++} with the technique introduced in Chapter 2. This is because although we can also smooth u^k first and inject the smoothed function into L^2 to get $\bar{u}^{L,k}$, we are not able to apply the idea of the proof of Lemma 5 to prove the similar result for J_{++},

$$J_{++}(\bar{u}^{L,k}) \leq J_{++}(u^k),$$

due to the dependence on the symmetry of the operator $J_\star$ in our proof.

We assume throughout this chapter that u is the minimizer of the continuous energy E and u^k is the minimizer of the discrete energy E_k when there is no ambiguity. The value of u^k at pixel (i,j) is denoted by $u^k_{i,j}$, with $0 \le i,j \le k-1$.

The proof of the error bound for the non-symmetric J_k shares most parts with the proof for the symmetric $J_\star$ in the previous chapter. First, It doesn't need any adaption on the proof of Proposition 2 to prove the following result.

Proposition 3 *If $g \in \mathrm{Lip}(\alpha, L^2(I))$ and u^k is the minimizer of E_k, then*

$$E_k(u^k) \le E(u) + \frac{C}{\lambda}\|g\|^2_{\mathrm{Lip}(\alpha,L^2)} h^{\alpha/(\alpha+1)}.$$

Corollary 1 *If $g \in L^\infty \cap \mathrm{Lip}(\beta, L^1(I))$ and u^k is the minimizer of E_k, then*

$$E_k(u^k) \le E(u) + \frac{C}{\lambda}(\|g\|^2 + \|g\|_\infty |g|_{\mathrm{Lip}(\beta,L^1)}) h^{\beta/(\beta+2)}.$$

Proof We only need to point out that $g \in L^\infty(I) \cap \mathrm{Lip}(\beta, L^1(I))$ implies $g \in \mathrm{Lip}(\beta/2, L^2(I))$. In fact, for any $w \in R^2$, $|w| \le h$,

$$\begin{aligned}
\int_{I_h} |g(x+w) - g(x)|^2 &\le \sup_{I_h} |g(x+w) - g(x)| \int_{I_h} |g(x+w) - g(x)| dx \\
&\le C \sup_I |g|\, \omega_1(g,h)_{L^1} \\
&\le C\|g\|_\infty |g|_{\mathrm{Lip}(\beta,L^1)} h^\beta,
\end{aligned}$$

where I_h is as defined in (1.10), thus

$$\begin{aligned}
\omega_1(g,h)_{L^2} &= \sup_{0<|w|\le h} \left\{ \int_{I_h} |g(x+w) - g(x)|^2 \right\}^{1/2} \\
&\le C\|g\|_\infty^{1/2} |g|^{1/2}_{\mathrm{Lip}(\beta,L^1)} h^{\beta/2},
\end{aligned}$$

and

$$\begin{aligned}
\|g\|_{\mathrm{Lip}(\beta/2,L^2)} &= \sup_{h>0} (h^{-\beta/2} \omega_1(g,h)_{L^2}) \\
&\le C\|g\|_\infty^{1/2} |g|^{1/2}_{\mathrm{Lip}(\beta,L^1)}.
\end{aligned}$$

■

We prove in the next section the bound in the other direction.

Proposition 4 *If $g \in L^\infty \cap \mathrm{Lip}(\beta, L^1(I))$, then*

$$E(u) \leq E(u^k) + \frac{C}{\lambda}(\|g\|^2 + \|g\|_\infty |g|_{\mathrm{Lip}(\beta,L^1)})h^{\beta/2}. \tag{3.3}$$

3.1 Estimate of the energy of the injected minimizer

Unlike the symmetric case, we do not use "smoothed" discrete function; instead we interpolate directly the discrete function to obtain its injection into L^2.

Definition 3.1.1 *Define $\bar{v}$ to be the injected function of v^k if*

$$\bar{v} = \sum_{i,j=0}^{k-1} v^k_{i,j}\phi^k_{i,j},$$

where $\phi^k_{i,j}$ is the basis function defined in (2.1) in chapter 2.

Assuming u^k is the minimizer for E_k, we now bound the energy $E(\bar{u})$.

For the BV term, by (2.2) in section 2.1.1 we have

$$\begin{aligned}\int_I |D\bar{u}| &= \frac{1}{2}(J_{++}(u^k) + J_{--}(u^k)) \\ &= J_k(u^k)\,.\end{aligned}$$

We use it directly as the BV estimate.

To estimate the L^2 term in $E(\bar{u})$, we expand it into a collapsing sum.

$$\begin{aligned}\|\bar{u} - g\|^2 &= \|\bar{u} - u^k + u^k - g^k + g^k - g\|^2 \\ &\leq \|u^k - g^k\|^2 + 2\|u^k - g^k\|\|\bar{u} - u^k + g^k - g\| + \|\bar{u} - u^k + g^k - g\|^2 \\ &\leq \|u^k - g^k\|^2 + 2\|u^k - g^k\|(\|\bar{u} - u^k\| + \|g^k - g\|) \\ &\quad + 2(\|\bar{u} - u^k\|^2 + \|g^k - g\|^2).\end{aligned} \tag{3.4}$$

Thus we shall bound $\|\bar{u}-u^k\|$, $\|g^k-g\|$ and $\|u^k-g^k\|$. By Lemma 11 in section 2.1.3, we have

$$\|u^k-g^k\|\le\|g\|.$$

We also need a slightly modified version of Lemma 7 of section 2.1.3 to bound $\|g^k-g\|$.

Lemma 19 *If $g\in L^\infty(I)\cap \mathrm{Lip}(\beta,L^1(I))$, then*

$$\|g^k-g\|\le C\|g\|_\infty^{1/2}|g|_{\mathrm{Lip}(\beta,L^1)}^{1/2}h^{\beta/2}.$$

Proof As shown in the proof of Corollary 1, $g\in L^\infty(I)\cap \mathrm{Lip}(\beta,L^1(I))$ implies that $g\in \mathrm{Lip}(\beta/2,L^2(I))$ and

$$|g|_{\mathrm{Lip}(\beta/2,L^2(I))}\le C\|g\|_\infty^{1/2}|g|_{\mathrm{Lip}(\beta,L^1)}^{1/2}.$$

By applying Lemma 7 in section 2.1.3, the result follows. ■

Finally we bound $\|\bar{u}-u^k\|$.

Lemma 20 *If u^k is the minimizer of E_k and $g\in \mathrm{Lip}(\beta,L^1(I))\bigcap L^\infty(I)$, then*

$$\|\bar{u}-u^k\|\le C\|g\|_\infty^{1/2}|g|_{\mathrm{Lip}(\beta,L^1)}^{1/2}h^{\beta/2}.$$

Proof Let $\tilde{u}=\sum_{i',j'}u_{i,j}^k\phi_{i',j'}^k$, where the sum is taken over all (i',j') such that supp $\phi_{i',j'}^k$ intersects supp $\phi_{i,j}^k$ non-trivially. We have shown previously the formula (2.18),

$$\int_I |u^k-\bar{u}|^2\le 36\sum_{i,j}(|u_{i+1,j}^k-u_{i,j}^k|^2+|u_{i,j+1}^k-u_{i,j}^k|^2)h^2,$$

Thus,

$$\begin{aligned}\int_I |u^k-\bar{u}|^2&\le 36\sum_{i,j}(|u_{i+1,j}^k-u_{i,j}^k|^2+|u_{i,j+1}^k-u_{i,j}^k|^2)h^2\\&\le C\|u^k\|_\infty\cdot h\cdot J_a(u^k),\end{aligned}\tag{3.5}$$

where $J_a(u^k)$ is the discretization of the anisotropic TV defined in (1.7) in section 1.1. Therefore,

$$
\begin{aligned}
\|\bar{u} - u^k\| &\le C\{\|u^k\|_\infty \cdot h \cdot J_a(u^k)\}^{1/2} \\
&\le C\|g^k\|_\infty^{1/2} h^{1/2} J_a^{1/2}(u^k) \;\; \text{(by maximum principle Lemma 3)} \\
&\le C\|g\|_\infty^{1/2}\omega_1^{1/2}(g,h)_{L^1} \quad \text{(by (2.15))} \\
&\le C\|g\|_\infty^{1/2}|g|_{\mathrm{Lip}(\beta,L^1)}^{1/2} h^{\beta/2}.
\end{aligned}
$$

■

Now we can prove Proposition 4.

Proof By (2.2), we have

$$\int_I |D\bar{u}| = J_k(u^k).$$

Since u is the minimizer of E,

$$
\begin{aligned}
E(u) &\le E(\bar{u}) \\
&= \int_I |D\bar{u}| + \frac{1}{2\lambda}\|\bar{u} - g\|^2 \\
&= J_k(u^k) + \frac{1}{2\lambda}\|\bar{u} - g\|^2 \\
&= J_k(u^k) + \frac{1}{2\lambda}\|\bar{u} - u^k + u^k - g^k + g^k - g\|^2 \\
&\le J_k(u^k) + \frac{1}{2\lambda}\|u^k - g^k\|^2 + \frac{1}{\lambda}\|u^k - g^k\|\|\bar{u} - u^k + g^k - g\| \\
&\quad + \frac{1}{2\lambda}\|\bar{u} - u^k + g^k - g\|^2 \\
&\le E_k(u^k) + \frac{1}{\lambda}\|u^k - g^k\|(\|\bar{u} - u^k\| + \|g^k - g\|) \\
&\quad + \frac{1}{\lambda}(\|\bar{u} - u^k\|^2 + \|g^k - g\|^2) \quad \text{(by (3.4))}.
\end{aligned}
$$

By applying Lemma 7, Lemma 19 and Lemma 20 to the last inequality, we obtain

$$
\begin{aligned}
E(u) \le E_k(u^k) + \frac{C}{\lambda}(&\|g\|_\infty |g|_{\mathrm{Lip}(\beta,L^1)} h^\beta + \|g\|_\infty |g|_{\mathrm{Lip}(\beta,L^1)} h^\beta + \\
&\|g\|\|g\|_\infty^{1/2}|g|_{\mathrm{Lip}(\beta,L^1)}^{1/2} h^{\beta/2} + \|g\|\|g\|_\infty^{1/2}|g|_{\mathrm{Lip}(\beta,L^1)}^{1/2} h^{\beta/2})\,.
\end{aligned}
$$

When $h < 1$, the largest terms have order $\mathcal{O}(h^{\beta/2})$, thus

$$\begin{aligned}E(u) &\leq E_k(u^k) + \frac{C}{\lambda}(\|g\|\|g\|_\infty^{1/2}|g|_{\mathrm{Lip}(\beta,L^1)}^{1/2})h^{\beta/2}\\ &\leq E_k(u^k) + \frac{C}{\lambda}(\|g\|^2 + \|g\|_\infty|g|_{\mathrm{Lip}(\beta,L^1)})h^{\beta/2}.\end{aligned}$$

■

We combine Proposition 4 and Corollary 1, and notice

$$\beta/2 > \beta/(\beta+2),$$

to obtain the inequality (3.3) for the solution of (3.1).

Theorem 3.1.1 *If* $g \in \mathrm{Lip}(\beta, L^1) \cap L^\infty$ *and* u^k *is the minimizer of energy with a non-symmetric discrete* TV *operator* J_k*,*

$$J_k(u^k) = \frac{1}{2}(J_{++}(u^k) + J_{--}(u^k)),$$

and u *is the minimizer of* E*, then*

$$|E_k(u^k) - E(u)| \leq \frac{C}{\lambda}N(g)h^{\beta/(\beta+2)},$$

where

$$N(g) = \|g\|^2 + \|g\|_\infty|g|_{\mathrm{Lip}(\beta,L^1)}.$$

Along the same line as the argument in the proof of Theorem 2.2, we obtain the following error bound for the discrete minimizer.

Theorem 3.1.2 *If* u_k *is the minimizer of* E_k *defined in Theorem 3.1 and* u *is the minimizer of* E*, then*

$$\|u^k - u\| \leq C[N(g)]^{1/2}h^{\beta/2(\beta+2)},$$

where $N(g)$ *is as defined in* Theorem 3.1.

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VITA

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